

Simple stably projectionless C^* -algebras with generalized tracial rank one

George A. Elliott, Guihua Gong, Huaxin Lin and Zhuang Niu

Abstract. We study a class of stably projectionless simple C^* -algebras which may be viewed as having generalized tracial rank one in analogy with the unital case. A number of structural questions concerning these simple C^* -algebras are studied, pertinent to the classification of separable stably projectionless simple amenable Jiang–Su stable C^* -algebras.

Mathematics Subject Classification (2010). 46L35.

Keywords. Simple C^* -algebras, stably projectionless.

Contents

| | | |
|----|--|----|
| 1 | Introduction | 1 |
| 2 | Notation | 2 |
| 3 | Some results of Rørdam | 7 |
| 4 | Compact C^* -algebras | 15 |
| 5 | Continuous scale and fullness | 23 |
| 6 | Non-unital and non-commutative one dimensional complexes | 30 |
| 7 | Maps from 1-dimensional non-commutative complexes | 34 |
| 8 | Tracially one-dimensional complexes | 44 |
| 9 | Traces and comparison for C^* -algebras in the class \mathcal{D} | 57 |
| 10 | Tracial approximate divisibility | 64 |
| 11 | Stable rank one | 81 |
| 12 | The C^* -algebras \mathcal{W} and the class \mathcal{D}_0 | 91 |
| | References | 94 |

1. Introduction

Recent developments in the program of classification of simple amenable C^* -algebras led to the classification of unital simple separable C^* -algebras with finite nuclear

dimension in the UCT class (see, for example, [16, 22], and [52]). The isomorphism theorem was established first for those unital simple separable amenable C^* -algebras with generalized tracial rank at most one (see [22]). Unital simple C^* -algebras with generalized tracial rank at most one have good regularity properties. For example, these simple C^* -algebras have strict comparison for positive elements, have stable rank one, and are quasidiagonal. All quasitraces are traces. When they are separable and amenable, they are \mathcal{Z} -stable. These C^* -algebras have many other good properties which lead to the classification by the Elliott invariant when they satisfy the UCT. In [16], using [52], we showed that unital finite simple (separable) C^* -algebras with finite nuclear dimension which satisfy the UCT have generalized tracial rank at most one after tensoring with a UHF-algebra of infinite type. This completed the classification of unital simple separable C^* -algebras with finite nuclear dimension in the UCT class. In the present paper, we study the non-unital version of the notion of generalized tracial rank one.

Acknowledgements. The greater part of this research was done while the second and third named authors were at the Research Center for Operator Algebras at East China Normal University in the Summer of 2016. Much of the revision from the initial work was done when the last three authors were there in the Summer of 2017. All authors acknowledge the support of the Research Center which is partially supported by Shanghai Key Laboratory of PMMP, Science and Technology Commission of Shanghai Municipality (STCSM), grant #13dz2260400 and a NNSF grant (11531003). The first named author was partially supported by NSERC of Canada. The second named author was partially supported by the NNSF of China (Grant #11531003), the third named author was partially supported by NSF grants (DMS #1361431 and #1665183), and the fourth named author was partially supported by a Simons Collaboration Grant (Grant #317222) and an NSF grant (DMS-1800882).

2. Notation

Definition 2.1. Let A be a C^* -algebra. Denote by $\text{Ped}(A)$ the Pedersen ideal (see [38, Section 5.6]).

Denote by $\tilde{\mathbb{T}}(A)$ the topological convex cone of all densely defined lower semicontinuous positive traces equipped with the topology of point-wise convergence on elements of $\text{Ped}(A)$. Recall that, if $\tau \in \tilde{\mathbb{T}}(A)$ and $b \in \text{Ped}(A)_+$, then τ is a finite (equivalently, bounded) trace on \overline{bAb} . One checks easily that $\tilde{\mathbb{T}}(A)$ is a Hausdorff space.

Let $\mathbb{T}(A)$ denote the tracial state space of A . Set

$$\mathbb{T}_1(A) = \{\tau \in \tilde{\mathbb{T}}(A) : |\tau(a)| \leq 1, a \in \text{Ped}(A), \text{ and } \|a\| \leq 1\}.$$

If $\tau \in T_1(A)$, then it can be extended to a positive linear functional on A with norm at most one. Therefore, we may view $T_1(A)$ as a subset of the unit ball of A^* : $T_1(A)$ is the weak*-compact convex subset of all tracial positive linear functionals on A with norm at most one, and is a Choquet simplex. In this way, we may write that $T(A) \subset T_1(A)$. $T_1(A)$ is also a closed set in $\tilde{T}(A)$. If $A = \text{Ped}(A)$, then $T(A)$ generates $\tilde{T}(A)$ as a cone.

Suppose that A is σ -unital. In the case that $\text{Ped}(A)_+$ contains a full element a of A (in particular when A is simple), let us clarify the structure of $\tilde{T}(A)$. Put $A_1 = \overline{aAa}$. Then we may identify A with a σ -unital hereditary sub-C*-algebra of $A_1 \otimes \mathcal{K}$ by Brown's theorem [3]. By 2.1 of [51], $A_1 = \text{Ped}(A_1)$. Therefore, $T_1(A_1)$ generates $\tilde{T}(A) = \tilde{T}(A_1)$ as a cone.

Consider the closure S of $T(A)$ in $\tilde{T}(A)$ in the above mentioned topology. Let $\overline{T(A)}^w$ denote the weak* closure of $T(A)$ in the dual space of A . Clearly, as a set, $\overline{T(A)}^w \subset S$. Note that each element in $\overline{T(A)}^w$ is a trace of A with norm at most one. Let $\iota: \overline{T(A)}^w \rightarrow S$ be the embedding as a subset. So the map ι is one-to-one. Suppose $t \in S$. Then, t is a densely defined linear functional and $|t(b)| \leq 1$ for all $b \in \text{Ped}(A)$ with $\|b\| \leq 1$. Thus, it uniquely extends to an element of A^* with norm at most one. Choose a net (τ_α) in $T(A)$ such that $\tau_\alpha(b) \rightarrow t(b)$ for all $b \in \text{Ped}(A)$. Since $\text{Ped}(A)$ is dense in A , and $\|\tau_\alpha\| = 1$ and $\|t\| \leq 1$, one concludes that $\tau_\alpha(a) \rightarrow t(a)$ for all $a \in A$. This shows that $S = \overline{T(A)}^w$ as subsets. In other words, ι is a bijection. On the other hand, if $\tau_\beta(a) \rightarrow \tau(a)$ for all $a \in A$, where $\tau_\beta, \tau \in \overline{T(A)}^w$, then $\tau_\beta(b) \rightarrow \tau(b)$ for all $b \in \text{Ped}(A)$. In other words, ι is continuous. Moreover, $\overline{T(A)}^w \subset A^*$ is compact and $S \subset \tilde{T}(A)$ is Hausdorff. It follows that ι is a homeomorphism. In what follows, we will identify S with $\overline{T(A)}^w$. In particular, S is compact.

Definition 2.2. Let $1 > \varepsilon > 0$. Define

$$f_\varepsilon(t) = \begin{cases} 0, & \text{if } t \in [0, \varepsilon/2], \\ \frac{t-\varepsilon/2}{\varepsilon/2}, & \text{if } t \in (\varepsilon/2, \varepsilon], \\ 1, & \text{if } t \in (\varepsilon, \infty). \end{cases} \quad (\text{e.2.1})$$

Definition 2.3. Let A be a C*-algebra and let $a \in A_+$. Suppose that $\tilde{T}(A) \neq \emptyset$. Define

$$d_\tau(a) = \lim_{\varepsilon \rightarrow 0} \tau(f_\varepsilon(a)), \quad \tau \in \tilde{T}(A),$$

with possibly infinite values. Note that $f_\varepsilon(a) \in \text{Ped}(A)_+$, and so by definition $\tau \mapsto \tau(f_\varepsilon(a))$ is a continuous affine function on $\tilde{T}(A)$ (to $[0, +\infty)$). It follows that $\tau \mapsto d_\tau(a)$ is a lower semicontinuous affine function on $\tilde{T}(A)$ (to $[0, +\infty]$). Note that

$$d_\tau(a) = \lim_{n \rightarrow \infty} \tau(a^{1/n}), \quad \tau \in \tilde{T}(A).$$

Let $a \in A_+$ be a strictly positive element. Define

$$\Sigma_A(\tau) = d_\tau(a), \quad \tau \in \tilde{T}(A).$$

The lower semicontinuous affine function $\Sigma_A: \tilde{T}(A) \rightarrow [0, +\infty]$ is independent of the choice of a , and will be called the scale function (or just scale) of A .

Definition 2.4. Let A be a C^* -algebra and let $a, b \in A_+$. We write $a \lesssim b$ if there exists a sequence (x_n) in A such that $x_n^* b x_n \rightarrow a$ in norm. If $a \lesssim b$ and $b \lesssim a$, we write $a \sim b$ and say that a and b are Cuntz equivalent. It is known that \sim is an equivalence relation. Denote by $\text{Cu}(A)$ the set of Cuntz equivalence classes of positive elements of $A \otimes \mathcal{K}$. It is an ordered abelian semigroup ([7]). Denote by $\text{Cu}(A)_+$ the subset of those elements which cannot be represented by projections. We shall write $\langle a \rangle$ for the equivalence class represented by a . Thus, $a \lesssim b$ will be also written as $\langle a \rangle \leq \langle b \rangle$. Recall that we write $\langle a \rangle \ll \langle b \rangle$ if the following holds: for any increasing sequence $(\langle y_n \rangle)$, if $\langle b \rangle \leq \sup\{\langle y_n \rangle\}$ then there exists $n_0 \geq 1$ such that $\langle a \rangle \leq \langle y_{n_0} \rangle$. In what follows we will also use the objects $\text{Cu}^\sim(A)$ and $\text{Cu}^\sim(\varphi)$ introduced in [43].

Definition 2.5. If B is a C^* -algebra, we will use $\text{QT}(B)$ for the set of quasitraces τ with $\|\tau\| = 1$ (see [2]). Let A be a σ -unital C^* -algebra. Suppose that every quasitrace of every hereditary sub- C^* -algebra B of A is a trace.

If $\tau \in \tilde{T}(A)$, we will extend it to $(A \otimes \mathcal{K})_+$ by the rule

$$\tau(a \otimes b) = \tau(a) \text{Tr}(b),$$

for all $a \in A$ and $b \in \mathcal{K}$, where Tr is the canonical densely defined trace on \mathcal{K} .

Recall that A has the (Blackadar) property of strict comparison for positive elements, if for any two elements $a, b \in (A \otimes \mathcal{K})_+$ with the property that

$$d_\tau(a) < d_\tau(b) < +\infty$$

for all $\tau \in \tilde{T}(A) \setminus \{0\}$, necessarily $a \lesssim b$. In general (without knowing that quasitraces are traces), this property will be called *strict comparison for positive elements using traces*.

Let S be a topological convex set. Denote by $\text{Aff}(S)$ the set of all real continuous affine functions on S , and by $\text{Aff}_+(S)$ the set of all real continuous affine functions f with $f(s) > 0$ for all s , together with the zero function.

Recall $\overline{\text{T}(A)}^w$ denotes the closure of $\text{T}(A)$ in $\tilde{T}(A)$ with respect to pointwise convergence on $\text{Ped}(A)$ (see the end of Definition 2.1). Suppose that $0 \notin \overline{\text{T}(A)}^w$ and that $\text{T}(A)$ generates the cone $\tilde{T}(A)$. (By Lemma 4.5 below, these properties hold in the case that $A = \text{Ped}(A)$.) Then A has strict comparison for positive elements using traces if, and only if, $d_\tau(a) < d_\tau(b)$ for all $\tau \in \overline{\text{T}(A)}^w$ implies $a \lesssim b$, for any $a, b \in (A \otimes \mathcal{K})_+$.

Definition 2.6. Let A be a C^* -algebra such that $0 \notin \overline{T(A)}^w$. There is a positive linear contraction $r_{\text{aff}}: A_{\text{s.a.}} \rightarrow \text{Aff}(\overline{T(A)}^w)$, $a \mapsto \widehat{a}$, from $A_{\text{s.a.}}$ to the ordered vector space of continuous real affine functions on $\overline{T(A)}^w$, defined by

$$r_{\text{aff}}(a)(\tau) = \widehat{a}(\tau) = \tau(a) \quad \text{for all } \tau \in \overline{T(A)}^w$$

and for all $a \in A_{\text{s.a.}}$. Denote by A^q the space $r_{\text{aff}}(A_{\text{s.a.}})$ and by A_+^q the cone $r_{\text{aff}}(A_+)$ (see [9]).

Denote by $\text{Aff}_0(T_1(A))$ the set of all real continuous affine functions on $T_1(A)$ which vanish at zero, and denote by $\text{Aff}_{0+}(T_1(A))$ the subset of those $f \in \text{Aff}_0(T_1(A))$ such that $f(t) > 0$ for all $t \in T_1(A) \setminus \{0\}$, together with the zero function. Denote by $\text{LAff}_{0+}(\overline{T(A)}^w)$ the set of those functions f on $\overline{T(A)}^w$ (with values in $[0, +\infty]$) such that there exists an increasing sequence of continuous affine functions $f_n \in \text{Aff}_{0+}(T_1(A))$ with $f_n|_{\overline{T(A)}^w} \nearrow f$ (as $n \rightarrow \infty$), together with the zero function. Denote by $\text{LAff}_{b,0+}(\overline{T(A)}^w)$ the subset of bounded functions in $\text{LAff}_{0+}(\overline{T(A)}^w)$. In particular, if $f \in \text{Aff}_{0+}(T_1(A))$, then $f|_{\overline{T(A)}^w} \in \text{LAff}_{0+}(\overline{T(A)}^w)$.

In the general case, recall the definition of $\text{LAff}_+(\widetilde{T}(A))$ from [43]: the set of lower semicontinuous affine (linear) extended positive real-valued functions on the cone $\widetilde{T}(A)$, strictly positive except at 0, which are the limits of increasing sequences of continuous, finite-valued such functions. (Thus, in the preceding special case, one considers the restrictions of this set of functions to subsets of $\widetilde{T}(A)$.)

Definition 2.7 ([44]). Let A be a non-unital C^* -algebra. Following [44], we shall say that A almost has stable rank one if for any hereditary sub- C^* -algebra B of A , $B \subset \text{GL}(\widetilde{B})$, where $\text{GL}(\widetilde{B})$ is the group of invertible elements of \widetilde{B} . Suppose that $A \otimes \mathcal{K}$ is σ -unital, almost has stable rank one, and contains a full projection e . Then $B = e(A \otimes \mathcal{K})e$ is unital. Since B almost has stable rank one, it follows that B has stable rank one. By Theorem 6.4 of [42], $B \otimes \mathcal{K}$ has stable rank one. By Brown's stable isomorphism theorem ([3]), $A \otimes \mathcal{K}$ has stable rank one. Thus, a σ -unital stable simple C^* -algebra which almost has stable rank one but does not have stable rank one must be projectionless. (We know of no such example.)

Definition 2.8. Let A and B be C^* -algebras and let $\varphi_n: A \rightarrow B$ be completely positive contractive maps. We shall say that $(\varphi_n)_{n=1}^\infty$ is a sequence of approximately multiplicative completely positive contractive maps if

$$\lim_{n \rightarrow \infty} \|\varphi_n(a)\varphi_n(b) - \varphi_n(ab)\| = 0 \quad \text{for all } a, b \in A.$$

Definition 2.9. Let A be a C^* -algebra. Denote by A^1 the closed unit ball of A , and by $A_+^{q,1}$ the image of the intersection $A_+ \cap A^1$ in A_+^q .

The following fact will be useful.

Proposition 2.10. *Let A be a σ -unital C^* -algebra and B be another C^* -algebra. Suppose that $\varphi: A \rightarrow B$ is a positive linear map and suppose that $e \in A_+$ is a strictly positive element. Then $\overline{\varphi(e)B\varphi(e)} = \overline{\varphi(A)B\varphi(A)}$.*

Proof. Let $C = \overline{\varphi(A)B\varphi(A)}$. Consider an approximate identity (b_α) of B . Then

$$\varphi(e)b_\alpha\varphi(e) \rightarrow \varphi(e)^2.$$

It follows that $\varphi(e)^2 \in C$. Consequently $\varphi(e) \in C$. It follows that $\overline{\varphi(e)B\varphi(e)} \subset C$.

Let g be a state of C . Suppose that $g(\varphi(e)) = 0$. We will show that $g = 0$.

For any $n \geq 1$, there exists $\lambda_n > 0$ such that $f_{1/n}(e) \leq \lambda_n e$. It follows that

$$g(\varphi(f_{1/n}(e))) \leq \lambda_n g(\varphi(e)) = 0.$$

Fix $a \in A_+$. Then

$$g(\varphi(f_{1/n}(e)af_{1/n}(e))) \leq \|a\|g(\varphi(f_{1/n}(e)^2)) \leq \|a\|g(\varphi(f_{1/n}(e))) = 0.$$

Since

$$\lim_{n \rightarrow \infty} \|\varphi(a) - \varphi(f_{1/n}(e)af_{1/n}(e))\| = 0,$$

one concludes that $g(\varphi(a)) = 0$. This implies that $g(\varphi(x)) = 0$ for all $x \in A$.

We claim that, for any $a \in A_+ \setminus \{0\}$, $g(\varphi(a)b\varphi(y)) = 0$ for any $b \in B$ and $y \in A$.

In fact,

$$\begin{aligned} |g(\varphi(a)b\varphi(y))|^2 &\leq g(\varphi(a)\varphi(a))g(\varphi(y)^*b^*b\varphi(y)) \\ &= g(\varphi(a)^2)g(\varphi(y)^*b^*b\varphi(y)) \end{aligned} \quad (\text{e 2.2})$$

$$\begin{aligned} &= g\left(\|a\|^2\varphi\left(\frac{a}{\|a\|}\right)^2\right)g(\varphi(y)^*b^*b\varphi(y)) \\ &= \|a\|^2g\left(\varphi\left(\frac{a}{\|a\|}\right)^2\right)g(\varphi(y)^*b^*b\varphi(y)) \end{aligned} \quad (\text{e 2.3})$$

$$\leq \|a\|^2g\left(\varphi\left(\frac{a}{\|a\|}\right)\right)g(\varphi(y)^*b^*b\varphi(y)) = 0. \quad (\text{e 2.4})$$

Recall that, for any $x \in A$, we may write $x = (x_1 - x_2) + i(x_3 - x_4)$, where $x_i \in A_+$, $i = 1, 2, 3, 4$. Therefore, for any $b \in B$ and $y \in A$,

$$g(\varphi(x)b\varphi(y)) = g\left(\sum_{i=1}^4 \varphi(x_i)b\varphi(y)\right) = \sum_{i=1}^4 g(\varphi(x_i)b\varphi(y)) = 0. \quad (\text{e 2.5})$$

It follows that $g(z) = 0$ if $z = \sum_{j=1}^m \varphi(a_j)b_j\varphi(c_j)$, where $b_j \in B$ and $a_j, c_j \in A$, $j = 1, 2, \dots, m$. Since the set

$$\left\{ \sum_{j=1}^m \varphi(a_j)b_j\varphi(c_j) : b_j \in B, a_j, c_j \in A \right\}$$

is dense in C , one concludes that $g(c) = 0$ for all $c \in C$. In other words, $g = 0$.

This shows that $\varphi(e)$ is a strictly positive element of C . Therefore,

$$\overline{\varphi(e)C\varphi(e)} = C.$$

On the other hand,

$$C \supset \overline{\varphi(e)B\varphi(e)} \supset \overline{\varphi(e)C\varphi(e)} = C.$$

So $C = \overline{\varphi(e)B\varphi(e)}$. □

3. Some results of Rørdam

For convenience, we would like to have the following version of a lemma of Rørdam:

Lemma 3.1 (Rørdam [46, Lemma 2.2]). *Let $a, b \in A$ with $0 \leq a, b \leq 1$ be such that $\|a - b\| < \delta/2$. Then there exists $z \in A$ with $\|z\| \leq 1$ such that*

$$(a - \delta)_+ = z^*bz.$$

Proof. It follows from the hypothesis that there exists $2 > \alpha > 1$ such that

$$\delta_0 := \|a - b^\alpha\| < \delta/2. \quad (\text{e 3.1})$$

Put $c = b^\alpha$. By Lemma 2.2 of [46],

$$f_\delta(a)^{1/2}(a - \delta_0 \cdot 1)f_\delta(a)^{1/2} \leq f_\delta(a)^{1/2}cf_\delta(a)^{1/2},$$

where f_δ is as defined in 2.2. Therefore,

$$f_\delta(a)^{1/2}(a - \delta_0 \cdot 1)_+f_\delta(a)^{1/2} \leq f_\delta(a)^{1/2}cf_\delta(a)^{1/2}.$$

Thus,

$$(a - \delta)_+ \leq f_\delta(a)^{1/2}(a - \delta_0 \cdot 1)_+f_\delta(a)^{1/2} \leq f_\delta(a)^{1/2}cf_\delta(a)^{1/2}.$$

Choose $0 < \beta < 1$ such that $\beta\alpha > 1$. Put $a_1 = (a - \delta)_+$ and $b_1 = f_\delta(a)^{1/2}cf_\delta(a)^{1/2}$. Then, as in the proof of Lemma 2.3 of [46], by 1.4.5 of [38], there is $r_1 \in A$ such that

$$\|r_1\| \leq \|b_1^{1/2-\beta/2}\| \leq 1 \quad \text{and} \quad a_1^{1/2} = r_1b_1^{\beta/2}.$$

Therefore, $a_1 = r_1b_1^\beta r_1^*$. Note that

$$b_1^\beta = (f_\delta(a)^{1/2}cf_\delta(a)^{1/2})^\beta.$$

Write $y = f_\delta(a)^{1/2}c^{1/2}$. Then $yy^* = b_1$. Let $y = v(y^*y)^{1/2}$ be the polar decomposition of y in A^{**} (so that $v \in A^{**}$). It follows from 1.4 of [8] that $vx \in A$ for all $x \in \overline{(y^*y)A(y^*y)}$ and $v(y^*y)^\beta v^* = b_1^\beta$. Note that

$$(y^*y)^\beta = (c^{1/2}f_\delta(a)c^{1/2})^\beta \leq c^\beta = b^{\alpha\beta}. \quad (\text{e 3.2})$$

Put $\gamma = 1/(\alpha\beta)$. Then $0 < \gamma < 1$. Let

$$x = (c^{1/2} f_\delta(a) c^{1/2})^{\beta/2}.$$

Then $x^2 \leq c^\beta = b^{\alpha\beta}$. Put

$$u_n = x((1/n) + (b^{\alpha\beta})^{1/2})(b^{\alpha\beta})^{1/2-\gamma/2}, \quad n = 1, 2, \dots$$

Then, as in the proof of 1.4.5 of [38],

$$\|u_n\| \leq \|(b^{\alpha\beta})^{1/2-\gamma/2}\| \leq 1$$

and $(u_n)_{n \geq 1}$ converges to $u \in A$ in norm. Moreover, $x = u(b^{\alpha\beta})^{\gamma/2}$. It follows that that

$$(y^*y)^\beta = xx = xx^* = u(b^{\alpha\beta})^\gamma u^* = ubu^*.$$

Note that, since $x = (y^*y)^{\beta/2}$, $vx \in A$. Therefore $vu_n \in A$ for all n . It follows that $vu \in A$. Note also that $\|vu\| \leq 1$. Now

$$(a - \delta)_+ = a_1 = r_1 b_1^\beta r_1 = r_1 (v(y^*y)^\beta v^*) r_1^* = (r_1 vu) b (u^* v^* r_1) = z^* b z,$$

where $z = u^* v^* r_1 = (vu)^* r_1 \in A$ and $\|z\| \leq 1$. \square

Lemma 3.2 (cf. [45, Proposition 2.4(v)] and [7, Theorem 3]). *Suppose that A is a C^* -algebra which almost has stable rank one. Suppose that $a, b \in A_+$ are such that $a \lesssim b$. Then, for any $0 < \delta$, there exists a unitary $u \in \tilde{A}$ such that*

$$u^* f_\delta(a) u \in \overline{bAb}.$$

Moreover, there exists $x \in A$ such that

$$x^* x = a \quad \text{and} \quad xx^* \in \overline{bAb}.$$

Furthermore, if $0 \leq a_1, a_2, b \leq 1$ are in A , and $a_1 a_2 = a_1$, and $a_2 \lesssim b$, then there exists a unitary $u \in \tilde{A}$ such that

$$u^* a_1 u \in \overline{bAb}. \tag{e 3.3}$$

In addition, if $d \in (A \otimes \mathcal{K})_+$, and also $\widehat{A \otimes \mathcal{K}}$ almost has stable rank one, then, for any $\varepsilon > 0$, there exists a unitary $u \in \widehat{A \otimes \mathcal{K}}$ such that $u f_\varepsilon(d) u^* \in M_n(A)$ for some large n .

Proof. The first statement follows from the proof of part (v) of [45]. The second statement (also the first one) follows from Proposition 3.3 of [44] (see also [39, Corollary 6] and [33, Lemma 1.4]).

To see the third statement, note that, by the first statement, for any $\delta > 0$, there exists a unitary $u \in \tilde{A}$ such that

$$u^* f_\delta(a_2)u \in \overline{bAb}. \quad (\text{e 3.4})$$

Since $a_1 a_2 = a_1 = a_2 a_1$, one has $f_\delta(a_2)^{1/2} a_1 = a_1$. Therefore,

$$u^* a_1 u = u^* f_\delta(a_2)^{1/2} a_1 f_\delta(a_2)^{1/2} u \leq u^* f_\delta(a_2) u \in \overline{bAb}. \quad (\text{e 3.5})$$

To see the last statement, let $(e_n)_{n \geq 1}$ be an approximate identity of $A \otimes \mathcal{K}$ such that $e_n \in M_n(A)$, $n = 1, 2, \dots$. Without loss of generality, we may assume that $0 \leq d \leq 1$. Then, for any $\varepsilon > 0$, there exists $n \geq 1$ such that $\|d - e_n d e_n\| < \varepsilon/4$. By Lemma 3.1, $f_{\varepsilon/2}(d) \lesssim e_n d e_n$. Thus the last conclusion follows from the first statement. \square

We shall also need the following variant of Lemma 3.1.

Lemma 3.3. *Let $\varepsilon > 0$ and $\sigma > 0$ be given. There exists $\delta > 0$ satisfying the following condition: If A is a C*-algebra, and if $x, y \in A_+$ are such that*

$$0 \leq x, y \leq 1 \quad \text{and} \quad \|x - y\| < \delta,$$

*then there exists a partial isometry $w \in A^{**}$ with*

$$w w^* f_\sigma(x) = f_\sigma(x) w w^* = f_\sigma(x), \quad (\text{e 3.6})$$

and

$$w^* c w \in \overline{yAy}, \quad \|w^* c w - c\| < \varepsilon \|c\| \quad \text{for all } c \in \overline{f_\sigma(x) A f_\sigma(x)}. \quad (\text{e 3.7})$$

If A almost has stable rank one (see Definiton 2.7), then w may be chosen to be a unitary in \tilde{A} .

Proof. Let $\varepsilon/4 > \delta_1 > 0$ be such that, for any C*-algebra B , and any pair of positive elements $x', y' \in B$ with $0 \leq x', y' \leq 1$ such that

$$\|x' - y'\| < \delta_1,$$

then

$$\|f_{\sigma/2}(x') - f_{\sigma/2}(y')\| < \sigma \cdot \varepsilon/8. \quad (\text{e 3.8})$$

Put $\eta = (\sigma \delta_1 / 16)^2$. Define $g(t) = f_{\sigma/2}(t)/t$ for all $0 < t \leq 1$ and $g(0) = 0$. Then $g \in C_0((0, 1])$. Note that $\|g\| \leq 2/\sigma$. Set $\delta_2 = \eta \delta_1 / 16$ and choose $0 < \delta < \eta/3$ such that, for any C*-algebra B , and any pair of positive elements $x'', y'' \in B$ with $0 \leq x'', y'' \leq 1$ such that

$$\|x'' - y''\| < 2\delta,$$

one has

$$\|(x'')^{1/2} - (y'')^{1/2}\| < \delta_2. \quad (\text{e 3.9})$$

Now let A be a C^* -algebra and let $x, y \in A$ be such that $0 \leq x, y \leq 1$ and $\|x - y\| < \delta$.

Then

$$\|x^2 - y^2\| = \|x^2 - xy + xy - y^2\| < 2\delta. \quad (\text{e 3.10})$$

Set $z = yf_\eta(x^2)^{1/2}$. Then, by (e 3.9),

$$\|(z^*z)^{1/2} - x\| = \|(f_\eta(x^2)^{1/2}y^2f_\eta(x^2)^{1/2})^{1/2} - x\| \quad (\text{e 3.11})$$

$$< \delta_2 + \|(f_\eta(x^2)^{1/2}x^2f_\eta(x^2)^{1/2})^{1/2} - x\| \quad (\text{e 3.12})$$

$$< \delta_2 + \sqrt{\eta} < \sigma \cdot \delta_1/8. \quad (\text{e 3.13})$$

Also,

$$\|(z^*z)^{1/2} - z\| < \sigma \cdot \delta_1/8 + \|x - yf_\eta(x^2)^{1/2}\| \quad (\text{e 3.14})$$

$$< \sigma \cdot \delta_1/8 + \delta + \|x - xf_\eta(x^2)^{1/2}\| \quad (\text{e 3.15})$$

$$< \sigma \cdot \delta_1/8 + \delta + \sqrt{\eta} < \sigma \cdot \delta_1/4. \quad (\text{e 3.16})$$

Consider the polar decomposition $z = v(z^*z)^{1/2}$ of z in A^{**} . Then

$$\begin{aligned} & \|vf_{\sigma/2}(x) - f_{\sigma/2}(x)\| \\ & \leq \|vf_{\sigma/2}(x) - vf_{\sigma/2}((z^*z)^{1/2})\| + \|vf_{\sigma/2}((z^*z)^{1/2}) - f_{\sigma/2}(x)\| \\ & < \sigma \cdot \varepsilon/8 + \|v(z^*z)^{1/2}g((z^*z)^{1/2}) - f_{\sigma/2}(x)\| \quad (\text{using (e 3.8)}) \\ & = \sigma \cdot \varepsilon/8 + \|zg((z^*z)^{1/2}) - f_{\sigma/2}(x)\| \\ & \leq \sigma \cdot \varepsilon/8 + \|zg((z^*z)^{1/2}) - (z^*z)^{1/2}g((z^*z)^{1/2})\| \\ & \quad + \|(z^*z)^{1/2}g((z^*z)^{1/2}) - f_{\sigma/2}(x)\| \\ & < \varepsilon/8 + \delta_1/2 + \|(z^*z)^{1/2}g((z^*z)^{1/2}) - f_{\sigma/2}(x)\| \quad (\text{using (e 3.16)}) \\ & < \varepsilon/4 + \|f_{\sigma/2}((z^*z)^{1/2}) - f_{\sigma/2}(x)\| \\ & < \varepsilon/4 + \sigma \cdot \varepsilon/8 < \varepsilon/2 \quad (\text{using (e 3.8) and (e 3.13)}). \end{aligned}$$

Hence, for any $c \in \overline{f_\sigma(x)Af_\sigma(x)}$ with $\|c\| \leq 1$,

$$\|vcv^* - c\| = \|vf_{\sigma/2}(x)cf_{\sigma/2}(x)v^* - c\| \quad (\text{e 3.17})$$

$$< \varepsilon/2 + \|f_{\sigma/2}(x)cf_{\sigma/2}(x)v^* - c\| \quad (\text{e 3.18})$$

$$= \varepsilon/2 + \|vf_{\sigma/2}(x)c^*f_{\sigma/2}(x) - c^*\| \quad (\text{e 3.19})$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad (\text{e 3.20})$$

It follows from (the proof of) 2.2 of [46] that

$$(\eta - \|x^2 - y^2\|)f_\eta(x^2) \leq f_\eta(x^2)^{1/2}y^2f_\eta(x^2)^{1/2} \leq f_\eta(x^2).$$

So $\overline{f_\eta(x^2)Af_\eta(x^2)}$ is the same as the hereditary sub-C*-algebra generated by

$$z^*z = f_\eta(x^2)^{1/2}y^2f_\eta(x^2)^{1/2}.$$

Note also that the hereditary sub-C*-algebra generated by zz^* is contained in \overline{yAy} . It follows that

$$vcv^* \in \overline{yAy} \quad \text{for all } c \in \overline{f_\eta(x^2)Af_\eta(x^2)}. \quad (\text{e 3.21})$$

Choose $w = v^*$. Then, since $\sqrt{\eta} < \sigma/4$,

$$f_\sigma(x)f_\eta(x^2) = f_\sigma(x)$$

and hence

$$ww^*f_\sigma(x) = f_\sigma(x)ww^* = f_\sigma(x). \quad (\text{e 3.22})$$

Thus (e 3.21) holds for all $c \in \overline{f_\sigma(x)Af_\sigma(x)}$. If A almost has stable rank one, we can choose δ for $\varepsilon/2$ and $\sigma/4$ first. Then, for $b = vf_{\sigma/4}(x)$, by Theorem 5 of [39], there is a unitary $u \in \tilde{A}$ such that $b = u^*f_{\sigma/4}(x)$. Then, for any $c \in \overline{f_\sigma(x)Af_\sigma(x)}$, $u^*cu = vcv^*$ and so w can be replaced by u . \square

Lemma 3.4 ([46]). *Let A be a C*-algebra and $a \in A_+$ be a full element. Then, for any $b \in A_+$, any $\varepsilon > 0$ and any $g \in C_0((0, +\infty))$ whose support is in $[\varepsilon, +\infty)$, there are $x_1, x_2, \dots, x_m \in A$ such that*

$$g(b) = \sum_{i=1}^m x_i^* a x_i.$$

Proof. Fix $\varepsilon > 0$. Since a is full, and a and b are positive, there are $z_1, z_2, \dots, z_m \in A$ such that

$$\left\| \sum_{i=1}^m z_i^* a z_i - b \right\| < \varepsilon.$$

Therefore, by 2.2 and 2.3 of [46], there is $y \in B$ such that

$$f_\varepsilon(b) = y^* \left(\sum_{i=1}^m z_i^* a z_i \right) y.$$

Therefore, since $f_\varepsilon g = g$,

$$g(b) = g(b)^{1/2} y^* \left(\sum_{i=1}^m z_i^* a z_i \right) y g(b)^{1/2}. \quad \square$$

We shall also need the following slight variant of a result of Rørdam:

Theorem 3.5 (cf. [48, 4.6]). *Let A be an exact simple C^* -algebra which is \mathcal{Z} -stable. Then A has the strict comparison property for positive elements: If $a, b \in (A \otimes \mathcal{K})_+$ are two elements such that*

$$d_\tau(a) < d_\tau(b) < +\infty \quad \text{for all } \tau \in \tilde{\mathbb{T}}(A) \setminus \{0\}, \quad (\text{e 3.23})$$

then $a \lesssim b$.

Proof. Let $a, b \in (A \otimes \mathcal{K})_+$ be as in (e 3.23), and set

$$\{\tau \in \tilde{\mathbb{T}}(A) : d_\tau(b) = 1\} = S.$$

The assumption (e 3.23) implies that

$$d_\tau(a) < d_\tau(b) \quad \text{for all } \tau \in S. \quad (\text{e 3.24})$$

Since A is simple and $b \neq 0$, for every $\varepsilon > 0$, $\langle (a - \varepsilon)_+ \rangle \leq K \langle b \rangle$ for some integer $K \geq 1$. Hence,

$$f(a) < f(b) \quad \text{for all } f \in S(\text{Cu}(A), \langle b \rangle)$$

(see [48] for the notation). Since, by Theorem 4.5 of [48], $\text{Cu}(A) = \text{W}(A \otimes \mathcal{K})$ is almost unperforated, by 3.2 of [48], $a \lesssim b$. \square

Corollary 3.6. *Let A be an exact simple separable C^* -algebra which is \mathcal{Z} -stable. Then A has the following strict comparison property for positive elements: If $a, b \in (B \otimes \mathcal{K})_+$ are two elements such that*

$$d_\tau(a) < d_\tau(b) < +\infty \quad \text{for all } \tau \in \overline{\mathbb{T}(B)}^w,$$

where $B = \overline{cAc}$ for some $c \in \text{Ped}(A)_+ \setminus \{0\}$, then $a \lesssim b$.

Proof. The condition (e 3.23) of Theorem 3.5 holds since $\mathbb{R}_+ \overline{\mathbb{T}(B)}^w = \tilde{\mathbb{T}}(A)$. \square

We would like to include the following statement.

Lemma 3.7. *Let B be a separable semiprojective C^* -algebra and A be another C^* -algebra such that there is an isomorphism*

$$j: A \otimes \mathcal{K} \rightarrow B \otimes \mathcal{K}.$$

Then, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ with the following property: If D is a C^* -algebra and $L: A \rightarrow D$ is a \mathcal{G} - δ -multiplicative completely positive contractive map, then there exists a homomorphism $h: A \rightarrow D \otimes \mathcal{K}$ such that

$$\|h(a) - L(a) \otimes e_{1,1}\| < \varepsilon \quad \text{for all } a \in \mathcal{F},$$

where $\{e_{i,j}\}$ is a system of matrix units for \mathcal{K} .

Proof. Let us write $B \otimes M_n \subset B \otimes M_{n+1}$ for all n , so that $\bigcup_{n=1}^{\infty} B \otimes M_n$ is dense in $B \otimes \mathcal{K}$. Let $e_n = \sum_{i=1}^n e_{i,i}$. Define a contractive completely positive map

$$d_n: B \otimes \mathcal{K} \rightarrow B \otimes M_n$$

by sending $b \otimes c$ to $b \otimes (e_n c e_n)$ for all $b \in B$ and $c \in \mathcal{K}$. There is an integer N such that

$$\|j(a) - d_N(j(a))\| < \varepsilon/8 \quad \text{for all } a \in \mathcal{F} = F \otimes e_{1,1}. \quad (\text{e 3.25})$$

Write

$$d_N(j(a)) = \sum_{i=1}^{m(a)} b_{a,i} \otimes c_{a,i},$$

where $b_{a,i} \in B$ and $c_{a,i} \in M_N$, $1 \leq i \leq m(a)$. Put

$$\mathcal{F}_M = \{c_{a,i} : 1 \leq i \leq m(a), a \in \mathcal{F}\}$$

and

$$\mathcal{F}_1 = \left\{ \sum_{i=1}^{m(a)} b_{a,i} \otimes c_{a,i} : a \in \mathcal{F} \right\} = d_N(j(\mathcal{F})).$$

Set

$$\Lambda = \max \{ (\|b_{a,i}\| + 1)(\|c_{a,i}\| + 1)m(a) : 1 \leq i \leq m(a), a \in \mathcal{F} \}.$$

Since $B \otimes M_N$ is semiprojective, there are a finite subset $\mathcal{G}_1 \subset B \otimes M_N$ and $\delta_0 > 0$ satisfying the following condition: if $L': B \otimes M_N \rightarrow D'$ (for any C^* -algebra D') is a \mathcal{G}_1 - δ_0 -multiplicative completely positive contractive map, there exists a homomorphism $H: B \otimes M_N \rightarrow D'$ such that

$$\|H(b) - L'(b)\| < \varepsilon/8 \quad \text{for all } b \in \mathcal{F}_1. \quad (\text{e 3.26})$$

Since j is an isomorphism, there exist a finite subset $\mathcal{G} \subset A$ and $\delta > 0$ satisfying the following condition: if $L: A \rightarrow D$ (for any C^* -algebra D) is a \mathcal{G} - δ -multiplicative completely positive contractive map, then $(L \otimes \text{id}_{\mathcal{K}})(j^{-1})$ is a \mathcal{G}_1 - δ -multiplicative completely positive contractive map.

Let $\iota: M_N \otimes \mathcal{K} \rightarrow \mathcal{K} \otimes \mathcal{K}$ denote the inclusion and let $\varphi: \mathcal{K} \otimes \mathcal{K} \rightarrow M_N \otimes \mathcal{K}$ be an isomorphism. There is a unitary $U \in \widetilde{M_N \otimes \mathcal{K}}$ such that

$$\text{Ad } U \circ \varphi \circ \iota \approx_{\varepsilon/4\Lambda} \text{id}_{M_N \otimes \mathcal{K}} \quad \text{on } \mathcal{F}_M \otimes e_{1,1}.$$

Put $\varphi_1 = \text{Ad } U \circ \varphi$. Consider

$$\Psi = (\text{id}_B \otimes \varphi_1) \circ (j \otimes \text{id}_{\mathcal{K}}): A \otimes \mathcal{K} \otimes \mathcal{K} \rightarrow B \otimes M_N \otimes \mathcal{K}.$$

Thus, for all $a \in \mathcal{F}$ (we identify A with $A \otimes e_{1,1}$, the first corner of $A \otimes \mathcal{K}$), by (e 3.25) and the definition of φ_1 , and (e 3.25) again,

$$\begin{aligned}
\Psi(a \otimes e_{1,1} \otimes e_{1,1}) &= (\text{id}_B \otimes \varphi_1)(j(a) \otimes e_{1,1}) \approx_{\varepsilon/8} (\text{id}_B \otimes \varphi_1)(d_N(j(a)) \otimes e_{1,1}) \\
&= (\text{id}_B \otimes \varphi_1) \left(\sum_{i=1}^{m(a)} b_{a,i} \otimes \iota(c_{a,i} \otimes e_{1,1}) \right) \\
&\approx_{\varepsilon/4} \sum_{i=1}^{m(a)} b_{a,i} \otimes c_{a,i} \otimes e_{1,1} \\
&\approx_{\varepsilon/8} j(a) \otimes e_{1,1}.
\end{aligned} \tag{e 3.27}$$

Now let $L: A \rightarrow D$ be a \mathcal{G} - δ -multiplicative completely positive contractive map. Consider the maps

$$L \otimes \text{id}_{\mathcal{K}}: A \otimes \mathcal{K} \rightarrow D \otimes \mathcal{K} \quad \text{and} \quad (L \otimes \text{id}_{\mathcal{K}})(j^{-1}): B \otimes \mathcal{K} \rightarrow D \otimes \mathcal{K},$$

and the restriction

$$\Phi := (L \otimes \text{id}_{\mathcal{K}})(j^{-1})|_{B \otimes M_N}: B \otimes M_N \rightarrow D \otimes \mathcal{K}.$$

By construction, Φ is a \mathcal{G}_1 - δ_0 -multiplicative completely positive contractive map. Therefore (by (e 3.26)) there is a homomorphism $h_0: B \otimes M_N \rightarrow D \otimes \mathcal{K}$ such that

$$\|h_0(b) - \Phi(b)\| < \varepsilon/8 \quad \text{for all } b \in \mathcal{F}_1. \tag{e 3.28}$$

Then, for all $a \in \mathcal{F}$, by (e 3.27), (e 3.25), (e 3.28), and (e 3.25) again,

$$\begin{aligned}
(h_0 \otimes \text{id}_{\mathcal{K}}) \circ \Psi(a \otimes e_{1,1} \otimes e_{1,1}) &\approx_{\varepsilon/2} (h_0 \otimes \text{id}_{\mathcal{K}})(j(a) \otimes e_{1,1}) \\
&\approx_{\varepsilon/4} (\Phi \otimes \text{id}_{\mathcal{K}})(d_N(j(a)) \otimes e_{1,1}) \\
&\approx_{\varepsilon/8} (L \otimes \text{id}_{\mathcal{K}})(a \otimes e_{1,1} \otimes e_{1,1}) = L(a) \otimes e_{1,1}.
\end{aligned}$$

Define $h: A \rightarrow D \otimes \mathcal{K}$ as the composed map $(\text{id}_D \otimes \psi) \circ h_0 \circ \Psi \circ \iota_A$, where $\iota_A(a) = a \otimes e_{11} \otimes e_{1,1}$ for all $a \in A$ and $\psi: \mathcal{K} \otimes \mathcal{K} \rightarrow \mathcal{K}$ is any isomorphism. From the last estimate the lemma follows. The above proof may be summarized by the following non-commutative diagram with the upper triangle approximately commutative on \mathcal{F} , and the lower right one approximately commutative on $\mathcal{F}_1 \otimes e_{1,1}$:

$$\begin{array}{ccccc}
& & A & & \\
& \swarrow (j \otimes \text{id}_{\mathcal{K}}) \circ \iota_A & \downarrow d_N \circ j \circ \iota_A & & \\
B \otimes \mathcal{K} \otimes \mathcal{K} & \xrightarrow{\text{id}_B \otimes \varphi_1} & B \otimes M_N \otimes \mathcal{K} & \xrightarrow{h_0 \otimes \text{id}_{\mathcal{K}}} & D \otimes \mathcal{K} \otimes \mathcal{K} \xrightarrow{\text{id}_D \otimes \psi} D \otimes \mathcal{K}. \\
& \searrow j^{-1} \otimes \text{id}_{\mathcal{K}} & \downarrow j^{-1} \otimes \text{id}_{\mathcal{K}} & \nearrow L \otimes \text{id}_{\mathcal{K}} \otimes \mathcal{K} & \\
& & A \otimes \mathcal{K} \otimes \mathcal{K} & &
\end{array}$$

where $h = (\text{id}_D \otimes \psi) \circ (h_0 \otimes \text{id}_{\mathcal{K}}) \circ (\text{id}_B \otimes \varphi_1) \circ (j \otimes \text{id}_{\mathcal{K}}) \circ \iota_A$. \square

4. Compact C*-algebras

Definition 4.1. A σ -unital C*-algebra A is said to be *compact*, if there is $e \in (A \otimes \mathcal{K})_+$ with $0 \leq e \leq 1$ and a partial isometry $w \in (A \otimes \mathcal{K})^{**}$ such that

$$w^*a, w^*aw \in A \otimes \mathcal{K}, \quad ww^*a = aww^* = a, \quad ew^*aw = w^*awe = w^*aw,$$

for all $a \in A$, where A is identified with $A \otimes e_{1,1}$.

Proposition 4.2. Let C be a σ -unital C*-algebra and let $c \in C_+ \setminus \{0\}$ with $0 \leq c \leq 1$ be a full element of C . Suppose that there is $e_1 \in C$ with $0 \leq e_1 \leq 1$ such that $e_1c = ce_1 = c$. Then \overline{cCc} is compact.

Proof. Set $\overline{cCc} = B$. Consider the sub-C*-algebra

$$E = \{(a_{ij})_{2 \times 2} : a_{11} \in B, a_{12} \in \overline{BC}, a_{21} \in \overline{CB}, a_{22} \in C\} \subset M_2(C),$$

containing $B_1 = B \otimes e_{1,1}$ and $C_2 = C \otimes e_{2,2}$ as full corners, where

$$\{e_{i,j} : 1 \leq i, j \leq 2\}$$

is a system of matrix units for M_2 . We may then in a natural way view $B_1 \otimes \mathcal{K}$ and $C_2 \otimes \mathcal{K}$ as full corners of $E \otimes \mathcal{K}$. We may write

$$E \otimes \mathcal{K} = \{(a_{ij})_{2 \times 2} : a_{11} \in B_1 \otimes \mathcal{K}, a_{12} \in \overline{BC} \otimes \mathcal{K}, \\ a_{21} \in \overline{CB} \otimes \mathcal{K}, a_{22} \in C_1 \otimes \mathcal{K}\} \subset M_2(C \otimes \mathcal{K}).$$

We also write

$$B_2 = B \otimes e_{2,2}.$$

Moreover, let p_1 denote the range projection of $B_1 \otimes \mathcal{K}$ and p_2 the range projection of $C_1 \otimes \mathcal{K}$; then $p_1, p_2 \in M(E \otimes \mathcal{K})$. Put $U = (a_{ij})_{2 \times 2}$, where $a_{11} = a_{12} = a_{22} = 0$ and $a_{21} = p_1$. Then,

$$Uc \in E \otimes \mathcal{K}$$

for all $c \in E \otimes \mathcal{K}$, and

$$UxU^* \in B_2 \otimes \mathcal{K} \subset C_2 \otimes \mathcal{K}$$

for all $x \in B_1 \otimes \mathcal{K}$.

By 2.8 of [3], there is a partial isometry $W \in M(E \otimes \mathcal{K})$ such that

$$W^*(B_1 \otimes \mathcal{K})W = C_2 \otimes \mathcal{K}, \quad WW^* = p_1, \quad \text{and} \quad W^*W = p_2.$$

Since $W \in M(E \otimes \mathcal{K})$, $Wb \in E \otimes \mathcal{K}$ for all $b \in B_2 \otimes \mathcal{K}$. It follows that

$$Up_1Wb = UWb \in B_2 \otimes \mathcal{K} \quad \text{for all } b \in B_2 \otimes \mathcal{K}.$$

In what follows we identify B with B_2 . Put

$$w = W^* p_1 U^* = W^* U^*.$$

Then, for any $b \in B (= B_2)$,

$$(w)^* b = U p_1 W b \in B_2 \otimes \mathcal{K}, \quad w^* b w = U W b W^* U^* \in B_2 \otimes \mathcal{K} (= B \otimes \mathcal{K}).$$

This also implies that w^* is a left multiplier of $B_2 \otimes \mathcal{K}$. So we may write $w \in (B_2 \otimes \mathcal{K})^{**}$ (which we identify with $(B \otimes \mathcal{K})^{**}$). Moreover, with

$$e = U p_1 W e_1 W^* p_1 U^* \in B_2 \otimes \mathcal{K} (= B \otimes \mathcal{K})$$

(where $e_1 := e_1 \otimes e_{22}$),

$$\begin{aligned} e w^* b w &= U W e_1 W^* U^* U W b W^* U^* \\ &= U W e_1 p_2 b W^* U^* = U W e_1 b W^* U^* = U W b W^* U^* = w^* b w \end{aligned}$$

for all $b \in B_2$. We also have

$$w^* b w e = e w^* b w, \quad w w^* b = W^* U^* U W b = p_2 b = b, \quad \text{and} \quad b w w^* = b$$

for all $b \in B_2 (= B)$.

Thus, B is compact. \square

Corollary 4.3. *A σ -unital full hereditary sub- C^* -algebra of a σ -unital compact C^* -algebra is compact.*

Proof. Let A be a compact C^* -algebra and let $b \in A_+$ be a full element. Put $B = \overline{bAb}$. Let e and w be as in Definition 4.1. Put $c = w^* b w$, $B_1 = w^* B w$, and $C = A \otimes \mathcal{K}$. Then $B \cong B_1$. Moreover, $B_1 = \overline{cCc}$ and $ec = ce = c$. So, by Proposition 4.2, B_1 is compact. \square

Lemma 4.4. *Let A be a σ -unital C^* -algebra which is compact. Then, there exists an integer $N \geq 1$, a partial isometry $w \in M_N(A)^{**}$ and $e \in M_N(A)$ with $0 \leq e \leq 1$ such that*

$$w^* a, w^* a w \in M_N(A), \quad w w^* a = a w w^* = a, \quad w^* a w e = e w^* a w = w^* a w$$

for all $a \in A$.

Proof. If A is unital, then we may choose $N = 1$ and $e = 1_A$.

Let $b \in A$ be a strictly positive element with $0 \leq b \leq 1$. We may assume, by Definition 4.1, that A is a full hereditary sub- C^* -algebra of a σ -unital C^* -algebra C such

that there is $e_1 \in C$ with $0 \leq e_1 \leq 1$ such that $be_1 = b = e_1b$. Moreover, $\overline{bCb} = A$. Since b is full in C , by Lemma 3.4, there exist $x_1, x_2, \dots, x_m \in C$ such that

$$\sum_{i=1}^m x_i^* b x_i = f_{1/4}(e_1).$$

Note that

$$f_{1/4}(e_1)b = b = bf_{1/4}(e_1).$$

Consider $X^* := (x_1^* b^{1/2}, x_2^* b^{1/2}, \dots, x_m^* b^{1/2})$ as a 1-row element of $M_m(C)$. Then

$$X^*X = f_{1/4}(e_1) \quad \text{and} \quad XX^* \in M_m(\overline{b^{1/2}Cb^{1/2}}) = M_m(A).$$

Consider the polar decomposition $X = v|X^*X|^{1/2}$ of X in $M_m(C)^{**}$. Then

$$vav^* = v|X^*X|^{1/2}a|X^*X|^{1/2}v^* = XaX^* \in M_m(A) \quad \text{for all } a \in A.$$

Note that $Xb^{1/n} \in M_m(A)$ for all n . Denote by p the range projection of b . Then $Xp \in M_m(A)^{**}$. Note also that

$$Xp = v|X^*X|^{1/2}p = vp.$$

Set $w = (Xp)^*$. Then

$$w^* = vp \quad \text{and} \quad ww^* = pX^*Xp = pf_{1/4}(e_1)p = p.$$

So w is a partial isometry. Note that, for any $a \in A$,

$$w^*b^{1/n}a = vpb^{1/n}a = Xb^{1/n}a \in A.$$

It follows that $w^*a \in A$. Then

$$w^*aw = XaX^* \in M_m(A) \quad \text{for all } a \in A.$$

Set $e = XX^*$. Then,

$$w^*awe = XaX^*XX^* = Xaf_{1/4}(e_1)X^* = XaX^* = w^*aw$$

and $ew^*aw = XX^*XaX^* = Xf_{1/4}(e_1)aX^* = XaX^* = w^*aw. \quad (\text{e 4.1})$

□

Lemma 4.5. *Let A be a σ -unital compact C^* -algebra. Then $0 \notin \overline{\mathbb{T}(A)}^w$. Hence, if $a \in A$ with $0 \leq a \leq 1$ is full, then there is $d > 0$ such that*

$$d_\tau(a) \geq d \quad \text{for all } \tau \in \overline{\mathbb{T}(A)}^w.$$

Proof. We may assume that $T(A) \neq \emptyset$. As in the proof of Lemma 4.4, without loss of generality, we may assume that A is a full hereditary sub-C*-algebra of B for some (stable) σ -unital C*-algebra B such that there is $e_1 \in B_+$ with $0 \leq e_1 \leq 1$ and $e_1 x = x e_1 = x$ for all $x \in A$. Note that $A \subset \text{Ped}(B)$. Since A is full in B , we may also assume that each $\tau \in T(A)$ has been extended to a unique element of $\tilde{T}(B)$. Let $e_0 \in A_+$ be a strictly positive element of A . Recall from Definition 2.1 that $\overline{T(A)}^w$ is a weak*-compact subset of $T_1(A)$. Consider the set

$$S = \{\tau \in T_1(A) : \tau(f_{1/4}(e_1)) \geq 1\}.$$

Then S is closed and $0 \notin S$. Note that

$$T(A) = \{\tau \in \tilde{T}(C) : d_\tau(e_0) = 1\}.$$

Then, for any $\tau \in T(A)$,

$$\tau(f_{1/4}(e_1)) \geq d_\tau(e_0) = 1. \quad (\text{e.4.2})$$

It follows that $T(A) \subset S$, and so $\overline{T(A)}^w \subset S$. Therefore, $0 \notin \overline{T(A)}^w$.

Since a is full in A , $\tau(a) > 0$ for all non-zero traces. In particular, $\tau(a) > 0$ for all $\tau \in \overline{T(A)}^w$. Thus, the lower semicontinuous function $\tau \mapsto d_\tau(a)$ is strictly positive on the compact set $\overline{T(A)}^w$. Therefore

$$d := \inf \{d_\tau(a) : \tau \in \overline{T(A)}^w\} > 0. \quad \square$$

Corollary 4.6. *Let A be a σ -unital C*-algebra which is compact and let $a \in A$ be a strictly positive element with $0 \leq a \leq 1$. Suppose that*

$$d := \inf \{d_\tau(a) : \tau \in \overline{T(A)}^w\} > 0.$$

Then, for any $0 < d_0 < d$, there exists an integer $n \geq 1$ such that, for all $m \geq n$,

$$\tau(f_{1/m}(a)) \geq d_0 \quad \text{and} \quad \tau(a^{1/m}) \geq d_0 \quad \text{for all } \tau \in \overline{T(A)}^w.$$

Proof. This holds as both increasing sequences $(\tau(f_{1/m}(a)))_{m=1}^\infty$ and $(\tau(a^{1/m}))_{m=1}^\infty$ converge pointwise to $d_\tau(a)$, and $\overline{T(A)}^w$ is compact. \square

Theorem 4.7. *Let A be a σ -unital C*-algebra. Then A is compact if and only if $\text{Ped}(A) = A$.*

Proof. Let $a \in A_+$ be a strictly positive element.

First assume that A is compact. We will identify A with $A \otimes e_{11} \subset A \otimes \mathcal{K}$. Then there exists $e \in (A \otimes \mathcal{K})_+$ and a partial isometry $w \in (A \otimes \mathcal{K})^{**}$ such that

$$w^* x w \in A \otimes \mathcal{K}, \quad w w^* x = x w w^* = x, \quad w^* x w e = e w^* x w = w^* x w$$

for all $x \in A$.

Set $z = w^*a^{1/2}$. Then $zz^* \in \text{Ped}(A \otimes \mathcal{K})_+$. Hence, by 5.6.2 of [38],

$$a = z^*z \in \text{Ped}(A \otimes \mathcal{K})_+.$$

Therefore, the hereditary sub-C*-algebra generated by a is contained in $\text{Ped}(A \otimes \mathcal{K})$. In other words, $A \subset \text{Ped}(A \otimes \mathcal{K})$. By Theorem 2.1 of [51], $A = \text{Ped}(A)$.

Conversely, assume that $\text{Ped}(A) = A$. Then there are $b_i \in A_+$ with $\|b_i\| \leq 1$, $i = 1, 2, \dots, m$, such that

$$a^{1/2} \leq \sum_{i=1}^m g_i(b_i),$$

where $g_i \in C_0((0, +\infty))$ and the support of g_i is in $[\sigma, +\infty)$ for some $1/2 > \sigma > 0$. Note that for each i , $g_i = \sum_{j=1}^K g_{i,j}$ for some $0 \leq g_{i,j} \leq 1$ with the support of $g_{i,j}$ still in $[\sigma, \infty)$. Thus, without loss of generality, we may assume that $0 \leq g_i \leq 1$.

Let $c_i = (g_i(b_i))^{1/2}$, $i = 1, 2, \dots, m$. Define

$$Z = (c_1, c_2, \dots, c_m),$$

which we view as a $m \times m$ matrix with zero rows other than the first row. Define

$$E = \text{diag}(f_{\sigma/2}(b_1), f_{\sigma/2}(b_2), \dots, f_{\sigma/2}(b_m)) \in M_m(A).$$

Note that

$$ZZ^* = \sum_{i=1}^m c_i^2 \geq a \quad \text{and} \quad Z^*Z = (d_{i,j})_{m \times m},$$

where

$$d_{i,j} = c_i c_j, \quad i, j = 1, 2, \dots, m.$$

It follows that

$$E(Z^*Z) = E(c_i c_j)_{m \times m} = (c_i c_j)_{m \times m} = (Z^*Z)E.$$

Consider the polar decomposition $Z^* = V|Z^*|$. Then $Vx \in M_m(A)$,

$$VV^*|Z| = |Z|VV^* = |Z| \quad \text{and} \quad (VxV^*)E = E(VxV^*) = (VxV^*)$$

for all $x \in \overline{(ZZ^*)M_m(A)(ZZ^*)}$. Note that $A \subset \overline{(ZZ^*)M_m(A)(ZZ^*)}$. This shows that A is compact. \square

Lemma 4.8. *Let A be a σ -unital C*-algebra with $0 \notin \overline{\mathbb{T}(A)}^w$ and $\tilde{\mathbb{T}}(A) = \mathbb{R}_+ \mathbb{T}(A)$. Suppose that $A \otimes \mathcal{K}$ almost has stable rank one and A has strict comparison (see Definition 2.5). Then A is compact. Moreover, let $a \in A$ with $0 \leq a \leq 1$ be a strictly positive element, set*

$$d = \inf \{d_\tau(a) : \tau \in \overline{\mathbb{T}(A)}^w\},$$

and (see Lemma 4.5) let n be an integer such that $nd > 1$. There exist elements $e_1, e_2 \in M_n(A)$ with $0 \leq e_1, e_2 \leq 1$, $e_1e_2 = e_2e_1 = e_1$, and $w \in (M_n(A))^{**}$ such that

$$w^*c, cw \in M_n(A), \quad ww^*c = cww^* = c \quad \text{for all } c \in A, \quad (\text{e 4.3})$$

$$\text{and} \quad w^*cwe_1 = e_1w^*cw = w^*cw \quad \text{for all } c \in A. \quad (\text{e 4.4})$$

Furthermore, there exist a full element $b_0 \in \text{Ped}(A)$ with $0 \leq b_0 \leq 1$ and $e_0 \in \text{Ped}(A)_+$ such that $b_0e_0 = e_0b_0 = b_0$.

Proof. Let $a \in A_+$ with $0 \leq a \leq 1$ be a strictly positive element. Since $0 \notin \overline{T(A)}^w$, and $\overline{T(A)}^w$ is compact (see the end of Definition 2.1), and $d_\tau(a)$ is lower semicontinuous, as stated in Lemma 4.5,

$$\inf \{d_\tau(a) : \tau \in \overline{T(A)}^w\} = d > 0.$$

Let n be an integer such that $nd > 1$. By Corollary 4.6 there exists $\varepsilon > 0$ such that

$$\inf \{\tau(f_\varepsilon(a)) : \tau \in \overline{T(A)}^w\} = d_0 > 2d/3$$

and $nd_0 > 1$. So, for any $\tau \in \overline{T(A)}^w$,

$$d_\tau(a) \leq 1 < nd_0 \leq n\tau(f_\varepsilon(a)) \leq d_\tau(f_\varepsilon(a)).$$

Therefore,

$$a \lesssim f_\varepsilon(a) \otimes e_n \in M_n(A),$$

where $e_n = 1_{M_n}$.

Put $b = f_\varepsilon(a) \otimes e_n \in M_n(A)$. Since $M_n(A)$ is assumed almost to have stable rank one, by Lemma 3.2, there exists $x \in M_n(A)$ such that

$$x^*x = a^{1/2} \quad \text{and} \quad xx^* \in \overline{b(M_n(A))b}.$$

By considering the polar decomposition of x in $(M_n(A))^{**}$, one obtains a partial isometry $w \in (M_n(A))^{**}$ such that $wA, Aw^* \subset M_n(A)$,

$$w^*wc = cw^*w = c \quad \text{for all } c \in A,$$

and $wAw^* \subset \overline{b(M_n(A))b}$. Put $e = f_{\varepsilon/2}(a) \otimes e_n \in M_n(A)$. Then $0 \leq e \leq 1$ and

$$ewcw^* = wcw^*e = wcw^* \quad \text{for all } c \in A.$$

This shows that A is compact. The second part of the statement with (e 4.3) and (e 4.4) also holds.

For the last part of the statement, choose $b_0 = f_\varepsilon(a)$ and $e_0 = f_{\varepsilon/2}(a)$. \square

Definition 4.9. Let A be a σ -unital C^* -algebra and let $e \in A$ be a strictly positive element. Set

$$\lambda_s = \inf \{d_\tau(e) : \tau \in T(A)\}. \quad (\text{e 4.5})$$

By Definition 2.3, $0 \leq \lambda_s \leq 1$. Since $\overline{T(A)}^w$ is compact and $\tau \mapsto d_\tau(e)$ is lower semicontinuous,

$$\lambda_s = \min \{d_\tau(e) : \tau \in \overline{T(A)}^w\}.$$

In particular, $0 \notin \overline{T(A)}^w$ if and only if $\lambda_s > 0$. (So, by Lemma 4.5, if A is compact, $\lambda_s > 0$.)

Now let A be a σ -unital exact simple C^* -algebra. Let $e \in \text{Ped}(A)_+ \setminus \{0\}$. Consider the set of traces normalized on e ,

$$T_e(A) = \{\tau \in \tilde{T}(A) : \tau(e) = 1\}.$$

It is a compact convex set (see [51, 2.6] and [27, 2.6]). If A has strict comparison, A is said to have bounded scale if $d_\tau(a)$ is a bounded function on $T_e(A)$ (see [1]). In the absence of strict comparison, let us say that A has bounded scale if there exists an integer $n \geq 1$ such that $n\langle e \rangle \geq \langle a \rangle$ for any $a \in A_+$. As first noted in [1], this is equivalent to saying that A is algebraically simple, and this in turn (in view of Theorem 4.7 above) is equivalent to saying that A is compact.

Proposition 4.10. *Let A be a σ -unital C^* -algebra with $0 \notin \overline{T(A)}^w$ such that every trace in $\tilde{T}(A)$ is finite (equivalently, bounded) on A . Let $B \subset A$ be a σ -unital full hereditary sub- C^* -algebra. Then $0 \notin \overline{T(B)}^w$.*

Proof. Let $b \in B_+$ with $\|b\| = 1$ be such that $B = \overline{bBb}$. Let $e \in A_+$ with $\|e\| = 1$ such that $A = \overline{eAe}$.

Since b is full, $\tau(b) > 0$ for all $\tau \in \overline{T(A)}^w$. Then, by continuity and compactness,

$$1 > r_0 = \inf \{\tau(b) : \tau \in \overline{T(A)}^w\} > 0.$$

For any $t \in T(B)$, the unique extension $\tau \in \tilde{T}(A)$ is finite, i.e., bounded, by hypothesis. Set $\tau_0 = \tau/\|\tau\| \in T(A)$ and $t = \|\tau\| \cdot \tau_0|_B$. It follows (since $\tau_0(b) \geq r_0$ and $\|\tau\| \geq 1$) that

$$t(b) \geq \|\tau\| \cdot r_0 \geq r_0.$$

This shows that $0 \notin \overline{T(B)}^w$. □

Proposition 4.11. *Let A be a σ -unital simple C^* -algebra, let $c \in \text{Ped}(A)$ be a positive element and set $C = \overline{cAc}$. Then each $\tau \in \overline{T(C)}^w$ extends to a unique element $\iota(\tau) \in \tilde{T}(A)$. Moreover $\iota(\overline{T(C)}^w)$ is compact in $\tilde{T}(A)$ and $0 \notin \iota(\overline{T(C)}^w)$.*

Proof. Note that the extension exists and is unique. So ι is well defined. (Moreover, the map ι is one-to-one.) Note that, by Corollary 4.3 and Theorem 4.7, $C = \text{Ped}(C)$. Put $K = \overline{T(C)}^w$. Then, by Lemma 4.5, $0 \notin K$.

Consider $\tau_\alpha, \tau \in K$ such that $\tau_\alpha(b) \rightarrow \tau(b)$ for all $b \in C$. Let us show that

$$\iota(\tau_\alpha)(a) \rightarrow \iota(\tau)(a) \quad \text{for all } a \in \text{Ped}(A)_+.$$

By the definition of the Pedersen ideal,

$$a \leq \sum_{i=1}^n g(y_i),$$

where $y_i \in A_+$ and $g \in C_0((0, +\infty))_+$ has compact support, $i = 1, 2, \dots, n$. It follows from Lemma 3.4 that $\langle g(y_i) \rangle \leq m\langle c \rangle$ for some integer $m \geq 1$. (Recall that, for any $c_1, c_2 \in A_+$, $\langle c_1 + c_2 \rangle = \langle (1, 1)\text{diag}(c_1, c_2)(1, 1)^* \rangle \leq \langle c_1 \rangle + \langle c_2 \rangle$.) Therefore, $\langle a \rangle \leq nm\langle c \rangle$. It follows that

$$d_{\iota(\tau_\alpha)}(a), d_{\iota(\tau)}(a) \leq nm. \quad (\text{e 4.6})$$

Then, for any $b \in \overline{aAa}$ with $\|b\| \leq 1$, $|\iota(\tau_\alpha)(b)| \leq nm$ and $|\iota(\tau)(b)| \leq nm$. In other words,

$$\|\iota(\tau_\alpha)|_{\overline{aAa}}\| \leq nm \quad \text{and} \quad \|\iota(\tau)|_{\overline{aAa}}\| \leq nm. \quad (\text{e 4.7})$$

For any $\varepsilon > 0$, by Lemma 3.4, there are $z_1, z_2, \dots, z_N \in A$ such that

$$\sum_{i=1}^N z_i^* c z_i = f_\varepsilon(a). \quad (\text{e 4.8})$$

It follows that

$$\sum_{i=1}^N a^{1/2} z_i^* c z_i a^{1/2} = a^{1/2} f_\varepsilon(a) a^{1/2}. \quad (\text{e 4.9})$$

Consider the element

$$b = \sum_{i=1}^N a^{1/2} z_i^* c z_i a^{1/2} \in \overline{aAa}.$$

Set $z'_i = z_i a^{1/2}$, $i = 1, 2, \dots, N$. Then, since $c^{1/2} z'_i z_i'^* c^{1/2} \in C$,

$$\begin{aligned} \iota(\tau_\alpha)(b) &= \sum_{i=1}^N \iota(\tau_\alpha)(z_i'^* c z_i) = \sum_{i=1}^N \iota(\tau_\alpha)(c^{1/2} z'_i z_i'^* c^{1/2}) = \sum_{i=1}^N \tau_\alpha(c^{1/2} z'_i z_i'^* c^{1/2}) \\ &\rightarrow \sum_{i=1}^N \tau(c^{1/2} z'_i z_i'^* c^{1/2}) = \sum_{i=1}^N \iota(\tau)(c^{1/2} z'_i z_i'^* c^{1/2}) \\ &= \sum_{i=1}^N \iota(\tau)(z_i'^* c z_i) = \iota(\tau)(b). \end{aligned} \quad (\text{e 4.10})$$

Since $a^{1/2} f_\varepsilon(a) a^{1/2} \rightarrow a$ in norm, by (e.4.7) and by (e.4.9),

$$\iota(\tau_\alpha)(a) \rightarrow \iota(\tau)(a).$$

This shows that ι is continuous. Since K is compact, $\iota(K)$ is compact in $\tilde{T}(A)$. Since $0 \notin K$, $0 \notin \iota(K) = \iota(\overline{T(C)})$. \square

Definition 4.12. Let A be a σ -unital C^* -algebra with $\tilde{T}(A) \neq \{0\}$. Suppose that there is an element $e \in \text{Ped}(A)_+$ which is full in A .

Set $A_e = \overline{eAe}$. Since e is full, one has that $\text{Ped}(A_e) = A_e$ (see the fourth paragraph of Section 2.1). Then, by Lemma 4.7, A_e is compact. Consequently, by Lemma 4.5, $0 \notin \overline{T(A_e)}^w$. Assume that A is not unital. Each $\tau \in \overline{T(A_e)}^w$ extends uniquely to a tracial state on \tilde{A}_e . There is a canonical order-preserving homomorphism

$$\rho_{\tilde{A}_e} : K_0(\tilde{A}_e) \rightarrow \text{Aff}(\overline{T(A_e)}^w).$$

By [3], one may identify $K_0(A)$ with $K_0(A_e)$. The composition of maps from $K_0(A)$ to $K_0(A_e)$, then from $K_0(A_e)$ to $K_0(\tilde{A}_e)$ and then to $\text{Aff}(\overline{T(A_e)}^w)$ is a homomorphism which will be denoted by ρ_A . Denote by $\ker \rho_A$ the subgroup of $K_0(A)$ (independent of e) consisting of those $x \in K_0(A)$ such that $\rho_A(x) = 0$. Elements in $\ker \rho_A$ are called infinitesimal elements.

5. Continuous scale and fullness

Definition 5.1. The previous section discussed C^* -algebras with bounded scale. Let us recall the definition of continuous scale ([27] and [31]).

Let A be a σ -unital C^* -algebra. Fix an increasing approximate unit (e_n) for A with the property that

$$e_{n+1}e_n = e_n e_{n+1} = e_n \quad \text{for all } n \geq 1.$$

The C^* -algebra A is said to have continuous scale if, for any $b \in A_+ \setminus \{0\}$, there exists $N \geq 1$ such that

$$e_m - e_n \lesssim b, \quad m > n \geq N.$$

This definition does not depend on the choice of (e_n) above. By Theorem 5.3 below, if A has continuous scale and $T(A) \neq \emptyset$, then $\lambda_s = 1$. (See Definition 4.9.)

Remark 5.2. Let A be an exact simple C^* -algebra such that $A \otimes \mathcal{Z} \cong A$. Then, by 6.6 of [19] (see also [43, 6.2.3]), the map $\langle a \rangle \mapsto \langle \hat{a} \rangle$ is an isomorphism of the ordered semigroup of purely non-compact elements of $\text{Cu}(A)$ with $\text{LAff}_+(\tilde{T}(A))$. Hence Proposition 5.4 below implies that there exists a non-zero hereditary sub- C^* -algebra B of $A \otimes \mathcal{K}$ such that B has continuous scale.

In fact slightly more can be said. Let $a \in A_+$ be a strictly positive element. One finds a non-zero element $b \in (A \otimes \mathcal{K})_+$ such that $\langle \widehat{b} \rangle$ is continuous and $\langle \widehat{b} \rangle < \langle \widehat{a} \rangle$. Write $C = A \otimes \mathcal{K}$ and view A as a hereditary sub-C*-algebra of C . Note that C is also \mathcal{Z} -stable. It follows from part (ii) of Theorem 1.2 of [44] that there exists a non-zero positive element $b_1 \in C$ such that $\overline{b_1 C} \subset \overline{a C}$ such that $\langle b_1^2 \rangle = \langle b_1 \rangle = \langle b \rangle$. Note that $b_1^2 = b_1 b_1^* \in \overline{a C} = A$. In other words, A contains a positive element b_1 such that $\langle \widehat{b_1} \rangle$ is continuous.

Thus, Theorem 5.3 and Proposition 5.4 below imply that there exists a non-zero hereditary sub-C*-algebra B of A such that B has continuous scale (see Corollary 11.9 below).

Theorem 5.3 (cf. [31]). *Let A be a σ -unital non-elementary simple C*-algebra with continuous scale. Then:*

- (1) $T(A)$ is compact;
- (2) $d_\tau(a)$ is continuous on $\widetilde{T}(A)$ for any strictly positive element a of A ;
- (3) $d_\tau(a)$ is continuous on $\overline{T(A)}^w$ for any strictly positive element a of A .

Conversely, if A has strict comparison for positive elements using tracial states (see Definition 2.5), and A is algebraically simple, then (1), (2), and (3) are equivalent and also equivalent to each of the following conditions:

- (4) A has continuous scale;
- (5) $d_\tau(a)$ is continuous on $\overline{T(A)}^w$ for some strictly positive element a of A ;
- (6) $d_\tau(a)$ is continuous on $\widetilde{T}(A)$ for some strictly positive element a of A .

Proof. Most parts of the theorem are well known. That (1) holds is perhaps less well known.

Since A has continuous scale, A is algebraically simple ([27, 3.3]). In particular, $A = \text{Ped}(A)$. As noted in Definition 2.1, $K = \overline{T(A)}^w$ is compact (as a subspace of $\widetilde{T}(A)$). Let $a \in A$ be a strictly positive element. Fix an element $b \in A_+ \setminus \{0\}$ with $\|b\| = 1$. Put

$$B = \overline{f_{1/2}(b) A f_{1/2}(b)}.$$

Since A is not elementary, B_+ contains infinitely many mutually orthogonal non-zero elements $\{x_n\}$ with $0 \leq x_n \leq 1$, $n = 1, 2, \dots$. By repeatedly applying Lemma 3.5.4 of [29], one then finds, for each n , n non-zero mutually orthogonal positive elements $\{x_{n,1}, x_{n,2}, \dots, x_{n,n}\}$ in A with $0 \leq x_{n,j} \leq 1$ such that

$$x_{n,1} \lesssim x_{n,2} \lesssim \dots \lesssim x_{n,n}$$

(see also [27, 2.3]).

Note that $\tau(f_{1/8}(b))$ is bounded on K and

$$d_\tau(f_{1/4}(b)) \leq \tau(f_{1/8}(b)) \quad \text{for all } \tau \in K. \quad (\text{e.5.1})$$

Since $f_{1/4}(b)x_{n,j} = x_{n,j}$ for all $1 \leq j \leq n$ and all n , it follows that, for any $\varepsilon > 0$, there exists $x_{n(\varepsilon),1}$ such that $d_\tau(x_{n(\varepsilon),1}) < \varepsilon$ for all $\tau \in K$. Note that

$$d_\tau(a) = \lim_{n \rightarrow \infty} \tau(f_{1/2^n}(a)) \quad \text{for all } \tau \in K.$$

Note that, with $e_n = f_{1/2^n}(a)$, $n = 1, 2, \dots$, $(e_n)_{n=1}^\infty$ forms an approximate identity with $e_{n+1}e_n = e_n$ for all n . Since A has continuous scale, for any $\varepsilon > 0$, there exists $N \geq 1$ such that

$$e_m - e_n \lesssim x_{k(\varepsilon),1} \quad \text{for all } m > n \geq N.$$

In particular,

$$\tau(e_m) - \tau(e_n) < \varepsilon \quad \text{for all } \tau \in K. \quad (\text{e 5.2})$$

It follows that $d_\tau(a)$ is continuous on K . Since $d_\tau(a) = 1$ on $T(A)$ and $T(A)$ is dense in K , $d_\tau(a) = 1$ for all $\tau \in K$. This implies that $T(A) = K$. This proves (1) and (3). Note that (2) is equivalent to (3), as $A = \text{Ped}(A)$, and so $\tilde{T}(A) = \mathbb{R}_+T(A)$.

Conversely, suppose that A is as stated, suppose that $d_\tau(a)$ is continuous, and consider the increasing sequence $e_n = f_{1/2^n}(a)$, $n = 1, 2, \dots$. Then, by Dini's theorem, $\tau(e_n)$ converges to $d_\tau(a)$ uniformly on K . It follows that for any non-zero $b \in A_+$, there exists $N \geq 1$ such that

$$d_\tau(e_m - e_n) < d_\tau(b) \quad \text{for all } \tau \in T(A) \text{ and for all } m > n \geq N.$$

(Indeed, from $(e_{m+2} - e_n)(e_{m+1} - e_{n+1}) = e_{m+1} - e_{n+1}$ for all $m > n$ follows $d_\tau(e_{m+1} - e_{n+1}) \leq \tau(e_{m+2} - e_n) = \tau(e_{m+2}) - \tau(e_n) \rightarrow 0$ uniformly on K .) Since A has strict comparison for positive elements, it follows that,

$$e_m - e_n \lesssim b \quad \text{for all } m > n \geq N.$$

Thus, A has continuous scale.

In other words, in this case, if A does not have continuous scale, $d_\tau(a)$ is not continuous on K . In particular, $d_\tau(a)$ is not identically 1. This implies $K \neq T(A)$. The proof above that (1) and (3) hold shows that, under the assumption that A is as stated in the second part of the theorem, the statements (1), (4) and (5) are equivalent. Since (5) and (6) are equivalent, these are also equivalent to (6). Since the notion of continuous scale is independent of the choice of a , these conditions are also equivalent to (2) and (3). \square

Proposition 5.4. *Let A be a σ -unital exact simple C^* -algebra with strict comparison for positive elements. Suppose that $T(A) \neq \emptyset$. Let $a \in A_+$ be such that $d_\tau(a)$ is continuous on $\tilde{T}(A)$. Then \overline{aAa} has continuous scale. If, in addition, A is algebraically simple and $d_\tau(a)$ is just assumed to be continuous on $\overline{\tilde{T}(A)}^w$, then \overline{aAa} has continuous scale.*

Proof. Put $B = \overline{aAa}$. We may assume that $a \neq 0$. Choose a non-zero element $c \in \text{Ped}(A)$ with $0 \leq c \leq 1$. Put $C = \overline{cAc}$. By Corollary 4.3 and Theorem 4.7, $C = \text{Ped}(C)$. Put $K = \overline{T(C)}^w$. Then, by Lemma 4.5, $0 \notin K$.

Note that each $\tau \in K$ extends uniquely to an element of $\tilde{T}(A)$. Let $\iota: K \rightarrow \tilde{T}(A)$ denote this map as in Proposition 4.11. By Proposition 4.11, $0 \notin \iota(K)$ and $\iota(K)$ is compact. Therefore $d_\tau(a)$ is continuous on $\iota(K)$. Let $e_n = f_{1/2^n}(a)$, $n = 1, 2, \dots$. Then $(e_n)_{n=1}^\infty$ is an approximate identity for B such that $e_{n+1}e_n = e_n$ for all n . Then $d_\tau(e_n) \nearrow d_\tau(a)$ uniformly on the compact set K . For any $b_0 \in B_+ \setminus \{0\}$, there exists $N \geq 1$ such that, for all $m > n \geq N$,

$$d_\tau(e_m - e_n) < d_\tau(b_0) \quad \text{for all } \tau \in K. \quad (\text{e 5.3})$$

Since K generates $\tilde{T}(A)$ as a cone, (e 5.3) holds for all $\tau \in \tilde{T}(A) \setminus \{0\}$. It follows that, for all $m > n \geq N$,

$$e_m - e_n \lesssim b_0. \quad (\text{e 5.4})$$

Therefore B has continuous scale.

The last part of the statement follows since $\tilde{T}(A) = \mathbb{R}_+T(A)$. \square

Now we turn to the important concept of local uniform fullness.

Definition 5.5. Let A be a sub-C*-algebra of a C*-algebra B . An element $a \in A_+ \setminus \{0\}$ is said to be *uniformly full* in B , if there are a positive number $M(a) > 0$ and an integer $N(a) \geq 1$ such that, for any $b \in B_+$ with $\|b\| \leq 1$ and any $\varepsilon > 0$, there are $x_1(a), x_2(a), \dots, x_{n(a)}(a) \in B$ such that

$$\|x_i(a)\| \leq M(a), \quad n(a) \leq N(a),$$

and

$$\left\| \sum_{i=1}^{n(a)} x_i(a)^* a x_i(a) - b \right\| < \varepsilon.$$

In this case, we shall also say that a is $(N(a), M(a))$ full.

We shall say that a is *strongly uniformly full* in B , if the above property holds with $\varepsilon = 0$ and replacing “ $< \varepsilon$ ” by “ $= 0$ ”.

We shall say that A is *locally uniformly full*, if every element $a \in A_+ \setminus \{0\}$ is uniformly full; and we say A is *strongly locally uniformly full* if every $a \in A_+ \setminus \{0\}$ is strongly uniformly full.

If B is unital and A is full in B , then A is always strongly locally uniformly full. In fact, for each $a \in A \setminus \{0\}$, there are $x_1, x_2, \dots, x_m \in B$ such that

$$\sum_{i=1}^m x_i^* a x_i = 1_B.$$

Choose $M(a) = \max\{\|x_i\| : 1 \leq i \leq m\}$ and $N(a) = m$.

Let A be a C^* -algebra, let B be non-unital C^* -algebra, and let $L: A \rightarrow B$ be a positive linear map. Let $F: A_+ \setminus \{0\} \rightarrow \mathbb{N} \times (\mathbb{R}_+ \setminus \{0\})$. Suppose that $\mathcal{H} \subset A_+ \setminus \{0\}$ is a finite subset. We shall say that L is F - \mathcal{H} -full if, for any $a \in \mathcal{H}$, for any $b \in B_+$ with $\|b\| \leq 1$, and any $\varepsilon > 0$, there are $x_1, x_2, \dots, x_m \in B$ such that $m \leq N(a)$ and $\|x_i\| \leq M(a)$, where $(N(a), M(a)) = F(a)$, and

$$\left\| \sum_{i=1}^m x_i^* L(a) x_i - b \right\| \leq \varepsilon. \quad (\text{e.5.5})$$

We shall say L is exactly F - \mathcal{H} -full, if (e.5.5) holds for $\varepsilon = 0$.

Proposition 5.6. *Let A be a non-zero σ -unital sub- C^* -algebra of a C^* -algebra B . Suppose that B is σ -unital and algebraically simple. Then A is strongly locally uniformly full in B .*

Proof. Let $b \in A$ be a strictly positive element. Then \overline{bBb} is a full hereditary sub- C^* -algebra \underline{B} . It suffices to show that \overline{bBb} is strongly locally uniformly full in B . Put $B_1 = \overline{bBb}$. In what follows we will identify B with $B \otimes e_{11}$ in $M_n(B)$.

Since B is algebraically simple, $B = \text{Ped}(B)$. By Theorem 4.7, B is compact. Applying Lemma 4.4, let $e \in M_n(B)$ for some $n \geq 1$ with $0 \leq e \leq 1$ and $w \in M_n(B)^{**}$ be such that

$$w^*a, w^*aw \in M_n(B), \quad ww^*a = aww^* = a, \quad w^*awe = ew^*aw = w^*aw$$

for all $a \in B$. Note that also $aw \in M_n(B)$ for all $a \in B$.

By Lemma 3.4, for any $1/4 > \varepsilon > 0$ and any $a \in (B_1)_+ \setminus \{0\}$, there are $x_1, x_2, \dots, x_m \in M_n(B)$ such that

$$f_\varepsilon(e) = \sum_{i=1}^m x_i^* a x_i.$$

Let p denote the range projection of B . Then $p x_i \in M_n(B)$ for $i = 1, 2, \dots, m$. We may assume that $p x_i = x_i$, $i = 1, 2, \dots, m$.

Fix $x \in B_+$ with $\|x\| \leq 1$. Then

$$f_\varepsilon(e)w^*xw = w^*xw f_\varepsilon(e) = w^*xw.$$

Let $M(a) = \max\{\|x_i\| : 1 \leq i \leq m\}$ and $N(a) = m$. Then,

$$w^*x^{1/2}w f_\varepsilon(e)w^*x^{1/2}w = w^*xw.$$

Therefore,

$$x^{1/2}w f_\varepsilon(e)w^*x^{1/2} = w(w^*xw)w^* = x.$$

Put $z_i = x_i w^* x^{1/2}$, $i = 1, 2, \dots, m$. Then $z_i \in B$, $i = 1, 2, \dots, m$. Then $\|z_i\| \leq M(a)$ and

$$\begin{aligned} \sum_{i=1}^m z_i^* a z_i &= x^{1/2} w \left(\sum_{i=1}^m x_i^* a x_i \right) w^* x^{1/2} \\ &= x^{1/2} w f_\varepsilon(e) w^* x^{1/2} = x. \end{aligned} \quad (\text{e 5.6})$$

□

Theorem 5.7. *Let A be a σ -unital simple C^* -algebra with $A = \text{Ped}(A)$ and with $\text{T}(A) \neq \emptyset$. Fix an element $e \in A_+ \setminus \{0\}$ with $\|e\| = 1$ and*

$$0 < d < \min \{ \inf \{ \tau(e) : \tau \in \text{T}(A) \}, \inf \{ \tau(f_{1/2}(e)) : \tau \in \text{T}(A) \} \}.$$

Then there exists a map $T: A_+ \setminus \{0\} \rightarrow \mathbb{N} \times (\mathbb{R}_+ \setminus \{0\})$ with the following property: For any finite subset $\mathcal{H}_1 \subset A_+^I \setminus \{0\}$, there are a finite subset $\mathcal{G} \subset A$ and $\delta > 0$ satisfying the following conditions: For any C^ -algebra B with $\text{QT}(C) = \text{T}(C)$ for all hereditary sub- C^* -algebras C of B , and $\text{T}(B) \neq \emptyset$, and $0 \notin \overline{\text{T}(B)}^w$, with strict comparison and such that $B \otimes \mathcal{K}$ almost has stable rank one, and for any \mathcal{G} - δ -multiplicative completely positive contractive map $\varphi: A \rightarrow B$ such that*

$$\tau(f_{1/2}(\varphi(e))) > d/2 \quad \text{for all } \tau \in \text{T}(B),$$

necessarily φ is exactly T - \mathcal{H}_1 -full. Moreover, for any $c \in \mathcal{H}_1$,

$$\tau(f_{1/2}(\varphi(c))) \geq \frac{d}{8 \min \{ M(c)^2 \cdot N(c) : c \in \mathcal{H}_1 \}} \quad \text{for all } \tau \in \text{T}(B).$$

Proof. Since A is a simple C^* -algebra with $A = \text{Ped}(A)$, there is a map

$$T_1: A_+ \setminus \{0\} \rightarrow \mathbb{N} \times (\mathbb{R}_+ \setminus \{0\})$$

such that the identity map id_A is exactly T_1 - $A_+ \setminus \{0\}$ -full.

Write $T_1 = (N_1, M_1)$, where $N_1: A_+ \setminus \{0\} \rightarrow \mathbb{N}$ and $M_1: A_+ \setminus \{0\} \rightarrow (\mathbb{R}_+ \setminus \{0\})$.

Let $n \geq 2$ be an integer such that $nd/2 > 1$. Set $N = 2nN_1$ and $M = 2M_1$ and $T = (N, M)$. Let $\mathcal{H}_1 \subset A_+ \setminus \{0\}$ be a fixed finite subset.

Suppose that $x_{i,h}, \dots, x_{N_1(h),h} \in A$ with $\|x_{i,h}\| \leq M_1(h)$ are such that

$$\sum_{i=1}^{N_1(h)} x_{i,h}^* h^2 x_{i,h} = f_{1/64}(e) \quad \text{for all } h \in \mathcal{H}_1. \quad (\text{e 5.7})$$

Choose a large enough $\mathcal{G} \subset A$ and small enough $\delta > 0$ that, for any \mathcal{G} - δ -multiplicative

completely positive contractive map φ defined on A ,

$$\|\varphi(f_{1/64}(e)) - f_{1/64}(\varphi(e))\| < 1/64 \quad (\text{e 5.8})$$

and
$$\left\| \sum_{i=1}^{N_1(h)} \varphi(x_{i,h})^* \varphi(h)^2 \varphi(x_{i,h}) - f_{1/64}(\varphi(e)) \right\| < 1/64 \quad \text{for all } h \in \mathcal{H}_1. \quad (\text{e 5.9})$$

Now let $\varphi: A \rightarrow B$ (for any B that fits the description in the statement of the theorem) be a \mathcal{G} - δ -multiplicative completely positive contractive map such that

$$\tau(f_{1/2}(\varphi(e))) \geq d/2 \quad \text{for all } \tau \in \overline{\mathbb{T}(B)}^w. \quad (\text{e 5.10})$$

Applying Lemma 3.1 (using (e 5.9)), one finds $y_{i,h} \in B$ with $\|y_{i,h}\| \leq 2\|x_{i,h}\|$, $i = 1, 2, \dots, N_1(h)$, such that

$$\sum_{i=1}^{N_1(h)} y_{i,h}^* \varphi(h)^2 y_{i,h} = f_{1/16}(\varphi(e)) \quad \text{for all } h \in \mathcal{H}_1. \quad (\text{e 5.11})$$

By the hypotheses on B , and since A is σ -unital, applying Lemma 4.8, we may choose $e_1, e_2 \in M_n(B)_+$ and $w \in M_n(B)^{**}$ as described there. Put

$$E_0 = f_{1/8}(\varphi(e)) \otimes 1_{2n} \quad (\text{e 5.12})$$

and
$$E_1 = f_{1/16}(\varphi(e)) \otimes 1_{2n} \in M_{2n}(B)_+. \quad (\text{e 5.13})$$

Then, by strict comparison,

$$e_2 \lesssim E_0 \in M_{2n}(B).$$

Since B almost has stable rank one, there exists a unitary $u \in \widetilde{M_{2n}(B)}$ such that

$$u^* f_{1/16}(e_2) u \in \overline{E_0(M_{2n}(B))E_0}.$$

Then

$$u^* f_{1/16}(e_2) u E_1 = E_1 u^* f_{1/16}(e_2) u = u^* f_{1/16}(e_2) u. \quad (\text{e 5.14})$$

We then may write

$$\sum_{i=1}^{2nN_1(h)} (y'_{i,h})^* \varphi(h)^2 y'_{i,h} = E_1 \quad \text{for all } h \in \mathcal{H}_1,$$

where $y'_{i,h} \in M_{2n}(B)$ and $\|y'_{i,h}\| = \|y_{j,h}\|$ for some $j \in \{1, 2, \dots, N_1(h)\}$, $i = 1, 2, \dots, 2nN_1(h)$. Then

$$\sum_{i=1}^{2nN_1(h)} (f_{1/16}(e_2)^{1/2} u y'_{i,h}^*) \varphi(h)^2 (y'_{i,h} u^* f_{1/16}(e_2)^{1/2}) = f_{1/16}(e_2).$$

Therefore, for any $b \in B_+$ with $\|b\| \leq 1$,

$$\sum_{i=1}^{2nN_1(h)} (w^* b^{1/2} w) (f_{1/16}(e_2)^{1/2} u y'_{i,h}{}^* \cdot \varphi(h)^{1/2} \varphi(h) \varphi(h)^{1/2} (y'_{i,h} u^* f_{1/16}(e_2)^{1/2})) (w^* b^{1/2} w) = w^* b w.$$

Then

$$\sum_{i=1}^{2nN_1(h)} (b^{1/2} w) (f_{1/16}(e_2)^{1/2} u y'_{i,h}{}^* \varphi(h)^{1/2} \cdot \varphi(h) \varphi(h)^{1/2} (y'_{i,h} u^* f_{1/16}(e_2)^{1/2})) w^* b^{1/2} = b.$$

Note that $b^{1/4} w \in M_n(B)$ and $f_{1/16}(e_2) \in M_n(B)$. Therefore,

$$(b^{1/4} w) f_{1/16}(e_2) \in M_n(B).$$

It follows that

$$(b^{1/2} w) (f_{1/16}(e_2)^{1/2} u y'_{i,h}{}^* \varphi(h)^{1/2}) \in B \quad (\text{e 5.15})$$

$$\text{and } \|(b^{1/2} w) (f_{1/16}(e_2)^{1/2} u y'_{i,h}{}^* \varphi(h)^{1/2})\| \leq 2M(h) \quad \text{for all } h \in \mathcal{H}_1. \quad (\text{e 5.16})$$

This implies that φ is exactly T - \mathcal{H}_1 -full. \square

Remark 5.8. In the light of Proposition 6.3 below, Theorem 5.7 can be applied with C^* -algebras B in the class \mathcal{C}' defined just before Remark 6.2.

6. Non-unital and non-commutative one dimensional complexes

Definition 6.1. Let F_1 and F_2 be two finite dimensional C^* -algebras. Suppose that there are homomorphisms $\varphi_0, \varphi_1: F_1 \rightarrow F_2$. Consider the mapping torus M_{φ_1, φ_2} :

$$\begin{aligned} A &= A(F_1, F_2, \varphi_0, \varphi_1) \\ &:= \{(f, g) \in C([0, 1], F_2) \oplus F_1 : f(0) = \varphi_0(g) \text{ and } f(1) = \varphi_1(g)\}. \end{aligned}$$

For $t \in (0, 1)$, define

$$\pi_t: A \rightarrow F_2 \quad \text{by } \pi_t((f, g)) = f(t) \text{ for all } (f, g) \in A.$$

For $t = 0$, define

$$\pi_0: A \rightarrow \varphi_0(F_1) \subset F_2 \quad \text{by } \pi_0((f, g)) = \varphi_0(g) \text{ for all } (f, g) \in A.$$

For $t = 1$, define

$$\pi_1: A \rightarrow \varphi_1(F_1) \subset F_2 \quad \text{by } \pi_1((f, g)) = \varphi_1(g) \text{ for all } (f, g) \in A.$$

In what follows, we will call π_t a point evaluation of A at t . There is a canonical map $\pi_e: A \rightarrow F_1$ defined by $\pi_e(f, g) = g$ every $(f, g) \in A$. It is a surjective map.

The class of all C*-algebras described above will be denoted by \mathcal{C} . If $A \in \mathcal{C}$, then A is the pull-back of

$$\begin{array}{ccc} A & \dashrightarrow & C([0, 1], F_2) \\ \downarrow \pi_e & & \downarrow (\pi_0, \pi_1) \\ F_1 & \xrightarrow{(\varphi_0, \varphi_1)} & F_2 \oplus F_2. \end{array} \quad (\text{e 6.1})$$

Every such pull-back is an algebra in \mathcal{C} . Infinite dimensional C*-algebras in \mathcal{C} are sometimes called one-dimensional non-commutative finite CW complexes (NCCW) (see [12] and [13]) and Elliott–Thomsen building blocks (see [15]).

Suppose that

$$F_1 = M_{R_1}(\mathbb{C}) \oplus M_{R_2}(\mathbb{C}) \oplus \cdots \oplus M_{R_l}(\mathbb{C})$$

and

$$F_2 = M_{r_1}(\mathbb{C}) \oplus M_{r_2}(\mathbb{C}) \oplus \cdots \oplus M_{r_k}(\mathbb{C}).$$

In what follows we may write

$$C([0, 1], F_2) = \bigoplus_{j=1}^k C([0, 1]_j, M_{r_j}),$$

where $[0, 1]_j$ denotes the j -th interval.

Denote by \mathcal{C}_0 the class of all C*-algebras A in \mathcal{C} which satisfy the following conditions:

- (1) $K_1(A) = \{0\}$; (2) $K_0(A)_+ = \{0\}$; (3) $0 \notin \overline{T(A)}^w$.

C*-algebras in \mathcal{C}_0 are stably projectionless. Condition (3) is equivalent to compact spectrum.

Examples of C*-algebras in \mathcal{C}_0 can be found in [41]. Let $F_1 = M_k$ for some $k \geq 1$ and $F_2 = M_{(m+1)k}$ for some $m \geq 1$. Define $\psi_0, \psi_1: F_1 \rightarrow F_2$ by

$$\psi_0(a) = \text{diag}(\overbrace{a, a, \dots, a}^m, 0) \quad \text{and} \quad \psi_1(a) = \text{diag}(\overbrace{a, a, \dots, a}^{m+1})$$

for all $a \in F_1$. Let us write

$$A = A(F_1, F_2, \psi_0, \psi_1) := R(k, m, m+1). \quad (\text{e 6.2})$$

Then, as shown in [41], $K_0(A) = \{0\} = K_1(A)$ and it is easy to check that $0 \notin \overline{T(A)}^w$. Let $e \in R(k, m, m+1)$ be a strictly positive element. Then (see Definition 4.9)

$$\lambda_s(R(k, m, m+1)) = \inf \{d_\tau(e) : \tau \in T(R(k, m, m+1))\} = m/(m+1).$$

Denote by \mathcal{R}_{az} the class of C^* -algebras which are finite direct sums of C^* -algebras as in (e 6.2).

Denote by \mathcal{C}_0^0 the subclass of C^* -algebras in \mathcal{C}_0 which also satisfy the stronger condition (2') $K_0(A) = \{0\}$.

Let $F_1 = \mathbb{C} \oplus \mathbb{C}$, $F_2 = M_{2n}(\mathbb{C})$. For $(a, b) \in \mathbb{C} \oplus \mathbb{C} = F_1$, define

$$\psi_0(a, b) = \text{diag}(\underbrace{a, a \dots a}_{n-1}, \underbrace{b, b \dots b}_{n-1}, 0, 0)$$

and

$$\psi_1(a, b) = \text{diag}(\underbrace{a, a \dots a}_n, \underbrace{b, b \dots b}_n).$$

Then $A(F_1, F_2, \psi_0, \psi_1) = A$ has the property that $K_0(A)$ is equal to $\{(k, -k) \in \mathbb{Z} \oplus \mathbb{Z}\}$ (which is isomorphic to \mathbb{Z}) but $K_0(A)_+ = \{0\}$. Also, $K_1(A) = \{0\}$. Thus $A \in \mathcal{C}_0$ but $A \notin \mathcal{C}_0^0$.

Let \mathcal{C}' denote the class of all full hereditary sub- C^* -algebras of C^* -algebras in \mathcal{C} , let \mathcal{C}'_0 denote the class of all full hereditary sub- C^* -algebras of C^* -algebras in \mathcal{C}_0 , and let $\mathcal{C}'_0{}^0$ denote the class of all full hereditary sub- C^* -algebras of C^* -algebras in \mathcal{C}_0^0 .

Remark 6.2. Let $A = A(F_1, F_2, \psi_0, \psi_1) \in \mathcal{C}_0$. Then $\tilde{A} \in \mathcal{C}$. Moreover, $\tilde{A} = A(F'_1, F_2, \psi'_0, \psi'_1)$ with both ψ'_0 and ψ'_1 unital, defined as follows:

Let $F'_1 = F_1 \oplus \mathbb{C}$ and let $p = \psi_0(1_{F_1}) \in F_2$ and $q = \psi_1(1_{F_1}) \in F_2$. Define $\psi'_0, \psi'_1: F'_1 \rightarrow F_2$ by

$$\psi'_0((a, \lambda)) = \psi_0(a) \oplus \lambda \cdot (1_{F_2} - p) \quad \text{and} \quad \psi'_1((a, \lambda)) = \psi_1(a) \oplus \lambda \cdot (1_{F_2} - q)$$

for all $a \in F_1$ and $\lambda \in \mathbb{C}$.

One checks that $K_0(\tilde{A})$ is finitely generated (see [22, Proposition 3.4]). In fact, $K_0(\tilde{A})_+$ is finitely generated (see [22, Theorem 3.15]). Let $\pi: \tilde{A} \rightarrow \mathbb{C}$ denote the quotient map. Suppose that $\{[p_i] : 1 \leq i \leq k\}$ generates the semigroup $K_0(\tilde{A})_+$. Let $x \in K_0(A) \subset K_0(\tilde{A})$. Then

$$x = \sum_{i=1}^k (m_i [p_i] - n_i [p_i]) = [p] - [q],$$

where $m_i \geq 0, n_i \geq 0$ and $p, q \in M_N(\tilde{A})$ (for some integer $N \geq 1$) are projections such that

$$[p] = \sum_{i=1}^k m_i [p_i] \quad \text{and} \quad [q] = \sum_{i=1}^k n_i [p_i].$$

One also has, since $x \in K_0(A)$, $\pi(p)$ and $\pi(q)$ are equivalent in M_N . Let n denote the rank of $\pi(p)$ and r_i the rank of $\pi(p_i)$, $1 \leq i \leq k$. Then

$$\sum_{i=1}^k m_i r_i = n = \sum_{i=1}^k n_i r_i.$$

Consequently,

$$\begin{aligned} & \left(\sum_{i=1}^k (m_i ([p_i] - r_i [1_{\tilde{A}}]) - (n_i [p_i] - r_i [1_{\tilde{A}}])) \right) \\ &= \left(\sum_{i=1}^k m_i [p_i] - n [1_{\tilde{A}}] \right) - \left(\sum_{i=1}^k n_i [p_i] - n [1_{\tilde{A}}] \right) = \sum_{i=1}^k (m_i [p_i] - n_i [p_i]) = x. \end{aligned} \tag{e 6.3}$$

It follows that $K_0(A)$ is generated by

$$\{([p_i] - r_i [1_{\tilde{A}}]) : 1 \leq i \leq k\}.$$

In other words, $K_0(A)$ is finitely generated.

Since $A \in \mathcal{C}_0$, either ψ_0 or ψ_1 is not unital. Hence at least one of ψ'_0 and ψ'_1 is non-zero on the second direct summand \mathbb{C} in $F'_1 = F_1 \oplus \mathbb{C}$.

Proposition 6.3. (1) *Let $A \in \mathcal{C}'$. Then, for any $a_1, a_2 \in A_+$, $a_1 \lesssim a_2$ if and only if*

$$d_{\text{tr} \circ \pi}(\pi(a_1)) \leq d_{\text{tr} \circ \pi}(\pi(a_2))$$

for every irreducible representation π of A , where we use “tr” for the tracial state on matrix algebras.

(2) *Let $A \in \mathcal{C}'$, and let $c \in A_+ \setminus \{0\}$. Then c is full if and only if $\tau(c) > 0$ for any $\tau \in T(A)$.*

Proof. (1) We first consider the case that $A \in \mathcal{C}$. By considering \tilde{A} , one sees that this case follows from 3.18 of [22].

Since a C^* -algebra $A \in \mathcal{C}'$ is a hereditary sub- C^* -algebra of some B in \mathcal{C} , it is easy to see that A also has the above-mentioned comparison property.

(2) Let us first assume again that $A \in \mathcal{C}$. It is clear that if $c \in A_+$ and $\tau(c) = 0$, for some $\tau \in T(A)$ then c has zero value somewhere in $\text{Sp}(A) = \bigsqcup_j (0, 1)_j \cup \text{Sp}(F_1)$. Therefore c is in a proper closed two-sided ideal of A .

Now assume that $\tau(c) > 0$ for all $\tau \in T(A)$. It follows that $\pi(c) > 0$ for every finite dimensional irreducible representation of A . Therefore c is full in A . In general, let A be a full hereditary sub- C^* -algebra of $B \in \mathcal{C}$. Let $c \in A_+$. Then $c \in A_+$ is full if and only if it is full in B . Therefore, the general case follows from the case that $A \in \mathcal{C}$. \square

Proposition 6.4. (1) Every C^* -algebra in \mathcal{C}' has stable rank one;

(2) If $A \in \mathcal{C}$ and A is unital, then the exponential rank of A is at most $2 + \varepsilon$. If $A \in \mathcal{C}$ and A is not unital, then \tilde{A} has exponential rank at most $2 + \varepsilon$;

(3) Every C^* -algebra in \mathcal{C} is semiprojective;

(4) Let $A \in \mathcal{C}$ and let $k \geq 1$ be an integer. Suppose that every irreducible representation of A has dimension at least k . Then, for any $f \in \text{LAff}_{0+}(\overline{T(A)}^w)$ with $f \leq 1$, there exists a positive element $a \in M_2(A)$ such that

$$\max_{\tau \in T(A)} |d_\tau(a) - f(\tau)| \leq 2/k.$$

Proof. (1) follows from 3.3 of [22].

(2) follows from 3.16 of [22] (see also [36, 5.19]).

(3) It was shown in [12] that every C^* -algebra in \mathcal{C} is semiprojective.

(4) follows the same proof as 10.4 of [22]. \square

7. Maps from 1-dimensional non-commutative complexes

Lemma 7.1 ([6, Lemma 2.1]). *Let A be a simple C^* -algebra with $A = \text{Ped}(A)$ and $n \geq 1$ be an integer.*

Let $a \in M_n(\tilde{A})_+ \setminus \{0\}$ be such that 0 is a limit point of the spectrum of a . Then, for any $\varepsilon > 0$, there exist $\delta > 0$ and a continuous affine function $f: T_1(A) \rightarrow \mathbb{R}_+$ with $f(0) = 0$ such that

$$d_\tau((a - \varepsilon)_+) < f(\tau) < d_\tau((a - \delta)_+) \quad \text{for all } \tau \in \overline{T(A)}^w.$$

Proof. This is essentially proved in the proof of Lemma 2.1 of [6]. Note that $d_\tau(b) > 0$ for any $b \in M_n(\tilde{A})_+ \setminus \{0\}$ and for any $\tau \in T_1(A) \setminus \{0\}$. As the proof there, there are

$$0 < \delta < \eta_1 < \eta_2 < \varepsilon$$

such that

$$d_\tau((a - \varepsilon)_+) < d_\tau((a - \eta_1)_+) < d_\tau((a - \eta_2)_+) < d_\tau((a - \delta)_+)$$

for all $\tau \in \overline{T(A)}^w$. Let $f_n(\tau) = \tau(f_{1/n}((a - \eta_2)_+))$ for all $\tau \in T_1(A)$. Then

$$f_n \nearrow \widehat{(a - \eta_2)_+}$$

on the compact set $\overline{T(A)}^w$ and $f_n \in \text{Aff}_{0+}(T_1(A))$. The rest of the proof is a compactness argument and an application of the Portmanteau theorem (and $f = f_{n_0}$ for some large n_0). \square

Lemma 7.2. *Let A be a (non-unital) simple C^* -algebra which almost has stable rank one. Suppose that $A = \text{Ped}(A)$ and the canonical map*

$$\iota: W(A) \rightarrow \text{LAff}_{b,0+}(\overline{T(A)}^w)$$

is surjective.

Let $0 \leq a \leq 1$ be a non-zero element of A which is not Cuntz equivalent to a projection. Then, for any $\varepsilon > 0$ there exist $\delta > 0$ and an element $e \in A$ with

$$0 \leq f_\varepsilon(a) \leq e \leq f_\delta(a) \tag{e7.1}$$

such that the function $\tau \mapsto d_\tau(e)$ is continuous on $\overline{T(A)}^w$.

Proof. Fix $\varepsilon > 0$. By Lemma 7.1, there are continuous affine functions $g_1, g_2 \in \text{Aff}_0(T_1(A))$ such that

$$d_\tau(f_{\varepsilon/8}(a)) < g_1(\tau) < d_\tau(f_{\delta_1}(a)) < g_2(\tau) < d_\tau(f_{\delta_2}(a)) \quad \text{for all } \tau \in \overline{T(A)}^w, \tag{e7.2}$$

where $0 < \delta_2 < \delta_1 < 1$. Since ι is surjective, there is $c \in M_m(A)_+$ for some $m \geq 1$ such that $d_\tau(c) = g_2(\tau)$ for all $\tau \in \overline{T(A)}^w$. It follows from Lemma 3.2 and (e7.2) that there exists $x \in M_m(A)$ such that

$$x^*x = c \quad \text{and} \quad xx^* \in \overline{f_{\delta_2}(a)Af_{\delta_2}(a)}.$$

Put $c_0 = xx^*$. Then $0 \leq c_0 \leq 1$. Note that

$$d_\tau(c_0) = d_\tau(c) \quad \text{for all } \tau \in \overline{T(A)}^w. \tag{e7.3}$$

Since g_1 is continuous, there is $m \geq 2$ such that

$$g_1(\tau) < \tau(f_{1/m}(c_0)) \quad \text{for all } \tau \in \overline{T(A)}^w. \tag{e7.4}$$

By (e7.2) and Lemma 3.2 again, there is a unitary u in the unitization of $\overline{f_{\delta_2}(a)Af_{\delta_2}(a)}$ such that

$$u^* f_{\varepsilon/8}(f_{\varepsilon/8}(a))u \in \overline{f_{1/m}(c_0)Af_{1/m}(c_0)}. \tag{e7.5}$$

Set $c_1 = uc_0u^*$. Then

$$f_{\varepsilon/8}(f_{\varepsilon/8}(a)) \in \overline{f_{1/m}(c_1)Af_{1/m}(c_1)} \subset \overline{c_1Ac_1}. \tag{e7.6}$$

There is a $g \in C_0((0, 1])$ with $0 \leq g \leq 1$ such that $g(t) \neq 0$ for all $t \in (0, 1]$,

$$g(t)f_{1/m} = f_{1/m}.$$

Put $e = g(c_1)$. Then

$$\langle e \rangle = \langle c_1 \rangle = \langle c_0 \rangle = \langle c \rangle.$$

Moreover,

$$d_\tau(e) = d_\tau(c_1) = g_2(\tau) \quad \text{for all } \tau \in \overline{T(A)}^w \quad (\text{e7.7})$$

and

$$f_\varepsilon(a) \leq f_{\varepsilon/8}(f_{\varepsilon/8}(a)) \leq e \leq f_{\delta_2/2}(a). \quad (\text{e7.8})$$

Choose $\delta = \delta_2/2$. □

The following theorem is a restatement of a result of Robert.

Theorem 7.3 ([43, Proposition 6.2.3]). *Let A be a stably projectionless simple C^* -algebra with stable rank one. Suppose that $A = \text{Ped}(A)$ and the canonical map*

$$\iota: \text{Cu}(A) \rightarrow \text{LAff}_{0+}(\overline{T(A)}^w)$$

is an isomorphism of ordered semigroups. Then the map defined in (6.6) of [43] is an isomorphism of ordered semigroups.

Moreover, if $a, b \in (\tilde{A} \otimes \mathcal{K})_+$ with $\langle \pi(a) \rangle = k < +\infty$, $\langle \pi(b) \rangle = m < +\infty$, where $\pi: \tilde{A} \rightarrow \mathbb{C}$ is the quotient map, are such that

$$d_\tau(a) + m < d_\tau(b) + k \quad \text{for all } \tau \in \overline{T(A)}^w, \quad (\text{e7.9})$$

then

$$\langle a \rangle + m \langle 1_{\tilde{A}} \rangle \leq \langle b \rangle + k \langle 1_{\tilde{A}} \rangle. \quad (\text{e7.10})$$

Furthermore, if either $\langle a \rangle$, or $\langle b \rangle$ is not represented by a projection, and

$$d_\tau(a) + m \leq d_\tau(b) + k \quad \text{for all } \tau \in \overline{T(A)}^w, \quad (\text{e7.11})$$

then $\langle a \rangle \leq \langle b \rangle$.

Proof. The proof of 6.2.3 of [43] applies since we assume that A has stable rank one and the conclusion of 6.2.1 of [43] holds for $A \otimes \mathcal{K}$. Denote the map defined in (6.6) of [43] by Γ . For the reader's convenience we include a detailed proof that the inverse of Γ is order preserving, since we will use this in an important way. Let us first check that the inverse of Γ restricted to the elements $\text{LAff}_+^{\sim}(\tilde{T}(A))$ is order preserving.

We will use some notation from [43] (but recall that our $\tilde{T}(A)$ is $T_0(A)$ in [43]). Let $a_1 \in \text{Cu}(\tilde{A})$, $\langle a_1 \rangle \neq \langle p \rangle$ for any projection and $\langle \pi(a_1) \rangle = k$. Suppose also that $\langle a_2 \rangle \in \text{Cu}(\tilde{A})$ such that $\langle \pi(a_2) \rangle = m$, where k and m are integers, and

$$\widehat{\langle a_1 \rangle} - k \widehat{\langle 1_{\tilde{A}} \rangle} \leq \widehat{\langle a_2 \rangle} - m \widehat{\langle 1_{\tilde{A}} \rangle}. \quad (\text{e7.12})$$

There are $\beta_1, \beta_2 \in \text{Aff}_+(\tilde{T}(A))$ and $\gamma_1, \gamma_2 \in \text{LAff}_+(\tilde{T}(A))$ such that

$$\langle a_1 \rangle + \beta_1 = k \langle 1_{\tilde{A}} \rangle + \gamma_1, \quad (\text{e7.13})$$

$$\langle a_2 \rangle + \beta_2 = m \langle 1_{\tilde{A}} \rangle + \gamma_2, \quad (\text{e7.14})$$

and

$$\gamma_1 - \beta_1 \leq \gamma_2 - \beta_2. \quad (\text{e7.15})$$

Note that we have used the notation in the proof of 6.2.3, and in particular, we identify $\beta_1, \beta_2, \gamma_1, \gamma_2$ with elements of $\text{Cu}(A)$. Thus,

$$\gamma_1 + \beta_2 \leq \gamma_2 + \beta_1 \quad (\text{e7.16})$$

$$\text{and} \quad \langle a_1 \rangle + \beta_1 + \beta_2 + m\langle 1_{\tilde{A}} \rangle = (k+m)\langle 1_{\tilde{A}} \rangle + \gamma_1 + \beta_2 \quad (\text{e7.17})$$

$$\leq (k+m)\langle 1_{\tilde{A}} \rangle + \gamma_2 + \beta_1 \quad (\text{e7.18})$$

$$= k\langle 1_{\tilde{A}} \rangle + \langle a_2 \rangle + \beta_2 + \beta_1. \quad (\text{e7.19})$$

Put $\beta = \beta_1 + \beta_2$. We have

$$\langle a_1 \rangle + \beta + m\langle 1_{\tilde{A}} \rangle \leq k\langle 1_{\tilde{A}} \rangle + \langle a_2 \rangle + \beta. \quad (\text{e7.20})$$

Exactly as proved in 6.2.3 of [43], one has

$$\langle (a_1 - \varepsilon)_+ \rangle + \beta \ll \langle a_1 \rangle + \beta \quad (\text{e7.21})$$

which implies that, also,

$$\langle (a_1 - \varepsilon)_+ \rangle + \beta + m\langle 1_{\tilde{A}} \rangle \ll \langle a_1 \rangle + \beta + m\langle 1_{\tilde{A}} \rangle. \quad (\text{e7.22})$$

Therefore,

$$\langle (a_1 - \varepsilon)_+ \rangle + \beta + m\langle 1_{\tilde{A}} \rangle \ll k\langle 1_{\tilde{A}} \rangle + \langle a_2 \rangle + \beta. \quad (\text{e7.23})$$

Since A has stable rank one, by weak cancellation ([49, 4.3]),

$$\langle (a_1 - \varepsilon)_+ \rangle + m\langle 1_{\tilde{A}} \rangle \leq \langle a_2 \rangle + k\langle 1_{\tilde{A}} \rangle. \quad (\text{e7.24})$$

It follows that

$$\langle a_1 \rangle + m\langle 1_{\tilde{A}} \rangle \leq \langle a_2 \rangle + k\langle 1_{\tilde{A}} \rangle. \quad (\text{e7.25})$$

In particular, this shows that Γ is injective.

Note that, above, we do not assume that $\langle a_2 \rangle$ is not represented by a projection. Therefore it remains to show the following:

If $(\langle a \rangle - k\langle 1_{\tilde{A}} \rangle)^\wedge < (\langle b \rangle - m\langle 1_{\tilde{A}} \rangle)^\wedge$ on $\overline{\text{T}(A)}^w$, then

$$\langle a \rangle - k\langle 1_{\tilde{A}} \rangle \leq \langle b \rangle - m\langle 1_{\tilde{A}} \rangle$$

for all $\langle a \rangle - k\langle 1_{\tilde{A}} \rangle, \langle b \rangle - m\langle 1_{\tilde{A}} \rangle \in \text{Cu}^\sim(A)$.

We only need to consider the case that $\langle a \rangle$ is represented by a projection. Then $\langle \hat{a} \rangle$ is continuous. It follows that there are non-zero elements $\beta_0, \beta \in \text{Aff}_+(\tilde{\text{T}}(A))$ such that

$$(\langle a \rangle + \beta + \beta_0 + m\langle 1_{\tilde{A}} \rangle)^\wedge < (\langle b \rangle + \beta + k\langle 1_{\tilde{A}} \rangle)^\wedge \quad \text{on } \overline{\text{T}(A)}^w. \quad (\text{e7.26})$$

Since A is stably projectionless, from what has been proved,

$$\langle a \rangle + \beta + \beta_0 + m\langle 1_{\tilde{A}} \rangle \leq \langle b \rangle + \beta + k\langle 1_{\tilde{A}} \rangle. \quad (\text{e7.27})$$

Then, since $\langle a \rangle$ is represented by a projection,

$$\begin{aligned} \langle a \rangle + \beta + (1/2)\beta_0 + m\langle 1_{\tilde{A}} \rangle &\ll \langle a \rangle + \beta + \beta_0 + m\langle 1_{\tilde{A}} \rangle \\ &\leq \langle b \rangle + \beta + k\langle 1_{\tilde{A}} \rangle. \end{aligned} \quad (\text{e7.28})$$

By the weak cancellation,

$$\langle a \rangle + (1/2)\beta_0 + m\langle 1_{\tilde{A}} \rangle \leq \langle b \rangle + k\langle 1_{\tilde{A}} \rangle. \quad (\text{e7.29})$$

It follows that

$$\langle a \rangle + m\langle 1_{\tilde{A}} \rangle \leq \langle b \rangle + k\langle 1_{\tilde{A}} \rangle. \quad (\text{e7.30})$$

□

Lemma 7.4. *With the same assumptions on A as in Theorem 7.3, we have the following statement: Let $0 \leq a \leq 1$ be a non-zero element of $\tilde{A} \otimes \mathcal{K}$ with $\pi(a)$ a projection of rank m for some integer $m \geq 1$ and a not Cuntz equivalent to a projection. Then, for any $1/2 > \varepsilon > 0$ there exist $1 > \delta > 0$ and an element $e \in \tilde{A} \otimes \mathcal{K}$ with*

$$0 \leq f_\varepsilon(a) \leq e \leq f_{\delta/2}(a) \quad (\text{e7.31})$$

such that the function $\tau \mapsto d_\tau(e)$ is continuous on $\overline{\mathbb{T}(A)}^w$.

Proof. Note that this statement is similar to that of Lemma 7.2 (the case $m = 0$). By the last statement of Lemma 3.2, we may assume that, for any $\varepsilon > 0$, there exists $n \geq 1$, $f_\varepsilon(a) \in M_n(\tilde{A})$. By Lemma 7.1, there exists $f \in \text{Aff}_{0+}(\mathbb{T}_1(A))$ such that, for some $\delta > 0$,

$$d_\tau(f_\varepsilon(a)) < f(\tau) < d_\tau(f_\delta(a)) \quad \text{for all } \tau \in \overline{\mathbb{T}(A)}^w.$$

Since we assume that $\pi(a)$ is a projection, $\pi(f_\varepsilon(a)) = \pi(f_\delta(a)) = \pi(a)$, where $\pi: \tilde{A} \rightarrow \mathbb{C}$ is the quotient map. The surjectivity of Γ in Theorem 7.3 implies that there is $c \in (\tilde{A} \otimes \mathcal{K})_+$ such that $\pi(c) = \pi(a)$,

$$d_\tau(c) = f(\tau) \quad \text{and} \quad \langle f_{\varepsilon/8}(a) \rangle \ll \langle c \rangle \ll \langle f_\delta(a) \rangle. \quad (\text{e7.32})$$

It remains to show that we can find e with $\langle e \rangle = \langle c \rangle$ but also satisfies (e7.31). For this we will use the same argument used in the proof of Lemma 7.2. Since \tilde{A} has stable rank one, the proof may be completed as in Lemma 7.2. □

We shall need the following two lemmas.

Lemma 7.5 ([6, Lemma 2.2]). *Let A and $a \in (\tilde{A} \otimes \mathcal{K})_+$ be as in Lemma 7.4. Then there exists a sequence $(a_n)_{n=1}^\infty$ of elements in $(\tilde{A} \otimes \mathcal{K})_+$ which satisfies the following:*

$$(1) \quad \langle a \rangle = \sup_n \langle a_n \rangle;$$

- (2) $a_n \in M_{n(k)}(\tilde{A})$ for some $n(k) \in \mathbb{N}$ and $\langle \pi(a_n) \rangle = \langle \pi(a) \rangle$, where $\pi: \tilde{A} \rightarrow \mathbb{C}$ is the quotient map;
- (3) the function $\tau \mapsto d_\tau(a_n)$ is continuous on $\overline{T(A)}^w$ for each $n \in \mathbb{N}$; and
- (4) $d_\tau(a_n) < d_\tau(a_{n+1})$ for all $\tau \in \overline{T(A)}^w$ and $n \in \mathbb{N}$.

Lemma 7.6. *Let A be as in Theorem 7.3. Suppose that $a, b \in \text{Ped}(\tilde{A} \otimes \mathcal{K})_+$ (with $0 \leq a \leq 1$ and $0 \leq b \leq 1$) such that neither are Cuntz equivalent to a projection. Suppose that $\langle a \rangle \ll \langle b \rangle$. Then there exist $\delta > 0$ and $c \in \text{Ped}(\tilde{A} \otimes \mathcal{K})_+$ with $0 \leq c \leq 1$ such that*

$$\langle a \rangle \leq \langle f_\delta(c) \rangle, \quad f_{\delta/2}(c) \leq f_{\delta/4}(b), \quad \text{and} \quad \inf \{ \tau(f_\delta(c)) - d_\tau(a) : \tau \in \overline{T(A)}^w \} > 0. \quad (\text{e 7.33})$$

Proof. By Lemma 7.5, choose $b_n \in (\tilde{A} \otimes \mathcal{K})_+$ such that (b_n) satisfies (1)–(4) in Lemma 7.5. Since $\langle a \rangle \ll \langle b \rangle$, there is $n_0 \geq 1$ such that $\langle a \rangle \leq \langle b_n \rangle$ for all $n \geq n_0$. Therefore we have

$$d_\tau(a) \leq d_\tau(b_{n_0}) < d_\tau(b_{n_0+1}) < d_\tau(b_{n_0+2}) < d_\tau(b_{n_0+3}) \leq d_\tau(b). \quad (\text{e 7.34})$$

Note that

$$\tau(f_{1/n}(b)) \nearrow d_\tau(b) \quad \text{and} \quad \tau(f_{1/n}(b_{n_0+1})) \nearrow d_\tau(b_{n_0+1}).$$

It follows from 5.4 of [34], for example, that there exists $n_1 \geq 1$ such that, for all $n \geq n_1$,

$$\tau(f_{1/n}(b)) > d_\tau(b_{n_0+2}) \quad \text{and} \quad \tau(f_{1/n}(b_{n_0+1})) > d_\tau(b_{n_0}) \quad \text{for all } \tau \in \overline{T(A)}^w.$$

Note that $\pi(f_{1/2n}(b)) = \pi(b)$ and $\langle \pi(b_n) \rangle = \langle \pi(b) \rangle$. By Theorem 7.3, we conclude that

$$\langle f_{1/2n}(b) \rangle \geq \langle b_{n_0+2} \rangle \quad \text{and} \quad \langle f_{1/2n}(b_{n_0+1}) \rangle \geq \langle b_{n_0} \rangle.$$

Put $c = b_{0+1}$. Since A has stable rank one, one may assume that $f_{1/2n}(c) \leq f_{1/4n}(b)$. Thus we may choose $0 < \delta < 1/2n_1$.

Since $\overline{T(A)}^w$ is compact and both functions in the above inequality are continuous, together with (e 7.34), we obtain

$$\inf \{ \tau(f_\delta(b)) - d_\tau(a) : \tau \in \overline{T(A)}^w \} > 0. \quad \square$$

In what follows, $\mathcal{C}^{(1)}$ is the collection of all C^* -algebras which are inductive limits of full hereditary sub- C^* -algebras of 1-dimensional non-commutative CW complexes with trivial K_1 groups whose connecting maps are injective.

Definition 7.7. Fix a C*-algebra $C \in \mathcal{C}^{(1)}$. A C*-algebra A is said to have the property (R) associated with C , if the following condition holds: For any finite subset $\mathcal{F} \subset C$ and $\varepsilon > 0$ there exists a finite subset $G \subset \text{Cu}^\sim(C)$ such that for any two homomorphisms $\varphi, \psi: C \rightarrow A$, if

$$\text{Cu}^\sim(\varphi)(g') \leq \text{Cu}^\sim(\psi)(g) \quad \text{and} \quad \text{Cu}^\sim(\psi)(g') \leq \text{Cu}^\sim(\varphi)(g) \quad (\text{e7.35})$$

for all $g', g \in G$ with $g' \ll g$, then there exists a unitary $u \in \tilde{A}$ such that

$$\|u^* \varphi(f)u - \psi(f)\| < \varepsilon \quad \text{for all } f \in \mathcal{F}. \quad (\text{e7.36})$$

This definition is taken from 3.3.1 of [43] and we adapt the notation from there. Note, by 3.3.1 of [43], every C*-algebra with stable rank one has the property (R) associated with C .

Theorem 7.8 (See [43, Theorem 3.3.1], [36, Theorem 5.2.7], and [22, Theorem 8.4]). *Let C be in $\mathcal{C}^{(1)}$ and assume that $\text{Ped}(C) = C$ and let $\Delta: C^{q \cdot I} \setminus \{0\} \rightarrow (0, 1)$ be an order preserving map. Then, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exist a finite subset $\mathcal{G} \subset C$, a finite subset $\mathcal{P} \subset \mathbf{K}_0(C)$, a finite subset $\mathcal{H}_1 \subset C_+^I \setminus \{0\}$, a finite subset $\mathcal{H}_2 \subset C_{\text{s.a.}}$, $\delta > 0$, and $\gamma > 0$ satisfying the following condition: for any two \mathcal{G} - δ -multiplicative contractive completely positive maps $\varphi_1, \varphi_2: C \rightarrow A$ for some C*-algebra A with $A = \text{Ped}(A)$ which is σ -unital, simple, stably projectionless, almost has stable rank one, and has the property that the map*

$$\text{Cu}_+(A) \rightarrow \text{LAff}_{0+}(\overline{\mathbf{T}(A)}^w)$$

is an isomorphism of ordered semigroups, such that

$$[\varphi_1]_{\mathcal{P}} = [\varphi_2]_{\mathcal{P}}, \quad (\text{e7.37})$$

$$\tau(\varphi_i)(a) \geq \Delta(\hat{a}) \quad \text{for all } a \in \mathcal{H}_1 \text{ and for all } \tau \in \overline{\mathbf{T}(A)}^w, \quad (\text{e7.38})$$

$$|\tau(\varphi_1(b)) - \tau(\varphi_2(b))| < \gamma \quad \text{for all } b \in \mathcal{H}_2 \text{ and for all } \tau \in \overline{\mathbf{T}(A)}^w, \quad (\text{e7.39})$$

there exists a unitary $u \in \tilde{A}$ such that

$$\|u^* \varphi_2(f)u - \varphi_1(f)\| < \varepsilon \quad \text{for all } f \in \mathcal{F}.$$

Proof. We will use 3.3.1 of [43].

Let $\varepsilon > 0$ be given. There exists ε_0 with $\varepsilon/16 > \varepsilon_0 > 0$ satisfying the following condition: In any C*-algebra, if $0 \leq x \leq 1$ is an element in the C*-algebra and $\|xg - g\| < \varepsilon_0$ and $\|gx - g\| < \varepsilon_0$ for any $\|g\| \leq 1$ in the C*-algebra, then

$$\|x^{1/2}g - gx^{1/2}\| < \varepsilon/64. \quad (\text{e7.40})$$

Let us first assume that C is a single full hereditary sub-C*-algebra of a 1-dimensional non-commutative CW complex. Fix $\varepsilon_0 > 0$ as above and $\mathcal{F} \subset C$. Let

$G \subset \text{Cu}^\sim(C)$ be as required by Property (R) associated with C for $\varepsilon_0/16$ (in place of ε) and \mathcal{F} . Without loss of generality, we may assume that \mathcal{F} is contained in the unit ball of A .

Recalling that C has stable rank one, as shown in [43], we may assume that G consists of a finite subset $\mathcal{P} \subset \text{K}_0(C)$ and a finite subset

$$\{[a_1] - k_1[1_{\tilde{A}}], [a_2] - k_2[1_{\tilde{C}}], \dots, k_m[1_{\tilde{C}}]\}$$

of the Cuntz semigroup of $\text{Cu}^\sim(C)$ such that $[a_i]$ can be represented by positive elements $0 \leq a_i \leq 1$ in $\tilde{C} \otimes \mathcal{K}$ which are not Cuntz equivalent to a projection, and k_i are non-negative integers, $i = 1, 2, \dots, m$. Write

$$\mathcal{P} = \{z_1 - k'_1[1_{\tilde{C}}], z_2 - k'_2[1_{\tilde{C}}], \dots, z_{m_0} - k'_{m_0}[1_{\tilde{C}}]\},$$

where the elements z_i are represented by projections in $\tilde{C} \otimes \mathcal{K}$. Note here we assume that $\langle \pi(a_i) \rangle = k_i \langle 1 \rangle$ and $[\pi_*(z_i)] = k'_i[1]$, where $\pi: \tilde{C} \rightarrow \mathbb{C}$ is the quotient map, $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, m_0$.

Suppose that

$$\langle a_i \rangle + k_j[1_{\tilde{C}}] \ll \langle a_j \rangle + k_i[1_{\tilde{C}}].$$

For each of these pairs i, j , put $a_{ij} = a_i \oplus 1_{M_{k_j}}$ and $a_{ji} = a_j \oplus 1_{M_{k_i}}$. Then, since $\tilde{C} \otimes \mathcal{K}$ also has stable rank one, there are a number $1/4 > \eta(i, j) > 0$ and an element $0 \leq c_{i,j} \leq 1$ in $(\tilde{C} \otimes \mathcal{K})_+$ such that

$$\langle a_{ij} \rangle \leq \langle f_{\eta_{i,j}}(c_{i,j}) \rangle \quad \text{and} \quad f_{\eta_{i,j}/2}(c_{i,j}) \leq f_{\eta_{i,j}/4}(a_{ji}). \quad (\text{e7.41})$$

Since a_{ij} is not Cuntz equivalent to a projection, we may choose $\eta(i, j)$ so that

$$f_{\eta_{i,j}/4}(a_{ji}) - f_{\eta_{i,j}/2}(c_{i,j}) \neq 0.$$

Choose a finite subset $\mathcal{H}_1 \subset C_+$ which contains non-zero positive elements $b_{i,j}$ such that

$$b_{i,j} \lesssim f_{\eta_{i,j}/4}(a_{ji}) - f_{\eta_{i,j}/2}(c_{i,j})$$

for all possible pairs of i and j such that $\langle a_{ij} \rangle \ll \langle a_{ji} \rangle$.

Let

$$\delta_0 = \inf \{ \Delta(\hat{g}) : g \in \mathcal{H}_1 \}. \quad (\text{e7.42})$$

Choose a finite subset \mathcal{H}'_2 of $(\tilde{C} \otimes \mathcal{K})_+$ which contains $f_{\eta_{i,j}}(c_{i,j})$, $f_{\eta_{i,j}/2}(c_{i,j})$, $f_{\eta_{i,j}/4}(a_{ji})$ for all possible i, j as described above.

Let the finite subset $\mathcal{H}_2 \subset C_{s.a.}$ containing \mathcal{H}_1 and $\delta_1 > 0$ be such that

$$|\tau(h_1^\sim(g)) - \tau(h_2^\sim(g))| < \delta_0/16 \quad \text{for all } g \in \mathcal{H}_1 \cup \mathcal{H}'_2 \quad (\text{e7.43})$$

and for all $\tau \in \overline{\text{T}(B)}^w$, whenever $h_1, h_2: C \rightarrow B$ are homomorphisms with B any C^* -algebra with $\text{T}(B) \neq \emptyset$ and $0 \notin \overline{\text{T}(B)}^w$ such that

$$|\tau \circ h_1(f) - \tau \circ h_2(f)| < \delta_1 \quad \text{for all } f \in \mathcal{H}_2 \text{ and } \tau \in \overline{\text{T}(B)}^w, \quad (\text{e7.44})$$

where $h_i^\sim: \tilde{C} \rightarrow \tilde{B}$ is the unital extension of h_i , $i = 1, 2$.

Put $\gamma = \min\{\delta_0/16, \delta_1/4\}$. Since 1-dimensional NCCW complexes are semiprojective ([12]), by Lemma 3.7, with suitable \mathcal{G} and δ , there are homomorphisms $\psi_i: C \rightarrow A \otimes \mathcal{K}$ such that

$$(\psi_i)_{*0}|_{\mathcal{P}} = [\varphi_i]|_{\mathcal{P}} \quad \text{and} \quad \|\psi_i(g) - \varphi_i(g)\| < \min\{\varepsilon_0/16, \gamma\}, \quad i = 1, 2, \quad (\text{e 7.45})$$

for all $g \in \mathcal{F} \cup \mathcal{H}_1 \cup \mathcal{H}_2$, where φ_1 and φ_2 are \mathcal{G} - δ -multiplicative completely positive contractive maps from C to a C^* -algebra A satisfying the assumptions of the theorem.

Since $\text{Ped}(C) = C$, $\psi_i(C) \subset \text{Ped}(A \otimes \mathcal{K})$.

Assume that $\varphi_1, \varphi_2: C \rightarrow A$ have the described properties for the above defined $\mathcal{G}, \delta, \mathcal{P}, \mathcal{H}_1, \mathcal{H}_2, \gamma$.

With $\psi_i: C \rightarrow A \otimes \mathcal{K}$ chosen satisfying (e 7.45), $i = 1, 2$, then

$$(\psi_1)_{*0}|_{\mathcal{P}} = (\psi_2)_{*0}|_{\mathcal{P}}, \quad (\text{e 7.46})$$

$$\tau \circ \psi_i(g) \geq \delta_0/2 \quad \text{for all } g \in \mathcal{H}_1, \quad (\text{e 7.47})$$

$$|\tau \circ \psi_1(b) - \tau \circ \psi_2(b)| < \delta_0/2 \quad \text{for all } b \in \mathcal{H}_2 \quad (\text{e 7.48})$$

for all $\tau \in \overline{\text{T}(A)}^w$. In particular, if $\langle a_{ij} \rangle \ll \langle a_{ji} \rangle$, then, by the choice of $\mathcal{H}_1, \mathcal{H}_2$, and γ above,

$$d_\tau(\psi_1(a_{ij})) \leq \tau(\psi_1(f_{\eta_{i,j}}(c_{i,j}))) < \delta_0/2 + \tau(\psi_2(f_{\eta_{i,j}}(c_{i,j}))) \quad (\text{e 7.49})$$

$$\leq \tau(\psi_2(f_{\eta_{i,j}/4}(a_{ij})) - \tau(\psi_2(f_{\eta_{i,j}}(c_{i,j})))) + \tau(\psi_2(f_{\eta_{i,j}}(c_{i,j}))) \quad (\text{e 7.50})$$

$$\leq d_\tau(a_{ji}) \quad (\text{e 7.51})$$

for all $\tau \in \overline{\text{T}(A)}^w$. Therefore, if $\langle a_{ij} \rangle \ll \langle a_{ji} \rangle$, then

$$\langle \psi_1(a_{ij}) \rangle \leq \langle \psi_2(a_{ji}) \rangle. \quad (\text{e 7.52})$$

Note also, if $\langle a_i \rangle + k'_i \langle 1_{\tilde{C}} \rangle \ll z_l + k'_i \langle 1_{\tilde{C}} \rangle$, then

$$\langle \psi_1(a_i) \rangle + k_l [1_{\tilde{A}}] \ll \text{Cu}^\sim(\psi_1)(z_l + k'_i \langle 1_{\tilde{C}} \rangle) = \text{Cu}^\sim(\psi_2)(z_l + k'_i \langle 1_{\tilde{C}} \rangle).$$

Combining these with (e 7.46), we conclude that, using the terminology of [43],

$$\text{Cu}^\sim(\psi_1(g)) \leq \text{Cu}^\sim(\psi_2(g')) \quad \text{and} \quad \text{Cu}^\sim(\psi_2(g)) \leq \text{Cu}^\sim(\psi_1(g')) \quad (\text{e 7.53})$$

for all $g, g' \in G$ and $g \ll g'$. Since A has the property (R) associated with C , by the choice of G , there exists a unitary $v \in (A \otimes \mathcal{K})^\sim$ such that

$$\|v\psi_2(f)v^* - \psi_1(f)\| < \varepsilon_0/16 \quad \text{for all } f \in \mathcal{F}.$$

From this and (e 7.45), we obtain that

$$\|v\varphi_2(f)v^* - \varphi_1(f)\| < \varepsilon_0/16 + \varepsilon_0/16 \quad \text{for all } f \in \mathcal{F}.$$

Choose $0 \leq e_1, e_2 \leq 1$ in A such that

$$\|\varphi_i(f)e_i - \varphi_i(f)\| < \varepsilon_0/32 \quad \text{and} \quad \|e_i\varphi_i(f) - \varphi_i(f)\| < \varepsilon_0/32 \quad \text{for all } f \in \mathcal{F}. \quad (\text{e7.54})$$

Put $y = e_1v^*e_2$ and $x = y^*y = e_2ve_1e_1v^*e_2$. Then

$$\|\varphi_2(f)x - \varphi_2(f)\| = \|v^*v(\varphi_2(f)x - \varphi_2(f))\| \quad (\text{e7.55})$$

$$< \|v^*v\varphi_2(f)y^*v - v^*v\varphi_2(f)v^*v\| + \varepsilon_0/32 \quad (\text{e7.56})$$

$$< \varepsilon_0/32 + \|v^*(v\varphi_2(f)v^*)e_2 - v^*v\varphi_2(f)v^*v\| + \varepsilon_0/32 < \varepsilon_0/2 \quad (\text{e7.57})$$

for all $f \in \mathcal{F}$. Similarly,

$$\|x\varphi_2(f) - \varphi_2(f)\| < \varepsilon_0/2 \quad \text{for all } f \in \mathcal{F}. \quad (\text{e7.58})$$

Consider the polar decomposition $y = W|y| = Wx^{1/2}$ of y in A^{**} . Since A almost has stable rank one, by Theorem 5 of [39], there exists a unitary $u \in \tilde{A}$ such that $uf_{\varepsilon/16}(x^{1/2}) = Wf_{\varepsilon/16}(x^{1/2})$. By the choice of ε_0 , we have

$$\|u^*\varphi_2(f)u - \varphi_1(f)\| < \varepsilon \quad \text{for all } f \in \mathcal{F}. \quad (\text{e7.59})$$

(Note that if C is a 1-dimensional non-commutative CW complex, then φ_i can be chosen to map C into A , so that v can be chosen in \tilde{A} .)

For the general case, given a finite subset $\mathcal{F} \subset C$, we may assume that $\mathcal{F} \subset C_n$ for some C_n which is a full hereditary sub-C*-algebra of a 1-dimensional non-commutative CW complex. Then the above argument applies. \square

Proposition 7.9. *Let $C \in \mathcal{C}'_0$ and let A be a stably projectionless simple exact C*-algebra with $\mathbf{K}_0(A) = \{0\}$, with stable rank one, and with continuous scale. Suppose that $\text{Cu}(A) = \text{LAff}_{0+}(\overline{\mathbf{T}(A)}^w)$. Let $\varphi: C \rightarrow A$ be a homomorphism. Then, for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset C$, and any integer $n \geq 1$, there is another homomorphism*

$$\varphi_0: C \rightarrow B = B \otimes e_{11} \subset M_n(B) \subset A,$$

where B is a hereditary sub-C*-algebra of A , such that

$$\|\varphi(x) - \varphi_0(x) \otimes 1_n\| < \varepsilon \quad \text{for all } x \in \mathcal{F}. \quad (\text{e7.60})$$

Proof. Fix a strictly positive element $e \in A_+$ with $\|e\| = 1$. We may assume that A is infinite dimensional. There are mutually orthogonal non-zero elements $e_1, e_2, \dots, e_n \in A_+$ such that $\langle e_i \rangle = \langle e_1 \rangle$ in $\text{Cu}(A)$ and $\langle \sum_{i=1}^n e_i \rangle = \langle e \rangle$ (see also the proof of Theorem 5.3). Let $B = \overline{e_1 A e_1} \subset A$. Then, with

$$D := \overline{\left(\sum_{i=1}^n e_i \right) A \left(\sum_{i=1}^n e_i \right)},$$

we have $D \cong M_n(B) \subset A$. Note that $K_0(A) = \{0\}$. So

$$\text{Cu}^\sim(A) = \text{LAff}_+^\sim(\tilde{T}(A))$$

(see Theorem 7.3 above and [43, 6.2.3]). Let $j: \text{LAff}_+^\sim(\tilde{T}(A)) \rightarrow \text{LAff}_+^\sim(\tilde{T}(A))$ be defined by $j(f) = (1/n)f$. Define $\lambda: \text{Cu}^\sim(C) \rightarrow \text{Cu}^\sim(B)$ by $\lambda = j \circ (\text{Cu}^\sim(\varphi))$. By Theorem 1.0.1 of [43], there exists a homomorphism $\varphi'_0: C \rightarrow B$ such that

$$\text{Cu}^\sim(\varphi'_0) = \lambda.$$

Define $\psi: C \rightarrow M_n(B)$ by $\psi(a) = \varphi'_0(a) \otimes 1_n$ for all $a \in C$. Then

$$\text{Cu}^\sim(\psi) = \text{Cu}^\sim(\varphi).$$

It follows from Theorem 1.0.1 of [43] that φ and ψ are approximately unitarily equivalent, as desired. \square

8. Tracially one-dimensional complexes

Definition 8.1. Let A be a simple C^* -algebra with a strictly positive element $a \in A$ with $\|a\| = 1$. Suppose that there exists $1 > f_a > 0$, for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any $b \in A_+ \setminus \{0\}$, there are \mathcal{F} - ε -multiplicative completely positive contractive maps $\varphi: A \rightarrow A$ and $\psi: A \rightarrow D$, with $\varphi(A) \perp D$, i.e., $\varphi(A)D = \{0\}$, for some sub- C^* -algebra $D \subset A$, such that:

$$\|x - (\varphi + \psi)(x)\| < \varepsilon \quad \text{for all } x \in \mathcal{F} \cup \{a\}, \quad (\text{e 8.1})$$

$$D \in \mathcal{C}_0^{0'} \quad (\text{or } D \in \mathcal{C}'_0), \quad (\text{e 8.2})$$

$$\varphi(a) \lesssim b, \quad (\text{e 8.3})$$

$$t(f_{1/4}(\psi(a))) \geq f_a \quad \text{for all } t \in T(D). \quad (\text{e 8.4})$$

Then we shall say $A \in \mathcal{D}_0$ (or $A \in \mathcal{D}$).

Proposition 8.2. Let A be a σ -unital simple C^* -algebra in \mathcal{D} (\mathcal{D}_0). Then, in Definition 8.1, we may further require that $\|\psi(x)\| \geq (1 - \varepsilon)\|x\|$ for all $x \in \mathcal{F}$ and that $\psi(a)$ be strictly positive in D (and so full in D). Moreover, (e 8.3) may be replaced by $c \lesssim b$ for some strictly positive element c of $\varphi(A)A\varphi(A)$.

Proof. Fix a strictly positive element $a \in A$ with $\|a\| = 1$. Let $\varepsilon > 0$, let $\mathcal{F} \subset A$ be a finite subset, and let $b_0 \in A_+ \setminus \{0\}$ be given. Without loss of generality, we may assume that there is $1/16 > \eta > 0$ such that

$$f_\eta(a)x = xf_\eta(a) = x \quad \text{for all } x \in \mathcal{F}.$$

By hypothesis, there exist a sequence of algebras $D_n \in \mathcal{C}'_0$ (or $D_n \in \mathcal{C}^{0'}_0$), and two sequences of completely positive contractive maps $\varphi_n: A \rightarrow A_n$ and $\psi_n: A \rightarrow D_n$, with $D_n \perp \text{Im}\varphi_n$, such that

$$\lim_{n \rightarrow \infty} \|\varphi_n(xy) - \varphi_n(x)\varphi_n(y)\| = 0, \quad (\text{e 8.5})$$

$$\lim_{n \rightarrow \infty} \|\psi_n(xy) - \psi_n(x)\psi_n(y)\| = 0 \quad \text{for all } x, y \in A, \quad (\text{e 8.6})$$

$$\lim_{n \rightarrow \infty} \|x - (\varphi_n(x) + \psi_n(x))\| = 0 \quad \text{for all } x \in A, \quad (\text{e 8.7})$$

$$\varphi_n(a) \lesssim b_0, \quad (\text{e 8.8})$$

$$\tau(f_{1/4}(\psi_n(a))) \geq f_a \quad \text{for all } \tau \in \mathbf{T}(D_n). \quad (\text{e 8.9})$$

Put

$$D'_n = \overline{f_{\eta/2}(\psi_n(a))D_n f_{\eta/2}(\psi_n(a))}, \quad n = 1, 2, \dots$$

By (e 8.9) and Proposition 6.3, $f_{1/4}(\psi_n(a))$ is full in D_n . Therefore $f_{\eta/2}(\psi_n(a))$ is also full in D_n . This implies that $D'_n \in \mathcal{C}'_0$ or $D'_n \in \mathcal{C}^{0'}_0$. Define $\psi_{n,0}: A \rightarrow D'_n$ by

$$\psi_{n,0}(x) = (f_{\eta/2}(\psi_n(a)))^{1/2} \psi_n(x) (f_{\eta/2}(\psi_n(a)))^{1/2} \quad \text{for all } x \in A.$$

It follows that $\psi_{n,0}(a)$ is full in D'_n . Note that

$$f_{1/4}(\psi_{n,0}(a)) = f_{1/4}(\psi_n(a)).$$

Therefore,

$$\tau(f_{1/4}(\psi_{n,0}(a))) \geq f_a \quad \text{for all } \tau \in \mathbf{T}(D'_n).$$

Replacing D by $\overline{\psi(a)D\psi(a)}$, we may add the condition that $\psi(a)$ is a strictly positive element of D .

By Proposition 2.10, $\varphi_n(a)$ is strictly positive in $\overline{\varphi_n(A)A\varphi_n(A)}$. Therefore, one can replace (e 8.3) by the condition that $c \lesssim b$ for any other strictly positive element of $\overline{\varphi_n(A)A\varphi_n(A)}$.

To get the inequality $\|\psi(x)\| \geq (1 - \varepsilon)\|x\|$ for all $x \in \mathcal{F}$, we note that, by (e 8.7) and (e 8.9)

$$\lim_{n \rightarrow \infty} \|\psi_n\| \geq f_a. \quad (\text{e 8.10})$$

Then, by (e 8.6) and (e 8.10), since A is simple,

$$\lim_{n \rightarrow \infty} \|\psi_n(x)\| = \|x\| \quad \text{for all } x \in A. \quad (\text{e 8.11})$$

This implies that, choosing $\psi = \psi_n$ with sufficiently large n , we may always assume that $\|\psi(x)\| \geq (1 - \varepsilon)\|x\|$ for all $x \in \mathcal{F}$. \square

Theorem 8.3. *Let A be a σ -unital simple C^* -algebra in \mathcal{D} (or in \mathcal{D}_0). Then the following statement holds. Fix a strictly positive element $a \in A$ with $\|a\| = 1$ and*

let $1 > \mathfrak{f}_a > 0$ be a positive number associated with a as in Definition 8.1. There is a map $T: A_+ \setminus \{0\} \rightarrow \mathbb{N} \times \mathbb{R}$ ($c \mapsto (N(c), M(c))$ for all $c \in A_+ \setminus \{0\}$) with the following property: For any finite subset $\mathcal{F}_0 \subset A_+ \setminus \{0\}$, any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$, and any $b \in A_+ \setminus \{0\}$, there are \mathcal{F} - ε -multiplicative completely positive contractive maps $\varphi: A \rightarrow A$ and $\psi: A \rightarrow D$ for some sub-C*-algebra $D \subset A$ with $D \perp \varphi(A)$ such that:

$$\|x - (\varphi + \psi)(x)\| < \varepsilon \quad \text{for all } x \in \mathcal{F} \cup \{a\}, \quad (\text{e 8.12})$$

$$D \in \mathcal{C}_0^{0'} \quad (\text{or } \mathcal{C}'_0), \quad (\text{e 8.13})$$

$$\varphi(a) \lesssim b, \quad (\text{e 8.14})$$

$$\|\psi(x)\| \geq (1 - \varepsilon)\|x\| \quad \text{for all } x \in \mathcal{F}, \quad (\text{e 8.15})$$

and $\psi(a)$ is strictly positive in D . Moreover, ψ may be chosen to be T - $\mathcal{F}_0 \cup \{f_{1/4}(a)\}$ -full in \overline{DAD} .

Furthermore, we may ensure that

$$t(f_{1/4}(\psi(a))) \geq \mathfrak{f}_a$$

$$\text{and} \quad t(f_{1/4}(\psi(c))) \geq \frac{\mathfrak{f}_a}{4 \inf\{M(c)^2 \cdot N(c) : c \in \mathcal{F}_0 \cup \{f_{1/4}(a)\}\}}$$

for all $c \in \mathcal{F}_0$ and for all $t \in T(D)$.

Proof. Since A is simple, and $f_{1/32}(a) \in \text{Ped}(A)$, for any $b \in A_+ \setminus \{0\}$, there exist $N_0(b) \in \mathbb{N}$, $M_0(b) > 0$, and $x_1(b), x_2(b), \dots, x_{N_0(b)}(b) \in A$ such that

$$\|x_i(b)\| \leq M_0(b),$$

and

$$\sum_{i=1}^{N_0(b)} x_i(b)^* b x_i(b) = f_{1/32}(a). \quad (\text{e 8.16})$$

Choose an integer n_0 such that $n_0 \mathfrak{f}_a \geq 4$. Set $N(b) = n_0 N_0(b)$ and $M(b) = 2M_0(b)$ for all $b \in A_+ \setminus \{0\}$. Let $T: A_+ \setminus \{0\} \rightarrow \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ be defined by $T(b) = (N(b), M(b))$ for $b \in A_+ \setminus \{0\}$.

Choose $\delta_0 > 0$ and a finite subset $\mathcal{E}_0 \subset A$ such that

$$\left\| \sum_{i=1}^{N_0} \psi(x_i(b))^* \psi(b) \psi(x_i(b)) - f_{1/32}(\psi(a)) \right\| < 1/2^{10} \quad \text{for all } b \in \mathcal{F}_0, \quad (\text{e 8.17})$$

whenever ψ is a \mathcal{E}_0 - δ_0 -multiplicative completely positive contractive map from A into a C*-algebra.

Let $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset A$ be given. Set $\delta = \min\{\varepsilon/4, \delta_0/2\}$ and $\mathcal{E} = \mathcal{F} \cup \mathcal{E}_0 \cup \{a, f_{1/4}(a)\}$. Let $n \geq 1$ be an integer and let $b_0 \in A_+ \setminus \{0\}$.

By the assumption and by Proposition 8.2, there are \mathcal{G} - δ -multiplicative completely positive contractive maps $\varphi: A \rightarrow A$ and $\psi: A \rightarrow D$ for some sub-C*-algebra $D \subset A$ with $\varphi(A) \perp D$ such that $D \in \mathcal{C}_0$ (or $D \in \mathcal{C}_0^{0'}$), $\psi(a)$ is strictly positive in D , and

$$\|x - (\varphi + \psi)(x)\| < \varepsilon \quad \text{for all } x \in \mathcal{G}, \quad (\text{e 8.18})$$

$$D \in \mathcal{C}_0^{0'} \quad (\text{or } \mathcal{C}_0'), \quad (\text{e 8.19})$$

$$\varphi(a) \lesssim b_0, \quad (\text{e 8.20})$$

$$\|\psi(x)\| \geq (1 - \varepsilon)\|x\| \quad \text{for all } x \in \mathcal{F}, \quad (\text{e 8.21})$$

$$\tau(f_{1/4}(\psi(a))) \geq f_a \quad \text{for all } \tau \in \mathbf{T}(D). \quad (\text{e 8.22})$$

At this point, we can apply (the proof of) Theorem 5.7 and Remark 5.8 to conclude that ψ is $T\text{-}\mathcal{F}_0 \cup \{f_{1/4}(a_0)\}$ -full. The last part of the conclusion then follows. \square

Corollary 8.4. *In Definition 8.1, for any integer $k \geq 1$, one may assume that every irreducible representation of D has dimension at least k .*

Proof. Let T be as in the statement of Theorem 8.3. Fix an integer $k \geq 1$. This corollary can be seen by taking \mathcal{F}_0 containing k mutually orthogonal non-zero positive elements e_1, e_2, \dots, e_k with $\|e_i\| = 1$ in Theorem 8.3, as follows.

When \mathcal{F}_0 is chosen. Set

$$\sigma_0 = \frac{f_a}{4 \inf\{M(c)^2 \cdot N(c) : c \in \mathcal{F}_0 \cup \{f_{1/4}(a)\}\}}.$$

There exists $\eta_0 > 0$ such that, if $0 < b_1, b_2 \leq 1$ are in any C*-algebra with $\|b_1 - b_2\| < \eta_0$, then

$$\|f_{1/4}(b_1) - f_{1/4}(b_2)\| < \sigma_0/2. \quad (\text{e 8.23})$$

By 10.1.12 of [37], there exists $\delta_0 > 0$ satisfying the following property: if $0 \leq h_i \leq 1$ and $\|h_i h_j\| < \delta_0$ ($i \neq j : 1 \leq i, j \leq k$) are in a C*-algebra, then there are mutually orthogonal h'_1, h'_2, \dots, h'_k in that C*-algebra such that $\|h_i - h'_i\| < \eta_0$, $i = 1, 2, \dots, k$.

Choose any finite subset \mathcal{F} containing \mathcal{F}_0 and $\delta > 0$ with $\delta < \delta_0$. We apply Theorem 8.3. Then

$$t(f_{1/4}(\psi(e_i))) > \sigma_0 \quad \text{for all } t \in \mathbf{T}(D), \quad i = 1, 2, \dots, k. \quad (\text{e 8.24})$$

By the choice of δ_0 and applying 10.1.12 of [37], there are mutually orthogonal non-zero elements $d_1, d_2, \dots, d_k \in D$ such that

$$\|d_i - \psi(e_i)\| < \eta_0, \quad i = 1, 2, \dots. \quad (\text{e 8.25})$$

It follows that

$$\|f_{1/4}(d_i) - f_{1/4}(\psi(e_i))\| < \sigma_0/2, \quad i = 1, 2, \dots, k.$$

We then estimate that

$$t \circ f_{1/4}(d_i) > t \circ f_{1/4}(\psi(e_i)) - \sigma_0/2 \geq \sigma_0/2 \quad \text{for all } t \in \mathsf{T}(D), \quad i = 1, 2, \dots, k. \quad (\text{e 8.26})$$

Thus, $\pi(D)$ admits k mutually orthogonal non-zero elements in each irreducible representation π which implies $\pi(D)$ has dimension at least k . \square

Note that, if D is in $\mathcal{C}_0^{0'}$ or in \mathcal{C}'_0 , then $M_k(D)$ is in $\mathcal{C}_0^{0'}$ or in \mathcal{C}'_0 for every integer $k \geq 1$. Therefore, the following proposition follows immediately from the definition.

Proposition 8.5. *Let A be a σ -unital simple C^* -algebra in the class \mathcal{D} (or in \mathcal{D}_0). Then $M_k(A)$ is in the class \mathcal{D} (or in \mathcal{D}_0) for every integer $k \geq 1$.*

Proposition 8.6. *Let A be a separable simple C^* -algebra and let $B \subset A$ be a hereditary sub- C^* -algebra. Then, if A is in \mathcal{D} (or in \mathcal{D}_0), so also is B . Moreover, if $A \neq \{0\}$, then $\mathsf{T}(A) \neq \emptyset$.*

Proof. Let \mathcal{S} denote \mathcal{C}'_0 or $\mathcal{C}_0^{0'}$. We may assume neither A nor B is zero. Let $b \in A_+$ with $\|b\| = 1$ and $B = \overline{bAb}$. Let $e \in A_+$ be a strictly positive element with $\|e\| = 1$ and let \mathfrak{f}_e be as given by Definition 8.1, as A is in \mathcal{D} or in \mathcal{D}_0 . Fix $b_0 \in B_+ \setminus \{0\}$.

By Theorem 8.3, there exists a sequence of sub- C^* -algebras D_n of A in \mathcal{S} and two sequences of completely positive contractive maps $\varphi_n: A \rightarrow A$ and $\psi_n: A \rightarrow D_n$ with $\varphi_n(A) \perp D_n$ such that

$$\lim_{n \rightarrow \infty} \|\varphi_n(xy) - \varphi_n(x)\varphi_n(y)\| = 0, \quad (\text{e 8.27})$$

$$\lim_{n \rightarrow \infty} \|\psi_n(xy) - \psi_n(x)\psi_n(y)\| = 0 \quad \text{for all } x, y \in A, \quad (\text{e 8.28})$$

$$\lim_{n \rightarrow \infty} \|x - (\varphi_n + \psi_n)(x)\| = 0 \quad \text{for all } x \in A, \quad (\text{e 8.29})$$

$$\varphi_n(e) \lesssim b_0, \quad (\text{e 8.30})$$

$$\lim_{n \rightarrow \infty} \|\psi_n(x)\| = \|x\| \quad \text{for all } x \in A, \quad (\text{e 8.31})$$

$f_{1/4}(\psi_n(b))$ is full in D_n , and $\psi_n(e)$ is a strictly positive element of D_n , $n = 1, 2, \dots$. Moreover, we may also assume that

$$t \circ f_{1/4}(\psi_n(e)) \geq \mathfrak{f}_e, \quad t \circ f_{1/4}(\psi_n(b)) \geq r_0 \quad (\text{e 8.32})$$

for all $t \in \mathsf{T}(D_n)$ and n , where r_0 is as previously defined $\left(\frac{\mathfrak{f}_e}{4 \inf\{M(c)^2 \cdot N(c) : c = \{b, f_{1/4}(e)\}\}}\right)$.

By (e 8.31),

$$\lim_{n \rightarrow \infty} \|\psi_n|_B\| = 1.$$

We also have

$$\lim_{j \rightarrow \infty} \|b - f_{1/2j}(b)^{1/2} b f_{1/2j}(b)^{1/2}\| = 0, \quad (\text{e 8.33})$$

whence $\lim_{j \rightarrow \infty} \|x - f_{1/2j}(b)^{1/2} x f_{1/2j}(b)^{1/2}\| = 0$ for all $x \in B$. (e 8.34)

Put $L_n(x) = \varphi_n(x) + \psi_n(x)$ for all $x \in A$. By (e 8.29), applying Lemma 3.3, for any $j \geq 2$, we obtain $n(j) \geq j$ and a partial isometry $v_j \in A^{**}$ such that:

$$v_j v_j^* f_{1/2j}(L_{n(j)}(b)) = f_{1/2j}(L_{n(j)}(b)) v_j v_j^* = f_{1/2j}(L_{n(j)}(b)), \quad (\text{e 8.35})$$

$$v_j^* c v_j \in B \quad \text{for all } c \in \overline{f_{1/2j}(L_{n(j)}(b)) A f_{1/2j}(L_{n(j)}(b))}, \quad (\text{e 8.36})$$

$$\lim_{j \rightarrow \infty} \left(\sup \{ \|v_j^* c v_j - c\| : 0 \leq c \leq 1 \text{ and } c \in \overline{f_{1/2j}(L_{n(j)}(b)) A f_{1/2j}(L_{n(j)}(b))} \} \right) = 0. \quad (\text{e 8.37})$$

Note that $f_{1/2j}(\psi_{n(j)}(b)) \leq f_{1/2j}(L_{n(j)}(b))$, $j = 1, 2, \dots$. It follows that

$$v_j^* c v_j \in B$$

for all $c \in \overline{f_{1/2j}(\psi_{n(j)}(b)) A f_{1/2j}(\psi_{n(j)}(b))}$. Since $f_{1/4}(\psi_{n(j)}(b))$ is full in $D_{n(j)}$, $f_{1/2j}(\psi_{n(j)}(b))$ is full in $D_{n(j)}$ for all $j \geq 2$. Consider the hereditary sub-C*-algebra of $D_{n(j)}$

$$E'_{n(j)} = \overline{f_{1/2j}(\psi_n(b)) D_{n(j)} f_{1/2j}(\psi_n(b))}, \quad j = 2, 3, \dots$$

Then $E'_{n(j)} \in \mathcal{S}$, $j = 2, 3, \dots$. Put

$$E_j = v_j^* E'_{n(j)} v_j, \quad j = 3, 4, \dots$$

Then $E_j \in \mathcal{S}$ and $E_j \subset B$, $j = 3, 4, \dots$

Define $\Phi_j: B \rightarrow B$ by $\Phi_j(a) = v_j^* \varphi_{n(j)}(a) v_j$ for all $a \in B$, and $\Psi_j: B \rightarrow E_j$ by $\Psi_j(x) = v_j^* f_{1/2j}(\psi_{n(j)}(b)) \psi_{n(j)}(x) f_{1/2j}(\psi_{n(j)}(b)) v_j$, $j = 3, 4, \dots$. For $j > 4$,

$$f_{1/4}(f_{1/2j}(\psi_{n(j)}(b)) \psi_{n(j)}(b) f_{1/2j}(\psi_{n(j)}(b))) = f_{1/4}(\psi_{n(j)}(b)) \quad (\text{e 8.38})$$

$$= f_{1/2j}(\psi(b)) f_{1/4}(\psi_{n(j)}(b)) f_{1/2j}(\psi_{n(j)}(b)). \quad (\text{e 8.39})$$

It follows that $f_{1/4}(\Psi_j(b))$ is full in E_j , $j = 4, 5, \dots$. We have

$$\lim_{j \rightarrow \infty} \|\Phi_j(xy) - \Phi_j(x)\Phi_j(y)\| = 0 \quad \text{for all } x, y \in B, \quad (\text{e 8.40})$$

$$\lim_{j \rightarrow \infty} \|\Psi_j(xy) - \Psi_j(x)\Psi_j(y)\| = 0 \quad \text{for all } x, y \in B. \quad (\text{e 8.41})$$

Moreover, applying (e 8.29), (e 8.37), and (e 8.34), we have

$$\lim_{j \rightarrow \infty} \|x - (\Phi_j + \Psi_j)(x)\| = 0 \quad \text{for all } x \in B, \quad (\text{e 8.42})$$

$$\lim_{n \rightarrow \infty} \|\Psi_n(x)\| = \|x\| \quad \text{for all } x \in B. \quad (\text{e 8.43})$$

We also have

$$\Phi_j(b) \lesssim b_0.$$

Moreover, by (e 8.32) and (e 8.38),

$$t \circ f_{1/4}(\Psi_j(b)) \geq r_0/2 \quad \text{for all } t \in T(E_{n(j)}). \quad (\text{e 8.44})$$

The first part of the proposition follows on choosing a sufficiently large j .

To see that, if A is non-zero, then $T(A)$ is non-empty, in the preceding argument, take $B = A$ and choose $t_j \in T(E_{n(j)})$ for all j large enough that $E_{n(j)}$ is non-zero. Let t be a weak* limit of $(t_j \circ \Psi_j)$. Then, by (e 8.44), t is a non-zero linear functional on A . Moreover, since $t_j \in T(E_{n(j)})$, by (e 8.41), t is a trace. This implies $T(A) \neq \emptyset$. \square

Lemma 8.7. *Let $B = A(F_1, F_2, \varphi_0, \varphi_1)$. Suppose that $g := (h, a) \in B_+$ is such that $h_j := h|_{[0,1]_j}$ has range projection P_j satisfying the following condition: There is a partition $0 = t_j^0 < t_j^1 < t_j^2 < \dots < t_j^{n_j} = 1$ such that:*

- (1) *on each open interval (t_j^l, t_j^{l+1}) , $P_j(t)$ is continuous and $\text{rank}(P_j(t)) = r_{j,l}$ is a constant;*
- (2) *for each t_j^l ,*

$$P_j((t_j^l)^+) = \lim_{t \rightarrow (t_j^l)^+} P_j(t) \quad (\text{if } t_j^l < 1)$$

$$\text{and} \quad P_j((t_j^l)^-) = \lim_{t \rightarrow (t_j^l)^-} P_j(t) \quad (\text{if } t_j^l > 0)$$

exist;

$$(3) \quad P_j(t_j^l) \leq P_j((t_j^l)^+) \quad \text{and} \quad P_j(t_j^l) \leq P_j((t_j^l)^-);$$

$$(4) \quad \pi^j(\varphi_0(p)) = P_j(t_j^0) = P_j(0) = P_j(0^+)$$

$$\text{and} \quad \pi^j(\varphi_1(p)) = P_j(t_j^{n_j}) = P_j(1) = P_j(1^-),$$

where p is the range projection of $a \in F_1$.

Then $\overline{gBg} \in \mathcal{C}$.

Proof. For each closed interval $[t_j^l, t_j^{l+1}]$, since the limits

$$P_j((t_j^l)^+) = \lim_{t \rightarrow (t_j^l)^+} P_j(t) \quad \text{and} \quad P_j((t_j^{l+1})^-) = \lim_{t \rightarrow (t_j^{l+1})^-} P_j(t)$$

exist, we can extend $P_j|_{(t_j^l, t_j^{l+1})}$ to the closed interval $[t_j^l, t_j^{l+1}]$, and denote this projection by P_j^l . Then we can identify $P_j^l C([t_j^l, t_j^{l+1}], M_{r_j}) P_j^l$ with $C([0, 1], M_{r_{j,l}})$ by identifying t_j^l with 0 and t_j^{l+1} with 1, where $r_{j,l} = \text{rank}(P_j^l)$. Set

$$E_2^{j,l} := M_{r_{j,l}} \quad \text{and} \quad E_1^{j,l} := P_j(t_j^l) M_{r_j} P_j(t_j^l) \cong M_{R_{j,l}}.$$

Since $P_j(t_j^l) \leq P_j((t_j^l)^+)$, we may identify $E_1^{j,l}$ with a unital hereditary sub-C*-algebra of $E_2^{j,l}$. Denote this identification by

$$\psi_0^{j,l}: E_1^{j,l} \rightarrow E_2^{j,l}.$$

Similarly since $P_j(t_j^l) \leq P_j((t_j^l)^-)$, we obtain a homomorphism

$$\psi_1^{j,l}: E_1^{j,l} \rightarrow E_2^{j,l-1}$$

which identifies $E_1^{j,l}$ with a unital hereditary sub-C*-algebra of $E_2^{j,l-1}$.

Set

$$E_1 := pF_1p \oplus \bigoplus_{j=1}^k \left(\bigoplus_{l=1}^{n_j-1} E_1^{j,l} \right)$$

(note we do not include $E_1^{j,l}$ for $l = 0$ and $l = n_j$. Instead, we include pF_1p) and let

$$E_2 = \bigoplus_{j=1}^k \left(\bigoplus_{l=0}^{n_j-1} E_2^{j,l} \right).$$

Let $\psi_0, \psi_1: E_1 \rightarrow E_2$ be defined by

$$\psi_0|_{pF_1p} = \varphi_0|_{pF_1p}: pF_1p \rightarrow \bigoplus_{j=1}^k E_2^{j,0},$$

$$\psi_1|_{pF_1p} = \varphi_1|_{pF_1p}: pF_1p \rightarrow \bigoplus_{j=1}^k E_2^{j,n_j-1},$$

$$\psi_0|_{E_1^{j,l}} = \psi_0^{j,l}: E_1^{j,l} \rightarrow E_2^{j,l} \quad \text{and} \quad \psi_1|_{E_1^{j,l}} = \psi_1^{j,l}: E_1^{j,l} \rightarrow E_2^{j,l-1}.$$

We then check

$$A' = \overline{gBg} \cong A(E_1, E_2, \psi_0, \psi_1) \in \mathcal{C}.$$

Namely, each element $(f, a) = ((f_1, f_2, \dots, f_k), a) \in \overline{gBg}$ corresponds to an element

$$(F, b) \in \{C([0, 1], E_2) \oplus E_1 : F(0) = \psi_0(b), F(1) = \psi_1(b)\} \\ = A(E_1, E_2, \psi_0, \psi_1),$$

where

$$F = (f_1^0, f_1^1, \dots, f_1^{n_1-1}, f_2^0, f_2^1, \dots, f_2^{n_2-1}, \dots, f_k^0, f_k^1, \dots, f_k^{n_k-1})$$

$$\text{and } b = (a, f_1(t_1^1), f_1(t_1^2), \dots, f_1(t_1^{n_1-1}), f_2(t_2^1), f_2(t_2^2), \dots, f_2(t_2^{n_2-1}), \dots \\ \dots, f_k(t_k^1), f_k(t_k^2), \dots, f_k(t_k^{n_k-1}))$$

and where $f_j^l(t) \in E_2^{j,l}$ is defined by

$$f_j^l(t) = f_j((t_j^{l+1} - t_j^l)t + t_j^l) \quad \text{for all } t \in [0, 1],$$

$$j \in \{1, 2, \dots, k\}, l \in \{0, 1, \dots, n_j - 1\}. \quad \square$$

8.8. Let $A = A(F_1, F_2, \varphi_0, \varphi_1) \in \mathcal{C}$ be as in Definition 6.1. Let $h = (f, a) \in A_+$ with $\|h\| = 1$. Recall that we may write

$$C([0, 1], F_2) = \bigoplus_{j=1}^k C([0, 1]_j, M_{r_j}),$$

where $[0, 1]_j$ denotes the j -th interval. For each fixed j , consider $f_j = f|_{[0,1]_j}$. By a simple application of Weyl's theorem, one can write the eigenvalues of $f_j(t)$ as continuous functions of t ,

$$\{0 \leq \lambda_{1,j}(t) \leq \lambda_{2,j}(t) \leq \dots \leq \lambda_{r_j,j}(t) \leq 1\}.$$

Let e_1, e_2, \dots, e_{r_j} be mutually orthogonal rank one projections and put

$$f_j' = \sum_{i=1}^{r_j} \lambda_{i,j} e_i.$$

Then, on each $[0, 1]_j$, f_j and f_j' have exactly the same eigenvalues at each point $t \in [0, 1]_j$. Let $p \in F_1$ denote the range projection of $a \in (F_1)_+$. By using a unitary in $C([0, 1]_j, M_{r_j})$, it is easy to construct a set of mutually orthogonal rank one projections $p_1, p_2, \dots, p_i, \dots, p_{r_j} \in C([0, 1], M_{r_j})$ such that $g_j(t) = \sum_{i=1}^{r_j} \lambda_i(t) p_i$ satisfies $g_j(0) = f_j(0)$ and $g_j(1) = f_j(1)$. In particular,

$$\sum_{\{i, \lambda_i(0) > 0\}} p_i(0) = \pi^j(\varphi_0(p)) \in M_{r_j} \quad \text{and} \quad \sum_{\{i, \lambda_i(1) > 0\}} p_i(1) = \pi^j(\varphi_1(p)) \in M_{r_j},$$

where $\pi^j: F_2 \rightarrow F_2^j = M_{r_j}$ is the canonical quotient map to the j th summand. Then, with $g|_{[0,1]_j} = g_j$, $(g, a) \in A_+$. By a result of Thomsen (see [50, Theorem 1.2]) (or [43]), for each j there is a sequence of unitaries $u_n^j \in C([0, 1], M_{r_j})$ with $u_n^j(0) = u_n^j(1) = \mathbf{1}_{r_j}$ (note that as $g(0) = f(0)$ and $g(1) = f(1)$, we can choose $u_n^j(0) = u_n^j(1) = 1$) such that

$$g_j = \lim_{n \rightarrow \infty} u_n^j f_j (u_n^j)^*.$$

Since $u_n^j(0) = u_n^j(1) = \mathbf{1}_{r_j}$, we can put $u_n^j \in C([0, 1], M_{r_j})$ together to define unitary $u_n \in \tilde{A}$ and get

$$(g, a) = \lim_{n \rightarrow \infty} u_n (f, a) u_n^*.$$

In other words, $(g, a) \sim_{a.u} (f, a)$ in A . Note this, in particular, implies that

$$\langle (f, a) \rangle = \langle (g, a) \rangle.$$

Lemma 8.9. *Let $c = (g, a) \in A(F_1, F_2, \varphi_0, \varphi_1)_+$ with $\|(g, a)\| = 1$ (see Definition 6.1). Suppose that*

$$g_j := g|_{[0,1]_j} = \sum_{i=1}^{r_j} \lambda_{i,j}(t) p_{i,j}(t),$$

where $\lambda_{i,j} \in C([0, 1])_+$, and $p_{i,j} \in C([0, 1], M_{r_j})$ are mutually orthogonal rank one projections. Then, for any $\varepsilon > 0$, there exists $0 \leq h \leq g$ such that $\|h - g\| < \varepsilon$, $(h, a) \in A(F_1, F_2, \varphi_0, \varphi_1)$, and $h_j := h|_{[0,1]_j}$ satisfies the condition described in Lemma 8.7.

Proof. Fix $\varepsilon_1 > 0$ and j . Let $g_j = g|_{[0,1]_j}$. Let $G_{i,j} = \{t \in [0, 1] : \lambda_{i,j}(t) = 0\}$. Since all $G_{i,j}$ are closed sets, there is $\delta_0 > 0$ such that if $0 \notin G_{i,j}$ (or $1 \notin G_{i,j}$, respectively), then $\text{dist}(0, G_{i,j}) > 2\delta_0$ (or $\text{dist}(1, G_{i,j}) > 2\delta_0$, respectively). Fix $\delta > 0$ such that $\delta < \delta_0$. For each i , there is a closed set $S_{i,j}$ which is a union of finitely many closed interval containing the set $G_{i,j}$ such that

$$\text{dist}(s, G_{i,j}) < \delta/4 \quad \text{for all } s \in S_{i,j}. \quad (\text{e 8.45})$$

Hence, $\text{dist}(0, S_{i,j}) > \delta$ (and $\text{dist}(1, S_{i,j}) > \delta$) if $G_{i,j}$ does not contain them. Choose $f_{i,j} \in C([0, 1])_+$ such that $f_{i,j}|_{S_{i,j}} = 0$, $1 \geq f_{i,j}(t) > 0$, if $t \notin S_{i,j}$ and $f_{i,j}(t) = 1$ if $\text{dist}(t, S_{i,j}) > \delta/2$. Put

$$\lambda'_{i,j} = f_{i,j} \lambda_{i,j}.$$

Then $0 \leq \lambda'_{i,j} \leq \lambda_{i,j}$. Define

$$h_j = \sum_{i=1}^{r_j} \lambda'_{i,j} p_{i,j}.$$

Then $h_j \leq g_j$. We can choose δ sufficiently small to begin with so that

$$\|h_j - \sum_{i=1}^{r_j} \lambda_{i,j} p_{i,j}\| < \varepsilon. \quad (\text{e 8.46})$$

Put $h \in C([0, 1], F_2)$ such that $h|_{[0,1]_j} = h_j$, $j = 1, 2, \dots, k$. Therefore,

$$\|h - g\| < \varepsilon. \quad (\text{e 8.47})$$

From the construction, we have $h_j(0) = g_j(0)$ and $h_j(1) = g_j(1)$ (note that if $0 \notin G_{i,j}$ (or $1 \notin G_{i,j}$), then $f_{i,j}(0) = 1$ (or $f_{i,j}(1) = 1$)). It follows that

$$h(0) = g(0) \quad \text{and} \quad h(1) = g(1).$$

Therefore $(h, a) \in A(F_1, F_2, \varphi_0, \varphi_1)$. Moreover, $(h, a) \leq (g, a)$.

Let $q_{i,j}(t) = p_{i,j}(t)$ if $\lambda'_{i,j}(t) \neq 0$ and $q_{i,j}(t) = 0$ if $\lambda'_{i,j}(t) = 0$. For each i , there is a partition

$$0 = t_{i,j}^{(0)} < t_{i,j}^{(1)} < \cdots < t_{i,j}^{(l_j)} = 1$$

such that $q_{i,j}$ is continuous on $(t_{i,j}^{(l)}, t_{i,j}^{(l+1)})$. Namely, on each interval $(t_{i,j}^{(l)}, t_{i,j}^{(l+1)})$, $q_{i,j}(t)$ either constant zero projection or rank one projection $p_{i,j}(t)$ and therefore both

$$\lim_{s \rightarrow t_{i,j}^{(l)+} } q_{i,j}(s) \quad \text{and} \quad \lim_{s \rightarrow t_{i,j}^{(l+1)-} } q_{i,j}(s)$$

exist. Furthermore, if $q_{i,j}(t)$ is zero on the open interval $(t_{i,j}^{(l)}, t_{i,j}^{(l+1)})$, then $q_{i,j}(t)$ is also zero on the boundary (since $\lambda'_{i,j}(t)$ is continuous). Hence we have

$$q_{i,j}((t_{i,j}^{(l)})^+) := \lim_{s \rightarrow t_{i,j}^{(l)+} } q_{i,j}(s) \geq q_{i,j}(t_{i,j}^{(l)})$$

and

$$q_{i,j}((t_{i,j}^{(l+1)})^-) := \lim_{s \rightarrow t_{i,j}^{(l+1)-} } q_{i,j}(s) \geq q_{i,j}(t_{i,j}^{(l+1)}).$$

Define $P_j(t) = \sum_{i=1}^{r_j} q_{i,j}(t)$. Then P_j satisfies the conditions described in Lemma 8.7. \square

We shall need the following fact.

Corollary 8.10. *Let $A \in \mathcal{C}$ (or \mathcal{C}_0 , or \mathcal{C}_0^0), let $a \in A_+$ be a full element, and set $B = \overline{aAa}$. Then, for any $\varepsilon > 0$, there is $0 \leq b \leq a$ such that b is full in A , $\|a - b\| < \varepsilon$, and $\overline{bBb} = \overline{bAb} \in \mathcal{C}$ (or \mathcal{C}_0 , or \mathcal{C}_0^0).*

Proof. Let us first assume that a satisfies the condition on c in Lemma 8.9. Then, by Lemma 8.9 and by Lemma 8.7, there is $0 \leq b \leq a$ such that, for any $\varepsilon > 0$,

$$\|a - b\| < \varepsilon \quad \text{and} \quad \overline{bAb} \in \mathcal{C}.$$

Write $a = (f, d) \in A = A(F_1, F_2, \varphi_0, \varphi_1)_+$, where $f \in C([0, 1], F_2)$ and $d \in F_1$. Note that, if a is full, then $\|f(t)\| \neq 0$ for all $t \in [0, 1]_j$, $j = 1, 2, \dots, k$, and d is full in F_1 . Since $f(t)$ is also continuous on each $[0, 1]_j$,

$$\inf \{ \|f(t)\| : t \in [0, 1]_j \} > 0.$$

Therefore, $b = (g, d_1) \in A(F_1, F_2, \varphi_0, \varphi_1)$ can be chosen so that $\|g(t)\| \neq 0$ for all $t \in [0, 1]_j$, $j = 1, 2, \dots, k$, and also d_1 is full in F_1 . In other words, b (close

enough to a) can also be chosen to be full, i.e., such that $\tau(b) > 0$ for all $\tau \in T(A)$. Since $0 \notin \overline{T(A)}^w$, it follows that

$$\inf \{ \tau(b) : \tau \in T(A) \} > 0.$$

This implies that $0 \notin \overline{T(\overline{bAb})}^w$. Then $C := \overline{bAb}$ is stably isomorphic to A by Brown's theorem ([3]). It follows that

$$K_i(C) \cong K_i(A) \quad \text{and} \quad K_0(C)_+ = K_0(A)_+.$$

Thus, if A is in \mathcal{C}_0 (or is in \mathcal{C}_0^0), then C is in \mathcal{C}_0 (or is in \mathcal{C}_0^0).

In general, by 8.8, a is approximately unitarily equivalent to $a' \in A$ which satisfies the condition for c in Lemma 8.9. Therefore there is an isomorphism

$$\varphi: \overline{a'Aa'} \rightarrow \overline{aAa}.$$

Let $b' \leq a'$ be as given by the first part of the proof. Choose $b = \varphi(b')$. The conclusion then holds for b . \square

Remark 8.11. Let $C \in \mathcal{C}'_0$ (or $\mathcal{C}_0^{0'}$), let $e \in C_+$ be such that $\tau(f_{1/2}(e)) > \mathfrak{f} > 0$ for all $\tau \in T(C)$, let \mathcal{F} be a finite subset in the unit ball of C , and let $\varepsilon > 0$. Put $\varepsilon_0 = \min\{\mathfrak{f}/4, \varepsilon/4\}$. Choose $\eta > 0$ such that

$$\|f_{1/2}(a') - f_{1/2}(b')\| < \varepsilon_0 \tag{e 8.48}$$

if $0 \leq a', b' \leq 1$ and $\|a' - b'\| < \eta$. We may assume that $\eta < \varepsilon_0$. Let $e_C \in C$ be a strictly positive element such that $\|e_C\| = 1$ and

$$\|e_C f e_C - f\| < \eta/4 \quad \text{for all } f \in \mathcal{F} \cup \{e, f_{1/2}(e)\}. \tag{e 8.49}$$

By Corollary 8.10, there exists $b \in C_+$ with $b \leq e_C$ and $\|b - e_C\| < \eta/4$ such that $B := \overline{bCb}$ in \mathcal{C}_0 (or in \mathcal{C}_0^0). Define $\psi: C \rightarrow B$ by $\psi(c) = bcb$ for all $c \in C$. Then, for all $f \in \mathcal{F} \cup \{e, f_{1/2}(e)\}$,

$$\|\psi(f) - f\| < \eta < \varepsilon, \quad \text{and} \quad \tau(f_{1/2}(\psi(e))) > \mathfrak{f}/2 \quad \text{for all } \tau \in T(C). \tag{e 8.50}$$

Consequently, as B is a hereditary sub- C^* -algebra of C ,

$$\tau(f_{1/2}(\psi(e))) > \mathfrak{f}/2 \quad \text{for all } \tau \in T(B). \tag{e 8.51}$$

It follows that, in the definition of \mathcal{D} and \mathcal{D}_0 , we may assume that $D \in \mathcal{C}_0$ (or $D \in \mathcal{C}_0^0$).

Corollary 8.12. *Let A be a simple C^* -algebra which is an inductive limit of C^* -algebras in \mathcal{C}'_0 (or in $\mathcal{C}_0^{0'}$). Then A can be also written as an inductive limit of C^* -algebras in \mathcal{C}_0 (or in \mathcal{C}_0^0).*

Proof. Let $C \in \mathcal{C}'_0$ (or $C \in \mathcal{C}_0^{0'}$). Then, by Corollary 8.10,

$$C = \overline{\bigcup_{k=1}^{\infty} C_k},$$

where each $C_k \in \mathcal{C}_0$ (or $C_k \in \mathcal{C}_0^0$), $C_k \subset C_{k+1}$ and C_k is a hereditary sub-C*-algebra of C . $k = 1, 2, \dots$

Suppose that $A = \lim_{n \rightarrow \infty} (A_n, \varphi_n)$, where $A_n \in \mathcal{C}'_0$ (or $A \in \mathcal{C}_0^{0'}$) and $\varphi_n: A_n \rightarrow A_{n+1}$ is a homomorphism, $n = 1, 2, \dots$. If $m > n$, put

$$\varphi_{n,m} = \varphi_m \circ \varphi_{m-1} \circ \dots \circ \varphi_n: A_n \rightarrow A_m$$

and $\varphi_{n,\infty}: A_n \rightarrow A$ the homomorphism induced by the inductive system. Choose a dense sequence $\{x_n\}$ in the unit ball of A such that

$$\{x_1, x_2, \dots, x_n\} \subset \varphi_{n,\infty}(A_n), \quad n = 1, 2, \dots$$

Write

$$A_n = \overline{\bigcup_{k=1}^{\infty} C_{n,k}},$$

where $C_{n,k} \in \mathcal{C}_0$ (or $C_{n,k} \in \mathcal{C}_0^0$), $C_{n,k} \subset C_{n,k+1}$, $k = 1, 2, \dots$. Without loss of generality, we may assume that $x_j \in \varphi_{j,\infty}(C_{j,j})$, $j = 1, 2, \dots$. Let $y_{j,i} \in C_{j,j}$ such that $\varphi_{j,\infty}(y_{j,i}) = x_i$, $i = 1, 2, \dots, j$ and $j = 1, 2, \dots$.

Let $\{z_{n,k}\}$ be a dense sequence in A_n . We may assume that $y_{j,i} \in \{z_{j,k}\}$ for $i = 1, 2, \dots, j$, and $j = 1, 2, \dots$. We may also assume $\varphi_n(z_{n,k}) \subset \{z_{n+1,k}\}$, $n = 1, 2, \dots$. Put

$$\mathcal{F}_n = \{y_{n,i} : 1 \leq i \leq n\} \cup \{z_{n,i} : 1 \leq i \leq n\}, \quad n = 1, 2, \dots$$

By the semiprojectivity of C*-algebras in \mathcal{C} (see 6.4), one easily produces a sequence of homomorphisms $\psi_n: C_{n,k_n} \rightarrow C_{n+1,k_{n+1}}$ for some $k_n \geq n$ such that

$$\|\psi_n(a) - \varphi_n(a)\| < 1/2^n$$

for all $a \in \mathcal{F}_n$, $n = 1, 2, \dots$.

Let $B = \lim_{n \rightarrow \infty} (C_{n,k_n}, \psi_n)$. Then B is an inductive limit of C*-algebras in \mathcal{C}_0 (or in \mathcal{C}_0^0). Let $C = \overline{\bigcup_{n=1}^{\infty} \varphi_{n,\infty}(C_{n,k_n})}$. Then C is a sub-C*-algebra of A . Since $\{x_n\} \subset C$, $C = A$. Let $\iota_n: C_{n,k_n} \rightarrow C_{n,k_n}$ be the identity map. Then,

$$\|\varphi_n \circ \iota_n(a) - \iota_{n+1} \circ \psi_n(a)\| < 1/2^n \quad \text{for all } a \in \mathcal{F}_n. \quad (\text{e 8.52})$$

By the Elliott approximate intertwining argument, there is an isomorphism $j: B \rightarrow C$, which is induced by $\{\iota_n\}$. It follows that A is an inductive limit of C*-algebras in \mathcal{C}_0 (or in \mathcal{C}_0^0). \square

9. Traces and comparison for C*-algebras in the class \mathcal{D}

Proposition 9.1. *Let A be a non-zero separable C*-algebra in the class \mathcal{D} . Then $\text{QT}(A) = \text{T}(A) \neq \emptyset$. Moreover, $0 \notin \overline{\text{T}(A)}^w$.*

Proof. Let $a_0 \in A$ be a strictly positive element of A with $\|a_0\| = 1$. Let $\mathfrak{f}_{a_0} > 0$ be as in Definition 8.1. Fix any $b_0 \in A_+ \setminus \{0\}$. Choose a sequence of positive elements $(b_n)_{n \geq 1}$ which has the following property: $b_{n+1} \lesssim b_{n,1}$, where $b_{n,1}, b_{n,2}, \dots, b_{n,n}$ are mutually orthogonal positive elements in $\overline{b_n A b_n}$ such that

$$b_n b_{n,i} = b_{n,i} b_n = b_{n,i}, \quad i = 1, 2, \dots, n \quad \text{and} \quad \langle b_{n,i} \rangle = \langle b_{n,1} \rangle, \quad i = 1, 2, \dots, n.$$

One obtains (from Theorem 8.3) two sequences of sub-C*-algebras $A_{0,n}, D_n$ of A , where $D_n \in \mathcal{C}'_0$, and two sequences of completely positive contractive maps

$$\varphi_{0,n}: A \rightarrow A_{0,n} \quad \text{and} \quad \varphi_{1,n}: A \rightarrow D_n$$

with $A_{0,n} \perp D_n$ with the following properties:

$$\lim_{n \rightarrow \infty} \|\varphi_{i,n}(ab) - \varphi_{i,n}(a)\varphi_{i,n}(b)\| = 0 \quad \text{for all } a, b \in A, \quad (\text{e 9.1})$$

$$\lim_{n \rightarrow \infty} \|a - (\varphi_{0,n} + \varphi_{1,n})(a)\| = 0 \quad \text{for all } a \in A, \quad (\text{e 9.2})$$

$$c_n \lesssim b_n, \quad (\text{e 9.3})$$

$$\lim_{n \rightarrow \infty} \|\varphi_{1,n}(x)\| = \|x\| \quad \text{for all } x \in A, \quad (\text{e 9.4})$$

$$\tau(f_{1/4}(\varphi_{1,n}(a_0))) \geq \mathfrak{f}_{a_0} \quad \text{for all } \tau \in \text{T}(D_n), \quad (\text{e 9.5})$$

and $\varphi_{1,n}(a_0)$ is a strictly positive element of D_n , where c_n is a strictly positive element of $A_{0,n}$. Since quasitraces are norm continuous ([2, Corollary II 2.5]), by (e 9.2),

$$\lim_{n \rightarrow \infty} \left(\sup \{ |\tau(a) - \tau((\varphi_{0,n} + \varphi_{1,n})(a))| : \tau \in \text{QT}(A) \} \right) = 0 \quad \text{for all } a \in A. \quad (\text{e 9.6})$$

Since $\varphi_{0,n}(a)\varphi_{1,n}(a) = \varphi_{1,n}(a)\varphi_{0,n}(a) = 0$, for any $\tau \in \text{QT}(A)$,

$$\tau((\varphi_{0,n} + \varphi_{1,n})(a)) = \tau(\varphi_{0,n}(a)) + \tau(\varphi_{1,n}(a)) \quad \text{for all } a \in A. \quad (\text{e 9.7})$$

Note that, by (e 9.3),

$$\lim_{n \rightarrow \infty} \left(\sup \{ \tau(\varphi_{0,n}(a)) : \tau \in \text{QT}(A) \} \right) = 0 \quad \text{for all } a \in A. \quad (\text{e 9.8})$$

Therefore

$$\lim_{n \rightarrow \infty} \left(\sup \{ |\tau(a) - \tau \circ \varphi_{1,n}(a)| : \tau \in \text{QT}(A) \} \right) = 0 \quad \text{for all } a \in A. \quad (\text{e 9.9})$$

Since D_n is exact, $\tau|_{D_n}$ extends to a trace, say t_n . Then, for any $a, b \in A$,

$$\tau \circ \varphi_{1,n}(a+b) = t_n \circ \varphi_{1,n}(a+b) = t_n \circ \varphi_{1,n}(a) + t_n \circ \varphi_{1,n}(b) \quad (\text{e 9.10})$$

$$= \tau \circ \varphi_{1,n}(a) + \tau \circ \varphi_{1,n}(b) \quad \text{for all } a, b \in A. \quad (\text{e 9.11})$$

It follows from (e 9.9) that, for every $\tau \in \text{QT}(A)$,

$$\tau(a+b) = \tau(a) + \tau(b) \quad \text{for all } a, b \in A.$$

Thus τ extends to a trace on A . This proves $\text{QT}(A) = \text{T}(A)$.

Choose $s_n \in \text{T}(D_n)$, $n = 1, 2, \dots$. Consider a positive linear functional $F_n: A \rightarrow \mathbb{C}$ defined by $F_n(a) = t_n \circ \varphi_{1,n}$, $n = 1, 2, \dots$. Let F_0 be a weak*-limit of $(F_n)_{n \geq 1}$. Note that, by (e 9.5),

$$F_n(8a_0) = s_n(8\varphi_{1,n}(a_0)) \geq s_n(f_{1/4}(\varphi_{1,n}(a_0))) \geq f_{a_0} \quad \text{for all } n. \quad (\text{e 9.12})$$

It follows that $F_0 \neq 0$. By (e 9.1), since s_n is a trace on D_n , F_0 is a non-zero trace on A . It follows that $\text{T}(A) \neq \emptyset$.

Now let $\tau_k \in T(A)$ such that, for some positive linear functional τ ,

$$\lim_{n \rightarrow \infty} \tau_k(a) = \tau(a) \quad \text{for all } a \in A.$$

Then, for each k , by (e 9.9),

$$\lim_{n \rightarrow \infty} \|\tau_k|_{D_n}\| = 1.$$

Consider the restriction $t_{k,n} = (\|\tau_k|_{D_n}\|^{-1})\tau|_{D_n}$ for large n . Then $t_{k,n} \in \text{T}(D_n)$ for all k . It follows from (e 9.5) that

$$t_{k,n}(f_{1/4}(\varphi_{1,n}(a_0))) \geq f_{a_0}, \quad n = 1, 2, \dots \quad (\text{e 9.13})$$

By (e 9.9) and (e 9.1),

$$\tau_k(f_{1/4}(a_0)) = \lim_{n \rightarrow \infty} \tau_{k,n}(f_{1/4}(\varphi_{1,n}(a_0))) \geq f_{a_0}, \quad k = 1, 2, \dots \quad (\text{e 9.14})$$

Therefore $\tau \neq 0$. This implies that $0 \notin \overline{\text{T}(A)}^w$. \square

Remark 9.2. Let $A \in \mathcal{D}$ and let $a \in A_+$ be a strictly positive element with $\|a\| = 1$. In view of Proposition 9.1,

$$r_0 := \inf \{ \tau(f_{1/4}(a)) : \tau \in \text{T}(A) \} > 0.$$

The proof above shows that we may choose $f_a = r_0/2$. In fact in the case that $A = \text{Ped}(A)$, one may choose f_a arbitrarily close to

$$\lambda_s(A) = \inf \{ \tau(a) : \tau \in \overline{\text{T}(A)}^w \}.$$

In the case that A has continuous scale, we may choose the strictly positive element in such a way that r_0 is arbitrarily close to 1.

Proposition 9.3. *Every C^* -algebra in \mathcal{D} is stably projectionless.*

Proof. Let $A \in \mathcal{D}$. Since, by Propositions 8.5 and 8.6, $A \in \mathcal{D}$ if and only if $M_n(A) \in \mathcal{D}$ for each n , we only need to show that A itself has no non-zero projections. Let $p \in A$ be a non-zero projection. By Proposition 9.1,

$$r := \inf \{ \tau(p) : \tau \in \overline{\mathbb{T}(A)}^w \} > 0.$$

Choose $r/4 > \varepsilon > 0$. Then, by Definition 8.1,

$$\|p - (x_1 + x_2)\| < \varepsilon/2, \quad (\text{e9.15})$$

where $x_1 \in (A_0)_+$ and $x_2 \in D_+$, where $A_0 = \overline{bAb}$ for some $b \in A_+$ with $d_\tau(b) < r/4$ for all $\tau \in \overline{\mathbb{T}(A)}^w$, $D \in \mathcal{C}'_0$, and $A_0 \perp D$. If ε is chosen to be small enough, there are projections $p_1 \in A_0$ and $p_2 \in D$ such that

$$\|p - (p_1 + p_2)\| < \varepsilon.$$

Since D is projectionless, $p_2 = 0$. This implies that $\tau(p) < \tau(p_1) + \varepsilon < r/2$ for all $\tau \in \mathbb{T}(A)$, in contradiction with (e9.15). \square

Theorem 9.4. *Let $A \in \mathcal{D}$. Suppose that $A = \text{Ped}(A)$. Let $a, b \in (A \otimes \mathcal{K})_+$ be such that $d_\tau(a) \leq d_\tau(b)$ for all $\tau \in \overline{\mathbb{T}(A)}^w$. Then*

$$a \lesssim b.$$

Proof. Recall that, by Theorem 4.7 and Lemma 4.5 (or Proposition 9.1), $0 \notin \overline{\mathbb{T}(A)}^w$. Let us first prove the case that $a, b \in A_+$ and $d_\tau(a) < d_\tau(b)$ for all $\tau \in \overline{\mathbb{T}(A)}^w$.

Fix a strictly positive element $a_0 \in A$ with $0 \leq a_0 \leq 1$. We may assume that $\|a\| = \|b\| = 1$. Let $1/2 > \varepsilon > 0$. By Proposition 9.3, A is projectionless, and so zero is not an isolated point of $\text{sp}(a)$. Therefore, there is a non-zero element $c' \in \overline{aAa}_+$ such that $c'c = cc' = 0$, where $c = f_{\varepsilon/32}(a)$. It follows that

$$r_0 := \inf \{ d_\tau(b) - d_\tau(c) : \tau \in \overline{\mathbb{T}(A)}^w \} > 0.$$

(Note that $d_\tau(b) - d_\tau(c) \geq d_\tau(c')$ and use $0 \notin \overline{\mathbb{T}(A)}^w$.)

Set $c_1 = f_{\varepsilon/16}(a)$, so that $cc_1 = c_1$. Then a compactness argument (cf. [34, Lemma 5.4]) shows that there is $1 > \delta_1 > 0$ such that

$$\tau(f_{\delta_1}(b)) > \tau(c) \quad (\geq d_\tau(c_1)) \quad \text{for all } \tau \in \overline{\mathbb{T}(A)}^w.$$

Put $b_1 = f_{\delta_1}(b)$. Then

$$r := \inf \{ \tau(b_1) - d_\tau(c_1) : \tau \in \mathbb{T}(A) \} \geq \inf \{ \tau(b_1) - \tau(c) : \tau \in \mathbb{T}(A) \} > 0. \quad (\text{e9.16})$$

Note that $\|b\| = 1$. Choosing a smaller δ_1 , we may assume that there exist non-zero elements $e, e' \in f_{2\delta_1}(b)Af_{2\delta_1}(b)$ with $0 \leq e \leq e' \leq 1$ and $e'e = ee' = e$ such that

$$\tau(e') < r/8 \quad \text{for all } \tau \in \overline{\mathbf{T}(A)}^w.$$

Set $r_1 = \inf\{\tau(e) : \tau \in \mathbf{T}(A)\}$. Note that, as above, since A is simple and $0 \notin \overline{\mathbf{T}(A)}^w$, $r_1 > 0$. Set $b_2 = (1 - e')b_1(1 - e')$. Thus (cf. above), there is $0 < \delta_2 < \delta_1/2 < 1/2$ such that

$$7r/8 < \inf\{\tau(f_{\delta_2}(b_2)) - d_\tau(c_1) : \tau \in \mathbf{T}(A)\} < r - r_1. \quad (\text{e9.17})$$

Since $f_{\delta_2}(b_2)f_{3/4}(b_2) = f_{3/4}(b_2)$ and since $\overline{f_{3/4}(b_2)Af_{3/4}(b_2)}$ is non-zero, there is $e_1 \in A_+$ with $\|e_1\| = 1$ with $e_1f_{\delta_2}(b_2) = e_1$ and $d_\tau(e_1) < r/18$ for all $\tau \in \mathbf{T}(A)$. Choose $\eta < 1/4$ and set $e_2 = f_{\eta/4}(e_1)$ and $e_3 = f_\eta(e_1)$. Note that $f_{\delta_2}(b_2)e_2 = e_2$. Let $\sigma_0 = \inf\{\tau(e_2) : \tau \in \mathbf{T}(A)\} > 0$.

By Lemma 3.4, there are $x_1, x_2, \dots, x_m \in A$ such that

$$\sum_{i=1}^m x_i^* e_3 x_i = f_{1/16}(a_0). \quad (\text{e9.18})$$

Choose a non-zero element $e_0 \in \overline{Ae}_+$ such that $d_\tau(e_0) < \sigma_0/16$ for all $\tau \in \mathbf{T}(A)$.

Let $\mathfrak{f}_{a_0} > 0$ be as in Definition 8.1. Set

$$\sigma = \mathfrak{f}_{a_0} \cdot \min\{\varepsilon^2/2^{17}(m+1), \delta_2/8, r_1/2^7(m+1), \sigma_0/16\}.$$

By (e9.17),

$$\begin{aligned} \tau(f_{\varepsilon^2/2^{12}}(c_1)) + \tau(e_2) &< \tau(f_{\varepsilon^2/2^{12}}(c_1)) + r/18 \\ &< \tau(f_{\delta_2}(b_2)) \quad \text{for all } \tau \in \mathbf{T}(A). \end{aligned} \quad (\text{e9.19})$$

Then, by 7.5 of [9], there are $z_1, z_2, \dots, z_K \in A$ and $b' \in A_+$ such that

$$\left\| f_{\varepsilon^2/2^{12}}(c_1) - \sum_{j=1}^K z_j^* z_j \right\| < \sigma/4 \quad (\text{e9.20})$$

and

$$\left\| f_{\delta_2}(b_2) - \left(b' + e_2 + \sum_{j=1}^K z_j z_j^* \right) \right\| < \sigma/4.$$

Since $A \in \mathcal{D}$, there exist sub-C*-algebras $A_0, D \subset A$, with $D \in \mathcal{C}'_0$ and $A_0 \perp D$, such that:

$$\|f_{\varepsilon^2/2^{14}}(c_1) - (f_{\varepsilon^2/2^{14}}(c_{0,2}) + f_{\varepsilon^2/2^{14}}(c_2))\| < \sigma, \quad (\text{e9.21})$$

$$\|f_{\delta_1/2}(b_2) - (f_{\delta_1/2}(b_{0,3}) + f_{\delta_1/2}(b_3))\| < \sigma, \quad (\text{e9.22})$$

$$\|f_{\delta_2/4}(b_2) - (f_{\delta_2/4}(b_{0,3}) + f_{\delta_2/4}(b_3))\| < \sigma, \quad (\text{e9.23})$$

$$\|e_i - (e_{0,i} + e_{1,i})\| < \sigma, \quad i = 1, 2, \quad \text{and} \quad e_{2,1}e_{1,1} = e_{1,1} \quad (\text{e9.24})$$

$$c_0 \lesssim e_0, \quad (\text{e9.25})$$

$$\|a_0 - (a_{0,0} + a_{1,0})\| < \sigma, \quad (\text{e9.26})$$

$$\tau(f_{1/4}(a_{1,0})) \geq \mathfrak{f}_{a_0} \quad \text{for all } \tau \in \overline{\mathbf{T}(D)}^w, \quad (\text{e9.27})$$

where $c_0 \in A_0$ is a strictly positive element of A_0 , and where $a_{0,0}, b_{0,3}, c_{0,2}, e_{0,1} \in (A_0)_+$ and $a_{1,0}, b_3, c_2, e_{1,1} \in D_+$. By (e9.20), we also obtain $z'_j, z'_j, x'_i \in D$ and $b'' \in D_+, i = 1, 2, \dots, m$ and $j = 1, 2, \dots, K$, such that:

$$\left\| \sum_{i=1}^m (x'_i)^* e_{1,1} x'_i - f_{1/16}(a_{1,0}) \right\| < \sigma, \quad (\text{e9.28})$$

$$\left\| f_{\varepsilon^2/2^{14}}(c_2) - \sum_{j=1}^K (z'_j)^* z'_j \right\| < \sigma, \quad (\text{e9.29})$$

and
$$\left\| f_{\delta_2}(b_3) - \left(\sum_{j=1}^K z'_j (z'_j)^* + e_{2,1} + b'' \right) \right\| < \sigma. \quad (\text{e9.30})$$

Note that, by (e9.28) and by Lemma 3.1,

$$\langle f_{1/4}(a_{1,0}) \rangle \leq \left\langle \sum_{i=1}^m (x'_i)^* e_{1,1} x'_i \right\rangle \leq m \langle e_{1,1} \rangle. \quad (\text{e9.31})$$

Then, by (e9.27),

$$t(e_{2,1}) \geq \mathfrak{d}_t(e_{1,1}) \geq \mathfrak{f}_{a_0}/(m+1) > 2\sigma \quad \text{for all } t \in \mathbf{T}(D). \quad (\text{e9.32})$$

Therefore, by (e9.29), (e9.32), and (e9.30),

$$\begin{aligned} \mathfrak{d}_t(f_{\varepsilon^2/2^{13}}(c_2)) &\leq t(f_{\varepsilon^2/2^{14}}(c_2)) \leq \sigma + \sum_{j=1}^K t((z'_j)^* z'_j) \\ &= \sigma + \sum_{j=1}^K t(z'_j (z'_j)^*) < t(e_{2,1}) - \sigma + \sum_{j=1}^K t(z'_j (z'_j)^*) \\ &\leq t(f_{\delta_2}(b_3)) \leq \mathfrak{d}_t(f_{\delta_2/2}(b_3)) \end{aligned}$$

for all $t \in T(D)$. It follows by Proposition 6.3 that

$$f_{\varepsilon^2/2^{13}}(c_2) \lesssim f_{\delta_2/2}(b_3).$$

By (e 9.23) and by Lemma 3.1 (see [45, Lemma 2.2]) (note $\sigma < \delta_2/8$ and $b_{0,3} \perp b_3$),

$$f_{\delta_2/2}(b_3) \lesssim (f_{\delta_2/4}(b_3) - 2\sigma)_+ \lesssim f_{\delta_2/4}(b_2) \leq b_2.$$

It then follows (also by Lemma 2.2 of [45]) that

$$\begin{aligned} f_{\varepsilon/2}(c) &\lesssim f_{\varepsilon^2/2^{11}}(c_{0,2} + c_2) \lesssim c_0 \oplus f_{\varepsilon^2/2^{11}}(c_2) \\ &\lesssim e + b_2 \lesssim b_1 \lesssim b. \end{aligned}$$

We also have

$$f_{\varepsilon}(a) \lesssim f_{\varepsilon/2}(f_{\varepsilon/16}(a)) \lesssim f_{\varepsilon/2}(c) \lesssim b.$$

Since this holds for all $1 > \varepsilon > 0$, by 2.4 of [45], we conclude that $a \lesssim b$.

If we only have $d_{\tau}(a) \leq d_{\tau}(b)$ for all $\tau \in \overline{T(A)}^w$, then, since A is stably projectionless as mentioned above, as shown at the beginning of the proof, for any $\varepsilon > 0$,

$$\tau(f_{\varepsilon/2}(a)) < d_{\tau}(b) \quad \text{for all } \tau \in \overline{T(A)}^w.$$

From what has been proved, $f_{\varepsilon}(a) \lesssim b$ for all $\varepsilon > 0$. Therefore, $a \lesssim b$. This proves the case that $a, b \in A_+$. For $a, b \in M_n(A)_+$ for some $n \geq 1$, one notes that, by Proposition 8.5, $M_n(A) \in D$. Therefore this case is easily reduced to the case that $n = 1$. In general, if $a, b \in (A \otimes \mathcal{K})_+$, then, for any $\varepsilon > 0$, by the last part of Lemma 3.2, we may assume that $f_{\varepsilon/16}(a)$ is in $M_n(A)$ for some $n \geq 1$. Hence $\tau \mapsto \tau(f_{\varepsilon/16}(a))$ is bounded and continuous. Since $\overline{T(A)}^w$ is compact, one concludes, as shown above, for some small $\delta_1 > 0$, that

$$\tau(f_{\varepsilon/16}(a)) < \tau(f_{\delta_1}(b)) \quad \text{for all } \tau \in \overline{T(A)}^w.$$

As mentioned above, we may also assume that $f_{\delta}(b) \in M_n(A)$ (with possibly larger n). Thus, we conclude that $f_{\varepsilon}(a) \lesssim f_{\delta}(b) \lesssim b$. It follows that $a \lesssim b$. \square

Definition 9.5. Let us denote by \mathcal{M}_0 the class of (non-unital) simple C^* -algebras which are inductive limits of sequences of C^* -algebras in \mathcal{C}_0^0 . We stipulate that the maps in the sequence be injective and preserve strictly positive elements, i.e., each map should send strictly positive elements to strictly positive elements. In fact, a decomposition with such maps can always be chosen.

Every algebraically simple C^* -algebra A in \mathcal{M}_0 is in \mathcal{D}_0 . To see this, write $A = \lim_{n \rightarrow \infty} (C_n, \varphi_n)$, where each C_n is in \mathcal{C}_0^0 and $\varphi_n: C_n \rightarrow C_{n+1}$ is a homomorphism which preserves strictly positive elements and is injective.

Let $a_1 \in C_1$ be a strictly positive element with $\|a_1\| = 1$. Then $a_n = \varphi_{1,n}(a_1) \in C_n$ is a strictly positive element of C_n , $n = 1, 2, \dots$. Then $a = \varphi_{n,\infty}(a_n)$ is a strictly positive element with $\|a\| = 1$. For any n , since $0 \notin \overline{\mathrm{T}(C_n)}^w$ (see Definition 6.1),

$$r_n := \inf \{ \tau(a_n) : \tau \in \overline{\mathrm{T}(C_n)}^w \} > 0.$$

Since $\varphi_{m,n}$ is a homomorphism preserving strictly positive elements, $t \circ \varphi_{m,n} \in \mathrm{T}(C_m)$ for all $t \in \mathrm{T}(C_n)$ and for all $n \geq m$. Thus, $r_n \geq r_m$ for all $n \geq m$.

Since A is algebraically simple and $f_{1/4}(a) \neq 0$, there are $x_1, x_2, \dots, x_k \in A$ such that

$$\sum_{i=1}^k x_i^* f_{1/4}(a) x_i = a.$$

Set $M = 2k \max\{\|x_i\| : 1 \leq i \leq k\}$. For some $m \geq 1$, there are $y_1, y_2, \dots, y_k \in C_m$ such that

$$\left\| \sum_{i=1}^k y_i^* \varphi_{1,m}(f_{1/4}(a_1)) y_i - a_m \right\| < r_1/2.$$

We may assume that $\|y_i\| \leq 2\|x_i\|$, $i = 1, 2, \dots, k$. Since $r_1 \leq r_m$, this implies that

$$\tau(\varphi_{1,m}(f_{1/4}(a_1))) \geq (r_m/2)/2M \quad \text{for all } \tau \in \mathrm{T}(C_m).$$

Put

$$f_a = \inf \{ \tau(\varphi_{1,m}(f_{1/4}(a_1))) : \tau \in \mathrm{T}(C_m) \}.$$

Note since $t \circ \varphi_{m,n} \in \mathrm{T}(C_n)$ for all $t \in \mathrm{T}(C_m)$,

$$t(\varphi_{m,n}(f_{1/4}(a_1))) \geq f_a \quad \text{for all } t \in \mathrm{T}(C_n).$$

From this, one concludes that $A \in \mathcal{D}_0$ (with $\varphi = 0$ and $\psi = \mathrm{id}_A$ in Definition 8.1).

Definition 9.6. Recall that \mathcal{W} is an inductive limit of C^* -algebras as described in (e 6.2) (see [41, 53], and [24]) which has $\mathrm{K}_0(\mathcal{W}) = \mathrm{K}_1(\mathcal{W}) = \{0\}$ and has a unique tracial state. Moreover, $\mathcal{W} = \overline{\bigcup_{n=1}^{\infty} C_n}$, where $C_n \subset C_{n+1}$ and each C_n is in $\mathcal{R}_{\mathrm{az}}$ (in fact as in (e 6.2)) and inclusion preserves the strictly positive elements. In particular, $\mathcal{W} \in \mathcal{M}_0$ and $\mathcal{W} \in \mathcal{D}_0$. Furthermore, we may also assume that (see Definitions 6.1 and 4.9 for λ_s)

$$\lambda_s(C_n) = \inf \{ \tau(e_{C_n}) : \tau \in \mathrm{T}(C_n) \} \rightarrow 1,$$

where e_{C_n} is a strictly positive element of C_n (for example, $\lim_{i \rightarrow \infty} a_i / (a_i + 1) = 1$ as shown in [24]). By [41], \mathcal{W} is unique with these properties.

10. Tracial approximate divisibility

Definition 10.1. Let A be a (non-unital) σ -unital simple C^* -algebra. Let us say that A is (non-unital) tracially approximately divisible if the following property holds:

For any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$, any $b \in A_+ \setminus \{0\}$, and any integer $n \geq 1$, there are σ -unital sub- C^* -algebras $A_0, A_1 \otimes e_{1,1} \subset M_n(A_1) \subset A$ such that $A_0 \perp M_n(A_1)$,

$$\text{dist}(x, A_0 + A_1 \otimes 1_n) < \varepsilon \quad \text{for all } x \in \mathcal{F},$$

and $a_0 \lesssim b$, where a_0 is a strictly positive element of A_0 .

In the unital case, this definition is equivalent to 5.3 of [32]. (Note, as can be seen in the proof of 5.4 of [32], the unit of the finite dimensional sub- C^* -algebra should be required to be $1 - q$ there.)

Lemma 10.2. *Let D be a (non-unital) separable simple C^* -algebra which can be written as $D = \lim_{k \rightarrow \infty} (D_k, \varphi_k)$, where each $D_k \in \mathcal{C}_0^{0'}$. Let $K \geq 1$ be an integer, let $\varepsilon > 0$, and let \mathcal{F} be a finite subset of D_n for some $n \geq 1$. There exist an integer $m \geq n$, a sub- C^* -algebra $D'_m = M_K(D''_m) \subset D_m$, where D''_m is a hereditary sub- C^* -algebra of D_m , and a finite subset $\mathcal{F}_1 \subset D''_m$ such that*

$$\text{dist}(\varphi_{n,m}(f), \mathcal{F}_1 \otimes 1_K) < \varepsilon \quad \text{for all } f \in \mathcal{F}. \quad (\text{e } 10.1)$$

If each D_k is just assumed to belong to \mathcal{C}'_0 , then there exist an integer $m \geq n$, a sub- C^ -algebra $D'_m = M_K(D''_m) \subset D_m$, where D''_m is a hereditary sub- C^* -algebra of D_m , and a finite subset $\mathcal{F}_1 \subset D''_m$ such that*

$$\|\varphi_{n,m}(f) - (r(f) + g_f \otimes 1_K)\| < \varepsilon \quad \text{for all } f \in \mathcal{F}, \quad (\text{e } 10.2)$$

where $r(f) \in \overline{eD_m e}$ and $g_f \in \mathcal{F}_1$ for all $f \in \mathcal{F}$, $e \in (D_m)_+$, and $e \lesssim e_d$, where e_d is a strictly positive element of D''_m .

Proof. We may assume that \mathcal{F} is in the unit ball of D . Consider first the case $D_k \in \mathcal{C}_0^{0'}$ for all k . By Corollary 8.12, without loss of generality, we may assume that $D_k \in \mathcal{C}_0^0$. One notes that $K_0(D) = K_1(D) = \{0\}$. One also notes that $D \otimes Q$ is an inductive limit of C^* -algebras in \mathcal{C}_0^0 . Moreover, D and $D \otimes Q$ have (in the natural sense) the same (lower semicontinuous) traces and the same tracial states. It follows from 6.2.4 of [43] (see [24, Theorem 1.2], also [41] and [53]) that $D \cong D \otimes Q$. Fix n such that $\mathcal{F} \subset D_n$. Without loss of generality, we may assume that $\mathcal{F} \subset D_n^1$. Note that Q is self-absorbing. Therefore, there exists a sub- C^* -algebra C of D with $C \cong D$ and $D = C \otimes M_K$ such that

$$\|\varphi_{n,\infty}(a) - c(a) \otimes 1_K\| < \varepsilon/4 \quad \text{for all } a \in \mathcal{F}, \quad (\text{e } 10.3)$$

for all $a \in \mathcal{F}$ and for some $c(a) \in C \subset C \otimes M_K \subseteq D$, where C is regarded as the corner $C \otimes e_{1,1}$ of D .

For each $a \in \mathcal{F}$, there exists $n_1 \geq n$ and $c(a)' \in D_{n_1}$ such that

$$\|\varphi_{n_1, \infty}(c(a)') - c(a) \otimes e_{11}\| < \varepsilon/16K^2.$$

Without loss of generality, we may assume that $\|c(a)\|, \|c(a)'\| \leq 1$. To simplify notation, without loss of generality, let us assume that there is $c_0 \in C_+$ with $\|c_0\| = 1$ such that $c_0 c(a) = c(a) c_0 = c(a)$ for all $a \in \mathcal{F}$. Consider the sub-C*-algebra

$$B = C_0 \otimes M_K \subset C \otimes M_K = D,$$

where C_0 is the sub-C*-algebra of C generated by c_0 . Since C is stably projectionless, $\text{sp}(c_0) = [0, 1]$. Then $C_0 \otimes M_K \cong C_0((0, 1]) \otimes M_K$. Fix a finite subset $\mathcal{G} \subset B$ which contains $\{c_0 \otimes e_{ij} : 1 \leq i, j \leq 1\}$, and $0 < \delta < \varepsilon/8$. Since B is semiprojective (see, for example, [12]), there is a homomorphism $H: B \rightarrow D_{m_1}$ for some $m_1 \geq n_1 \geq n$ such that

$$\|\varphi_{m_1, \infty} \circ H(g) - g\| < \delta/K^2 \quad \text{for all } g \in \mathcal{G}. \quad (\text{e 10.4})$$

Set $c_{00} = H(c_0 \otimes e_{11})$.

Fix $m > m_1$. Define $D_m'' = \overline{\varphi_{m_1, m}(c_{00}) D_m \varphi_{m_1, m}(c_{00})}$. Then the sub-C*-algebra D_m' generated by D_m'' and $\varphi_{m_1, m}(H(B))$ is isomorphic to $D_m'' \otimes M_K \subset D_m$. Define

$$x_{1, j}(a) = (c_{00} \otimes e_{j1}) \varphi_{n_1, m_1}(c(a)')(c_{00} \otimes e_{1j}) \in H(B),$$

$$\text{and } y_{1, j}(a) = \varphi_{m_1, m}(c_{00} \otimes e_{j1}) \varphi_{n_1, m}(c(a)') \varphi_{m_1, m}(c_{00} \otimes e_{1, j}) \quad j = 1, 2, \dots, K, \quad (\text{e 10.5})$$

for all $a \in \mathcal{F}$. Note $y_{1, j} = \varphi_{m_1, m}(x_{1, j}(a))$. Moreover, one may write

$$\sum_{j=1}^N x_{1, j}(a) = \sum_{j=1}^N x_{11}(a) \otimes e_{ii} = x_{11}(a) \otimes 1_K.$$

By (e 10.4), for $a \in \mathcal{F}$,

$$\begin{aligned} \varphi_{m_1, \infty} \left(\sum_{j=1}^K x_{1, j}(a) \right) &\approx_{\delta} \sum_{j=1}^K (c_0 \otimes e_{j1}) \varphi_{n_1, \infty}(c(a)')(c_0 \otimes e_{1, j}) \\ &\approx_{\varepsilon/16} \sum_{j=1}^K (c_0 \otimes e_{j1})(c(a) \otimes e_{11})(c_0 \otimes e_{1j}) \\ &= \sum_{j=1}^K (c_0 c(a) c_0) \otimes e_{jj} = \sum_{j=1}^K c(a) \otimes e_{jj} = c(a) \otimes 1_K. \end{aligned} \quad (\text{e 10.6})$$

Define $\mathcal{F}_1 = \{y_{1,1}(a) : a \in \mathcal{F}\} \subset D_m'' \otimes M_K \subset D_m$. Then, by (e 10.3) and (e 10.6) above, without loss of generality, choosing a larger m if necessary, we may assume that

$$\|\varphi_{m_1,m}(\varphi_{n,m_1}(a)) - \varphi_{m_1,m}(x_{11}(a) \otimes 1_K)\| < \varepsilon/2 \quad \text{for all } a \in \mathcal{F}. \quad (\text{e 10.7})$$

It follows that

$$\text{dist}(\varphi_{n,m}(a), \mathcal{F}_1 \otimes 1_K) < \varepsilon \quad \text{for all } a \in \mathcal{F}. \quad (\text{e 10.8})$$

This proves the first part of the statement.

In the case $D_n \in \mathcal{C}'_0$, by [51], $D \otimes \mathcal{Z} \cong D$. In \mathcal{Z} (see the proof of Lemma 2.1 of [44], and also Lemma 4.2 of [48]), there are $e_1, e_2, \dots, e_K, d \in \mathcal{Z}_+$ such that

$$\sum_{j=1}^K e_j + d = 1_{\mathcal{Z}},$$

e_1, e_2, \dots, e_K are mutually orthogonal, $d \lesssim e_1$, and there exist $w_1, w_2, \dots, w_K \in \mathcal{Z}$ such that $e_j = w_j w_j^*$ and $e_{j+1} = w_j^* w_j$. Moreover, as in the proof of Lemma 4.2 of [48], since \mathcal{Z} has stable rank one, there is a unitary $v \in \mathcal{Z}$ such that $v^* d v \leq e_1$.

Without loss of generality, identifying D with $D \otimes \mathcal{Z}$, we may assume that

$$\varphi_{n,\infty}(x) = y \otimes 1$$

for some $y = y(x) \in D$ for every element $x \in \mathcal{F}$. Let

$$d' = c_0 \otimes d, \quad v' = c_0^{1/2} \otimes v, \quad e'_j = c_0 \otimes e_j, \quad w'_j = c_0^{1/2} \otimes w_j, \quad j = 1, 2, \dots, K.$$

Note that $d' + \sum_{j=1}^K e'_j = c_0$. With sufficiently large m and with a standard perturbation, we may assume that $d', v', e'_j, w'_j \in \varphi_{m,\infty}(D_m)$, $j = 1, 2, \dots, m$, and $\varphi_{n,\infty}(x)$ commutes with d', e'_j and w'_j for all $x \in \mathcal{F}$. With possibly even larger m , without loss of generality, there are $d'', v'', e''_j, w''_j \in D_m$ such that

$$d'' + \sum_{j=1}^K e''_j = c'_0, \quad d'' = v''(v'')^*, \quad (v'')^* v'' \leq e''_1,$$

$e''_1, e''_2, \dots, e''_K$ are mutually orthogonal, $(w''_j)(w''_j)^* = e'_j$ and $e''_{j+1} = (w''_j)^*(w''_j)$, where $c'_0 \in (D_m)_+$ is such that

$$c'_0 \varphi_{n,m}(x) = \varphi_{n,m}(x) c'_0 = \varphi_{n,m}(x) \quad \text{for all } x \in \mathcal{F}$$

and

$$\|[\varphi_{n,m}(x), y]\| < \varepsilon/16K^2 \quad \text{for all } x \in \mathcal{F} \quad (\text{e 10.9})$$

and

$$y \in \{d''^{1/2}, d'', v'', e''_j, w''_j, j = 1, 2, \dots, K\}.$$

Define

$$D''_m = \overline{e''_j D_m e''_j}, \quad r(f) = (d'')^{1/2} \varphi_{n,m} (d'')^{1/2}$$

and

$$\mathcal{F}_1 = \{e''_1 \varphi_{n,m}(x) e''_1 : x \in \mathcal{F}\},$$

and identify $d(e''_1 \varphi_{n,m}(x) e''_1)$ with

$$\sum_{j=1}^K e''_j \varphi_{n,m}(x) e''_j \in M_K(D''_m) \quad \text{for all } x \in \mathcal{F}.$$

The conclusion of the lemma follows. \square

Theorem 10.3. *Let $A \in \mathcal{D}$ (or $A \in \mathcal{D}_0$) be a separable C^* -algebra with $A = \text{Ped}(A)$. Then the following statement holds: Let a_0 be a strictly positive element of A with $\|a_0\| = 1$. There exists $1 > d_A > 0$ satisfying the following condition: For any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$, and any $b_0 \in A_+ \setminus \{0\}$, there exist a separable simple C^* -algebra $D = \lim_{n \rightarrow \infty} (D_n, \psi_n)$, where $D_n \in \mathcal{C}_0$ (or $D_n \in \mathcal{C}_0^0$) and an \mathcal{F} - ε -multiplicative completely positive contractive map $\varphi: A \rightarrow D_1$ such that, for any $n > 1$, there exist a completely positive contractive map $\Phi_n: A \rightarrow A$ and an embedding $j_n: D_n \rightarrow A$ with $\Phi_n(A) \perp j_n \circ (\psi_{1,n} \circ \varphi(A))$ such that*

$$\|x - (\Phi_n + j_n \circ \psi_{1,n} \circ \varphi)(x)\| < \varepsilon \quad \text{for all } x \in \mathcal{F}, \quad (\text{e 10.10})$$

$$c_n \lesssim b_0, \quad (\text{e 10.11})$$

$$\tau(f_{1/4}(\psi_{1,n} \circ \varphi(a_0))) > d_A \quad \text{for all } \tau \in T(D_n), \quad (\text{e 10.12})$$

where c_n is a strictly positive element of $\overline{\Phi_n(A)A\Phi_n(A)}$. Moreover, if $\mathbf{K}_0(A) = \{0\}$, we may assume that $(\psi_{1,n}|_{D_1})_* = 0$.

Proof. Let $1 > f_{a_0} > 0$ be as in Definition 8.1. Fix an integer $k_0 \geq 1$ such that $(f_{a_0})^2 > 2^{-k_0}$.

Replacing a_0 by $g(a_0)$ for some $g \in C_0((0, 1])$ with $0 \leq g \leq 1$, we may assume that

$$\tau(a_0) > f_{a_0} \quad \text{for all } \tau \in T(A) \quad (\text{e 10.13})$$

(see Remark 9.2). Fix any $b_0 \in A_+ \setminus \{0\}$. Choose a sequence of non-zero positive elements $(b_n)_{n \geq 1}$ in A with the following property: $b_1 \lesssim b_0$ and $b_{n+1} \lesssim \overline{b_{n,1}}$, where $b_{n,1}, b_{n,2}, \dots, b_{n,2^{n+k_0+5}}$ are mutually orthogonal positive elements in $\overline{b_n A b_n}$ such that

$$b_n b_{n,i} = b_{n,i} b_n = b_{n,i}, \quad i = 0, 1, 2, \dots, n$$

and

$$\langle b_{n,i} \rangle = \langle b_{n,1} \rangle, \quad i = 1, 2, \dots, 2^{n+k_0+3}.$$

It should be noted that

$$\sum_{j=m}^{\infty} \sup \{ \tau(b_j) : \tau \in \overline{T(A)}^w \} < (f_{a_0})^2 / 2^{m+5} \quad \text{for all } m \geq 1. \quad (\text{e 10.14})$$

One obtains (see also the end of Remark 8.11) two sequences of sub-C*-algebras $A_{0,n}$ and D_n of A , with $A_{0,n} \perp D_n$ and $D_n \in \mathcal{C}_0$ (or $D_n \in \mathcal{C}_0^0$), and two sequences of completely positive contractive maps $\varphi_n^{(0)}: A \rightarrow A_{0,n}$ and $\varphi_n^{(1)}: A \rightarrow D_n$ with $\|\varphi_n^{(i)}\| = 1$ ($i = 0, 1$) satisfying the following conditions:

$$\lim_{n \rightarrow \infty} \|\varphi_n^{(i)}(ab) - \varphi_n^{(i)}(a)\varphi_n^{(i)}(b)\| = 0 \quad \text{for all } a, b \in A, i = 0, 1, \quad (\text{e 10.15})$$

$$\lim_{n \rightarrow \infty} \|a - (\varphi_n^{(0)} + \varphi_n^{(1)})(a)\| = 0 \quad \text{for all } a \in A, \quad (\text{e 10.16})$$

$$c_n \lesssim b_n, \quad (\text{e 10.17})$$

$$\tau(f_{1/4}(\varphi_n^{(1)}(a_0))) \geq f_{a_0} \quad \text{for all } \tau \in T(D_n), \quad (\text{e 10.18})$$

and $\varphi_n^{(1)}(a_0)$ is a strictly positive element of D_n , where c_n is a strictly positive element of $A_{0,n}$ with $\|c_n\| = 1$. To avoid confusion, since the sequence of subalgebras (D_n) is not increasing, we shall denote the embedding of D_n in A by j_n . As in the proof of Proposition 9.1, it follows that

$$\lim_{n \rightarrow \infty} \left(\sup \{ |\tau(a) - \tau \circ \varphi_n^{(1)}(a)| : \tau \in T(A) \} \right) = 0 \quad \text{for all } a \in A. \quad (\text{e 10.19})$$

Consider the sequence (a_n) defined inductively by

$$a_1 := \varphi_1^{(1)}(a_0), \quad a_2 := \varphi_2^{(1)}(a_1), \dots, a_n := \varphi_n^{(1)}(a_{n-1}), \quad n = 1, 2, \dots$$

For fixed n , by (e 10.17), (e 10.14), (e 10.15), and (e 10.16),

$$\lim_{k \rightarrow \infty} \left(\sup \{ \tau(f_{1/4}(\varphi_k^{(1)}(\varphi_n^{(0)}(b)))) : \tau \in T(D_k) \} \right) \leq (f_{a_0})^2 / 2^n \quad (\text{e 10.20})$$

for any $0 \leq b \leq 1$. It follows that, for fixed n , and any fixed $m \geq n$,

$$\lim_{k \rightarrow \infty} \left(\sup \{ |\tau(f_{1/4}(\varphi_{m+k}^{(1)}(a_m - a_n)))| : \tau \in T(D_{m+k}) \} \right) \leq (f_{a_0})^2 / 2^{n-1}. \quad (\text{e 10.21})$$

Without loss of generality, passing to a subsequence if necessary, by (e 10.16), we may assume that, for all $m > n$,

$$\|a_n - (\varphi_m^{(0)} + \varphi_m^{(1)})(a_n)\| < \frac{(f_{a_0})^2}{2^{(n+4)^2}}, \quad (\text{e 10.22})$$

$$\|f_{1/4}(a_n) - f_{1/4}((\varphi_m^{(0)} + \varphi_m^{(1)})(a_n))\| < \frac{(f_{a_0})^2}{2^{(n+4)^2}}, \quad (\text{e 10.23})$$

$$\text{and } \|\varphi_m^{(1)} \circ \cdots \circ \varphi_{n+1}^{(1)}(f_{1/4}(a_n)) - f_{1/4}(a_{n+m})\| < \frac{(\mathfrak{f}_{a_0})^2}{2^{(n+4)^2}}, \quad n = 1, 2, \dots, \quad (\text{e 10.24})$$

and by (e 10.21) whenever $m + k \geq n + 1$,

$$\sup \{ |\tau(f_{1/4}(\varphi_{m+k}^{(1)}(a_{n+1} - a_n)))| : \tau \in \mathbb{T}(D_{m+k}) \} \leq \frac{(\mathfrak{f}_{a_0})^2}{2^{(n+4)^2}}. \quad (\text{e 10.25})$$

Claim 1.

$$\liminf_{n \rightarrow \infty} \left(\inf \{ \tau(f_{1/4}(\varphi_m^{(1)}(a_n))) : \tau \in \mathbb{T}(D_m) \text{ and } m > n \} \right) \geq \frac{(\mathfrak{f}_{a_0})^2}{8}. \quad (\text{e 10.26})$$

Claim 2. If we first take a subsequence $(N(k))$ and as above define

$$a_1 := \varphi_{N(1)}^{(1)}(a_0), \quad a_2 := \varphi_{N(2)}^{(1)}(a_1), \dots, a_n := \varphi_{N(n)}^{(1)}(a_{n-1}), \quad n = 0, 1, \dots,$$

then Claim 1 still holds, when m is replaced by $N(m)$.

Let us first explain that Claim 2 follows from Claim 1 since we may first pass to another subsequence in the construction above and then apply Claim 1.

Proof of Claim 1. Assume Claim 1 is false. Then there exists $\eta_0 > 0$ such that $\frac{(\mathfrak{f}_{a_0})^2}{8} - \eta_0 > 0$ and

$$\liminf_{n \rightarrow \infty} \left(\inf \{ \tau(f_{1/4}(\varphi_m^{(1)}(a_n))) : \tau \in \mathbb{T}(D_m) \text{ and } m > n \} \right) \leq \frac{(\mathfrak{f}_{a_0})^2}{8} - \eta_0. \quad (\text{e 10.27})$$

By (e 10.25), there is $n_0 \geq 1$ such that, for all $m \geq n \geq n_0$ and $k \geq 1$,

$$\tau(f_{1/4}(\varphi_{m+k}^{(1)}(a_m))) \leq \tau(f_{1/4}(\varphi_{m+k}^{(1)}(a_n))) + \eta_0/2 \quad \text{for all } \tau \in \mathbb{T}(D_{m+k}). \quad (\text{e 10.28})$$

Hence there exists a subsequence (n_k) which has the following property: if $k' \geq k$, then

$$t_{n_{k'}}(f_{1/4}(\varphi_{n_{k'}}^{(1)}(a_{n_k}))) \leq (\mathfrak{f}_{a_0})^2/8 - \eta_0/2. \quad (\text{e 10.29})$$

Consider the positive linear functional τ_k defined by $\tau_k(a) = t_k(\varphi_{n_k}^{(1)}(a))$ for all $a \in A$, $k = 1, 2, \dots$. Let τ be a weak* limit of $(\tau_k)_{k \geq 1}$. It follows (e 10.18) that $1 \geq \|\tau\| \geq \mathfrak{f}_{a_0}$. Moreover, by (e 10.15), τ is a trace. On the other hand, by (e 10.29) and (e 10.23),

$$\tau(f_{1/4}(a_{n_k})) < (\mathfrak{f}_{a_0})^2/8 \quad \text{for all } k. \quad (\text{e 10.30})$$

It follows from (e 10.23), (e 10.14) and (e 10.17) that, if $m > n \geq 1$,

$$\tau(f_{1/4}(\varphi_m^{(1)}(a_n))) \geq \tau(f_{1/4}(a_n)) - (\mathfrak{f}_{a_0})^2/2^{m+5} - (\mathfrak{f}_{a_0})^2/2^{(n+4)^2}. \quad (\text{e 10.31})$$

Therefore, also using (e 10.13), for all k ,

$$\tau(f_{1/4}(a_{n_k})) \geq \tau(f_{1/4}(a_0)) - \left(\sum_{j=1}^{n+k} ((f_{a_0})^2/2^{j+5} - (f_{a_0})^2/2^{(j+1)^2}) \right) > (f_{a_0})^2/4. \quad (\text{e 10.32})$$

This contradicts with (e 10.30) and so Claim 1 is proved. \square

Since $A = \text{Ped}(A)$, by Theorem 5.7, we obtain a map $T': A_+ \setminus \{0\} \rightarrow \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$, which has the properties described by Theorem 5.7. Set $T(a) = (N(a), M(a))$ and $\lambda(a) = (N(a) + 1)(M(a) + 1)$ for each $a \in A_+ \setminus \{0\}$.

Now fix a finite subset $\mathcal{F} \subset A_+ \setminus \{0\}$ and $1/16 > \varepsilon > 0$. We may assume that $\|a\| \leq 1$ for all $a \in \mathcal{F}$, $a_0 \in \mathcal{F}$. Let e_n^D be a strictly positive element of D_n with $0 \leq e_n^D \leq 1$. Let $(\mathcal{F}'_{k,n})$ be an increasing sequence of finite subsets of $(D_k)_+$ such that the union of these subsets is dense in $(D_k)_+$. We may assume that $\varphi_k^{(1)}(\mathcal{F}) \subset \mathcal{F}'_{k,1}$. By (e 10.16) and (e 10.18), without loss of generality, by choosing larger k , if necessary, we may also assume that the elements of $\varphi_k^{(1)}(\mathcal{F})$ are non-zero, and therefore that those of $\mathcal{F}'_{k,n}$ are non-zero. Choose $1/4 > \eta_0 > 0$ such that

$$\|f_{1/4}(a') - f_{1/4}(a'')\| < \min\{\varepsilon, f_{a_0}^2\}/64 \quad (\text{e 10.33})$$

whenever $0 \leq a', a'' \leq 1$ and $\|a' - a''\| < \eta_0$. We may assume that

$$\|x - (\varphi_1^{(0)} + \varphi_1^{(1)})(x)\| < \min\{\eta_0, \varepsilon\}/16^2 \quad \text{for all } x \in \mathcal{F}. \quad (\text{e 10.34})$$

Set

$$\mathcal{F}''_{1,n} = \{(a - \|a\|/2)_+ : a \in \mathcal{F}'_{1,n}\} \quad \text{and} \quad \mathcal{F}'''_{1,n} = \mathcal{F}'_{1,n} \cup \mathcal{F}''_{1,n}. \quad (\text{e 10.35})$$

Choose $\sigma_1 > 0$ such that

$$\|f_{\sigma_1}(e_1^D)x f_{\sigma_1}(e_1^D) - x\| < \min\{\varepsilon, \eta_0\}/16^2 \quad \text{for all } x \in \varphi_1^{(1)}(\mathcal{F}), \quad (\text{e 10.36})$$

$$\|f_{\sigma_1}(c_1)\varphi_1^{(0)}(x)f_{\sigma_1}(c_1) - \varphi_1^{(0)}(x)\| < \min\{f_{a_0}, \eta_0, \varepsilon\}/16^3 \quad \text{for all } x \in \mathcal{F}. \quad (\text{e 10.37})$$

Define $\Phi_1: A \rightarrow A_{0,1}$ by $\Phi_1(x) = f_{\sigma_1}(c_1)\varphi_1^{(0)}(x)f_{\sigma_1}(c_1)$. Put

$$\varepsilon_0 = \min\{f_{a_0}^2, \varepsilon/16, \eta_0/16, \sigma_1/2\}.$$

Let $\mathcal{F}_{1,n}$ be a finite subset which also contains $\mathcal{F}'''_{1,n} \cup \{a_1, e_1^D, f_{\sigma_1}(e_1^D), f_{\sigma_1/2}(e_1^D)\}$. By Theorem 5.7, since D_1 is assumed to be in the class \mathcal{C}_0 , there exist an integer $n_2 \geq 2$ and a homomorphism $\psi_1: D_1 \rightarrow D_{n_2}$ such that

$$\sum_{i=1}^{m_1(a)} \varphi_{n_2}^{(1)}(x(a)_{i,1})^* \varphi_{n_2}^{(1)}(a) \varphi_{n_2}^{(1)}(x(a)_{i,1}) = f_{1/16}(\varphi_{n_2}^{(1)}(a_1)), \quad (\text{e 10.38})$$

where $x(a)_{i,1} \in A$, $m_1(a) \leq N(a)$ and $\|x(a)_{i,1}\| \leq M(a)$, $i = 1, 2, \dots, m_1(a)$, and

$$\|\psi_1(a) - \varphi_{n_2}^{(1)}(a)\| < \varepsilon_0/4 \cdot 16\lambda(a) \quad (\text{e 10.39})$$

for all $a \in \mathcal{F}_{1,1}'''$. It follows that

$$\left\| \sum_{i=1}^{m_1(a)} \varphi_{n_2}^{(1)}(x(a)_{i,1})^* \psi_1(a) \varphi_{n_2}^{(1)}(x(a)_{i,1}) - f_{1/16}(\varphi_{n_2}^{(1)}(a_1)) \right\| < \varepsilon_0/32 \quad (\text{e 10.40})$$

for all $a \in \mathcal{F}_{1,1}''$. Therefore, applying Lemma 3.1, one obtains $y(a)_{i,n_2} \in D_{n_2}$ with $\|y(a)_{i,n_2}\| \leq \|x(a)_{i,1}\| + (f_{a_0})^2/16$ such that

$$\sum_{i=1}^{m_2(a)} y(a)_{i,n_2}^* \psi_1(a) y(a)_{i,n_2} = f_{1/8}(\varphi_{n_2}^{(1)}(a_1)) \quad \text{for all } a \in \mathcal{F}_{1,1}'''. \quad (\text{e 10.41})$$

By (e 10.15), we may assume, choosing n_2 large, that

$$2\|x - (\varphi_{n_2}^{(0)} + \varphi_{n_2}^{(1)})(x)\| < \varepsilon_0/16^2 \quad \text{for all } x \in \mathcal{F} \cup \{c_1\} \cup \mathcal{F}_{1,1}, \quad (\text{e 10.42})$$

$$\|\varphi_{n_2}^{(1)}(f_{\sigma_1/2}(c_1))\varphi_{n_2}^{(1)}(f_{\sigma_1/2}(e_1^D))\| < \varepsilon_0/16, \quad (\text{e 10.43})$$

$$\|f_{\sigma'}(\varphi_{n_2}^{(1)}(c_1)) - \varphi_{n_2}^{(1)}(f_{\sigma'}(c_1))\| < \varepsilon_0/16, \quad \sigma' \in \{\sigma_1, \sigma_1/2, \sigma_1/4\}, \quad (\text{e 10.44})$$

$$\|f_{\sigma_1/2}(\varphi_{n_2}^{(1)}(c_1))^{1/2} \varphi_{n_2}^{(1)}(\Phi_1(x)) f_{\sigma_1/2}(\varphi_{n_2}^{(1)}(c_1))^{1/2} - \varphi_{n_2}^{(1)}(\Phi_1(x))\| < \varepsilon_0/16 \quad (\text{e 10.45})$$

for all $x \in \mathcal{F}$. Put $\varphi_1: A \rightarrow D_1$ by

$$\varphi_1(x) = f_{\sigma_1}(e_1^D) \varphi_1^{(1)}(x) (f_{\sigma_1}(e_1^D))$$

for $x \in A$. Define $\varphi_{n_2}^{(1)'}$ by

$$\varphi_{n_2}^{(1)'}(x) = f_{\sigma_1/2}(\varphi_{n_2}^{(1)}(c_1))^{1/2} \varphi_{n_2}^{(1)}(\Phi_1(x)) f_{\sigma_1/2}(\varphi_{n_2}^{(1)}(c_1))^{1/2}$$

for all $x \in A$. Define $\Phi_2: A \rightarrow A_{0,n_2}$ by (note, below, the two sums are orthogonal sums)

$$\Phi_2(x) = (1 - \psi_1(f_{\sigma_1/2}(e_1^D))) (\varphi_{n_2}^{(0)}((\Phi_1 + \varphi_1^{(1)})(x)) + \varphi_{n_2}^{(1)'}(x)) (1 - \psi_1(f_{\sigma_1/2}(e_1^D)))$$

for all $x \in A$. Note that Φ_2 is a completely positive contractive map and $\Phi_2(A) \perp j_2(\psi_1 \circ \varphi_1(A))$. Also (note the sum is orthogonal)

$$\psi_1(f_{\sigma_1/2}(e_1^D)) (\varphi_{n_2}^{(0)}(\Phi_1 + \varphi_1^{(1)})(x)) = 0 \quad \text{for all } x \in A. \quad (\text{e 10.46})$$

By (e 10.39) and (e 10.43),

$$\varphi_{n_2}^{(1)'}(x) \psi_1(f_{\sigma_1/2}(e_1^D)) \approx_{\varepsilon_0/16} \varphi_{n_2}^{(1)'}(x) \varphi_{n_2}^{(1)} f_{\sigma_1/2}(e_1^D) \approx_{\varepsilon_0/16} 0. \quad (\text{e 10.47})$$

Thus, on \mathcal{F} , by (e 10.45) and (e 10.37),

$$\begin{aligned}\Phi_2(x) &\approx_{\varepsilon_0/8} \varphi_{n_2}^{(0)}((\Phi_1 + \varphi_1^{(1)})(x)) + \varphi_{n_2}^{(1)'}(x) \\ &\approx_{\varepsilon_0/16} \varphi_{n_2}^{(0)}((\Phi_1 + \varphi_1^{(1)})(x)) + \varphi_{n_2}^{(1)}(\Phi_1(x)) \\ &\approx_{\varepsilon/16^2} \varphi_{n_2}^{(0)}((\varphi_1^{(0)} + \varphi_1^{(1)})(x)) + \varphi_{n_2}^{(1)}(\varphi_1^{(0)}(x)).\end{aligned}\quad (\text{e 10.48})$$

We then verify

$$\|x - (\Phi_2(x) + j_2(\psi_1 \circ \varphi_1(x)))\| < \varepsilon/16 \quad \text{for all } x \in \mathcal{F}. \quad (\text{e 10.49})$$

Note that, by (e 10.42),

$$\|(\varphi_{n_2}^{(0)} + \varphi_{n_2}^{(1)})(c_1) - c_1\| < \varepsilon_0/16. \quad (\text{e 10.50})$$

By Lemma 3.1, since $\varepsilon_0 < \sigma_1/2$ and $\varphi_{n_2}^{(0)}(c_1) \perp \varphi_{n_2}^{(1)}(c_1)$,

$$f_{\sigma_1/2}(\varphi_{n_2}^{(1)}(c_1)) \lesssim f_{\sigma_1/2}((\varphi_{n_2}^{(0)} + \varphi_{n_2}^{(1)})(c_1)) \lesssim c_1. \quad (\text{e 10.51})$$

Note, by the definition of Φ_2 , for any $x \in A_+$,

$$\begin{aligned}\langle \Phi_2(x) \rangle &\leq \langle (\varphi_{n_2}^{(0)}((\Phi_1 + \varphi_1^{(1)})(x)) + \varphi_{n_2}^{(1)'}(x)) \rangle \\ &\leq \langle c_{n_2} \rangle + \langle f_{\sigma_1/2}(\varphi_{n_2}^{(1)}(c_1)) \rangle \\ &\leq \langle b_{n_2} \rangle + \langle c_1 \rangle \leq \langle b_2 \rangle + \langle b_{1,2} \rangle \leq \langle b_{1,1} + b_{1,2} \rangle.\end{aligned}$$

It follows that, if c'_2 is a strictly positive element of $\overline{\Phi_2(A)A\Phi_2(A)}$ (see Proposition 2.10), then

$$c'_2 \lesssim b_{1,1} + b_{1,2}. \quad (\text{e 10.52})$$

To simplify notation, passing to a subsequence, if necessary, without loss of generality, we may assume that $n_2 = 2$.

Put $e'_2 = \psi_1(e_1^D)$. Set

$$\mathcal{F}'_{2,n} = \{(a - \|a\|/2)_+ : a \in \mathcal{F}'_{2,n}\} \quad \text{and} \quad \mathcal{F}''_{2,n} = \mathcal{F}'_{2,n} \cup \tilde{\mathcal{F}}''_{2,n}. \quad (\text{e 10.53})$$

Choose $\sigma_2 > 0$ such that:

$$\|f_{\sigma_2}(e'_2)x f_{\sigma_1}(e'_2) - x\| < \varepsilon/16^3 \quad \text{for all } x \in \mathcal{F}'_{2,2} \cup \psi_1(\mathcal{F}_{1,2}) \cup \varphi_2^{(1)}(\mathcal{F}), \quad (\text{e 10.54})$$

$$\|f_{\sigma_2}(c'_2)\Phi_2(x)f_{\sigma_2}(c'_2) - \Phi_2(x)\| < \varepsilon/16^3 \quad \text{for all } x \in \mathcal{F}. \quad (\text{e 10.55})$$

Recall that $\mathbf{K}_0(D_1)$ is finitely generated (see Remark 6.2), say by $[p_j] - n_j[1]$, where $p_j \in \mathbf{M}_m(\tilde{D}_1)$ is a projection with $[\pi_d(p_j)] = n_j[1]$, $\pi_d: \mathbf{M}_m(\tilde{D}_1) \rightarrow \mathbf{M}_m$ is the

quotient map, $j = 1, 2, \dots, k_1$. We may write $p_j = q_j + h_j$, where $h_j = (h_j^{(i,k)})$ with $h_j^{(i,k)} \in D_1$, where

$$q_j = \text{diag}(\overbrace{1, 1, \dots, 1}^{n_j}, 0, \dots, 0).$$

If $K_0(A) = \{0\}$, without loss of generality, we may assume that there exists $v_j \in M_m(\tilde{A})$ such that

$$v_j^* v_j = p_j \quad \text{and} \quad v_j v_j^* = q_j, \quad j = 1, 2, \dots, k_1, \quad (\text{e 10.56})$$

where we identify q_j with the matrix in $M_m(\mathbb{C} \cdot 1_{\tilde{A}})$. Write

$$v_j = \lambda_j + s_j,$$

where $s_j = (s_j^{(i,k)})$ with $s_j^{(i,k)} \in D_1$, $\lambda_j \in M_m(\mathbb{C} \cdot 1_{\tilde{A}})$ is a partial isometry and $s_j \in M_m(D_1)$, $1 \leq j \leq k_1$. Put

$$\varepsilon_1 = \min \{ \varepsilon_0 / (m16)^2, \sigma_2 / (m16)^2 \}.$$

Let $\mathcal{F}_{2,n}$ be a finite subset which also contains

$$\mathcal{F}_{2,n}''' \cup \{a_2, e'_2, f_{\sigma_2}(e'_2), f_{\sigma_2/2}(e'_2)\} \cup \psi_1(\mathcal{F}_{1,2}) \cup \varphi_2^{(1)}(\mathcal{F})$$

as well as $h_j^{(i,k)}$ and $s_j^{(i,k)}$. Applying Theorem 5.7, since $D_2 \in \mathcal{C}_0$, as in the previous step, we obtain a homomorphism $\psi_2: D_2 \rightarrow D_{n_3}$ such that

$$\|\psi_2(a) - \varphi_{n_3}^{(1)}(a)\| < \varepsilon_1 / 4 \cdot 16^3 \lambda(a) \quad \text{for all } a \in \mathcal{F}_{2,2} \quad (\text{e 10.57})$$

and

$$\left\| \sum_{i=1}^{m_2(a)} \varphi_{n_3}^{(1)}(x(a)_{i,2})^* \psi_2(a) \varphi_{n_3}^{(1)}(x(a)_{i,2}) - f_{1/16}(\varphi_{n_3}^{(1)}(a_2)) \right\| < \varepsilon_1 / 16^2 \quad (\text{e 10.58})$$

for all $a \in \mathcal{F}_{2,2}$, where $m_2(a) \leq N(a)$ and $\|x(a)_{i,2}\| \leq M(a)$ for all $a \in \mathcal{F}_{2,2}'''$. By Lemma 3.1, there are $y(a)_{i,n_3} \in D_{n_3}$ with $\|y(a)_{i,n_3}\| \leq \|x(a)_{i,2}\| + (f_{a_0})^2 / 16^2$ such that

$$\sum_{i=1}^{m_2(a)} y(a)_{i,n_3}^* \psi_2(a) y(a)_{i,n_3} = f_{1/8}(\varphi_{n_3}^{(1)}(a_2)) \quad \text{for all } a \in \mathcal{F}_{2,2}'''. \quad (\text{e 10.59})$$

By (e 10.15), choosing n_3 large, we may assume that:

$$\|x - (\varphi_{n_3}^{(0)} + \varphi_{n_3}^{(1)})(x)\| < \varepsilon_1 / 16^3 \quad \text{for all } x \in \mathcal{F} \cup \{c'_2\} \cup \mathcal{F}_{2,2}, \quad (\text{e 10.60})$$

$$\|\varphi_{n_3}^{(1)}(f_{\sigma_2/2}(c'_2)) \varphi_{n_3}^{(1)}(f_{\sigma_2/2}(e'_2))\| < \varepsilon_1 / 16^3, \quad (\text{e 10.61})$$

$$\|f_{\sigma'}(\varphi_{n_2}^{(1)}(c'_2)) - \varphi_{n_2}^{(1)}(f_{\sigma'}(c'_2))\| < \varepsilon_1/16^3, \quad \sigma' \in \{\sigma_2, \sigma_2/2, \sigma_2/4\}, \quad (\text{e } 10.62)$$

$$\|f_{\sigma_2/2}(\varphi_{n_3}^{(1)}(c'_2))^{1/2} \varphi_{n_3}^{(1)}(\Phi_2(x)) f_{\sigma_2/2}(\varphi_{n_3}^{(1)}(c'_2))^{1/2} - \varphi_{n_3}^{(1)}(\Phi_2(x))\| < \varepsilon_1/16^2 \quad (\text{e } 10.63)$$

for all $x \in \mathcal{F}$. In particular (we continue to use $\varphi_k^{(i)}$ for $\varphi_k^{(i)} \otimes \text{id}_{M_m}$),

$$\|h_j - (\varphi_{n_3}^{(0)}(h_j) + \varphi_{n_3}^{(1)}(h_j))\| < \varepsilon_1/16^3$$

and

$$\|s_j - (\varphi_{n_3}^{(0)}(s_j) + \varphi_{n_3}^{(1)}(s_j))\| < \varepsilon_1/16^3$$

(when $K_0(A) = 0$). By choosing smaller ε_1 , we may assume that (for $1 \leq j \leq k_1$) there is a partial isometry $u_j \in M_m(\tilde{D}_{n_3})$ such that

$$u_j^* u_j = \psi_2^{\sim}(p_j) \quad \text{and} \quad u_j u_j^* = q_j, \quad (\text{e } 10.64)$$

where

$$\|(\psi_2^{\sim}(p_j) + \varphi_{n_3}^{(0)}(h_j)) - p_j\| < \varepsilon_1, \quad \|(u_j + \varphi_{n_3}^{(0)}(s_j)) - v_j\| < \varepsilon_1,$$

where we identify $M_m(\mathbb{C} \cdot 1_{\tilde{D}_1})$ with $M_m(\mathbb{C} \cdot 1_{\tilde{A}})$, and where ψ_2^{\sim} is the extension of ψ_2 on $M_m(\tilde{D}_1)$. In particular, $(\psi_2)_{*0} = 0$, when $K_0(A) = \{0\}$. Define $\varphi_{n_3}^{(1)'}$ by

$$\varphi_{n_3}^{(1)'}(x) = f_{\sigma_2/2}(\varphi_{n_3}^{(1)}(c'_2))^{1/2} \varphi_{n_3}^{(1)}(\Phi_2(x)) f_{\sigma_2/2}(\varphi_{n_3}^{(1)}(c'_2))^{1/2}$$

for all $x \in A$. Define $\Phi_3: A \rightarrow A$ by

$$\begin{aligned} \Phi_3(x) = & (1 - \psi_2(f_{\sigma_2/2}(e'_2))) (\varphi_{n_3}^{(0)}((\Phi_2 + j_2 \circ \psi_1 \circ \varphi_1)(x)) + \varphi_{n_3}^{(1)'}(x)) \\ & \cdot (1 - \psi_2(f_{\sigma_2/2}(e'_2))) \end{aligned}$$

for all $x \in A$. Note Φ_3 is a completely positive contractive map and $\Phi_3(A) \perp j_2(\psi_{1,2} \circ \varphi_1(A))$, where $\psi_{1,2} = \psi_2 \circ \psi_1$. Also

$$\psi_2(f_{\sigma_2/2}(e'_2)) \varphi_{n_3}^{(0)}(\Phi_2 + j_2 \circ \psi_1 \circ \varphi_1)(x) = 0 \quad \text{for all } x \in A. \quad (\text{e } 10.65)$$

By (e 10.57) and (e 10.61),

$$\varphi_{n_3}^{(1)'}(x) \psi_2(f_{\sigma_1/2}(e'_2)) \approx_{\varepsilon_1/16^2} \varphi_{n_3}^{(1)'}(x) \varphi_{n_3}^{(1)}(f_{\sigma_2/2}(e'_2)) \approx_{\varepsilon_1/16^2} 0. \quad (\text{e } 10.66)$$

Thus, on \mathcal{F} , by (e 10.63) and (e 10.49) (using also the orthogonality of the sum),

$$\begin{aligned} \Phi_3(x) + \varphi_{n_3}^{(1)}(j_2 \circ \psi_1(\varphi_1(x))) & \\ \approx_{\varepsilon_1/2^7} \varphi_{n_3}^{(0)}((\Phi_2 + j_2 \circ \psi_1 \circ \varphi_1)(x)) + \varphi_{n_3}^{(1)'}(x) + \varphi_{n_3}^{(1)}(j_2 \circ \psi_1(\varphi_1(x))) & \\ \approx_{\varepsilon_1/2^8} \varphi_{n_3}^{(0)}((\Phi_2 + j_2 \circ \psi_1 \circ \varphi_1)(x)) + \varphi_{n_3}^{(1)}(\Phi_2 + j_2 \circ \psi_1)(x) & \\ \varepsilon_1/16 \approx \varphi_{n_3}^{(0)}(x) + \varphi_{n_3}^{(1)}(x) \approx_{\varepsilon_1/16^3} x. & \end{aligned}$$

Therefore, since $\varepsilon_1/2^7 + \varepsilon_1/2^8 + \varepsilon_1/16^3 + \varepsilon_1/16^3 < \varepsilon/16^2$, by (e 10.57),

$$\|x - (\Phi_3(x) + j_3(\psi_{1,2} \circ \varphi_1(x)))\| < \varepsilon/16 + \varepsilon/16^2 \quad (\text{e 10.67})$$

for all $x \in \mathcal{F}$. Note that, by (e 10.42),

$$\|(\varphi_{n_3}^{(0)} + \varphi_{n_3}^{(1)})(c'_2) - c'_2\| < \varepsilon_0/16. \quad (\text{e 10.68})$$

By Lemma 3.1 (recall that $\varphi_{n_3}^{(0)}(c'_2) \perp \varphi_{n_3}^{(1)}(c'_2)$),

$$f_{\sigma_2/2}(\varphi_{n_3}^{(1)}(c'_2)) \lesssim c'_2 \lesssim b_{1,1} + b_{1,2}. \quad (\text{e 10.69})$$

By the definition of Φ_3 above, for any $x \in A_+$,

$$\langle \Phi_3(x) \rangle \leq \langle c_{n_3} \rangle + \langle f_{\sigma_2/2}(\varphi_{n_3}^{(1)}(c'_2)) \rangle \leq \langle b_{n_3,1} \rangle + \langle c'_2 \rangle. \quad (\text{e 10.70})$$

Let c'_3 be a strictly positive element of $\overline{\Phi_3(A)A\Phi_3(A)}$. Then, by (e 10.52),

$$\langle c'_3 \rangle \leq \langle b_{n_3,1} \rangle + \langle b_{1,1} + b_{1,2} \rangle. \quad (\text{e 10.71})$$

To simplify notation, by passing to a subsequence, if necessary, without loss of generality, we may assume that $n_3 = 3$.

Continuing this process, one then obtains a sequence of homomorphisms $\psi_n: D_n \rightarrow D_{n+1}$ such that

$$\|\psi_n(a) - \varphi_n^{(1)}(a)\| < \varepsilon_0/4 \cdot 16^n \lambda(a) \quad \text{for all } a \in \mathcal{F}_{n,n} \quad (\text{e 10.72})$$

and

$$\sum_{i=1}^{m_n(a)} y(a)_{i,n+1}^* \psi_n(a) y(a)_{i,n+1} = f_{1/8}(\varphi_{n+1}^{(1)}(a_n)) \quad \text{for all } a \in \mathcal{F}_{n,n}, \quad (\text{e 10.73})$$

where $\mathcal{F}_{n,n}$ contains

$$\mathcal{F}_{n,n}''' \cup \{a_n, e'_n, f_{\sigma_n}(e'_n), f_{\sigma_n/2}(e'_n)\} \cup \psi_n(\mathcal{F}_{n-1,n-1}) \cup \varphi_n^{(1)}(\mathcal{F}),$$

where (for $m \geq n$)

$$\mathcal{F}_{n,m}'' = \{(a - \|a\|/2)_+ : a \in \mathcal{F}'_{n,m}\} \quad \text{and} \quad \mathcal{F}_{n,m}''' = \mathcal{F}'_{n,m} \cup \mathcal{F}''_{n,m}. \quad (\text{e 10.74})$$

Moreover, $m_n(a) \leq N(a)$, $\|y(a)_{i,n+1}\| \leq M(a) + 1$ for all n . Furthermore, there is a completely positive contractive map $\Phi_n: A \rightarrow A$ such that $\Phi_n(A) \perp j_n(\psi_{1,n} \circ \varphi_1(A))$,

$$\|x - (\Phi_n(x) + j_n(\psi_{1,n} \circ \varphi_1(x)))\| < \sum_{k=1}^n \varepsilon/16^k < \varepsilon \quad \text{for all } x \in \mathcal{F} \quad (\text{e 10.75})$$

and

$$c'_n \lesssim b_{1,1} + b_{1,2} + b_{2,1} + \cdots + b_{n,1} \lesssim b_0, \quad n = 1, 2, \dots \quad (\text{e } 10.76)$$

Consider the inductive limit C*-algebra $D = \lim_{n \rightarrow \infty} (D_n, \psi_n)$. (Again, one should note that we have passed to a subsequence to simplify notation.) Note that, if $A \in \mathcal{D}_0$, then each $D_n \in \mathcal{C}_0^0$. Note also that, while, by construction, $D_n \in A$, for each $n = 1, 2, \dots$, this is not true for D . Let us verify that D is simple. Fix a non-zero positive element $d_0 \in (D)_+$ with $\|d_0\| = 1$. Since each D_n is stably projectionless, so is D . Fix $1/64 > \varepsilon_1 > 0$. There is $d \in (D)_+$ such that $d = \psi_{m,\infty}(d')$ for $d' \in (D'_m)_+$ with $\|d'\| = 1$ and

$$\|d - d_0\| < \varepsilon_1/32. \quad (\text{e } 10.77)$$

It follows from Lemma 3.1 that there is $z \in D$ such that

$$(d - \varepsilon_1/16)_+ = z^* d_0 z. \quad (\text{e } 10.78)$$

By construction, there is $d'' \in \mathcal{F}_{m',m'}$ for some $m' \geq m + 16$ such that

$$\|\psi_{m,m'}((d' - \varepsilon_1/16)_+) - d''\| < \varepsilon_1/64. \quad (\text{e } 10.79)$$

There is $y \in D_{m'}$ such that

$$(d'' - \varepsilon_1/8)_+ = y^* \psi_{m,m'}((d' - \varepsilon_1/4)_+) y. \quad (\text{e } 10.80)$$

Note that $\varepsilon_1/2 \leq \|d''\|/8$.

By (e 10.73), there are $x_1, x_2, \dots, x_L \in D_{m'+1}$ such that

$$\sum_{i=1}^L x_i^* \psi_{m',m'+1}((d'' - \varepsilon_1/2)_+) x_i = f_{1/8}(a_{m'+1}). \quad (\text{e } 10.81)$$

Claim 3. The element $a_{00} := \psi_{m'+1,\infty}(a_{m'+1})$ is full in D . In fact, for any $m'' > m' + 1$, it follows from (e 10.72), (e 10.24), and Claim 2 that

$$\begin{aligned} \tau(\psi_{m'+1,m''}(f_{1/8}(a_{m'+1}))) &\geq \tau(\varphi_{m''}^{(1)} \circ \cdots \circ \varphi_{m'+2}^{(1)}(f_{1/8}(a_{m'+1}))) - \sum_{j=m'+2}^{m''} (f_{a_0})^2 / 16^j \\ &\geq \tau(\varphi_{m''}^{(1)} \circ \cdots \circ \varphi_{m'+2}^{(1)}(f_{1/4}(a_{m'+1}))) - (f_{a_0})^2 / 16^{m'+1} \\ &\geq \tau(f_{1/4}(a_{m''})) - \frac{(f_{a_0})^2}{2^{(m'+1+4)^2}} - \frac{(f_{a_0})^2}{16^{m'+1}} > \frac{(f_{a_0})^2}{16} \end{aligned} \quad (\text{e } 10.82)$$

for all $\tau \in \mathbf{T}(D_{m''})$. By Proposition 6.3, we conclude that $\psi_{m'+1,m''}(a_{m'+1})$ is full in $D_{m''}$. Therefore a_{00} is full in $\psi_{m'',\infty}(D_{m''})$ for all $m'' > m' + 1$. Hence, the closed two-sided ideal generated by a_{00} contains $\bigcup_{m'' > m'+1} \psi_{m'',\infty}(D_{m''})$. This implies that a_{00} is full in D , which proves Claim 3.

It follows from (e 10.81) that $\psi_{m',\infty}((d'' - \varepsilon_1/2)_+)$ is full in D . By (e 10.80), the element $\psi_{m,\infty}((d' - \varepsilon_1/4)_+)$ is full in D . Then, by (e 10.78), d_0 is full in D . Since d_0 is arbitrarily chosen in $(D)_+ \setminus \{0\}$ with $\|d_0\| = 1$, this implies that D is indeed simple.

By (e 10.33), and then as in the estimate of (e 10.82),

$$\begin{aligned}
& \tau(f_{1/4}(\psi_{1,n}(\varphi_1(a_0)))) \\
& > \tau(f_{1/4}(\psi_{1,n}(\varphi_1^{(0)}(a_0)))) - f_{a_0}/64 \\
& = \tau(\psi_{1,n}(f_{1/4}(a_1))) - f_{a_0}/64 \\
& > \tau(\varphi_n^{(1)} \circ \cdots \circ \varphi_2^{(1)}(f_{1/4}(a_1))) - \sum_{j=2}^n (f_{a_0})^2/16^j - f_{a_0}^2/64 \\
& \geq \tau(f_{1/4}(a_n)) - (f_{a_0})^2/2^{(1+4)^2} - (f_{a_0})^2/16^2(16/15) - f_{a_0}^2/64 \\
& \geq f_{a_0}^2/8 - f_{a_0}^2/32 = f_{a_0}^2/16 =: d_A \quad \text{for all } \tau \in \mathbf{T}(D_n).
\end{aligned}$$

We also note, since $(\psi_2)_{*0} = 0$, that $(\psi_{1,n})_{*0} = 0$ for all $n \geq 2$. \square

Theorem 10.4. *Let A be a separable C^* -algebra in \mathfrak{D}_0 . Then A is tracially approximately divisible in the sense of Definition 10.1.*

Proof. Let $\varepsilon > 0$, $\mathcal{F} \subset A$ be a finite subset, $b \in A_+ \setminus \{0\}$, and let $K \geq 1$ be an integer. Let $e_A \in A$ be a strictly positive element with $0 \leq e_A \leq 1$. Choose $1/2 > \sigma > 0$ such that

$$\|f_\sigma(e_A)af_\sigma(e_A) - a\| < \varepsilon/4 \quad \text{for all } a \in \mathcal{F}. \quad (\text{e 10.83})$$

Set $\mathcal{F}_1 = \{f_\sigma(e_A)af_\sigma(e_A) : a \in \mathcal{F}\}$ and $A' = \overline{f_\sigma(e_A)Af_\sigma(e_A)}$. Then A' is algebraically simple (as A is), and so $A' = \text{Ped}(A')$. Choose $b_0 \in (A')_+ \setminus \{0\}$ such that $\langle b_0 \rangle \leq \langle b \rangle$.

We apply Theorem 10.3 to A' , \mathcal{F}_1 , $\varepsilon/4$ and b_0 . Let $D = \varinjlim (D_n, \psi_n)$, $j_n: D_n \rightarrow A$, and $\varphi: A \rightarrow D_1$ be as in Theorem 10.3. Put

$$C_k = \overline{\psi_{1,k}(\varphi(A))D_k\psi_{1,k}(\varphi(A))}, \quad k = 1, 2, \dots,$$

and

$$C = \lim_{k \rightarrow \infty} (C_k, \psi_k|_{C_{k-1}}).$$

Then C is a hereditary sub- C^* -algebra of D and $C_k \in \mathfrak{C}_0^{0'}$. By Lemma 10.2, there exist $n \geq 1$ and sub- C^* -algebras $D'_n = \mathbf{M}_K(D''_n) \subset C_n$ such that

$$\text{dist}(\psi_{1,n} \circ \varphi(a), \mathcal{F}_2 \otimes 1_K) < \varepsilon/4 \quad \text{for all } a \in \mathcal{F}_1, \quad (\text{e 10.84})$$

where D''_n is a hereditary sub- C^* -algebra of C_n and $\mathcal{F}_2 \subset D''_n$ is a finite subset.

Let $\Phi_n: A \rightarrow A$ be as in Theorem 10.3. Then $\Phi_n(A) \perp j_n(\psi_{1,n} \circ \varphi_1(A))$ and $c_n \lesssim b_0$, where c_n is a strictly positive element of $\overline{\Phi_n(A)A\Phi_n(A)}$. Recall that, in Theorem 10.3, D_n is embedded in A by means of j_n . Define $A_0 = \overline{\Phi_n(A)A\Phi_n(A)}$. Then $A_0 \perp j_n(C_n)$ and $B = A_0 \oplus M_K(j_n(D_n'')) \subset A$. Then

$$\text{dist}(\Phi(x) + j_n \circ \psi_{1,n}(\varphi(x)), A_0 + D_m'' \otimes 1_K) < \varepsilon/4 \quad \text{for all } x \in \mathcal{F}_1. \quad (\text{e 10.85})$$

However, as part of the conclusion of Theorem 10.3,

$$\|x - (\Phi(x) + j_n \circ \psi_{1,n}(\varphi(x)))\| < \varepsilon/4 \quad \text{for all } x \in \mathcal{F}_1. \quad (\text{e 10.86})$$

By (e 10.83),

$$\text{dist}(a, A_0 + D_m'' \otimes 1_K) < \varepsilon \quad \text{for all } a \in \mathcal{F}. \quad (\text{e 10.87})$$

Note we also have $c_n \lesssim b_0 \lesssim b$. \square

Remark 10.5. In fact, one can also prove the conclusion of Theorem 10.4 by replacing the condition $A \in \mathcal{D}_0$ by $A \in \mathcal{D}$ and $\mathbf{K}_0(A) = 0$, and applying Proposition 7.9, since we will show that a C^* -algebra in \mathcal{D} has stable rank one (which will be done in Theorem 11.5). This will be carried out in Proposition 11.10 below.

Corollary 10.6. *Let A be a separable C^* -algebra in the class \mathcal{D} . Then A has the following property: For any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$, any $a_0 \in A_+ \setminus \{0\}$, and any integer $n \geq 1$, there are mutually orthogonal elements $e_0, e_{00}, e_{01} \in A_+$, completely positive contractive maps $\varphi_0: A \rightarrow E_0$, $\varphi_1: A \rightarrow E_1$, and $\varphi_2: A \rightarrow E_2$, where E_0, E_1, E_2 are sub- C^* -algebras of A , $E_0 = \overline{e_0 A e_0}$, $e_{00} \in E_1$, $e_{01} \in E_2$, $E_0 \perp E_1$, $M_n(E_2) \subset E_1$, with $E_2 \in \mathcal{C}'_0$ and $E_2 \subset e_{01} A e_{01}$, such that*

$$\|x - (\varphi_0 + \varphi_1)(x)\| < \varepsilon/2 \quad (\text{e 10.88})$$

$$\text{and} \quad \|\varphi_1(x) - (r(x) + \varphi_2(a) \otimes 1_n)\| < \varepsilon/2, \quad (\text{e 10.89})$$

$$r(x) \in \overline{e_{00} A e_{00}} \quad \text{for all } x \in \mathcal{F}, \quad (\text{e 10.90})$$

and

$$e_0 + e_{00} \lesssim a_0 \quad \text{and} \quad e_{00} \lesssim e_{01}.$$

Proof. The proof is almost the same as that of Theorem 10.4. One replaces \mathcal{C}'_0 by \mathcal{C}'_0 and instead of applying the first part of Lemma 10.2, one applies the second part of Lemma 10.2. We omit the repetition. \square

Theorem 10.7. *Let A be a separable C^* -algebra in \mathcal{D}_0 . Let $a \in A_+$ with $\|a\| = 1$ be a strictly positive element. Then the following statement is true.*

There exists $1 > \mathfrak{f}_a > 0$ such that, for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any $b \in A_+ \setminus \{0\}$ and any integer $n \geq 1$, there are \mathcal{F} - ε -multiplicative completely positive

contractive maps $\varphi: A \rightarrow A$ and $\psi: A \rightarrow D$ for some sub-C*-algebra $D \in \mathcal{C}_0^{0'}$ with $M_n(D) \subset A$ and $M_n(D) \perp \varphi(A)$ such that $\|\psi\| = 1$,

$$\|x - (\varphi(x) + \psi(x) \otimes 1_n)\| < \varepsilon \quad \text{for all } x \in \mathcal{F} \cup \{a\}, \quad (\text{e } 10.91)$$

$$c \lesssim b, \quad (\text{e } 10.92)$$

$$t(f_{1/4}(\psi(a))) \geq \mathfrak{f}_a, \quad t \in \mathbf{T}(D), \quad (\text{e } 10.93)$$

and $\psi(a)$ is strictly positive in D , where c is a strictly positive element of $A_0 := \varphi(a)A\varphi(a)$.

Proof. Fix a strictly positive element $a \in A_+$ with $\|a\| = 1$. It follows from Proposition 9.1 that $0 \notin \overline{\mathbf{T}(A)}^w$. Let

$$r_0 = \inf \{ \tau(f_{1/2}(a)) : \tau \in \overline{\mathbf{T}(A)}^w \} > 0. \quad (\text{e } 10.94)$$

Let $\mathfrak{f}_{a_0} = r_0/6$. Choose an integer $k_0 \geq 1$ such that $r_0/16 > 1/k_0$.

Let $1 > \varepsilon > 0$ and $\mathcal{F} \subset A$ be a finite subset. Choose $\varepsilon_1 = \min\{\varepsilon/16, r_0/128\}$. Let $\mathcal{F}_1 \supset \mathcal{F} \cup \{a, f_{1/4}(a)\}$ be a finite subset of A . Let $b \in A_+ \setminus \{0\}$, and any integer $n \geq 1$ be given.

Choose $b'_0, b'_1, \dots, b'_{n+2k_0} \in \overline{bAb}$ such that $b'_0, b'_1, \dots, b'_{n+2k_0}$ are mutually orthogonal and mutually equivalent in the sense of Cuntz and there are non-zero and mutually orthogonal elements $b_0, b_1, \dots, b_{n+2k_0} \in A_+$ such that $b_i b'_0 = b_i$, $i = 0, 1, \dots, n + 2k_0$.

By Theorem 10.4, A has the property of tracial approximate divisibility. Therefore there are sub-C*-algebras A_0, A_1 and $M_n(A_1)$ of A such that $A_0 \perp M_n(A_1)$,

$$\text{dist}(x, A_0 + A_1 \otimes 1_n) < \varepsilon_1/2 \quad \text{for all } x \in \mathcal{F}_1,$$

and $a_0 \lesssim b_0$, where a_0 is a strictly positive element of A_0 . Moreover, there are $y_0 \in A_0$ and $y_1 \in A_1$ such that

$$\|a - (y_0 + y_1 \otimes 1_n)\| < \varepsilon/2 \quad (\text{e } 10.95)$$

$$\text{and} \quad \|f_{1/4}(a) - (f_{1/4}(y_0) + f_{1/4}(y_1) \otimes 1_n)\| < \varepsilon_1/2. \quad (\text{e } 10.96)$$

Note that

$$\tau(f_{1/4}(y_1) \otimes 1_n) \geq r_0 - 1/(n + 2k_0) - \varepsilon_1/2 > r_0/3 \quad (\text{e } 10.97)$$

for all $\tau \in \mathbf{T}(A)$.

Let $A'_0 = \overline{a_0 A a_0}$ and $A'_1 = \overline{y_1 A y_1}$. Note that $0 \notin \overline{\mathbf{T}(A'_1)}^w$ by Proposition 4.10. Moreover, if $\tau \in \mathbf{T}(A)$, then $\|\tau|_{A_1}\| \geq r_0/3$. We also have

$$\tau(f_{1/4}(y_1)) \geq r_0/3 \quad \text{for all } \tau \in \mathbf{T}(A'_1). \quad (\text{e } 10.98)$$

Note, by Remark 9.2, in Definition 8.1 the constant \mathfrak{f}_{y_1} can be chosen to be $r_0/6$.

Let $\mathcal{G} \subset A_1$ be a finite subset such that the following holds:

$$\text{dist}(f, \{(x_0 + x_1 \otimes 1_n) : x_0 \in A_0, x_1 \in \mathcal{G}\}) < \varepsilon_1/2 \quad \text{for all } f \in \mathcal{F}_1 \quad (\text{e 10.99})$$

and $y_1 \in \mathcal{G}$.

Note that A'_1 is a hereditary sub-C*-algebra of A . By Proposition 8.6, $A'_1 \in \mathcal{D}_0$. Thus, there exist two sub-C*-algebras B_0 and D of A'_1 , where $D \in \mathcal{C}_0^0$ and two \mathcal{G} - ε_1 -multiplicative completely positive contractive maps $\varphi_0: A'_1 \rightarrow B_0$ and $\psi_0: A'_1 \rightarrow D$ such that

$$\|x - (\varphi_0 + \psi_0)(x)\| < \varepsilon_1/2 \quad \text{for all } x \in \mathcal{G}, \quad (\text{e 10.100})$$

$$\varphi_0(c_0) \lesssim b_1, \quad \|\psi_0\| = 1, \quad (\text{e 10.101})$$

$$\tau \circ f_{1/4}(\psi_0(y_1)) \geq r_0/6 \quad \text{for all } \tau \in \text{T}(D), \quad (\text{e 10.102})$$

and $\psi_0(y_1)$ is a strictly positive element in D , where c_0 is a strictly positive element of A'_1 .

Set $A_{00} = A_0 \oplus A'_0 \otimes 1_n$ and let $c = a_0 + c_0 \otimes 1_n$. Choose a function $g \in C_0((0, 1])$, define $\varphi_{00}: A \rightarrow A_{00}$ by

$$\varphi_{00}(x) = g(a_0)xg(a_0) + \varphi_0(x) \otimes 1_n \quad \text{for all } x \in A.$$

Then, with a choice of g , we have

$$\|x - (\varphi_{00}(x) + \psi_0(x) \otimes 1_n)\| < \varepsilon \quad \text{for all } x \in \mathcal{F}. \quad (\text{e 10.103})$$

Moreover,

$$\langle c \rangle \leq \langle b_0 \rangle + \langle b_1 \rangle + \cdots + \langle b_n \rangle \leq \langle b \rangle.$$

Now let $\varphi = \varphi_{00}$. Then $\varphi(a) = \varphi_0(a) \lesssim c \lesssim b$. Put $\psi = \psi_0$. Note also (e 10.102) holds. It follows that φ , ψ , and D meet the requirements. \square

The following corollary follows from the combination of Theorems 8.3 and 10.7.

Corollary 10.8. *Let A be a separable algebraically simple C*-algebra in \mathcal{D}_0 (cf. Remark 11.4). Then the following property holds. Fix a strictly positive element $a \in A$ with $\|a\| = 1$ and let $1 > \mathfrak{f}_a > 0$ be as in Definition 8.1 (see also Remark 9.2). There is a map*

$$T: A_+ \setminus \{0\} \rightarrow \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$$

($a \mapsto (N(a), M(a))$ for all $a \in A_+ \setminus \{0\}$) satisfying the following condition: For any finite subset $\mathcal{F}_0 \subset A_+ \setminus \{0\}$, for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any $b \in A_+ \setminus \{0\}$ and any integer $n \geq 1$, there are \mathcal{F} - ε -multiplicative completely positive contractive maps $\varphi: A \rightarrow A$ and $\psi: A \rightarrow D = D \otimes e_{11}$ for some sub-C*-algebra $M_n(D) \subset A$ with $\varphi(A) \perp M_n(D)$ such that

$$\|x - (\varphi(x) + \psi(x) \otimes 1_n)\| < \varepsilon, \quad x \in \mathcal{F} \cup \{a\}, \quad (\text{e 10.104})$$

$$D \in \mathcal{C}_0^0, \quad (\text{e 10.105})$$

$$a_0 \lesssim b, \quad \|\psi\| = 1, \quad (\text{e 10.106})$$

and $\psi(a)$ is strictly positive in D , where $a_0 \in \overline{\varphi(a)A\varphi(a)}$ is a strictly positive element. Moreover, ψ is $T\text{-}\mathcal{F}_0 \cup \{f_{1/4}(a)\}$ -full in D .

Furthermore, we may assume that

$$t(f_{1/4}(\psi(a))) \geq \mathfrak{f}_a \quad (\text{e 10.107})$$

$$\text{and} \quad t(f_{1/4}(\psi(c))) \geq \frac{\mathfrak{f}_a}{4 \inf\{M(c)^2 \cdot N(c) : c \in \mathcal{F}_0 \cup \{f_{1/4}(a)\}\}} \quad (\text{e 10.108})$$

for all $c \in \mathcal{F}_0$ and for all $t \in \mathsf{T}(D)$, and, we may also require that

$$\varphi(a) \lesssim \psi(a). \quad (\text{e 10.109})$$

Proof. Note that the existence of the map T and the fact that ψ can be required to be $T\text{-}\mathcal{F}_0 \cup \{f_{1/4}(a)\}$ -full in D , and that (e 10.108) holds, are applications of Theorem 5.7.

To see the last part of conclusion, i.e., (e 10.109), let $1/2 > \eta > 0$ be such that $\tau(f_\eta(a)) > \mathfrak{f}_a/2$ for all $\tau \in \overline{\mathsf{T}(A)}^w$ (see Remark 9.2) and choose $b \in A_+ \setminus \{0\}$ such that $d_\tau(b) < \mathfrak{f}_a/4(n+1)$ for all $\tau \in \mathsf{T}(A)$. Then, with $\varepsilon < \eta/4$, (e 10.104) implies that

$$f_\eta(a) \lesssim \varphi(a) + \psi(a) \otimes 1_n. \quad (\text{e 10.110})$$

It follows that $d_\tau(\psi(a)) > \mathfrak{f}_a/2(n+1)$ for all $\tau \in \overline{\mathsf{T}(A)}^w$ or

$$d_\tau(\psi(a)) > d_\tau(b) \geq d_\tau(\varphi(a)) \quad \text{for all } \tau \in \overline{\mathsf{T}(A)}^w. \quad (\text{e 10.111})$$

It follows by (1) of Theorems 5.3 and 9.4 that $\varphi(a) \lesssim \psi(a)$. \square

Remark 10.9. It is clear from the proof that, for $n = 1$, both Theorem 10.7 and Corollary 10.8 hold if $A \in \mathcal{D}$ (with now D in \mathcal{C}_0' or \mathcal{C}_0).

11. Stable rank one

The proof of the following result is very similar to that of Lemma 2.1 of [44].

Lemma 11.1. *Let A be a separable, simple, and stably projectionless C^* -algebra such that every hereditary sub- C^* -algebra B has comparison for positive elements as formulated in the conclusion of Theorem 9.4, and satisfies the conclusion of Corollary 10.6 (without even assuming that E_2 belongs to a specific class of C^* -algebras). Then A almost has stable rank one (see Definition 2.7), for any hereditary sub- C^* -algebra B of A ,*

$$B \subset \overline{\text{GL}(\tilde{B})}.$$

Proof. It is clearly sufficient to consider the case $B = A$.

Fix an element $x \in A$ and $\varepsilon > 0$. Let $e \in A$ with $0 \leq e \leq 1$ be a strictly positive element. Upon replacing x by $f_\eta(e)x f_\eta(e)$ for some small $1/8 > \eta > 0$, we may assume that $x \in \overline{f_\eta(e)A f_\eta(e)}$. Put $B_1 = \overline{f_\eta(e)A f_\eta(e)}$.

By the assumption, we know that e is not a projection. We obtain a positive element $b_0 \in B_1^\perp \setminus \{0\}$.

Note that

$$B_1^\perp = \{a \in A : ab = ba = 0 \text{ for all } b \in B_1\}$$

is a non-zero hereditary sub-C*-algebra of A . Since we assume that A is infinite dimensional, $\overline{b_0 A b_0}$ contains non-zero positive elements $b_{0,1}, b'_{0,1}, b_{0,2}, b'_{0,2} \in B_1^\perp$ such that

$$b'_{0,1} \lesssim b'_{0,2}, \quad b_{0,1} b'_{0,1} = b_{0,1}, \quad b_{0,2} b'_{0,2} = b_{0,2}, \quad \text{and} \quad b'_{0,1} b'_{0,2} = 0.$$

Since A has comparison for positive elements as described in the conclusion of Theorem 9.4, we can choose a large integer $n \geq 2$ which has the following property: if $a_1, a_2, \dots, a_n \in A_+$ are n mutually orthogonal and mutually equivalent positive elements, then

$$a_1 + a_2 \lesssim b_{0,1}.$$

There is $B'_1 \subset B_1$ which has the form

$$B'_1 = B_{1,1} + D \otimes 1_n,$$

where $B_{1,1}$ is a hereditary sub-C*-algebra with a strictly positive element $b_{11} \lesssim b_{0,1}$ and there are $x_0 \in B_{1,1}$ and $x_1 \in D \setminus \{0\}$ such that

$$\|x - (x_0 + x_1 \otimes 1_n)\| < \varepsilon/16. \quad (\text{e 11.1})$$

 Let $d_0 \in D$ be a strictly positive element. By the choice of n , $d_0 \lesssim b_{0,1}$. Choose $0 < \eta_1 < 1/4$ such that 

$$\|f_{\eta_1}(d_0)x_1 f_{\eta_1}(d_0) - x_1\| < \varepsilon/16. \quad (\text{e 11.2})$$

Put $x'_1 = f_{\eta_1}(d_0)x_1 f_{\eta_1}(d_0)$. Note that

$$f_{\eta_1/8}(d_0) \lesssim b'_{0,2}. \quad (\text{e 11.3})$$

There are $w_i \in A$ such that:

$$w_i w_i^* = \text{diag}(\overbrace{0, 0, \dots, 0}^{i-1}, f_{\eta_1/4}(d_0), 0, \dots, 0), \quad i = 1, 2, \dots, n, \quad (\text{e 11.4})$$

$$w_i^* w_i = \text{diag}(\overbrace{0, 0, \dots, 0}^i, f_{\eta_1/4}(d_0), 0, \dots, 0), \quad i = 1, 2, \dots, n-1, \quad (\text{e 11.5})$$

$$w_n^* w_n \in \overline{b'_{0,2} A b'_{0,2}}. \quad (\text{e 11.6})$$

There is $v \in A$ such that

$$v^*v = 0 + \text{diag}(x'_1, 0, \dots, 0) \quad \text{and} \quad vv^* \in \overline{(b'_{0,1} + b'_{0,2})A(b'_{0,1} + b'_{0,2})}. \quad (\text{e 11.7})$$

Put

$$x''_i = \text{diag}(\overbrace{0, \dots, 0}^{i-1}, x'_1, 0, \dots, 0), \quad i = 1, 2, \dots, n, \quad (\text{e 11.8})$$

$$y''_i = \text{diag}(\overbrace{0, 0, \dots, 0}^{i-1}, f_{\eta_1/4}(d_0), 0, \dots, 0), \quad i = 1, 2, \dots, n, \quad (\text{e 11.9})$$

$$z_1 = v^*, \quad z_2 = v, \quad z_3 = \sum_{i=1}^{n-1} w_i^* x''_i, \quad \text{and} \quad z_4 = \sum_{i=1}^{n-1} y''_i w_i. \quad (\text{e 11.10})$$

Note that $z_3 z_2 = 0$ and $z_1 z_4 = 0$. Therefore,

$$(z_1 + z_3)(z_2 + z_4) = z_1 z_2 + z_3 z_4 \quad (\text{e 11.11})$$

$$= v^*v + \text{diag}(0, x'_1, x'_1, \dots, x'_1) \quad (\text{e 11.12})$$

$$= x_0 + \text{diag}(\overbrace{x'_1, x'_1, \dots, x'_1}^n) = x_0 + x'_1 \otimes 1_n. \quad (\text{e 11.13})$$

On the other hand,

$$z_1^2 = v^*v^* = 0, \quad z_1 z_3 = 0. \quad (\text{e 11.14})$$

We also compute that

$$z_3^2 = \sum_{i,j} w_i^* x''_i w_j^* x''_j = \sum_{i=2}^{n-1} w_i^* x''_i w_{i-1}^* x''_{i-1}. \quad (\text{e 11.15})$$

Inductively, we compute that

$$z_3^n = 0. \quad (\text{e 11.16})$$

Thus, by (e 11.14),

$$(z_1 + z_3)^k = \sum_{i=1}^k z_3^i z_1^{k-i} \quad \text{for all } k. \quad (\text{e 11.17})$$

Therefore, by (e 11.14), and (e 11.16), for $k = n+1$, $(z_1 + z_3)^{n+1} = 0$. We also have that $z_2 z_4 = 0$ and $z_2^2 = 0$. A similar computation shows that $z_4^n = 0$. Therefore, as above, $(z_2 + z_4)^{n+1} = 0$. One has the estimate

$$\|x - (z_1 + z_3)(z_2 + z_4)\| < \varepsilon/4.$$

Suppose that $\|z_i\| \leq M$ for $i = 1, \dots, 4$. Consider the elements of \tilde{A}

$$z_5 = z_1 + z_3 + \varepsilon/16(M + 1) \quad \text{and} \quad z_6 = z_2 + z_4 + \varepsilon/16(M + 1).$$

Since $(z_1 + z_3)$ and $(z_2 + z_4)$ are nilpotent, both z_5 and z_6 are invertible in \tilde{A} . We also estimate that, by (e 11.1),

$$\|x - z_5 z_6\| < \varepsilon. \quad \square$$

Corollary 11.2. *Let $A \in \mathcal{D}$ be a separable C^* -algebra. Then, for every $n = 1, 2, \dots$, $M_n(A)$ almost has stable rank one (see Definition 2.7).*

Proof. This follows from Propositions 8.5, 8.6, 9.3, Corollary 10.6, and Lemma 11.1. \square

Corollary 11.3. *Let $A \in \mathcal{D}$. Suppose that A is separable. Then $A = \text{Ped}(A)$.*

Proof. By Proposition 8.5, $M_n(A)$ is in \mathcal{D} . The corollary then follows from the combination of Proposition 9.1, Corollary 11.2, Theorem 9.4, Lemma 4.8, and Theorem 4.7. \square

Remark 11.4. Note, by Corollary 11.3, the assumption that $A = \text{Ped}(A)$ in Theorem 10.3 can be removed.

Theorem 11.5. *Let A be a separable C^* -algebra in \mathcal{D} . Then A has stable rank one.*

Proof. Let $x \in \tilde{A}$. We must show that $x \in \overline{\text{GL}(\tilde{A})}$. Applying 3.2 and 3.5 of [46], without loss of generality, we may assume that there exists a non-zero positive element $e'_0 \in \tilde{A}$ with $\|e'_0\| = 1$ such that $x e'_0 = e'_0 x = 0$. We may further assume that there exists $e_0 \in \tilde{A}_+$ with $\|e_0\| = 1$ such that $e_0 e'_0 = e'_0 e_0 = e_0$. Define

$$\sigma = \inf \{ \tau(f_{1/4}(e_0)) : \tau \in \overline{\text{T}(A)}^w \}. \quad (\text{e 11.18})$$

Multiplying by a scalar multiple of the identity, without loss of generality, we may assume that $x = 1 + a$, where $a \in A$.

Let $0 < \varepsilon_0 < \varepsilon$ be given and set $\varepsilon_1 = \min\{\varepsilon_0/(\|x\| + 1), \sigma\}$. Since $A \in \mathcal{D}$, there exist a hereditary sub- C^* -algebra $B_0 \subset A$ and a sub- C^* -algebra $D \subset A$ with $D \in \mathcal{C}_0$ such that:

$$\|a - (x_0 + x_1)\| < \varepsilon_1/64, \quad \|e_0 - (e_{0,0} + e_{0,1})\| < \varepsilon_1/64, \quad (\text{e 11.19})$$

$$\|f_{\delta'}(e_0) - (f_{\delta'}(e_{00}) + f_{\delta'}(e_{0,1}))\| < \varepsilon_1/64, \quad \delta' \in \{1/2^k : 2 \leq k \leq 6\}, \quad (\text{e 11.20})$$

where $x_0, e_{0,0} \in B_0$ and $x_1, e_{0,1} \in D$, $B_0 D = D B_0 = \{0\}$,

$$d_\tau(b_0) < \min\{\varepsilon_1/64, \sigma/64\} \quad \text{for all } \tau \in \overline{\text{T}(A)}^w, \quad (\text{e 11.21})$$

where b_0 is a strictly positive element of B_0 . Let p_{B_0} denote the open projection associated with B_0 . Then, for $\delta' \in \{1/2^k : 2 \leq k \leq 6\}$,

$$\begin{aligned} \|(1+x_1)f_{\delta'}(e_{0,1})\| &= \|(1+x_1)(1-p_{B_0})f_{\delta'}(e_{0,1})\| & (\text{e 11.22}) \\ &= \|(1+x_1+x_0)(1-p_{B_0})(f_{\delta'}(e_{0,0})+f_{\delta'}(e_{0,1}))\| & (\text{e 11.23}) \\ &< \|(1+x_1+x_0)(1-p_{B_0})f_{\delta'}(e_0)\| + (\|x\| + \varepsilon_1/64)\varepsilon_1/64 & (\text{e 11.24}) \\ &= \|(1-p_{B_0})(1+x_1+x_0)f_{\delta'}(e_0)\| + \varepsilon_0/64 & (\text{e 11.25}) \\ &< \|(1-p_{B_0})xf_{\delta'}(e_0)\| + \varepsilon_1/64 + \varepsilon_0/64 & (\text{e 11.26}) \\ &= \varepsilon_1/64 + \varepsilon_0/64 < \varepsilon_0/32. & (\text{e 11.27}) \end{aligned}$$

Put

$$x'_1 = (-2f_{1/64}(e_{0,1}) + f_{1/64}(e_{0,1})^2) + (1 - f_{1/64}(e_{0,1}))x_1(1 - f_{1/64}(e_{0,1})). \quad (\text{e 11.28})$$

Then $x'_1 \in D$. By the calculation above,

$$(1 - f_{1/64}(e_{0,1}))(1+x_1)(1 - f_{1/64}(e_{0,1})) = 1 + x'_1 \quad (\text{e 11.29})$$

and $\|(1+x'_1) - (1+x_1)\| < 3\varepsilon_0/30. \quad (\text{e 11.30})$

Moreover,

$$(1+x'_1)f_{1/64}(e_{0,1}) = (1 - f_{1/64}(e_{0,1}))(1+x_1)(1 - f_{1/64}(e_{0,1}))f_{1/16}(e_{0,1}) = 0. \quad (\text{e 11.31})$$

We also have, by (e 11.20),

$$\tau(f_{1/4}(e_{0,1})) \geq \tau(f_{1/4}(e_0)) - \varepsilon_1/64 \geq \sigma - \varepsilon_0/64 > \sigma/2 \quad \text{for all } \tau \in \overline{\mathbb{T}(A)}^w. \quad (\text{e 11.32})$$

Therefore, by (e 11.21),

$$d_\tau(b_0) < \tau(f_{1/4}(e_{0,1})) \quad \text{for all } \tau \in \overline{\mathbb{T}(A)}^w. \quad (\text{e 11.33})$$

By Theorem 9.4, $b_0 \lesssim f_{1/8}(e_{0,1})$. Note that $f_{1/16}(e_{0,1})f_{1/8}(e_{0,1}) = f_{1/8}(e_{0,1})$.

To simplify notation, choosing sufficiently small ε_0 and changing notations, we may assume that

$$\|a - (x_0 + x_1)\| < \varepsilon/16 \quad \text{and} \quad \|e_0 - (e_{0,0} + e_{0,1})\| < \varepsilon/16, \quad (\text{e 11.34})$$

where $x_0, e_{0,0} \in B_0$ and $x_1, e_{0,1} \in D$, and also

$$e_{0,1}(1+x_1) = (1+x_1)e_{0,1} = 0 \quad (\text{e 11.35})$$

and $b_0 \lesssim e'_{0,1}$, where $0 \leq e'_{0,1} \leq 1$ and $e'_{0,1}f_\delta(e_{0,1}) = e'_{0,1}$ for some $0 < \delta < 1/4$.

We may also assume, without loss of generality, that there are $b_{0,1}, b'_{0,1} \in B_0$ with $0 \leq b_{0,1}, b'_{0,1} \leq 1$ such that

$$b_{0,1}x_0 = x_0b_{0,1} = x_0, \quad f_{1/16}(b'_{0,1})b_{0,1} = b_{0,1}. \quad (\text{e 11.36})$$

Set $A_2 = \overline{(f_\delta(e_{0,1}) + b'_{0,1})A(f_\delta(e_{0,1}) + b'_{0,1})}$. Note that, and by Proposition 8.6, $A_2 \in \mathcal{D}$. Since $b'_{0,1} \lesssim b_0 \lesssim e'_{0,1}$, and by Lemma 11.1, A_2 almost has stable rank one, there is a unitary $u'_1 \in \tilde{A}_2$ (see Lemma 3.3) such that

$$(u'_1)^*b'_{0,1}(u'_1) \in \overline{f_\delta(e_{0,1})Af_\delta(e_{0,1})}. \quad (\text{e 11.37})$$

Let q_0 denote the open projection in A^{**} corresponding to $\overline{b'_{0,1}Ab'_{0,1}}$, and q be the open projection in A^{**} corresponding to the hereditary sub- C^* -algebra A_2 . Then $q_0 \leq q$. Note that

$$x_0q_0 = x_0q_0 = x_0 \quad (\text{e 11.38})$$

$$\text{and} \quad q_0u'_1q_0 = (u'_1)(u'_1)^*(q_0u'_1)q_0 = (u'_1)((u'_1)^*q_0u'_1)q_0 = 0. \quad (\text{e 11.39})$$

Note also that

$$\|x - (1 + x_0 + x_1)\| = \|a - (x_0 + x_1)\| < \varepsilon/16. \quad (\text{e 11.40})$$

Put $z = 1 + x_0 + x_1$. Then $z \in \tilde{A}$. Put

$$z_0 = zq_0 = (1 + x_0)q_0 = q_0(1 + x_0) \quad (\text{e 11.41})$$

$$\text{and} \quad z_1 = z(1 - q_0) = (1 - q_0)z = (1 - q_0) + x_1. \quad (\text{e 11.42})$$

Keep in mind that $z_0 + z_1 = z$.

Now write $u'_1 = \lambda 1_{\tilde{A}_2} + y$ for some $y \in A_2$ and for some scalar $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Set $u_1 = \lambda q + y$. Multiplying by $\bar{\lambda}$ and changing notation, we may assume that $u_1 = q + y$. Define $u = 1 + y = u_1 + (1 - q)$. Since $q_0 \leq q = 1_{\tilde{A}_2}$, we have

$$q_0u = q_0qu = q_0qu_1 = q_0qu'_1 = q_0u'_1. \quad (\text{e 11.43})$$

Then, by (e 11.39),

$$(z_0u)(z_0u) = (1 + x_0)q_0uq_0(1 + x_0)u = (1 + x_0)q_0u'_1q_0(1 + x_0) = 0. \quad (\text{e 11.44})$$

In other words, z_0u is a nilpotent in A^{**} .

On the other hand, by (e 11.35),

$$z_1f_\delta(e_{0,1}) = (1 - q_0)(1 + x_1)f_\delta(e_{0,1}) = 0.$$

Therefore,

$$z_1c = ((1 - q_0) + x_1)c = cz_1 = 0 \quad \text{for all } c \in A_2. \quad (\text{e 11.45})$$

Thus, as $y \in A_2$,

$$z_1 u = z_1(1 + y) = z_1 + z_1 y = z_1. \quad (\text{e 11.46})$$

Put $D_1 = D + \mathbb{C} \cdot (1 - q_0)$. Then $D_1 \in \mathcal{C}$. Let

$$D_2 = \{d' \in D_1 : d' f_\delta(e_{0,1}) = f_\delta(e_{0,1}) d' = 0\}. \quad (\text{e 11.47})$$

Then, by (e 11.45), $z_1 \in D_2$. Note that D_2 is a hereditary sub-C*-algebra of D_1 . If D_2 is unital, say e' is the unit, then $e' \neq 1 - q_0$. Then $(1 - q_0) - e'$ is also a non-zero projection. One of them must be in D . Since D is stably projectionless, that one has to be zero. Since $(1 - q_0) - e' \neq 0$, this leads a contradiction. So D_2 is not unital. Since D_1 has stable rank one (see, for example, 3.3 of [22]), so is D_2 . Let e_{D_2} be a strictly positive element of D_2 . In A^{**} , let $p_d = \lim_{n \rightarrow \infty} (e_{D_2})^{1/n}$ (converges in A^{**}). In particular, $p_d \leq 1 - q_0$. So $q_0 p_d = 0$. Moreover, since $e_{D_2} f_\delta(e_{0,1}) = 0$,

$$p_d f_\delta(e_{0,1}) = 0. \quad (\text{e 11.48})$$

We also have $p_d B_0 = B_0 p_d = 0$. It follows that $p_d q = 0$. Therefore,

$$z_0 u(1 - p_d) = z_0 u q(1 - p_d) = z_0 u q = z_0 u. \quad (\text{e 11.49})$$

Hence, $z_0 u \in (1 - p_d) A^{**} (1 - p_d)$.

Since $D_2 + \mathbb{C} p_d$ has stable rank one, there is an invertible element $z'_1 \in D_2 + \mathbb{C} p_d$ such that

$$\|z_1 - z'_1\| < \varepsilon/16. \quad (\text{e 11.50})$$

We may write $z'_1 = \lambda_1 p_d + y_d$, where $\lambda_1 \in \mathbb{C}$ and $y_d \in D_2$. We may also write $y_d = \lambda_2(1 - q_0) + d_0$, where $\lambda_2 \in \mathbb{C}$ and $d_0 \in D$.

Set $I = D \cap D_2$. Then we have the natural short exact sequence

$$0 \rightarrow I \rightarrow D_2 + \mathbb{C} p_d \xrightarrow{\pi} \mathbb{C} \oplus \mathbb{C} \rightarrow 0. \quad (\text{e 11.51})$$

Then $\|\pi(z_1 - z'_1)\| < \varepsilon/16$. Thus, $|\lambda_1| < \varepsilon/16$ and $|1 - \lambda_2| < \varepsilon/16$. Put

$$z''_1 := (1/\lambda_2) z'_1 = \eta p_d + (1 - q_0) + d'_0,$$

where $\eta = \lambda_1/\lambda_2$ and $d'_0 = d_0/\lambda_2 \in D$. Then $|\eta| < \varepsilon/8$ and

$$\|z_1 - z''_1\| < 3\varepsilon/16. \quad (\text{e 11.52})$$

Moreover, z''_1 is invertible in $D_2 + \mathbb{C} p_d$. Without loss of generality, we may require that $\eta \neq 0$ (since elements near z'_1 are invertible). We may also write

$$z''_1 = z_1 + (z''_1 - z_1) = z_1 + (\eta p_d + (1 - q_0) + d'_0 - (1 - q_0 + x_1)) \quad (\text{e 11.53})$$

$$= z_1 + \eta p_d + d'_0 - x_1. \quad (\text{e 11.54})$$

Therefore,

$$\eta p_d + d'_0 - x_1 = z''_1 - z_1. \quad (\text{e 11.55})$$

Since $z_0 u$ is a nilpotent, $z_0 u + \eta(1 - p_d)$ is invertible in $(1 - p_d)A^{**}(1 - p_d)$. Let ζ_1 denote the inverse of $z_0 u + \eta(1 - p_d)$ in $(1 - p_d)A^{**}(1 - p_d)$ and ζ_2 the inverse of z''_1 in $D_2 + \mathbb{C} \cdot p_d$. Then

$$(z_0 u + \eta(1 - p_d) \oplus z''_1)(\zeta_1 \oplus \zeta_2) = (1 - p_d) + p_d = 1. \quad (\text{e 11.56})$$

It follows that

$$z_2 := z_0 u + \eta(1 - p_d) + z''_1 \in \text{GL}(A^{**}). \quad (\text{e 11.57})$$

However, by (e 11.54) and (e 11.46),

$$\begin{aligned} z_2 &= z_0 u + \eta(1 - p_d) + z''_1 \\ &= z_0 u + \eta(1 - p_d) + z_1 + \eta p_d + (d'_0 - x_1) \end{aligned} \quad (\text{e 11.58})$$

$$= z_0 u + z_1 + (d'_0 - x_1) + \eta \cdot 1 = (z_0 + z_1)u + (d'_0 - x_1) + \eta \cdot 1 \quad (\text{e 11.59})$$

$$= z u + (d'_0 - x_1) + \eta \cdot 1 \in \tilde{A}. \quad (\text{e 11.60})$$

It follows that $z_2 \in \text{GL}(\tilde{A})$. We have (by (e 11.60), (e 11.55), and (e 11.52))

$$\|z u - z_2\| = \|(d'_0 - x_1) + \eta \cdot 1\| \quad (\text{e 11.61})$$

$$\leq \|d'_0 - x_1 + \eta p_d\| + \eta\|(1 - p_d)\| \quad (\text{e 11.62})$$

$$= \|z''_1 - z_1\| + \eta < 3\varepsilon/16 + \varepsilon/8 = 5\varepsilon/16. \quad (\text{e 11.63})$$

Therefore (using also (e 11.40)),

$$\|x u - z_2\| < \varepsilon \quad \text{or} \quad \|x - z_2 u^*\| < \varepsilon. \quad (\text{e 11.64})$$

Since z_2 is invertible so is $z_2 u^*$. However, $u \in \tilde{A}$. One concludes that $z_2 u^*$ is in $\text{GL}(\tilde{A})$. \square

At this point, we would like to introduce the following definition:

Definition 11.6. Let A be a simple C^* -algebra. Suppose that A is stably projectionless. We shall say that A has generalized tracial rank at most one, and write $\text{gTR}(A) \leq 1$, if for any $a \in \text{Ped}(A)_+$, $\overline{aAa} \in \mathcal{D}$. (This extends the definition of generalized tracial rank at most one in the unital case [22].)

Proposition 11.7. A separable stably projectionless simple C^* -algebra A has generalized tracial rank at most one, i.e., $\text{gTR}(A) \leq 1$, if, and only if, for some $a \in \text{Ped}(A)_+ \setminus \{0\}$, $\overline{aAa} \in \mathcal{D}$.

Proof. Let A be a separable stably projectionless simple C^* -algebra. Suppose that there is $a \in \text{Ped}(A)_+ \setminus \{0\}$ such that $\overline{aAa} \in \mathcal{D}$. We must show that, for any $b \in \text{Ped}(A)_+ \setminus \{0\}$, $\overline{bAb} \in \mathcal{D}$.

There are $b_1, b_2, \dots, b_k \in A_+$ and $g_i \in C_0((0, \infty)_+)$ such that

$$b \leq d := \sum_{i=1}^k g_i(b_i).$$

By repeated application of Lemma 3.4, one obtains $x_1, x_2, \dots, x_n \in A$ such that

$$\sum_{i=1}^n x_i^* a x_i = d. \quad (\text{e 11.65})$$

Let $Z = (x_1^* a^{1/2}, x_2^* a^{1/2}, \dots, x_n^* a^{1/2})$ be considered as an $n \times n$ matrix in $M_n(A)$ with zero rows except for the first row. Then

$$ZZ^* = \text{diag}(d, \overbrace{0, 0, \dots, 0}^{n-1}) \quad \text{and} \quad Z^*Z \in M_n(\overline{aAa}). \quad (\text{e 11.66})$$

Let $Z^* = U(ZZ^*)^{1/2}$ denote the polar decomposition of Z^* in $M_n(A)^{**}$. Then

$$UZZ^*U^* = Z^*Z \in M_n(\overline{aAa}).$$

It follows that the map $x \mapsto UxU^*$ from

$$\overline{dAd} \otimes e_{1,1} = \overline{ZZ^*M_n(A)ZZ^*} \quad \text{to} \quad \overline{Z^*ZM_n(A)Z^*Z} = \overline{Z^*ZM_n(\overline{aAa})Z^*Z}$$

is an isomorphism. Thus,

$$\overline{dAd} \cong \overline{ZZ^*M_n(\overline{aAa})ZZ^*}. \quad (\text{e 11.67})$$

It follows from Proposition 8.5 that $M_n(\overline{aAa}) \in \mathcal{D}$. Then, by Proposition 8.6,

$$\overline{ZZ^*M_n(\overline{aAa})ZZ^*} \in \mathcal{D}.$$

By (e 11.67), $\overline{dAd} \in \mathcal{D}$. Since $b \in \overline{dAd}$, by Proposition 8.6, $\overline{bAb} \in \mathcal{D}$, as desired. \square

Proposition 11.8. *Let $A \in \mathcal{D}$. Suppose that $A = \text{Ped}(A)$ (see Corollary 11.3). Then the map $\text{Cu}(A) \rightarrow \text{LAff}_{0+}(\overline{\text{T}(A)}^w)$ is an isomorphism of ordered semigroups.*

Proof. First, note that by Corollary 11.3 and Theorem 9.4, the map (taking a positive element and a given trace into the trace of the range projection) is an order isomorphism onto its image (with the pointwise order on affine functions).

So it remains to prove surjectivity. (The argument is similar to existing arguments in the unital case; see [5, 5.3] and [22, 10.5]. One can also just use the somewhat different proof of 6.2.1 of [43] using 10.6 instead of (D). In the present setting, we use Proposition 6.4(4).)

Since the map preserves suprema of increasing sequences (constructed pointwise in $\text{LAff}_+(\tilde{T}(A))$), to establish surjectivity it is enough by the definition of $\text{LAff}_+(\tilde{T}(A))$ to show the image of the map contains the subset $\text{Aff}_+(\tilde{T}(A))$ of continuous, finite-valued, affine (i.e., linear) functions on $\tilde{T}(A)$, strictly positive except at 0. Indeed, by the preceding paragraph, an increasing sequence of functions in $\text{Aff}_+(\tilde{T}(A))$ is the image of an increasing sequence of elements of $\text{Cu}(A)$, and then the supremum of the former is the image of the supremum of the latter.

In fact, it is enough to approximate the functions in $\text{Aff}_+(\tilde{T}(A))$ uniformly by functions in the image when restricted to the compact subset $K = \overline{T(A)}^w$.

To see that this is enough, recall that by Corollary 11.3 every trace in $\tilde{T}(A)$ is bounded, so K generates the cone $\tilde{T}(A)$. To obtain a given function, say f , in $\text{Aff}_+(\tilde{T}(A))$ as the supremum of an increasing sequence of elements of $\text{LAff}_+(\tilde{T}(A))$ in the image, consider the restrictions of all functions to K . Recall (Proposition 9.1) that 0 does not belong to K . Choose a strictly increasing sequence of scalars $\lambda(n)$ converging to 1, and note that the functions $\lambda(n)f = f_n$ increase strictly to f on the compact set K . Approximating f_n sufficiently well uniformly on K by a function g_n in the image for each n , we find that the sequence (g_n) increases to f , at each point of K and therefore at each point of $\tilde{T}(A)$, as desired.

Finally, the desired uniform approximation on $\overline{T(A)}^w$ (or on $T_1(A)$) follows almost immediately from Proposition 6.4(4) with large k , applied to a subalgebra B of A in the class \mathcal{C} arising from Definition 8.1, as modified in Corollary 8.4 to ensure k is large. This is seen as follows.

By Corollary 6.4 of [9], there exists a positive element h of A giving rise to the affine (linear) function f on the cone $\tilde{T}(A)$ (generating the Banach space of traces). Second, as A is in the class \mathcal{D} there exists a subalgebra B of A in the class \mathcal{C} , and two mutually orthogonal positive elements a and b of A with b in B such that $a + b$ is arbitrarily close to (the fixed element) h , and such that all traces of norm one on A are arbitrarily small on b . This of course implies that traces of norm one on A are (uniformly) approximately the same on a and on h . By Proposition 6.4(4) and Corollary 8.4, there is a positive element k of B such that $d_\tau(k)$ is (uniformly) close to $d_\tau(a)$ (which is always positive) for all traces on B of norm at most one. In particular, this includes the restriction to B of traces on A of at most one, i.e., in $T_1(A)$, and so (since such traces are small on b) traces in $T_1(A)$ are (uniformly) approximately equal on h and on a . Hence, $d_\tau(k)$ is close to $\tau(h) = f(\tau)$ uniformly for τ in $T_1(A)$, as desired. \square

Corollary 11.9. *Let $A \in \mathcal{D}$. Then there exists an element $a \in A_+ \setminus \{0\}$ such that \overline{aAa} has continuous scale.*

Proof. By Theorem 11.5, A has stable rank one. Hence by Proposition 11.8 (see [7]), there exists an element $a \in A_+ \setminus \{0\}$ such that $d_\tau(a)$ is continuous on $\overline{T(A)}^w$. Again by Proposition 11.8, and Proposition 5.4, one concludes that \overline{aAa} has continuous scale. \square

Proposition 11.10. *Let $A \in \mathcal{D}$ be a separable C^* -algebra with $K_0(A) = \{0\}$. Then A has the properties described in Theorem 10.7 (and Corollary 10.8) but replacing $\mathcal{C}_0^{0'}$ (and \mathcal{C}_0^0) by \mathcal{C}_0 .*

Proof. By Proposition 11.8, the map $\text{Cu}(A) \rightarrow \text{LAff}_{0+}(\overline{\text{T}(A)}^{\mathcal{W}})$ is surjective. By Theorem 11.5, A has stable rank one. Then, by Proposition 7.9 and by Definition 8.1, A is tracially approximately divisible. The proof of Theorem 10.7 then applies to A with $\mathcal{C}_0^{0'}$ replaced by \mathcal{C}_0 . One then also obtains the conclusion of Corollary 10.8 with \mathcal{C}_0^0 replaced by \mathcal{C}_0 . \square

Let us summarize some of the facts we have established.

Proposition 11.11. *Let A be a separable simple C^* -algebra. Suppose that A is stably projectionless and $\text{gTR}(A) \leq 1$. Then the following statements hold. (See Theorem 11.5, Propositions 9.1, 11.8, 11.7, and 8.6.)*

- (1) A has stable rank one;
- (2) Every quasitrace of A is a trace;
- (3) $\text{Cu}(A) = \text{LAff}_+(\tilde{\text{T}}(A))$;
- (4) If $A = \text{Ped}(A)$, then $A \in \mathcal{D}$;
- (5) If $B \subset A$ is a hereditary sub- C^* -algebra, then $\text{gTR}(B) \leq 1$;
- (6) $M_n(A)$ is stably projectionless and $\text{gTR}(M_n(A)) \leq 1$ for every integer $n \geq 1$.

12. The C^* -algebras \mathcal{W} and the class \mathcal{D}_0

Definition 12.1. Recall (see Definition 9.6) that \mathcal{W} is a unital separable simple C^* -algebra with $K_i(\mathcal{W}) = 0$, $i = 0, 1$, which is in both \mathcal{M}_0 and \mathcal{D}_0 .

Let A be a non-unital separable C^* -algebra, and let $\tau \in \text{T}(A)$. Let us say that τ is a \mathcal{W} -trace if there exists a sequence of completely positive contractive maps $\varphi_n: A \rightarrow \mathcal{W}$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| &= 0 \quad \text{for all } a, b \in A, \\ \tau(a) &= \lim_{n \rightarrow \infty} \tau_{\mathcal{W}}(\varphi_n(a)) \quad \text{for all } a \in A, \end{aligned} \quad (\text{e 12.1})$$

where $\tau_{\mathcal{W}}$ is the unique tracial state on \mathcal{W} .

Theorem 12.2. *Let A be a separable simple C^* -algebra with $A = \text{Ped}(A)$. If every tracial state $\tau \in \text{T}(A)$ is a \mathcal{W} -trace, then $K_0(A) = \ker \rho_A$ (see Definition 4.12 for the definition of ρ_A).*

Proof. Suppose that there are two projections $p, q \in M_k(\tilde{A})$ such that $x = [p] - [q] \in K_0(A)$ and $\rho_A(x) \neq 0$. In other words, $\tau(p) \neq \tau(q)$ for some $\tau \in T(A)$, where τ denotes still the canonical extension of τ to \tilde{A} and also to $M_k(\tilde{A})$. Recall that $[p] - [q] \in K_0(A)$ means that $\pi(p)$ and $\pi(q)$ have the same rank in $M_k(\mathbb{C})$, where $\pi: M_k(\tilde{A}) \rightarrow M_k(\mathbb{C})$ is the quotient map.

Set $d = |\tau(p) - \tau(q)|$. If τ were a \mathcal{W} -trace, then there would be a sequence (φ_n) of completely positive contractive maps from $M_k(A)$ into $M_k(\mathcal{W})$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\varphi_n(a)\varphi_n(b) - \varphi_n(ab)\| &= 0 \quad \text{for all } a, b \in M_k(A), \\ \tau(a) &= \lim_{n \rightarrow \infty} \tau_{\mathcal{W}} \circ \varphi_n(a) \quad \text{for all } a \in M_k(A). \end{aligned} \quad (\text{e 12.2})$$

Denote by $\tilde{\varphi}_n: M_k(\tilde{A}) \rightarrow M_k(\tilde{\mathcal{W}})$ the canonical unital extension of the completely positive contractive map φ_n . Then

$$\lim_{n \rightarrow \infty} \|\tilde{\varphi}_n(a)\tilde{\varphi}_n(b) - \tilde{\varphi}_n(ab)\| = 0 \quad \text{for all } a, b \in M_k(\tilde{A}).$$

Let $\tau_{\mathcal{W}}$ also denote the canonical extension of $\tau_{\mathcal{W}}$ to $M_k(\tilde{\mathcal{W}})$. Then we also have

$$\tau(a) = \lim_{n \rightarrow \infty} \tau_{\mathcal{W}} \circ \tilde{\varphi}_n(a) \quad \text{for all } a \in M_k(\tilde{A}).$$

Passing to a subsequence, we may assume that

$$|\tau_{\mathcal{W}} \circ \tilde{\varphi}_n(p) - \tau_{\mathcal{W}} \circ \tilde{\varphi}_n(q)| \geq d/2 \quad \text{for all } n. \quad (\text{e 12.3})$$

There are projections $p_n, q_n \in M_k(\tilde{\mathcal{W}})$ such that

$$\lim_{n \rightarrow \infty} \|\tilde{\varphi}_n(p) - p_n\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|\tilde{\varphi}_n(q) - q_n\| = 0. \quad (\text{e 12.4})$$

Since $\pi(p)$ and $\pi(q)$ have the same rank, there exists $v \in M_k(\tilde{A})$ such that $\pi(v^*v) = \pi(p)$ and $\pi(vv^*) = \pi(q)$. Denote by $\pi_{\mathcal{W}}: M_k(\tilde{\mathcal{W}}) \rightarrow M_k$ the quotient map. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\pi_{\mathcal{W}} \circ \varphi_n(v^*v) - \pi_{\mathcal{W}}(p_n)\| &= 0 \\ \text{and} \quad \lim_{n \rightarrow \infty} \|\pi_{\mathcal{W}} \circ \varphi_n(vv^*) - \pi_{\mathcal{W}}(q_n)\| &= 0. \end{aligned} \quad (\text{e 12.5})$$

It follows that $\pi_{\mathcal{W}}(p_n)$ and $\pi_{\mathcal{W}}(q_n)$ are equivalent projections in M_k for all large n . Since $K_0(\mathcal{W}) = 0$, it follows that $[p_n] - [q_n] = 0$ in $K_0(\mathcal{W})$, which means that p_n and q_n are equivalent in $M_k(\tilde{\mathcal{W}})$ since $\tilde{\mathcal{W}}$ has stable rank one. In particular,

$$\tau_{\mathcal{W}}(p_n) = \tau_{\mathcal{W}}(q_n)$$

for all sufficiently large n , in contradiction with (e 12.3) and (e 12.4). \square

Proposition 12.3. *Let A be a separable simple C^* -algebra with a \mathcal{W} -trace $\tau \in \mathsf{T}(A)$. Let $0 \leq a_0 \leq 1$ be a strictly positive element of A . Then there exists a sequence of completely positive contractive maps $\varphi_n: A \rightarrow \mathcal{W}$ such that $\varphi_n(a_0)$ is a strictly positive element, and*

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\varphi_n(a)\varphi_n(b) - \varphi_n(ab)\| &= 0 \quad \text{for all } a, b \in A \\ \tau(a) &= \lim_{n \rightarrow \infty} \tau_{\mathcal{W}} \circ \varphi_n(a) \quad \text{for all } a \in A. \end{aligned} \quad (\text{e 12.6})$$

Proof. We may assume that

$$\tau(a_0^{1/n}) > 1 - 1/2n, \quad n = 1, 2, \dots \quad (\text{e 12.7})$$

Since τ is a \mathcal{W} -trace (12.1), there exists a sequence of completely positive contractive maps $\psi_n: A \rightarrow \mathcal{W}$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\psi_n(a)\psi_n(b) - \psi_n(ab)\| &= 0 \quad \text{for all } a, b \in A \\ \tau(a) &= \lim_{n \rightarrow \infty} \tau_{\mathcal{W}} \circ \psi_n(a) \quad \text{for all } a \in A. \end{aligned} \quad (\text{e 12.8})$$

Put $b_n = \psi_n(a_0^{1/n})$. By (e 12.7) and (e 12.8), passing to a subsequence of (ψ_n) , we may assume

$$\tau_{\mathcal{W}}(b_n) \geq 1 - 1/n, \quad n = 1, 2, \dots \quad (\text{e 12.9})$$

Consider the non-zero hereditary sub- C^* -algebra

$$B_n = \overline{\psi_n(A)\mathcal{W}\psi_n(A)}.$$

Since $a_0^{1/n}$ is a strictly positive element, by Proposition 2.10, b_n is a strictly positive element of B_n . By [3], B_n is stably isomorphic to \mathcal{W} . (In fact, $B_n \cong \mathcal{W}$, as will be shown in [17].) Therefore (see Theorem 7.3),

$$\text{Cu}^{\sim}(B_n) = \text{Cu}^{\sim}(\mathcal{W}) = \mathbb{R} \cup \{+\infty\}.$$

Let $r_n = \langle b_n \rangle \in \mathbb{R}_+ \subset \mathbb{R} \cup \{+\infty\}$, $n = 1, 2, \dots$. Consider the map $r \mapsto r_n \cdot r$ (for $r \in \mathbb{R}$) and $+\infty \mapsto +\infty$. By Theorem 1.0.1 of [43] (see also [43, 6.2.4], [24, Theorem 1.2], or [41, Theorem 1.1], and Corollary 8.12), there is a homomorphism $h_n: B_n \rightarrow \mathcal{W}$ such that $h_n(b_n)$ is strictly positive. Since B_n has a unique trace, there is $\alpha_n > 0$ such that

$$\alpha_n \tau_{\mathcal{W}}(b) = \tau_{\mathcal{W}} \circ h_n(b) \quad \text{for all } b \in B_n, \quad n = 1, 2, \dots \quad (\text{e 12.10})$$

Since $h_n(b_n)$ is strictly positive,

$$\lim_{k \rightarrow \infty} \tau_{\mathcal{W}} \circ h_n(b_n^{1/k}) = 1. \quad (\text{e 12.11})$$

Since $B_n \subset \mathcal{W}$, and by (e 12.10), $\alpha_n \tau_{\mathcal{W}}|_{B_n} = \tau_{\mathcal{W}} \circ h_n$, (e 12.11) implies that $\alpha_n \geq 1$.

On the other hand, by (e 12.9), together with (e 12.10), we have that

$$1 - 1/n \leq \tau_{\mathcal{W}}(b_n) = \frac{\tau_{\mathcal{W}} \circ h_n(b_n)}{\alpha_n} \leq 1/\alpha_n, \quad n = 1, 2, \dots$$

Therefore,

$$1 \leq \alpha_n \leq \frac{1}{1 - 1/n}, \quad n = 1, 2, \dots,$$

from which it follows that $\lim_{n \rightarrow \infty} \alpha_n = 1$. Set $\varphi_n = h_n \circ \psi_n$. Since $\tau_{\mathcal{W}} \circ \varphi_n = \alpha_n \tau_{\mathcal{W}} \circ \psi_n$, (e 12.6) holds. Since $\varphi_n(a_0^{1/n}) = h_n(\psi_n(a_0^{1/n})) = h_n(b_n)$ is strictly positive (in B_n), by Proposition 2.10 for each n the element $\varphi_n(a_0)$ is strictly positive, and so the sequence (φ_n) meets the requirements. \square

The following two statements will be established in [17].

Theorem 12.4. *Let A be a separable simple C^* -algebra with finite nuclear dimension and with $A = \text{Ped}(A)$ such that $T(A) \neq \emptyset$, $K_0(A) = \ker \rho_A$, and every tracial state is a \mathcal{W} -trace. Suppose also that every hereditary sub- C^* -algebra of A with continuous scale is tracially approximately divisible. Then $A \in \mathcal{D}_0$.*

Theorem 12.5. *Let A be a separable simple C^* -algebra with finite nuclear dimension and with $A = \text{Ped}(A)$. Suppose that $T(A) \neq \emptyset$. Then $A \otimes \mathcal{W} \in \mathcal{D}_0$.*

References

- [1] B. E. Blackadar, Traces on simple AF C^* -algebras, *J. Funct. Anal.*, **38** (1980), no. 2, 156–168. Zbl 0443.46037 MR 587906
- [2] B. E. Blackadar and D. E. Handelman, Dimension functions and traces on C^* -algebras, *J. Funct. Anal.*, **45** (1982), no. 3, 297–340. Zbl 0513.46047 MR 650185
- [3] L. G. Brown, Stable isomorphism of hereditary subalgebras of C^* -algebras, *Pacific J. Math.*, **71** (1977), no. 2, 335–348. Zbl 0362.46042 MR 454645
- [4] L. G. Brown and G. K. Pedersen, On the geometry of the unit ball of a C^* -algebra, *J. Reine Angew. Math.*, **469** (1995), 113–147. Zbl 0834.46041 MR 1363827
- [5] N. Brown, F. Perera, and A. Toms, The Cuntz semigroup, the Elliott conjecture, and dimension functions on C^* -algebras, *J. Reine Angew. Math.*, **621** (2008), 191–211. Zbl 1158.46040 MR 2431254
- [6] N. P. Brown and A. S. Toms, Three applications of the Cuntz semigroup, *Int. Math. Res. Not. IMRN*, (2007), no. 19, Art. ID rnm068, 14pp. Zbl 1134.46040 MR 2359541
- [7] K. Coward, G. A. Elliott, and C. Ivanescu, The Cuntz semigroup as an invariant for C^* -algebras, *J. Reine Angew. Math.*, **623** (2008), 161–193. Zbl 1161.46029 MR 2458043
- [8] J. Cuntz, The structure of multiplication and addition in simple C^* -algebras, *Math. Scand.*, **40** (1977), no. 2, 215–233. Zbl 0372.46063 MR 500176

- [9] J. Cuntz and G. K. Pedersen, Equivalence and traces on C^* -algebras, *J. Funct. Anal.*, **33** (1979), no. 2, 135–164. Zbl 0427.46042 MR 546503
- [10] M. Dădărlat and S. Eilers, On the classification of nuclear C^* -algebras, *Proc. London Math. Soc. (3)*, **85** (2002), no. 1, 168–210. Zbl 1031.46070 MR 1901373
- [11] M. Dădărlat and T. Loring, A universal multicoefficient theorem for the Kasparov groups, *Duke Math. J.*, **84** (1996), no. 2, 355–377. Zbl 0881.46048 MR 1404333
- [12] S. Eilers, T. Loring, and G. K. Pedersen, Stability of anticommutation relations: an application of noncommutative CW complexes, *J. Reine Angew. Math.*, **499** (1998), 101–143. Zbl 0897.46056 MR 1631120
- [13] S. Eilers, T. Loring, and G. K. Pedersen, Fragility of subhomogeneous C^* -algebras with one-dimensional spectrum, *Bull. London Math. Soc.*, **31** (1999), no. 3, 337–344. Zbl 0958.46028 MR 1673413
- [14] G. A. Elliott, The classification problem for amenable C^* -algebras, in *Proceedings of the International Congress of Mathematicians. Vol. 1, 2 (Zürich, 1994)*, 922–932, Birkhäuser, Basel, 1995. Zbl 0946.46050 MR 1403992
- [15] G. A. Elliott, An invariant for simple C^* -algebras, in *Canadian Mathematical Society. 1945–1995, Vol. 3*, 61–90, Canadian Math. Soc., Ottawa, ON, 1996. Zbl 1206.46046 MR 1661611
- [16] G. A. Elliott, G. Gong, H. Lin, and Z. Niu, On the classification of simple amenable C^* -algebras with finite decomposition rank. II, 2016. arXiv:1507.03437
- [17] G. A. Elliott, G. Gong, H. Lin, and Z. Niu, The classification of simple separable KK-contractible C^* -algebras with finite nuclear dimension, 2017. arXiv:1712.09463
- [18] G. A. Elliott and Z. Niu, On tracial approximation, *J. Funct. Anal.*, **254** (2008), no. 2, 396–440. Zbl 1137.46035 MR 2376576
- [19] G. A. Elliott, L. Robert, and L. Santiago, The cone of lower semicontinuous traces on a C^* -algebra, *Amer. J. Math.*, **133** (2011), no. 4, 969–1005. Zbl 1236.46052 MR 2823868
- [20] G. Gong and H. Lin, Almost multiplicative morphisms and K -theory, *Internat. J. Math.*, **11** (2000), no. 8, 983–1000. Zbl 0965.46045 MR 1797674
- [21] G. Gong and H. Lin, On classification of non-unital simple amenable C^* -algebras. II, *in preparation*.
- [22] G. Gong, H. Lin, and Z. Niu, Classification of finite simple amenable \mathbb{Z} -stable C^* -algebras, 2015. arXiv:1501.00135
- [23] J. Hua and H. Lin, Rotation algebras and the Exel trace formula, *Canad. J. Math.*, **67** (2015), no. 2, 404–423. Zbl 1328.46042 MR 3314840
- [24] B. Jacelon, A simple, monotracial, stably projectionless C^* -algebra, *J. Lond. Math. Soc. (2)*, **87** (2013), no. 2, 365–383. Zbl 1275.46047 MR 3046276
- [25] E. Kirchberg and N. C. Phillips, Embedding of exact C^* -algebras in the Cuntz algebra \mathcal{O}_2 , *J. Reine Angew. Math.*, **525** (2000), 17–53. Zbl 0973.46048 MR 1780426
- [26] E. Kirchberg and W. Winter, Covering dimension and quasidiagonality, *Internat. J. Math.*, **15** (2004), no. 1, 63–85. Zbl 1065.46053 MR 2039212

- [27] H. Lin, Simple C^* -algebras with continuous scales and simple corona algebras, *Proc. Amer. Math. Soc.*, **112** (1991), no. 3, 871–880. Zbl 0744.46048 MR 1079711
- [28] H. Lin, Stable approximate unitary equivalence of homomorphisms, *J. Operator Theory*, **47** (2002), no. 2, 343–378. Zbl 1029.46097 MR 1911851
- [29] H. Lin, *An introduction to the classification of amenable C^* -algebras*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001. Zbl 1013.46055 MR 1884366
- [30] H. Lin, Traces and simple C^* -algebras with tracial topological rank zero, *J. Reine Angew. Math.*, **568** (2004), 99–137. Zbl 1043.46041 MR 2034925
- [31] H. Lin, Simple corona C^* -algebras, *Proc. Amer. Math. Soc.*, **132** (2004), no. 11, 3215–3224. Zbl 1049.46040 MR 2073295
- [32] H. Lin, Simple nuclear C^* -algebras of tracial topological rank one, *J. Funct. Anal.*, **251** (2007), no. 2, 601–679. Zbl 1206.46052 MR 2356425
- [33] H. Lin, Cuntz semigroups of C^* -algebras of stable rank one and projective Hilbert modules, 2010. arXiv:1001.4558
- [34] H. Lin, Locally AH algebras, *Mem. Amer. Math. Soc.*, **235** (2015), no. 1107, vi+109pp. Zbl 1327.46057 MR 3338301
- [35] H. Lin, Homomorphisms from AH-algebras, *J. Topol. Anal.*, **9** (2017), no. 1, 67–125. Zbl 1370.46035 MR 3594607
- [36] H. Lin, *From the basic homotopy lemma to the classification of C^* -algebras*, CBMS Regional Conference Series in Mathematics, 124, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2017. Zbl 1386.46001 MR 3699426
- [37] T. A. Loring, *Lifting solutions to perturbing problems in C^* -algebras*, Fields Institute Monographs, 8, American Mathematical Society, Providence, RI, 1997. Zbl 1155.46310 MR 1420863
- [38] G. K. Pedersen, *C^* -algebras and their automorphism groups*, London Mathematical Society Monographs, 14, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1979. Zbl 0416.46043 MR 548006
- [39] G. K. Pedersen, Unitary extensions and polar decompositions in a C^* -algebra, *J. Operator Theory*, **17** (1987), no. 2, 357–364. Zbl 0646.46053 MR 887230
- [40] N. C. Phillips, A classification theorem for nuclear purely infinite simple C^* -algebras, *Doc. Math.*, **5** (2000), 49–114. Zbl 0943.46037 MR 1745197
- [41] S. Razak, On the classification of simple stably projectionless C^* -algebras, *Canad. J. Math.*, **54** (2002), no. 1, 138–224. Zbl 1038.46051 MR 1880962
- [42] M. A. Rieffel, Dimension and stable rank in the K -theory of C^* -algebras, *Proc. London Math. Soc. (3)*, **46** (1983), no. 2, 301–333. Zbl 0533.46046 MR 693043
- [43] L. Robert, Classification of inductive limits of 1-dimensional NCCW complexes, *Adv. Math.*, **231** (2012), no. 5, 2802–2836. Zbl 1268.46041 MR 2970466
- [44] L. Robert, Remarks on \mathcal{Z} -stable projectionless C^* -algebras, *Glasg. Math. J.*, **58** (2016), no. 2, 273–277. Zbl 1339.14010 MR 3483583

- [45] M. Rørdam, On the structure of simple C^* -algebras tensored with a UHF-algebra, *J. Funct. Anal.*, **100** (1991), no. 1, 1–17. Zbl 0773.46028 MR 1124289
- [46] M. Rørdam, On the structure of simple C^* -algebras tensored with a UHF-algebra. II, *J. Funct. Anal.*, **107** (1992), no. 2, 255–269. Zbl 0810.46067 MR 1172023
- [47] M. Rørdam, Classification of certain infinite simple C^* -algebras, *J. Funct. Anal.*, **131** (1995), no. 2, 415–458. Zbl 0831.46063 MR 1345038
- [48] M. Rørdam, The stable and the real rank of \mathcal{Z} -absorbing C^* -algebras, *Internat. J. Math.*, **15** (2004), no. 10, 1065–1084. Zbl 1077.46054 MR 2106263
- [49] M. Rørdam and W. Winter, The Jiang–Su algebra revisited, *J. Reine Angew. Math.*, **642** (2010), 129–155. Zbl 1209.46031 MR 2658184
- [50] K. Thomsen, Homomorphisms between finite direct sums of circle algebras, *Linear Multilinear Algebra*, **32** (1992), no. 1, 33–50. Zbl 0783.46029 MR 1198819
- [51] A. Tikuisis, Nuclear dimension, \mathcal{Z} -stability, and algebraic simplicity for stably projectionless C^* -algebras, *Math. Ann.*, **358** (2014), no. 3-4, 729–778. Zbl 1319.46043 MR 3175139
- [52] A. Tikuisis, S. White, and W. Winter, Quasidiagonality of nuclear C^* -algebras, *Ann. of Math. (2)*, **185** (2017), no. 1, 229–284. Zbl 1367.46044 MR 3583354
- [53] K.-W. Tsang, On the positive tracial cones of simple stably projectionless C^* -algebras, *J. Funct. Anal.*, **227** (2005), no. 1, 188–199. Zbl 1093.46036 MR 2165091
- [54] W. Winter, On topologically finite-dimensional simple C^* -algebras, *Math. Ann.*, **332** (2005), no. 4, 843–878. Zbl 1089.46039 MR 2179780
- [55] W. Winter, Nuclear dimension and \mathcal{Z} -stability of pure C^* -algebras, *Invent. Math.*, **187** (2012), no. 2, 259–342. Zbl 1280.46041 MR 2885621
- [56] W. Winter, Localizing the Elliott conjecture at strongly self-absorbing C^* -algebras, *J. Reine Angew. Math.*, **692** (2014), 193–231. Zbl 1327.46058 MR 3274552
- [57] W. Winter, Classifying crossed product C^* -algebras, *Amer. J. Math.*, **138** (2016), no. 3, 793–820. Zbl 1382.46053 MR 3506386
- [58] W. Winter and J. Zacharias, The nuclear dimension of C^* -algebras, *Adv. Math.*, **224** (2010), no. 2, 461–498. Zbl 1201.46056 MR 2609012

Received 19 April, 2018; revised 01 October, 2018

G. A. Elliott, Department, University, PO Box or Street, City, Country

E-mail:

G. Gong, Department, University, PO Box or Street, City, Country

E-mail:

H. Lin, Department of Mathematics, University of Oregon,

Eugene, Oregon 97402, USA

E-mail: hlin@uoregon.edu

Z. Niu, Department, University, PO Box or Street, City, Country

E-mail: