Two–chains and square roots of Thompson's group F

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ABSTRACT. We study two–generated subgroups $\langle f, g\rangle < \mathrm{Homeo}^+(I)$ such that $\langle f^2, g^2\rangle$ is isomorphic to Thompson's group F, and such that the supports of f and g form a chain of two intervals. We show that this class contains uncountably many isomorphism types. These include examples with nonabelian free subgroups, examples which do not admit faithful actions by C^2 diffeomorphisms on 1–manifolds, examples which do not admit faithful actions by PL homeomorphisms on an interval, and examples which are not finitely presented. We thus answer questions due to M. Brin. We also show that many relatively uncomplicated groups of homeomorphisms can have very complicated square roots, thus establishing the behavior of square roots of F as part of a general phenomenon among subgroups of $\mathrm{Homeo}^+(I)$.

1. INTRODUCTION

Thompson's group F is a remarkable group of PL homeomorphisms of the interval that occurs naturally and abundantly as a group of homeomorphisms of the real line, and that has been extensively studied since the 1970s. The group F has been shown to satisfy various exotic properties, and has been proposed as a counterexample to well–known conjectures in group theory [7, 10]. Among the most well–known facts about Thompson's group F are the following:

Theorem 1.1 (Brin–Squier, [5]). *The group F satisfies no law and contains no nonabelian free subgroups.*

Theorem 1.2 (Ghys–Sergiescu, [17]). *The group F admits a faithful action by C^∞ diffeomorphisms of the circle.*

Theorem 1.3 (Thompson, see [11]). *The group F is finitely presented.*

In this article, we study a certain class of groups which we call *square roots of Thompson's group F*. These are two–generated subgroups $\langle f, g\rangle < \mathrm{Homeo}^+(I)$ of the group of orientation-preserving homeomorphisms of the interval, which satisfy

$$\langle f^2, g^2\rangle \cong \langle A, B \mid [A, (AB)^{-k}B(AB)^k] \text{ for } k \in \{1, 2\}\rangle \cong F,$$

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and for which the supports $\operatorname{supp} f$ and $\operatorname{supp} g$ of f and g respectively form a *two–chain* of intervals. That is, $\operatorname{supp} f$ and $\operatorname{supp} g$ are both open intervals, and the intersection $\operatorname{supp} f \cap \operatorname{supp} g$ is a proper subinterval of both $\operatorname{supp} f$ and $\operatorname{supp} g$.

Among other things, we demonstrate that (the second part of) Theorem 1.1, Theorem 1.2, and Theorem 1.3 all fail for square roots of F. In particular, we show that there are square roots of F which contain nonabelian free subgroups, that there are square roots of F which do not admit faithful actions by C^2 diffeomorphisms on the interval, circle, or real line, and that there are uncountably many isomorphism types of square roots of F.

1.1. **Main results.** We denote the set of isomorphism classes of square roots of F by $\mathscr{S}$. The goal of this paper is to produce interesting elements of $\mathscr{S}$. Note that $\mathscr{S}$ contains F for example, since squaring the generators in the standard presentation for F as given in the previous subsection results in a group isomorphic to F.

In this article we use two different finite presentations of the group F. The first presentation, which was mentioned in the previous section is:

$$\langle A, B \mid [A, (AB)^{-1}B(AB)], [A, (AB)^{-2}B(AB)^2]\rangle \cong F.$$

The second presentation is obtained by performing a Tietze transformation to produce generators $a = AB$, $b = B$, and is given by:

$$\langle a, b \mid [ab^{-1}, a^{-1}ba], [ab^{-1}, a^{-2}ba^2]\rangle \cong F.$$

Next, we describe a certain subgroup P of F, which will be needed to state and prove our results. We fix two copies of F:

$$F_1 = \langle p_1, p_2 \mid [p_1p_2^{-1}, p_1^{-1}p_2p_1], [p_1p_2^{-1}, p_1^{-2}p_2p_1^2]\rangle,$$
$$F_2 = \langle q_1, q_2 \mid [q_1q_2^{-1}, q_1^{-1}q_2q_1], [q_1q_2^{-1}, q_1^{-2}q_2q_1^2]\rangle.$$

We will write P for the subgroup of $F_1 \times F_2$ generated by (p_1, q_2) and (p_2, q_1). Note that the group P is a subgroup of F which itself contains F as a subgroup. The fact that $P < F$ is an elementary exercise that we leave to the reader, and that $F < P$ is a direct consequence of Brin's Ubiquity Theorem (see [4]).

We denote the free group on two generators by $\mathbf{F_2}$, and we call a group $H = \langle h_1, h_2\rangle$ a *marked extension* of P if there exists a sequence of surjective homomorphisms

$$\mathbf{F_2} \to H \to P,$$

where $h_1 \mapsto (p_1, q_2)$ and $h_2 \mapsto (p_2, q_1)$ under the second map. Even though the map $H \to P$ may be suppressed from the notation, we always think of a marked extension of P as equipped with such a homomorphism. A (countable) group is *left orderable* if it admits a left invariant total ordering, or equivalently if it admits a faithful action by orientation preserving homeomorphisms of the real line [14, 20]. Our main result is the following:

Theorem 1.4. *Let H be a marked, left orderable extension of P. Then there exists a square root of Thompson's group $G \in \mathscr{S}$ such that $H < G$.*

Since the free group $\mathbf{F_2}$ is left orderable and is naturally a marked extension of P, we immediately obtain the following:

Corollary 1.5. *There exists a square root $G \in \mathscr{S}$ such that $\mathbf{F_2} < G$.*

We will show that square roots of F can contain torsion–free nilpotent groups of arbitrary nilpotence degree. As a consequence of Theorem 1.4 and the Plante–Thurston Theorem [21], we have the following:

Corollary 1.6. *There exists a square root $G \in \mathscr{S}$ such that G does not admit a faithful action by C^2 diffeomorphisms on a compact one–manifold or on the real line.*

Corollary 1.6 gives an example of a subgroup $\langle f, g\rangle < \mathrm{Homeo}^+(I)$ which admits no faithful C^2 action on the interval, the circle, or the real line, but where $\langle f^2, g^2\rangle$ admits a faithful C^∞ action on every one–manifold (cf. [17, 18]).

Nonabelian nilpotent groups cannot act by piecewise–linear homeomorphisms on I or on S^1:

Corollary 1.7. *There exists a square root $G \in \mathscr{S}$ such that G does not admit a faithful action by PL homeomorphisms of a compact one–manifold.*

Corollary 1.7 stands in contrast to the standard definition of F, which is as a group of PL homeomorphisms of the interval. Corollary 1.7 answers a question due to M. Brin.

In order to show that square roots of F may not be finitely presented, we prove the following result which is similar in spirit to some of the methods in [18]:

Theorem 1.8. *The class $\mathscr{S}$ contains uncountably many distinct isomorphism types.*

Since there are only countably many isomorphism types of finitely presented groups, we immediately obtain the following:

Corollary 1.9. *There exists an element $G \in \mathscr{S}$ which admits no finite presentation.*

1.2. **Square roots of other groups.** Essential in the discussion of square roots of F in this paper is the *dynamical realization* of F on a two–chain of intervals, which is a dynamical setup in which F occurs naturally (see Subsection 2.2). If one abandons the dynamical framework of chains of intervals, the group theoretic diversity phenomena witnessed by Theorems 1.4 and 1.8 become so common as to be a general feature of homeomorphism groups.

To be precise, let $H = \langle h_1, \ldots, h_n\rangle < \mathrm{Homeo}^+(I)$ be a finitely generated subgroup. An n–generated subgroup $G = \langle g_1, \ldots, g_n\rangle < \mathrm{Homeo}^+(I)$ is called a *square*

root of H if

$$H \cong \langle g_1^2, \dots, g_n^2 \rangle.$$

If $H = \langle h_1, \dots, h_n \rangle$ is a generating set for a group H, we will define the *skew subdirect product* of H to be the subgroup of $H \times H$ generated by $\{(h_i, h_i^{-1})\}_{i=1}^n$, and we will denote this group by $\widehat{H}$.

Theorem 1.10. *Let $\mathbb{Z} = \langle t_1, \dots, t_{n+1} \mid t_1 = \cdots = t_{n+1} \rangle$, and let $H < \mathrm{Homeo}^+(I)$ be an n–generated group. Then there exists a square root G of $\mathbb{Z}$ such that $\widehat{H} < G$.*

Corollary 1.11. *There exist uncountably many isomorphism types of three–generated subgroup of $\mathrm{Homeo}^+(I)$ such that the squares of the generators generate a cyclic group. Moreover, there exists a three–generated subgroup of $\mathrm{Homeo}^+(I)$ such that the squares of the generators generate a cyclic group and which contain nonabelian free groups.*

Theorem 1.12. *Let $L = \mathbb{Z} \wr \mathbb{Z}$ be the lamplighter group. Then L has uncountably many isomorphism types of square roots.*

In Subsection 4.4, we will define the notion of a formal square root of a finitely generated group. We will show that formal square roots of left orderable groups are again left orderable, and generally contain nonabelian free groups.

1.3. **Notes and references.**

1.3.1. *Remarks on context.* The bulk of the present work could just as well be a discussion of the very general setup of two–generated subgroups of $\mathrm{Homeo}^+(I)$ whose generators are supported on intervals J_1 and J_2, which in turn form a chain. It is well–known that under suitable dynamical hypotheses (cf. Subsection 1.3.2 below), the resulting subgroup is isomorphic to F. The class $\mathscr{S}$ of square roots of F is merely the first instance of interesting algebraic behavior for such homeomorphism groups which does not follow from the properties of Thompson's group F. In particular, the results of this article apply to higher roots of F beyond the square root.

1.3.2. *Relation to other authors' work.* To the authors' knowledge, it was M. Brin who first asked what sorts of groups can occur as square roots of F, and in particular if square roots of F can contain nonabelian free groups, whether they can fail to be finitely presented, and whether they can fail to act by PL homeomorphisms on the interval. The main results of this paper form a natural complement to the joint work of the authors with S. Kim in [18]. In that paper, Kim and the authors introduced the notions of a *prechain group* and of a *chain group*. In the terminology of [18], square roots of F form a restricted subclass of 2–prechain groups, namely those which square to become 2–chain groups. The class of 2–chain groups

in turn consists of just one isomorphism type (i.e. Thompson's group F). Chain groups with "fast" dynamics also fall into very few isomorphism types (namely the Higman–Thompson groups $\{F_n\}_{n\geqslant 2}$, and their subgroup structure has been studied independently by Bleak–Brin–Kassabov–Moore–Zarmesky [2] (cf. [6]). For generalities on Thompson's group F, the reader is directed to the classical Cannon–Floyd–Parry notes [11], as well as Burillo's book [8].

2. Square roots of F

In this section we establish the main result, after gathering some relevant preliminary facts. All actions will be right actions, and the words will be read from left to right. In particular, for homeomorphisms $f, g \in \mathrm{Homeo}^+(\mathbb{R})$ and $x \in \mathbb{R}$, we fix the convention:

$$fg(x) = g^{-1} \circ f^{-1}(x) = g^{-1}(f^{-1}(x))$$

2.1. **Roots of homeomorphisms.** We record the following well–known fact, whose proof we recall for the convenience of the reader:

Lemma 2.1. *Let $f \in \mathrm{Homeo}^+(I)$. Then for all $n \in \mathbb{N}$, there exists an element $g = g_n \in \mathrm{Homeo}^+(I)$ such that $g^n = f$. Moreover, there are uncountably many possible choices of such a map g.*

Proof. By considering the components of the support of f separately, we may consider the case where f has no fixed points in the interval $(0,1)$. In this case, f is topologically conjugate to the homeomorphism of $\mathbb{R} \cup \{\pm\infty\}$ given by $x \mapsto x+1$.

We now define $g(z) = z + 1/n$ for $z \in \mathbb{Z}\langle 1/n\rangle$. We extend g to all of $\mathbb{R}$ in the following manner. First, we choose an arbitrary orientation-preserving homeomorphism

$$h_0 \colon [0, 1/n] \to [1/n, 2/n]$$

Next, we inductively define homeomorphisms

$$h_m : [\frac{m}{n}, \frac{m+1}{n}] \to [\frac{m+1}{n}, \frac{m+2}{n}]$$

for all $m \in \mathbb{Z}$, such that

$$h_{k+(n-1)} \circ \cdots \circ h_k = x + 1$$

for each $k \in \mathbb{Z}$. It is clear then that the homeomorphisms h_m piece together to give a homeomorphism g of $\mathbb{R} \cup \{\pm\infty\}$, whose n^{th} power is translation by one. Moreover, the arbitrariness of the choices made guarantees that there are uncountably many choices for g. □

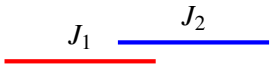

FIGURE 1. A chain of two intervals.

2.2. **2–prechain groups and variations thereupon.** Let $\mathscr{J} = \{J_1, J_2\}$ be two nonempty open subintervals of $\mathbb{R}$. We call $\mathscr{J}$ a *chain of intervals* if $J_1 \cap J_2$ is a proper nonempty subinterval of J_1 and of J_2. See Figure 1.

If $f \in \mathrm{Homeo}^+(\mathbb{R})$, we write $\operatorname{supp} f = \{x \in \mathbb{R} \mid f(x) \neq x\}$. Let f and g satisfy $\operatorname{supp} f = J_1$ and $\operatorname{supp} g = J_2$. In the terminology of [18], the group $\langle f, g\rangle$ is a 2–prechain group. Note that, up to replacing f and g by their inverses, we may assume $f(x), g(x) \geqslant x$ for $x \in \mathbb{R}$.

Writing $J_1 = (a, c)$ and $J_2 = (b, d)$ with $a < b < c < d$, we have the following basic dynamical stability result, which can also be found in [18]:

Lemma 2.2. *Suppose $g \circ f(b) \geqslant c$. Then $\langle f, g\rangle \cong F$.*

Under the dynamical hypotheses of Lemma 2.2, the group $\langle f, g\rangle$ is a chain group. There is another configuration of intervals and homeomorphisms closely related to chain groups, which naturally gives rise to F, which we will need in the sequel, and which we will describe in the next subsection.

2.3. **Nested generators for F.** A natural generating set for F emerges as homeomorphisms supported on a nested pair of intervals, satisfying elementary dynamical conditions. This shall be useful in our construction to follow.

Lemma 2.3. *Let $[a, b_1]$ and $[a, b_2]$ be compact intervals in $\mathbb{R}$ such that $b_1 < b_2$. Let f, g be homeomorphisms satisfying:*

(1) The supports of gand f are contained in $[a, b_1]$ and $[a, b_2]$ respectively.
(2) f is a decreasing map on (a, b_2).
(3) f, g agree on the interval $[a, f(b_1)]$.

Then $\langle f, g\rangle \cong F$.

Lemma 2.4. *Let $[a_1, b]$ and $[a_2, b]$ be compact intervals in $\mathbb{R}$ such that $a_1 < a_2$. Let f, g be homeomorphisms satisfying:*

(1) The supports of f and g are contained in $[a_1, b]$ and $[a_2, b]$ respectively.
(2) f is an increasing map on (a_1, b).
(3) f, g agree on the interval $[f(a_2), b]$.

Then $\langle f, g\rangle \cong F$.

Proofs of Lemmas 2.3 and 2.4. The proofs of both lemmas above follow from checking that the homeomorphisms f and g in each lemma satisfy the relations

$$[fg^{-1}, f^{-1}gf] = 1 \qquad [fg^{-1}, f^{-2}gf^2] = 1$$

Since f and g do not commute, and since every proper quotient of F is abelian (see [11]), they generate a group isomorphic to F. □

2.4. **Orderable extensions of P.** We will use the following standard facts from the theory of orderable groups:

Lemma 2.5 (See [20, 14])**.** *Let $1 \to K \to G \to Q \to 1$ be an exact sequence of groups, and suppose that Q and K are left orderable. Then G admits a left ordering which agrees with any prescribed ordering on K. Moreover, any countable, left orderable group can be embedded in* $\mathrm{Homeo}^+(I)$.

The following lemma is obvious, after the observation that $F \times F$ is left orderable, applying the Brin–Squier Theorem [5], and Brin's Ubiquity Theorem [4]:

Lemma 2.6. *The group P is a two–generated sub-direct product of $F \times F$. It is a left orderable group which contains no free subgroups.*

Let $\mathscr{R} < \mathbf{F_2} = \langle A, B\rangle$ be such that $\mathbf{F_2}/\mathscr{R} \cong P$. Note that since $P < F \times F$, we have $\mathscr{R} \neq 1$. Let $\mathscr{R}_k$ denote the k^{th} term of the derived series of $\mathscr{R}$ and let $\mathscr{R}^k$ denote the k^{th} term of the lower central series of $\mathscr{R}$, with the convention $\mathscr{R}_1 = \mathscr{R}^1 = \mathscr{R}$.

Lemma 2.7. *For each $k \geqslant 1$, the groups*

$$S_k = \langle A, B \mid \mathscr{R}_k\rangle \quad \textit{and} \quad N_k = \langle A, B \mid \mathscr{R}^k\rangle$$

are marked, left orderable extensions of P.

Proof. It is clear that for each k, the groups S_k and N_k are quotients of the free group $\mathbf{F_2}$ via the canonical map. Since $\mathscr{R}_k, \mathscr{R}^k \subset \mathscr{R}$, we have that S_k and N_k both surject to P simply by imposing the relations in $\mathscr{R}$. It therefore suffices to show that S_k and N_k are both left orderable, which since P is left orderable, reduces to showing that $\mathscr{R}/\mathscr{R}_k$ and $\mathscr{R}/\mathscr{R}^k$ are left orderable by Lemma 2.5.

Since $\mathscr{R}$ is an infinitely generated free group, these quotients are merely the universal k–step solvable and nilpotent quotients of the infinitely generated free group. We proceed by induction on k. The case $k = 1$ is trivial, and in the case $k = 2$, we obtain the group $\mathbb{Z}^\infty$ which is easily seen to be left orderable. By induction, $\mathscr{R}/\mathscr{R}_k$ (resp. $\mathscr{R}/\mathscr{R}^k$) is left orderable, and $\mathscr{R}_k/\mathscr{R}_{k+1}$ (resp. $\mathscr{R}^k/\mathscr{R}^{k+1}$) is again isomorphic to $\mathbb{Z}^\infty$, so the conclusion follows by applying Lemma 2.5 again. □

2.5. **Building square roots of F.** In this section we provide a recipe that produces a square root of F that contains a given group H as a subgroup, provided H is an orderable marked extension of P.

Step 1: Partition $[1, 2)$ into left closed, right open intervals $\{J_1, \ldots, J_{16}\}$ so that J_i occurs to the left of J_j in $\mathbb{R}$ whenever $i < j$. Moreover, we require that these

intervals are of the same length. For ease of notation, we denote by J_X for some $X \subset \{1, \dots, 16\}$ the union $\bigcup_{i \in X} J_i$. For example,

$$J_{\{1,\dots,4\}} = J_1 \cup J_2 \cup J_3 \cup J_4.$$

Step 2: Construct homeomorphisms f and g of the real line that satisfy the following:

(1) f and g are increasing maps on $(0,2)$ and on $(1,3)$, respectively, and equal the identity outside these respective intervals.
(2) f maps J_i isometrically onto J_{i+4} for $1 \leqslant i \leqslant 11$.
(3) g maps J_i isometrically onto J_{i+4} for $2 \leqslant i \leqslant 12$.
(4) The map gf^{-1} has two components of support, which are

$$[0,1] \cup J_1, \qquad J_{\{12,\dots,16\}} \cup [2,3].$$

It is elementary to construct homeomorphisms f and g that satisfy (1)–(3) above. If f and g satisfy (1)–(3), then it holds that gf^{-1} is the identity on $J_{\{2,\dots,11\}}$. Hence the support of gf^{-1} is contained in

$$[0,1] \cup J_1 \bigcup J_{\{12,\dots,16\}} \cup [2,3].$$

To ensure that the components of support of gf^{-1} are precisely as stated in (4), we choose g such that it is sufficiently slow on J_1, and f so that it is sufficiently slow on $J_{\{12,\dots,16\}}$. Note that gf^{-1} is decreasing on the interior of $[0,1] \cup J_1$ and increasing on the interior of $J_{\{12,\dots,16\}} \cup [2,3]$.

Step 3: Let $H = \langle h_1, h_2 \rangle$ be a marked, left orderable extension of P. We identify the elements h_1 and h_2 with their dynamical realizations, both supported on the interval J_6. Define a map h_3 as:

$$h_3 = g^{-1} h_2 g = f^{-1} h_2 f.$$

By definition, h_3 is supported on the interval J_{10}. Finally, we define homeomorphisms

$$\lambda_1 = h_1^{-1} h_3^{-1} f, \qquad \lambda_2 = g.$$

Our goal for the rest of this section will be to demonstrate the following:

Proposition 2.8. *The group $\langle \lambda_1, \lambda_2 \rangle$ is a marked square root of F which contains H as a subgroup.*

The group $\langle \lambda_1, \lambda_2 \rangle$ is manifestly orderable, since it is presented as a group of orientation preserving homeomorphisms of the interval. It is clear by our construction that λ_1^2 and λ_2^2 satisfy the dynamical condition of Lemma 2.2, and hence generate a copy of F. So it suffices to show that $H < \langle \lambda_1, \lambda_2 \rangle$.

Proposition 2.9. *The elements $\lambda_2 \lambda_1^{-1}$ and $\lambda_1^{-1} \lambda_2$ generate an isomorphic copy of H.*

Proof. The element $\lambda_2\lambda_1^{-1}$ has four components of support:

$$[0,1] \cup J_1, \qquad J_6, \qquad J_{10}, \qquad J_{\{12,\dots,16\}} \cup [2,3].$$

Note that:

$$\lambda_2\lambda_1^{-1}\restriction J_6 = h_1, \qquad \lambda_2\lambda_1^{-1}\restriction J_{10} = h_3 = f^{-1}h_2 f.$$

We denote by p_2 the following restriction:

$$\lambda_2\lambda_1^{-1}\restriction [0,1] \cup J_1 = gf^{-1}\restriction [0,1] \cup J_1.$$

We denote by q_1 the following restriction:

$$\lambda_2\lambda_1^{-1}\restriction J_{\{12,\dots,16\}} \cup [2,3] = gf^{-1}\restriction J_{\{12,\dots,16\}} \cup [2,3].$$

The element $\lambda_1^{-1}\lambda_2$ has four components of support:

$$[0,1] \cup J_{\{1,\dots,5\}}, \qquad J_{10}, \qquad J_{14}, \qquad J_{16} \cup [2,3].$$

Note that

$$\lambda_1^{-1}\lambda_2\restriction J_{10} = f^{-1}h_1 f.$$

Denote by p_1 the following restriction:

$$\lambda_1^{-1}\lambda_2\restriction [0,1] \cup J_{\{1,\dots,5\}} = f^{-1}g\restriction [0,1] \cup J_{\{1,\dots,5\}}.$$

Denote by q_2 the following restriction:

$$\lambda_1^{-1}\lambda_2\restriction J_{\{14,\dots,16\}} \cup [2,3].$$

First observe that the restrictions on J_{10} are:

$$\lambda_1^{-1}\lambda_2\restriction J_{10} = f^{-1}h_1 f\restriction J_{10}, \qquad \lambda_2\lambda_1^{-1}\restriction J_{10} = h_3\restriction J_{10} = f^{-1}h_2 f\restriction J_{10}.$$

It follows that this restriction to J_{10} corresponds to the isomorphism

$$H \to \langle \lambda_1^{-1}\lambda_2\restriction J_{10}, \lambda_2\lambda_1^{-1}\restriction J_{10}\rangle,$$

given by

$$h_2 \mapsto \lambda_2\lambda_1^{-1}\restriction J_{10}, \qquad h_1 \mapsto \lambda_1^{-1}\lambda_2\restriction J_{10},$$

since these restrictions generate a dynamical realization of H on J_{10}.

Next observe that

$$\lambda_2\lambda_1^{-1}\restriction J_6 = h_1\restriction J_6, \qquad \lambda_1^{-1}\lambda_2\restriction J_6 = id\restriction J_6.$$

Since H is a marked extension of P, every relation in H is necessarily a product of commutators. It follows that the abelianization of H is $\mathbb{Z}^2$. We have then that this restriction to J_6 corresponds to the quotient

$$H \to \langle \lambda_1^{-1}\lambda_2\restriction J_6, \lambda_2\lambda_1^{-1}\restriction J_6\rangle,$$

given by

$$h_2 \mapsto \lambda_2\lambda_1^{-1}\restriction J_6, \qquad h_1 \mapsto \lambda_1^{-1}\lambda_2\restriction J_6.$$

which is a homomorphism whose kernel is the normal closure of h_1 in H.

Next, we observe that by construction, the maps p_1, p_2 and q_1, q_2 satisfy the dynamical conditions described in Lemmas 2.3 and 2.4 respectively. Define

$$j_1 = \sup(J_1), \qquad j_2 = \inf(J_{14}).$$

By construction, we have

$$p_1(j_1) = \lambda_1^{-1}\lambda_2(j_1) = f^{-1}g(j_1) = g(f^{-1}(j_1)) < 1,$$

and

$$p_1 \restriction [0,1] = \lambda_1^{-1}\lambda_2 \restriction [0,1] = f^{-1} \restriction [0,1] = \lambda_2\lambda_1^{-1} \restriction [0,1] = p_2 \restriction [0,1].$$

It follows that:

$$\langle p_1, p_2 \rangle \cong \langle p_1, p_2 \mid [p_1 p_2^{-1}, p_1^{-1} p_2 p_1], [p_1 p_2^{-1}, p_1^{-2} p_2 p_1^2] \rangle \cong F.$$

Next, observe that by construction we have

$$q_1(j_2) = \lambda_2\lambda_1^{-1}(j_2) = gf^{-1}(j_2) = f^{-1}(g(j_2)) > 2,$$

and

$$q_2 \restriction [2,3] = \lambda_1^{-1}\lambda_2 \restriction [2,3] = g \restriction [2,3] = \lambda_2\lambda_1^{-1} \restriction [2,3] = q_1 \restriction [2,3].$$

It follows that:

$$\langle q_1, q_2 \rangle \cong \langle q_1, q_2 \mid [q_1 q_2^{-1}, q_1^{-1} q_2 q_1], [q_1 q_2^{-1}, q_1^{-2} q_2 q_1^2] \rangle \cong F.$$

In particular, the subgroup of $\langle p_1, p_2 \rangle \times \langle q_1, q_2 \rangle$ generated by the elements (p_1, q_2) and (p_2, q_1) is isomorphic to P.

Now we claim that the map

$$h_2 \mapsto \lambda_2\lambda_1^{-1}, \qquad h_1 \mapsto \lambda_1^{-1}\lambda_2,$$

extends to an embedding

$$\langle h_1, h_2 \rangle \to \langle \lambda_1, \lambda_2 \rangle.$$

This is true for the component J_{10}, where this is a dynamical realization of H. So it suffices to show that each relation in h_1 and h_2 is satisfied by the restrictions of $\lambda_1^{-1}\lambda_2$ and $\lambda_2\lambda_1^{-1}$ on other components. As we saw before, for J_6, this via the $\mathbb{Z}$–quotient given by killing the normal closure of the generator $h_1 \in H$, which factors through the abelianization map. For the components

$$[0,1] \bigcup J_{\{1,\dots,5\}}, \qquad J_{\{12,\dots,16\}} \bigcup [2,3],$$

the action of H is precisely as P, and since H is a marked extension of P, whence the desired conclusion. □

2.6. **Smoothability.** To construct square roots of F which are not conjugate into $\mathrm{Diff}^2(I)$ or $\mathrm{Diff}^2(\mathbb{R})$, the group of C^2 orientation-preserving diffeomorphisms of the interval and the real line respectively, we use the following result due to Plante–Thurston and its generalizations due to Farb–Franks:

Theorem 2.10 (See [21, 15])**.** *Let $N < \mathrm{Diff}^2(M)$ be a finitely generated nilpotent subgroup, where here M is a compact one–manifold. Then N is abelian. Moreover, any nilpotent subgroup of $\mathrm{Diff}^2(\mathbb{R})$ is metabelian.*

Proof of Corollary 1.6. Let $N_k = \mathbf{F_2}/\mathscr{R}^{\mathbf{k}}$ be as in Lemma 2.7. Then $\mathscr{R}/\mathscr{R}^k < N_k$. Taking a finite subset of a free generating set S for $\mathscr{R}$, we have that the image of S in N_k generates a nilpotent subgroup $\Gamma_S < N_k$. It is straightforward to check that Γ_S is a retract of $\mathscr{R}/\mathscr{R}^k$, and is therefore a nonabelian nilpotent subgroup of N_k whenever $k \geqslant 3$. Applying Theorem 1.4 and Theorem 2.10 gives the desired conclusion in the case where M is compact. Choosing a $k \gg 0$ such that N_k contains a nilpotent subgroup which is not metabelian, we get the desired conclusion for $\mathbb{R}$ as well. □

Corollary 1.7 similarly follows from Theorem 1.4 and Theorem 4.1 of [15].

3. Uncountability of $\mathscr{S}$ and infinitely presented examples

In this section, we prove Theorem 1.8. For this, we retain the notation from the previous discussion.

3.1. **Sources of uncountability.** A construction of B. Neumann on the existence of uncountably many distinct isomorphism classes of two–generated groups as outlined by de la Harpe in [12] has the advantage that the resulting groups are all left orderable, as observed in [18]. We summarize the relevant conclusions here:

Proposition 3.1. *There exists an uncountable class $\mathscr{N}$ of pairwise non–isomorphic groups such that if $N \in \mathscr{N}$ then N is two–generated, left orderable, and $N^{ab} = \mathbb{Z}^2$. In particular, N can be realized as a subgroup of $\mathrm{Homeo}^+(\mathbb{R})$.*

3.2. **Equations over $\mathrm{Homeo}^+(\mathbb{R})$.** In order to prove Theorem 1.8, we will construct an explicit orderable marked extension of P which contains a given element of $\mathscr{N}$ as a subgroup. To do this, we will need to solve equations over $\mathrm{Homeo}^+(\mathbb{R})$.

Let $\{f_1, \ldots, f_k, g\} \subset \mathrm{Homeo}^+(\mathbb{R})$ be given, and let $w \in \mathbf{F_n}$ be a reduced word in the free group on n fixed generators, where here $k < n$. An *equation* over $\mathrm{Homeo}^+(\mathbb{R})$ is an expression of the form

$$w(f_1, \ldots, f_k, x_1, \ldots, x_{n-k}) = g.$$

A tuple $\{y_1, \ldots, y_{n-k}\} \subset \mathrm{Homeo}^+(\mathbb{R})$ is a *solution* to the equation if this expression becomes an equality after substituting y_i for x_i for each i, and interpreting the expression in $\mathrm{Homeo}^+(\mathbb{R})$.

We will restrict out attention to the case where $n = 2$. Even here, equations may not admit solutions. A trivial example can be given by taking $f \neq g$ and setting w to be the first free generator. A slightly less trivial example can be given by taking f to be fixed point free, taking g to have at least one fixed point, and setting w to be a conjugate of the first free generator.

We will concern ourselves with a particular commutator word w with free generators s and t, so that under the map $\mathbf{F_2} \to P$ given by $s \mapsto (p_1, q_2)$ and $t \mapsto (p_2, q_1)$, the element w lies in the kernel.

The following lemma is key in proving Theorem 1.8:

Lemma 3.2. *Fix a group $N \in \mathscr{N}$ and let τ be the map $\tau(t) = t + 1$. There is a homeomorphism $\kappa \in \mathrm{Homeo}^+(I)$ and a nontrivial commutator word $w \in \ker\{\mathbf{F_2} \to P\}$ such that:*

(1) The group $\langle \kappa, \tau \rangle$ contains N as a subgroup.
(2) The equation $w(\tau, x) = \kappa$ admits a solution $y \in \mathrm{Homeo}^+(\mathbb{R})$.

We first show how Lemma 3.2 implies Theorem 1.8:

Proof of Theorem 1.8. We recall some of the notation and the construction in Subsection 2.5. We will use informal language below, since we already have a precise description in that subsection.

Given any $h_1, h_2 \in \mathrm{Homeo}^+(I)$, we can build a square root $G \in \mathscr{S}$ generated by λ_1, λ_2 such that the group $H = \langle \lambda_1^{-1}\lambda_2, \lambda_2\lambda_1^{-1} \rangle$ satisfies the following.

(1) H acts as a dynamical realization of P on
$$([0,1] \cup J_{\{1,\dots,5\}}) \bigcup (J_{\{12,\dots,16\}} \cup [2,3])$$
(2) $\langle \lambda_2\lambda_1^{-1} \rangle$ acts faithfully by $\mathbb{Z}$ on the interval J_6 and $\langle \lambda_1^{-1}\lambda_2 \rangle$ acts trivially on the interval J_6.
(3) $\langle \lambda_1^{-1}\lambda_2 \rangle$ acts as h_1 on J_{10} and $\langle \lambda_2\lambda_1^{-1} \rangle$ acts as h_2 on J_{10}.
(4) The action of H outside the above intervals is trivial.

Let τ, κ and y be the homeomorphisms of the real line from Lemma 3.2. For the rest of the proof, we fix dynamical realizations of τ, κ on J_{10} obtained from conjugating by a homeomorphism of $\mathbb{R}$ to the interior of J_{10}. We shall now denote by τ, κ, y as these homeomorphisms supported on J_{10}.

We use the input $h_1 = \tau$ and let $h_2 = y$ to produce a square root G of F. Consider the subgroup K of H generated by
$$k_1 = \lambda_1^{-1}\lambda_2, \qquad k_2 = w(\lambda_1^{-1}\lambda_2, \lambda_2\lambda_1^{-1}).$$
We check the following:

(1) $k_1 \restriction J_{10} = \tau$ and $k_2 \restriction J_{10} = \kappa$.

(2) k_2 acts trivially outside J_{10} since $w(s,t)$ represents the identity in P under the map $s \mapsto (p_1, q_2)$ and $t \mapsto (p_2, q_1)$.
(3) Any commutator vanishes on J_6.
(4) k_1 acts trivially on J_6 and by $\mathbb{Z}$ on

$$([0,1] \cup J_{\{1,\dots,5\}}) \bigcup (J_{\{12,\dots,16\}} \cup [2,3])$$

By our assumption, $N < \langle k_1, k_2 \rangle \restriction J_{10}$. We claim that in fact, $N < \langle k_1, k_2 \rangle$. This follows from the fact that the relations in N are elements of the commutator subgroup of the free group, and since $\langle k_1, k_2 \rangle$ acts by $\mathbb{Z}$ outside J_{10}. Therefore $N < G$ where G is the corresponding square root of F.

We thus obtain that if $N \in \mathscr{N}$ is given, then there is a square root $G_N \in \mathscr{S}$ which contains N as a subgroup. Since the class $\mathscr{N}$ contains uncountably many different isomorphism types and since any element of $\mathscr{S}$ is two–generated and hence countable, the class $\{G_N \mid N \in \mathscr{N}\} \subset \mathscr{S}$ consists of uncountably many different isomorphism types. □

Proof of Lemma 3.2. We shall use the commutator word

$$w(s,t) = [w_1(s,t), w_2(s,t)],$$

where

$$w_1 = [st^{-1}, s^{-2}ts^2], \qquad w_2 = t[st^{-1}, t^{-1}s^{-1}t]t^{-1}.$$

It is straightforward to check that for the map $\mathbf{F_2} \to P$ given by $s \mapsto (p_1, q_2)$ and $t \mapsto (p_2, q_1)$, the element w lies in the kernel.

Let $\phi, \psi \in \mathrm{Homeo}^+(I)$ be given generators of N. We first choose homeomorphisms $\mu, \nu \in \mathrm{Homeo}^+(I)$ such that $\psi = [\mu, \nu^{-1}]$ and homeomorphisms $\chi, \xi \in \mathrm{Homeo}^+(I)$ such that $\phi = [\xi, \chi^{-1}]$. Such choices are possible, since every element of $\mathrm{Homeo}^+(I)$ is a commutator (see Theorem 2.65 of [9], for instance).

Identify I with the unit interval $[0,1] \subset \mathbb{R}$. Recall that τ is translation by 1 on $\mathbb{R}$. We set $\kappa \in \mathrm{Homeo}^+(\mathbb{R})$ as

$$\kappa = (\tau^{-2}\psi\tau^2)(\tau^{-102}\phi\tau^{102})$$

Intuitively, κ acts by ψ on the interval $[2,3]$, by ϕ on the interval $[102,103]$, and by the identity otherwise. We now verify that κ witnesses the conditions of the lemma.

We set

$$y = (\tau^{-1}\mu\tau^1) \circ (\tau^{-2}\nu\tau^2)(\tau^{-101}\chi\tau^{101})(\tau^{-102}\xi\tau^{102}).$$

Intuitively, the homeomorphism y acts by μ on $[1,2]$, by ν on $[2,3]$, by χ on $[101,102]$, and by ξ on $[102,103]$. We check that y is the solution to the equation. We proceed by analysing the two inner commutators separately, and then considering the outer commutator.

Consider the commutator $[\tau y^{-1}, \tau^{-2} y \tau^2]$. First note that since y^{-1} has disjoint support from $\tau^{-2} y \tau^2$, they commute, and hence

$$[\tau y^{-1}, \tau^{-2} y \tau^2] = [\tau, \tau^{-2} y \tau^2]$$

We can now easily check that the action of the resulting homeomorphism is as follows:

(1) It acts by μ on $[2,3]$, by $\nu\mu^{-1}$ on $[3,4]$, and by ν^{-1} on $[4,5]$.
(2) It acts by χ on $[102,103]$, by $\xi\chi^{-1}$ on $[103,104]$, and by ξ^{-1} on $[104,105]$.

We denote this homeomorphism by α.

Next, consider the commutator

$$[\tau y^{-1}, y^{-1}\tau^{-1} y]$$

First, note that

$$[\tau y^{-1}, y^{-1}\tau^{-1} y] = [\tau, y^{-2}][\tau^{-1}, y^{-1}]$$

It is straightforward to check that the homeomorphism resulting from the product of these commutators is as follows:

(1) It acts by μ^{-2} on $[0,1]$, by $\nu^{-2}\mu^3$ on $[1,2]$, by $\nu^2\mu^{-1}\nu$ on $[2,3]$, and by ν^{-1} on $[3,4]$.
(2) It acts by χ^{-2} on $[100,101]$, by $\xi^{-2}\chi^3$ on $[101,102]$, by $\xi^2\chi^{-1}\xi$ on $[102,103]$, and by ξ^{-1} on $[103,104]$.

We denote by β the homeomorphism $t[\tau y^{-1}, y^{-1}\tau^{-1} y]t^{-1}$.

Finally, we consider the homeomorphism $[\alpha,\beta]$. Observe that the supports of α and β intersect in the intervals $[2,3]$ and $[102,103]$. Since $\psi = [\mu, \nu^{-1}]$, we see that $[\alpha,\beta]$ acts by ψ on $[2,3]$. Similarly, since $\phi = [\chi, \xi^{-1}]$, we have that $[\alpha,\beta]$ acts by ϕ on $[102,103]$. It follows that $[\alpha,\beta]$ agrees with κ, whence y is a solution to the equation as claimed.

Finally, we show that $N < \langle \kappa, \tau \rangle$. Indeed the group generated by $\tau^{-100}\kappa\tau^{100}, \kappa$ acts as N on $[102,103]$ and as $\mathbb{Z}$ outside this interval. Since the relations in N are elements of the commutator subgroup of the free group, it follows that this group is isomorphic to N. □

4. General square root phenomena

In this section, we pass to the completely general setup of finitely generated subgroups of $\mathrm{Homeo}^+(I)$ and address the results in Subsection 1.2.

4.1. **Free groups.** Classical result from combinatorial and geometric group theory show that there exist two–generated groups $\langle a, b \rangle$ which are not free, but such that $\langle a^2, b^2 \rangle$ is free. Moreover, one can arrange for these groups to be left orderable, and

hence to be realized as subgroups of $\mathrm{Homeo}^+(I)$. For instance, we take the braid group on three strands

$$B_3 = \langle a, b \mid aba = bab \rangle.$$

All braid groups are left orderable [13], and it is a standard fact that the squares of the standard braids generate a free group (cf. [16]).

4.2. **The lamplighter group.** Using square roots of F, we can produce many square roots of the lamplighter group $L = \mathbb{Z} \wr \mathbb{Z}$. Recall that

$$\mathbb{Z} \wr \mathbb{Z} \cong \mathbb{Z} \ltimes \Big(\bigoplus_{i \in \mathbb{Z}} \mathbb{Z}_i \Big),$$

where the natural action of $\mathbb{Z}$ is by translating the index $\mathbb{Z}_i \mapsto \mathbb{Z}_{i+1}$. The group L is naturally realized as a subgroup of $\mathrm{Homeo}^+(\mathbb{R}) < \mathrm{Homeo}^+(I)$ as follows. We choose an arbitrary homeomorphism ψ such that $\operatorname{supp} \psi = (0,1) \subset \mathbb{R}$, and then we consider the group generated by ψ and τ, where as before $\tau(x) = x + 1$. It is clear that $\langle \psi, \tau \rangle \cong L$. The following result clearly implies Theorem 1.12, in light of Theorem 1.8

Theorem 4.1. *Let G be a left orderable marked extension of P. Then there exists a square root of L containing an isomorphic copy of G.*

Sketch of proof. Let $T \in \mathrm{Homeo}^+(\mathbb{R})$ be given by $T(x) = x + 1/2$. Note that the intervals $(0,1)$ and $T((0,1))$ together form a chain of intervals. Following the setup in Subsection 2.5, it is clear that we can start with an arbitrary increasing homeomorphism ψ of $(0,1)$ which is "slow enough" and suitably modify it on $(0,1/2)$ and $(1/2,1)$ so that $\langle \psi, T\psi T^{-1} \rangle$ is a square root of F which contains a prescribed marked extension of P. □

4.3. **Square roots of $\mathbb{Z}$.** In this section, we give a recipe for producing many n–generated groups of homeomorphisms of the interval, so that the squares of the generators generate a cyclic group.

Proof of Theorem 1.10. Let $\tau_1, \ldots, \tau_{n+1}$ be $n+1$ copies of the translation $\tau(x) = x + 1$ viewed as a homeomorphism of $\mathbb{R}$, and let

$$\langle h_1, \ldots, h_n \rangle = H < \mathrm{Homeo}^+(I)$$

be an arbitrary n–generated subgroup. We set $T_{n+1}(x) = x + 1/2$.

Lemma 2.1 constructs all possible roots τ, and we follow the construction given there. We first scale down H to be a group of homeomorphisms of $[0,1/2]$, and we abuse notation and label the generators of H by $\{h_1, \ldots, h_n\}$. We now define T_i to be the homeomorphism $T_{n+1} \circ h_i$ on $[0,1/2]$. The requirement $T_i^2 = \tau$ determines the values of T_i on the rest of $\mathbb{R}$.

Now let $S_i = T_{n+1}^{-1} \circ T_i$ for $1 \leqslant i \leqslant n$. Observe that S_i acts by h_i on each interval of the form $[k, k+1/2]$ and by h_i^{-1} on each interval of the form $[k-1/2, k]$, where $k \in \mathbb{Z}$. It is clear then the $\widehat{H} \cong \langle S_1, \dots, S_n \rangle < \langle T_1, \dots, T_{n+1} \rangle$. □

Proof of Corollary 1.11. First, note that if H is free then the skew subdirect product $\widehat{H}$ is also free, which establishes the second part of the corollary. For the first part, we perform a mild modification of the Neumann groups as discussed in Proposition 3.1 (see [18] for a detailed discussion of these groups). All the Neumann groups quotients of a single two–generated group $\Gamma = \langle t, s_0 \rangle$ which is left orderable. The element of $\mathcal{N}$ are given as quotients of Γ by certain central normal subgroups $N_X < \Gamma$.

It is straightforward to check that the map given by $t \mapsto t^{-1}$ and $s_0 \mapsto s_0^{-1}$ extends to a well–defined automorphism of Γ, whence $\widehat{\Gamma} \cong \Gamma$. Moreover, the subgroups N_X are all stable under this automorphism of Γ. In particular, it follows that if $N \in \mathcal{N}$ is one of the Neumann groups, then $N \cong \widehat{N}$. The first claim of the corollary follows from Theorem 1.10. □

4.4. **General groups.** For a general finitely generated group $H = \langle x_1, \dots, x_n \mid R \rangle$, one can *formally take the square root* of H by setting

$$G = \langle y_1, \dots, y_n, x_1, \dots, x_n \mid R, x_1 = y_1^2, \dots, x_n = y_n^2 \rangle.$$

Note that this definition depends on the presentation of H which is given. If H is given as a free group with no relations then G will be free of the same rank. However, if H is not freely presented then G can be very complicated. We will call a presentation for a group H *reduced* if x_i is nontrivial in H for each i.

Theorem 4.2. *Let H be a left orderable finitely generated group with a reduced presentation. Then the formal square root G of H is left orderable.*

Proof. If $H = \langle x_1, \dots, x_n \mid R \rangle$, set

$$K = \langle y, x_1, \dots, x_n \mid R, x_1 = y^2 \rangle.$$

If we can prove that K is left orderable then the result will follow by induction on n.

To this end, note that K admits a description as an amalgamated product via

$$K = \mathbb{Z} *_{2\mathbb{Z} = \langle x_1 \rangle} H.$$

Since we can order $\mathbb{Z}$ either positively or negatively, we may assume that the isomorphism $2\mathbb{Z} \cong \langle x_1 \rangle$ is order preserving. Then, a result of Bludov–Glass [3] (cf. Bergman [1]) implies that the corresponding amalgamated product is again orderable. □

Note that the assumption that the presentation for H in Theorem 4.2 is reduced was essential, since otherwise the formal square root would contain torsion. Moreover, we need not assume that H be finitely generated in Theorem 4.2, and this hypothesis could be replaced by countable generation. We include this hypothesis since nearly all groups under consideration in this paper are finitely generated.

Finally, we show that formal square roots generally contain nonabelian free groups, so that free subgroups are in some precise sense a general phenomenon in square roots of groups of homeomorphisms:

Theorem 4.3. *Let $H = \langle x_1, \dots, x_n \mid R \rangle$ be a reduced presentation for a non–cyclic finitely generated left orderable group, and let $K = \langle y, x_1, \dots, x_n \mid R, x_1 = y^2 \rangle$. Then K contains a nonabelian free group.*

Thus, Theorem 4.3 implies that the formal square root of a non–cyclic group always contains nonabelian free groups.

Proof of Theorem 4.3. The result follows from general Bass–Serre theory. One can construct free subgroups explicitly using normal forms for amalgamated products (see [22, 19]). To do this, let $z \in H \backslash \langle x_1 \rangle$, which exists since H is assumed not to be cyclic. Note that z has infinite order since H is left orderable. Consider the group $\langle z, yzy^{-1} \rangle$. An arbitrary word in these generators will be of the form

$$z^{n_1} y z^{m_1} y^{-1} \cdots z^{n_k} y z^{m_k} y^{-1},$$

where all these exponents are nonzero except possibly n_1 and m_k. This word cannot collapse to the identity since it is in normal form. It follows that the group $\langle z, yzy^{-1} \rangle$ is free. □

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