Quality Engineering

ISSN: 0898-2112 (Print) 1532-4222 (Online) Journal homepage: https://www.tandfonline.com/loi/lqen20

Multi-dimensional Lévy processes with Lévy copulas for multiple dependent degradation processes in lifetime analysis

Yu Shi, Qianmei Feng, Yin Shu & Yisha Xiang

QUALITY ENGINEERING
https://doi.org/10.1080/08982112.2020.1757704

Taylor & Francis
Taylor & Francis Group

Check for updates

Multi-dimensional Lévy processes with Lévy copulas for multiple dependent degradation processes in lifetime analysis

Yu Shi[a], Qianmei Feng[a], Yin Shu[b], and Yisha Xiang[c]

[a]Department of Industrial Engineering, University of Houston, Houston, Texas; [b]Reliability Sciences Department, Corning Incorporated, Corning, New York; [c]Department of Industrial, Manufacturing & Systems Engineering, Texas Tech University, Lubbock, Texas

ABSTRACT

The analysis of multiple dependent degradation processes is a challenging research problem in the reliability field, especially for complex degradation processes with random jumps. To integrally handle the jumps with uncertainties and the dependence among degradation processes, we construct general multi-dimensional Lévy processes to describe multiple dependent degradation processes in engineering systems. The evolution of each degradation process can be modeled by a one-dimensional Lévy subordinator with a marginal Lévy measure. The dependence among all dimensions is described by Lévy copulas and the associated multiple-dimensional Lévy measure. The multi-dimensional Lévy measure is obtained from one-dimensional marginal Lévy measures and the Lévy copula. We develop the Fokker-Planck equations to describe the time evolution of the probability density for stochastic processes. The Laplace transforms of both reliability function and lifetime moments are then derived. Numerical examples are used to demonstrate our models in lifetime analysis. The results of this research are expected to provide a precise reliability prediction, help to avoid failures caused by multiple dependent degradation processes, and maintain the long-term operation of a system.

KEYWORDS

Multi-dimensional Lévy processes; multiple dependent degradation processes; Lévy copulas; degradation modeling; reliability function; lifetime analysis

Introduction

In engineering applications, it is common to observe more than one degradation process in a component (e.g., multiple crack growth on a metal surface), or in a multi-component system where several components are subjected to degradation due to wear, aging, fatigue, corrosion, etc. Typically, these degradation processes are dependent due to the complicated internal mechanisms (e.g., mechanical, thermal, electrical, or chemical) and/or the exposure to the common external conditions (e.g., temperature, pressure, humidity, or vibration). The analysis of multiple dependent degradation processes is a challenging research problem in the reliability field. They are commonly described using multi-dimensional stochastic processes that can characterize the physical degradation processes.

In reliability studies, stochastic processes have been widely used to model the temporal variability in degradation evolution, such as Wiener process, gamma process, and inverse Gaussian process. A Wiener process/Brownian motion with drift has been applied to model the non-monotonic degradation process with stationary and independent increments following a normal distribution (Ye et al. 2013). The class of gamma processes is another type of special stochastic processes with independent and nonnegative increments that are random variables following a gamma distribution. van Noortwijk (2009) provided an overview of applying gamma processes to model deterioration in civil infrastructures for reliability analysis. The inverse Gaussian (IG) process is another class of stochastic processes that has independent and nonnegative increments, which follow an IG distribution. Wang and Xu (2010) developed an attractive degradation model with monotonic paths based on IG processes.

These stochastic processes, however, limit themselves to certain-distributed and independent increments that cannot fit different types of degradation datasets in general. To relax the assumption of certain-distributed increments, researchers explored to use Lévy processes for constructing degradation models with uncertain jumps (Çinlar 1977; Abdel-Hameed 1984; Shu, Feng, and Coit 2015, Shu et al. 2016, Shu,

CONTACT Qianmei Feng qmfeng@uh.edu Department of Industrial Engineering, University of Houston, Houston, TX 77204, USA.

Feng, and Liu 2019). Abdel-Hammed (1984) used a Lévy process to model the wear degradation and studied its life distribution properties where the threshold is assumed to be random. Shu, Feng, and Coit (2015) gave a new closed-form reliability function for degradation described by Lévy subordinators, a class of non-decreasing Lévy processes. Using Fokker-Planck Equations (FPE), a tool for modeling stochastic processes without analytical forms of probability distributions, Shu et al. (2016) derived the explicit results for the reliability function and lifetime characteristics of degradation processes with uncertainty jumps modeled by Lévy subordinators and their extensions. In addition, the parameters in stochastic degradation models are estimated using linear programing estimators and empirical characteristic functions (Shu, Feng, and Liu 2019).

In engineering practices, however, multiple degradation processes can occur simultaneously in a system that affect the performance and reliability of the system. For example, for an LED lighting system, Sari et al. (2009) developed a bivariate constant stress degradation data model to describe two degradation failures that dominate the system reliability. To study and model multiple degradation processes, a challenge to be addressed is how to describe and model the dependence structures among those degradation processes. Liu et al. (2014) used a multi-dimensional Wiener process to model the degradation, in which the dependence among degradation was described by a covariance matrix. While the covariance can only describe the linear dependence, copulas can capture the whole dependence structure including various non-linear and linear forms of dependence among random variables.

Copula analysis is an effective tool to extend marginal probability distributions to multi-dimensional probability distributions. The concept of copulas was introduced to separate the dependence structure of a random vector from its univariate margins (Sklar 1996). By providing a complete characterization of possible dependence structures of a random vector with fixed margins, copulas can be used to construct multi-dimensional distributions with specified dependence and arbitrary marginal laws. Li (1999) developed a copula function to approach the default correlation of multiple-entities survival time. Based on a chosen copula function, they computed the pairwise correlation of survival times and obtained the default correlation of two discrete events. Embrechts et al. (2002) used the copula concept to represent the dependence of a random vector for risk management modeling. Recently, the copula method has been applied in the analysis of multi-dimensional degradation processes. Hao, Su, and Li (2015) monitored the reliability of LED lighting systems via multi-dimensional Gamma processes where the dependence is described by Frank copula. For multiple dependent performance characteristics, Wang et al. (2015) used a copula function in estimating the residual life. Sun et al. (2016) applied the copula method in Wiener processes to analyze multiple nonlinear accelerated degradation processes for highly reliable products. Peng et al. (2016) utilized Wiener process, IG process and copula functions to model a bivariate degradation process. Using stochastic process models with Gaussian copula, Peng, Ye, and Chen (2019) studied residual lifetime prediction of multiple degradation systems with measurement errors. More recently, Fang, Pan, and Hong (2020) used four Archimedean copulas in a multivariate degradation analysis for monitoring more than one performance characteristic of a product.

With similar properties to regular copulas, Lévy copulas are introduced to completely characterize the dependence among components of multi-dimensional Lévy processes by utilizing Lévy measures and separating the information around multiple components (Cont and Tankovs 2004, Barndorff-Nielsen and Lindner 2007). Extended from regular copulas, Lévy copula is a defined function that maps marginal probability functions to the joint probability function for multi-dimensional Lévy processes. It separates the dependence structure and univariate margins of multi-dimensional Lévy processes. As one of the major copula families, Archimedean copulas include a large number of member copula functions that can represent different types of dependency structures (Naifar 2011). The role of Lévy copulas is described in details by Sklar's theorem (Aas et al. 2009). For an m-dimensional Lévy process $X(t)$, Sklar's theorem characterizes the multi-dimensional Lévy measure ν using one-dimensional marginal Lévy measures ν_i ($i = 1, 2, \dots, m$) and the Lévy copula C. In addition, it also describes one-dimensional marginal Lévy measures ν_i as the projections of ν on m dimensions.

In this research, we intend to construct general multi-dimensional Lévy processes governed by Lévy copulas to model multiple degradation processes by integrally handling the jumps with uncertainties and the dependence among degradation processes in engineering systems. Although multi-dimensional Lévy processes governed by Lévy copulas have been applied in actuarial and insurance analysis (Bauerle, Blatter, and Muller 2008), they have not been widely used in the analysis of multiple degradation processes in engineering systems. Existing studies described multiple dependent degradation processes using multi-dimensional Wiener processes, Gamma processes, and IG processes. However, Wiener-based models cannot handle the jumps in degradation, while the Gamma- and Poisson-based models are not flexible in general. To precisely evaluate and predict reliability characteristics, it is critical to construct appropriate stochastic processes with appropriate copulas to handle internally-induced stochastic uncertainties, complex jumps, and dependence. One of the most important advantages of using Lévy processes to model degradation is that their jump parts represented by Lévy measures can model a great deal of jump mechanisms in degradation, which provides flexibility in fitting various degradation data series. More specifically, we use non-decreasing multi-dimensional Lévy processes, or multi-dimensional Lévy subordinators, to model the commonly observed degradation behaviors in this research.

The remainder of this paper is organized as follows. In section "Lévy copulas and Lévy measures", we construct multi-dimensional copulas for multiple dependent degradation processes described by Lévy subordinators. In section "Reliability function and lifetime moments", the multi-dimensional Lévy measures are derived based on the multi-dimensional Lévy copula obtained in section "Lévy copulas and Lévy measures". The reliability function and lifetime moments are also derived. Simulation studies are implemented in section "Simulation and numerical examples", while Section "Conclusions and discussion" provides the conclusion and future research directions.

Notations

- Euclidean space: R^m, $m \in N$
- Inner product: $\langle x, y\rangle = \sum_{i=1}^{m} x_i y_i$
- Euclidean norm: $|x| = \langle x, x\rangle^{1/2} = \left(\sum_{i=1}^{m} x_i x_i\right)^{1/2}$
- Brownian motion under covariance matrix A at time t: $B_A(t)$
- m-dimensional Lévy process: $X(t) = \{X_1(t), \ldots, X_m(t)\}$
- m-dimensional Lévy subordinators: $X_S(t) = \{X_S{}^1(t), \ldots, X_S{}^m(t)\}$
- Lévy symbol: $\eta(\mu)$
- Marginal Lévy measure: $\nu_i(dy_i)$, $i = 1, 2, \ldots, m$
- m-dimensional Lévy copula: $C(u_1, \ldots, u_m)$
- m-dimensional Lévy measure under copula C: $\nu C(dy_1 \ldots dy_m)$
- Reliability function at time t: $R(t)$
- Marginal reliability function at time t: $R_i(t)$, $i = 1, 2, \ldots, m$
- Laplace transform of reliability function at time t: $R^{LL}(u, w)$
- Laplace transform of lifetime moments: $M^L(T_X{}^n, x)$
- Poisson random measure at time t: $J(t, dy)$
- A volume function: U

Lévy copulas and Lévy measures

Lévy copulas for multi-dimensional Lévy processes are defined by extending the definition of regular copulas (Cont and Tankovs 2004). For references, some fundamental concepts and preliminaries of multi-dimensional Lévy processes and regular copulas are described in the appendix. The main difference between regular copulas and Lévy copulas lies in the domain and range.

Definition 1. (Cont and Tankovs 2004). *A function C defined on $R^m \to R$ is called Lévy copula if:*

1. $C(u_1, u_2, \ldots \ldots, u_m) \neq \infty$ *for* $(u_1, u_2, \ldots \ldots, u_m) \neq (\infty, \infty, \ldots, \infty)$;
2. $C(u_1, u_2, \ldots \ldots, u_m) = 0$ *if* $u_i = 0$ *for at least one* $i \in \{1, 2, \ldots, m\}$ *(grounded);*
3. C *is m-increasing;*
4. $C^{(i)}(u) = u$ *for* $i \in \{1, 2, \ldots, m\}$, $u \in R$ *(uniform marginal).*

In this research, all multivariate Lévy processes, Lévy measures and Lévy copulas are considered on R_+^m space. When we change the domain and range from $R^m \to R$ to $R^m{}_+ \to R_+$, it becomes a positive Lévy copula.

An important feature of Lévy copulas is that they can separate the marginal and internal dependence structures for multi-dimensional Lévy processes, as described in Sklar's theorem. The theorem describes the relationship between Lévy copula C, the volume function (also named the tail integral) U, and the marginal volume functions $\{U_1, \ldots, U_m\}$ in multi-dimensional Lévy processes.

Theorem 1. (Sklar's theorem, Barndorff-Nielsen and Lindner 2007). *For an m-dimensional Lévy measure ν that satisfies $\nu(\{0\}) = 0$ and $\int_{R^m} \min\{1, |y|^2\} \nu(dy) < \infty$ with a volume function U and marginal volume functions $\{U_1, \ldots, U_m\}$, there exists a (positive) Lévy copula C such that:*

$$F(x_1, \ldots, x_m) = C(F_1(x_1), \ldots, F_m(x_m)), \forall x_1, \ldots, x_m \; [0, \infty]. \tag{1}$$

Lévy copula C is uniquely determined on $Ran(U_1)\cdot \ldots \cdot Ran(U_m)$. Conversely, if C is a positive Lévy copula and $\{U_1, \ldots, U_m\}$ are volume functions of one-dimensional positive Lévy measures $v_1, \ldots, v_m$, then (1) defines a Lévy measure $v \in {L_+}^m$ with a volume function U and marginal Lévy measures $v_1, \ldots, v_m$.

The random jumps in multi-dimensional degradation processes are measured and represented by a multi-dimensional Lévy measure v_C associated with Lévy copula. Based on Sklar's theorem, the margins and internal dependence structures of v_C are separated by the corresponding Lévy copula C. Therefore, an important problem to be addressed is to derive the multi-dimensional Lévy measure v_C from the corresponding Lévy copula C.

Multi-dimensional Lévy measures from Lévy copulas

The mathematical relationship between Lévy copula C and the associated Lévy measure v_C can be considered as a mapping, where a bijection is defined under the multi-dimensional space (Barndorff-Nielsen and Lindner 2007):

$$Q_m: \ [0,\infty]^m \to [0,\infty]^m,\ (x_1,\ldots,\ x_m) \to \left({x_1}^{-1},\ldots,\ {x_m}^{-1}\right),$$

such that $Q_m = {Q_m}^{-1}$. Another measure χ is also defined such that v_C is the image of χ on an m-dimensional Boral set B:

$$v(B) = (Q\chi)(B) = \chi\left(Q^{-1}(B)\right),\ \forall B \subseteq [0,\infty]^m.$$

Using the fact that the Lévy copula C has a uniform margin, we can find the measure satisfying:

$$\begin{aligned}&\chi_C\left([0,\infty]^{k-1} \times [0,x_k] \times [0,\infty]^{m-k}\right) = \\ &C(\infty,\ldots,x_k,\ldots,\infty) = x_k,\end{aligned} \tag{2}$$

which implies that there is a Lévy copula C satisfying $\chi_C = \chi$. Since $Q_m = {Q_m}^{-1}$, we can get the relationship between Lévy measure v_C and Lévy copula C:

$$\begin{aligned}&v_C([x_1,\infty] \times \cdots \times [x_m,\infty]) = \chi_C \\ &\times Q_m^{-1}([x_1,\infty],\ldots,[x_m,\infty]) = C(x_1^{-1},\ldots,x_m^{-1}).\end{aligned} \tag{3}$$

Lévy measures for multiple-dimensional Lévy subordinators

Due to the non-decreasing path of most degradation processes in practice, we study the special class of Lévy processes, *Lévy subordinators* $X_S(t)$, which take values in $[0, \infty)$ with a non-decreasing path. Based on Sklar's theorem, Lévy copulas are an important way to analyze and organize the internal multivariate structures, and build bivariate and multivariate distributions with given margins (Joe 1993). One of the parametric copula families is the Archimedean copula that is easy to construct and has a wide application in several fields (Wang, Wu, and Lai 2013). The standard expression for this family in m-dimension is

$$C(u_1,\ldots,\ u_m) = \varphi^{-1}\left(\varphi(u_1) + \ldots + \varphi(u_m)\right), \tag{4}$$

where φ is called a generator of the copula family, which is a continuous, strictly decreasing function from $[0, 1]$ to $[0, \infty]$ such that $\varphi(1) = 0$.

A sub-family of Archimedean copula, Clayton copula family, is widely considered in modeling multi-dimensional Lévy subordinators (Jaworski et al. 2009). An m-dimensional Clayton copula C for an m-dimensional Lévy subordinator $X_S(t) = \{{X_S}^1(t), \ldots, {X_S}^m(t)\}$ is defined on ${R_+}^m \to R_+$:

$$C(u_1,\ldots,\ u_m) = \left(\sum_i^m u_i^{-\theta}\right)^{\frac{-1}{\theta}}, \tag{5}$$

and $\theta \in (0, \infty)$ is the dependence parameter, where a larger value of θ represents the stronger dependence.

Based on Eq. (1) and (5), we have:

$$\begin{aligned}F(x_1,\ldots,\ x_m) &= C\left(F_1(x_1),\ldots,F_m(x_m)\right) \\ &= \left[F_1(x_1)^{-\theta} + \ldots + F_m(x_m)^{-\theta}\right]^{\frac{-1}{\theta}}.\end{aligned} \tag{6}$$

For a Boral set $B = [x_1, \infty]\cdot \ldots \cdot[x_m, \infty]$ in $[0, \infty]^m$, we can simply plug it into Eq. (3) such that

$$\begin{aligned}v_C([x_1,\infty] \times \cdots \times [x_m,\infty]) &= \chi_C \times Q_m^{-1}([x_1,\infty],\ldots,[x_m,\infty]) \\ &= \left(\left[0,{x_1}^{-1}\right]\ldots\left[0,{x_m}^{-1}\right]\right) \\ &= C\left({x_1}^{-1},\ldots,{x_m}^{-1}\right) = F(x_1,\ldots,x_m),\end{aligned}$$

which is a *Lévy measure* for the m-dimensional Lévy subordinator $X_S(t) = \{{X_S}^1(t), \ldots, {X_S}^m(t)\}$, since there is no atom for v_C at zero. A continuous measure is said to have no atom (Sun and Duan 2012).

An m-dimensional Lévy measure v_C has m margins, $U_1, \ldots, U_m$, which are volume functions of one-dimensional positive Lévy measures. We define a one-dimensional bijection, Q_1: $[0, \infty] \to [0, \infty]$, to map $x_i \to {x_i}^{-1}$ ($i = 1, 2, \ldots, m$), and another measure χ_i where $\chi_i([0, x]) = C(x) = x$. Then we can get the one-dimensional Lévy measure:

$$v_i = \chi_i\ {Q_1}^{-1}, i = 1,\ 2,\ldots,\ m. \tag{7}$$

Therefore, for an m-dimensional Lévy subordinator $X_S(t) = \{{X_S}^1(t), \ldots, {X_S}^m(t)\}$, each marginal subordinator ${X_S}^i(t)$, $i = 1, 2, \ldots, m$, can be expressed by $v_i(dy)$ based on one-dimensional Lévy-Itô decomposition.

Reliability function and lifetime moments

For an m-dimensional degradation process modeled by a multi-dimensional Lévy subordinator $X_S(t) = \{X_S{}^1(t), \ldots, X_S{}^m(t)\}$, the failure time of a system can be defined as the time when any degradation process exceeds its failure threshold x_i, $i = 1, 2, \ldots, m$, i.e., a series system is considered in this research. Therefore, the failure time of a system can be defined as $T_x \equiv inf\{t: X_1(t) > x_1, \ldots, X_m(t) > x_m\}$. Accordingly, the reliability function can be defined as $R(t) = P(T_x \geq t) = \mathrm{P}(X_1(t) \leq x_1, \ldots, X_m(t) \leq x_m) = F_{Xs(t)}(x)$. At the initial time $t = 0$, we have $R(x, 0) = P(T_x \geq 0) = P(X_S(t) \leq x) \equiv 1$, $x = (x_1, x_2, \ldots x_m)$, as the system is new without cumulating degradation at $t = 0$. Since there are no closed-form distribution functions for Lévy subordinators, it is challenging to derive reliability functions and lifetime moments. Fokker-Planck equations (Sun and Duan 2012) provide us a way to overcome the challenge in analyzing probability laws for stochastic processes.

Fokker-Planck equations

In a stochastic dynamical system, stochastic differential equations (SDE) are a main tool to model the dynamical status. For stochastic differential equations, Fokker-Planck equations (FPE) provide an effective, deterministic tool to manage the transition probability density (Soize 1994). The FPE of stochastic differential equations with Lévy processes were derived in Sun and Duan (2012), which provides a new way to obtain the probability laws for a Lévy subordinator, and therefore, the reliability function.

The Fokker-Planck equation of general stochastic dynamical systems is given as (Sun and Duan 2012)

$$\begin{aligned}\frac{\partial p(x,t)}{\partial t} = &-\frac{\partial}{\partial x}(f(x,t)p(x,t) - b\frac{\partial}{\partial x}(\sigma(x,t)p(x,t)) \\ &+\frac{1}{2}A\frac{\partial^2}{\partial x^2}(\sigma^2(x,t)p(x,t)) \\ &+\int_{R\ \{0\}}\Bigg[\sum_{k=1}^{\infty}\frac{(-y)^k}{k!}\frac{\partial^k}{\partial x^k}(\sigma^k(x,t)p(x,t)) \\ &+I_{(-1,1)}(y)y\frac{\partial}{\partial x}(\sigma(x,t)p(x,t))\Bigg]v(dy).\end{aligned} \tag{8}$$

When $\sigma(x, t) = 1$, it is the Fokker-Planck equation for a stochastic dynamical system with an additive Lévy process (Soize 1994).

Based on the Lévy-Itô decomposition in Lemma A1 of the appendix, a multi-dimensional degradation process modeled by a Lévy process can be decomposed into four parts: (1) a drift term, bt, (2) a Brownian motion $B_A(t)$ with a covariance matrix A, (3) a jump part$\int_{|y|\geq 1} yJ(t,dy)$ that is a compound Poisson process, and (4) another jump part $\int_{|y|<1} y\big(J(t,dy) - v(t,dy)\big)$ that is the compensated version of Poisson process $\int_{0\leq|y|\leq 1} yJ(t,dy)$. The last part can have finite or infinite activities, where $(J(t, dy) - v(t, dy))$ is the compensated Poisson random measure. Using Eq. (8), we can derive the Fokker-Planck equation for a multi-dimensional Lévy subordinator.

Lemma 1. *For a multi-dimensional degradation process with random jumps modeled by Lévy subordinators, the Fokker-Planck equation is*

$$\begin{aligned}\frac{\partial p(x,t)}{\partial t} = &-b_1\frac{\partial p(x,t)}{\partial x_1} - \ldots\ldots - b_m\frac{\partial p(x,t)}{\partial x_m} \\ &+\int_{R_+^m}(p(x_1-y_1,\ldots,x_m-y_m,t) - p(x,t))v(dy) \\ &+\int_{R_+^m}(1_{|y|<1}y_1\frac{\partial p(x,t)}{\partial x_1} + \ldots + 1_{|y|<1}y_m\frac{\partial p(x,t)}{\partial x_m})v(dy),\end{aligned} \tag{9}$$

where $x = (x_1, \ldots, x_m)$ is the threshold vector and $v(dy) = v(dy_1, \ldots, dy_m)$ is the m-dimensional Lévy measure.

Lifetime characteristics for Lévy subordinators

For a one-dimensional Lévy subordinator $X_S(t)$, Shu et al. (2016) showed that the relationship between the Laplace transform of reliability function $R^{LL}(u, w)$ and the Laplace transform of the probability density function $p^{LL}(u, w)$ is:

$$R^{LL}(u, w) = u^{-1}p^{LL}(u, w).$$

The relationship between the Laplace transform of lifetime moments and the Laplace transform of the derivative of reliability function $\tilde{Q}(x,t) = -\frac{\partial}{\partial t}R(x,t)$ is:

$$M^L(T_X{}^n, u) = \tilde{Q}_n^{LL}(u,0).$$

Accordingly, we extend the results from the one-dimensional case to the multi-dimensional case as in Lemma 2 and Lemma 3. For a multi-dimensional degradation process modeled by a multi-dimensional Lévy subordinator, the multi-dimensional Laplace transform of both reliability function and lifetime moments $R_X(x, t)$ and $M(T_X{}^n, x)$, $x = (x_1, \ldots, x_m)$, can be derived from FPE and represented by Lévy measure v_C with copula C.

Lemma 2. *Let $R^{LL}(u,\ w)$ be the Laplace transform of an m-dimensional reliability function $R_X(x,\ t)$ where $x=(x_1,\dots,\ x_m)$ and $u=(u_1,\dots,\ u_m)$, then*

$$R^{LL}(u,\ w)\ =u^{-1}p^{LL}(u,\ w). \tag{10}$$

Lemma 3. *Let $\tilde{Q}(x,t)=-\frac{\partial}{\partial t}R(x,t)$ be the derivative of an m-dimensional reliability function, and $\tilde{Q}^{LL}(u,w)$ is the Laplace transform of $\tilde{Q}(x,t)$ and $\tilde{Q}_n^{LL}(u,w)=(-1)^n\frac{\partial^n\tilde{Q}^{LL}(u,w)}{\partial w^n}$, where $x=(x_1,\dots,\ x_m)$ and $u=(u_1,\dots,\ u_m)$,* then

$$M^L(T_X{}^n,u)=\tilde{Q}_n^{LL}(u,0). \tag{11}$$

Based on Lemma 2, we derive and present the closed form of reliability function $R_X(x,\ t)$ in terms of m-dimensional Laplace transform, as in Theorem 2.

Theorem 2. *For multiple dependent degradation processes with random jumps described by a multi-dimensional Lévy subordinator $X_S(t)=\{X_S{}^1(t),\dots,X_S{}^m(t)\}$, the multi-dimensional Laplace transform of reliability function $R_X(x,\ t)$ is*

$$R^{LL}(u,w)=u^{-1}\{w+\langle b^*,u\rangle-\int_{R_+^m}(e^{-\langle u,y\rangle}-1)v(dy)\}^{-1}, \tag{12}$$

where $b^*=b-\int_{|y|<1}yv(dy)=(b_1{}^*,\dots,\ b_m{}^*)$ is an m-dimensional constant vector, $u=(u_1,\dots,\ u_m)$, $v(dy)=v(dy_1,\dots,\ dy_m)$ is an m-dimensional Lévy measure and m is the number of dimensions.

Proof. Let $p(x,\ t)=p(x_1,\dots,\ x_m,\ t)$ be the probability density function for an m-dimensional Lévy subordinator $X_S(t)=\{X_S{}^1(t),\dots,\ X_S{}^m(t)\}$. Based on Lemma 1, the FPE of $X_S(t)$ can be presented by Eq. (9) . We can perform a Laplace transform of $p(x,\ t)$ with respect to (w.r.t.) t on both sides of Eq. (9)

$$\begin{aligned}&wp^L(x,w)-p(x,0)\\&=-b_1\frac{\partial p^L(x,w)}{\partial x_1}\dots\dots-b_m\frac{\partial p^L(x,w)}{\partial x_m}\\&\quad+\int_{R_+^m}\Big[p^L(x_1-y_1,\dots..,x_m-y_m,w)-p^L(x_1,\dots..,x_m,w)\\&\quad+I_{|y|<1}\left(y_1\frac{\partial p^L(x,w)}{\partial x_1}+\dots..+y_m\frac{\partial p^L(x,w)}{\partial x_m}\right)\Big]v(dy).\end{aligned}$$

Performing an m-dimensional Laplace transform w.r.t. $x=(x_1,\dots,\ x_m)$ on both sides:

$$\begin{aligned}wp^{LL}(u,w)-I=-\langle b,u\rangle p^{LL}(u,w)+\int_{R_+^m}e^{-\langle u,y\rangle}p^{LL}(u,w)\\-p^{LL}(u,w)+I_{|y|<1}\langle u,y\rangle p^{LL}(u,w)]v(dy).\end{aligned}$$

Let $b^*=b-\int_{|y|<1}yv(dy)=(b_1{}^*,\dots,\ b_m{}^*)$. Therefore, the double Laplace transform of probability density function $p(x,\ t)=p(x_1,\dots,\ x_m,\ t)$ is obtained as

$$p^{LL}(u,w)=\{w+\langle b^*,u\rangle-\int_{R_+^m}(e^{-\langle u,y\rangle}-1)v(dy)\}^{-1}.$$

Based on Lemma 2, the m-dimensional Laplace transform of reliability function can be written as

$$R^{LL}(u,w)=u^{-1}\{w+\langle b^*,u\rangle-\int_{R_+^m}(e^{-\langle u,y\rangle}-1)v(dy)\}^{-1}.$$

As we can observe in (12), the Laplace transform of reliability function is a monotonous function with time transform w such that it decreases monotonously as w increases when the thresholds are fixed.

In addition, the expression of lifetime moments $M^L(T_X{}^n,\ x)$ in terms of m-dimensional Laplace transform can be derived and presented as Theorem 3 based on Lemma 3.

Theorem 3. *For multiple dependent degradation processes with random jumps described by a multi-dimensional Lévy subordinator $X_S(t)=\{X_S{}^1(t),\dots,\ X_S{}^m(t)\}$, the multi-dimensional Laplace transform of lifetime moments $M^L(T_X^n,x)$is*

$$M^L(T_X^n,x)=n!u^{-1}\left\{\langle b^*,u\rangle-\int_{R_+^m}(e^{-\langle u,y\rangle}-1)v(dy)\right\}^{-n}, \tag{13}$$

where $b^*=b-\int_{|y|<1}yv(dy)=(b_1{}^*,\dots,\ b_m{}^*)$ is an m-dimensional constant vector, $u=(u_1,\dots,\ u_m)$, $v(dy)=v(dy_1,\dots,\ dy_m)$ is an m-dimensional Lévy measure and m is the number of dimensions.

Proof. Let $\tilde{Q}^{LL}(u,w)$ be the Laplace transform of $\tilde{Q}(x,t)$ with respect to x and t. Based on Theorem 2, we can take a double Laplace transform on this equation and obtain:

$$\begin{aligned}\tilde{Q}^{LL}(u,w)&=-wR^{LL}(u,w)+u^{-1}\\&=-wu^{-1}\{w+\langle b^*,u\rangle\\&\quad-\int_{R_+^m}(e^{-\langle u,y\rangle}-1)v(dy)\}^{-1}+u^{-1}.\end{aligned}$$

The n-order differential equation is

$$\frac{\partial^n \tilde{Q}^{LL}(u,w)}{\partial w^n}\bigg|_{w=0} = 0 + (-u^{-1}) w \left[\frac{\partial^n \{w + \langle b^*, u\rangle - \int_{R_+^m}(e^{-\langle u,y\rangle}-1)\nu(dy)\}^{-1}}{\partial w^n}\right]_{w=0}$$

$$-(u^{-1})\left[n\partial^{n-1}\{w+\langle b^*,u\rangle - \frac{\int_{R_+^m}(e^{-\langle u,y\rangle}-1)\nu(dy)\}^{-1}}{\partial w^{n-1}}\right]_{w=0},$$

and

$$\frac{\partial^{n-1}\{w+\langle b^*,u\rangle - \int_{R_+^m}(e^{-\langle u,y\rangle}-1)\nu(dy)\}^{-1}}{\partial w^{n-1}} = (-1)^{n-1}(n-1)![w+\langle b^*,u\rangle - \int_{R_+^m}(e^{-\langle u,y\rangle}-1)\nu(dy)]^{-n}.$$

It can be re-written as

$$\frac{\partial^n \tilde{Q}^{LL}(u,w)}{\partial w^n}\bigg|_{w=0} = (-u^{-1})(-1)^{n-1}(n-1)!\{w+\langle b^*,u\rangle - \int_{R_+^m}(e^{-\langle u,y\rangle}-1)\nu(dy)\}^{-n}_{w=0}$$
$$= (-u^{-1})(-1)^{n-1}(n-1)!\{\langle b^*,u\rangle - \int_{R_+^m}(e^{-\langle u,y\rangle}-1)\nu(dy)\}^{-n}.$$

Finally, from Eq. (11), we can obtain

$$M^L(T_X^n, u) = \tilde{Q}_n^{LL}(u,0) = (-1)^n \frac{\partial^n \tilde{Q}^{LL}(u,w)}{\partial w^n}\bigg|_{w=0} = u^{-1} n!\left\{\langle b^*,u\rangle - \int_{R_+^m}(e^{-\langle u,y\rangle}-1)\nu(dy)\right\}^{-n}.$$

Simulation and numerical examples

To demonstrate our theorems and models, we implement the algorithm by Cont and Tankovs (2004) to simulate the two-dimensional and three-dimensional subordinator series with dependent components. For a simple case to illustrate, the marginal Lévy subordinator is assumed to be a positive α-stable subordinator $PS_\alpha(t)$ that is a type of pure jump Lévy processes based on α-stable Lévy measure.

Two-dimensional Lévy subordinators

In a two-dimensional Lévy subordinator $X_S(t) = \{X_S{}^1(t), X_S{}^2(t)\}$, $X_S{}^1(t)$ and $X_S{}^2(t)$ are two dependent α-stable subordinators represented by their marginal α-stable Lévy measures and the two-dimensional Clayton Lévy copula $C(u_1,u_2,\theta) = (u_1{}^{-\theta} + u_2{}^{-\theta})^{\frac{-1}{\theta}}$, where the strength of dependence is described by the dependence parameter $\theta \in (0,\infty)$. The marginal Lévy measures are α-stable Lévy measures, $\nu(dy) = \frac{\kappa}{\Gamma(1-\kappa)}\frac{1}{y^{\kappa+1}}dy$, $0<\kappa<1$. Based on Eq. (3), the two-dimensional Lévy measure can be described as the second derivative of the Lévy copula

$$\nu(y_1,y_2) = \frac{\partial C(z_1,z_2)}{\partial z_1 \partial z_2}\bigg|_{z_1=F_1(x_1),\, z_2=F_2(x_2)} \nu_1\nu_2$$
$$= (1+\theta)\left[\int_{x_1}^{\infty}\nu_1 dy_1\right]^{-1-\theta}\left[\int_{x2}^{\infty}\nu_2 dy_2\right]^{-1-\theta}\left[\int_{x_1}^{\infty}\nu_1 dy_1{}^{-\theta} + \int_{x2}^{\infty}\nu_2 dy_2{}^{-\theta}\right]^{-\frac{1}{\theta}-2}\nu_1\nu_2,$$

where $\nu_1(dy_1) = \frac{\kappa_1}{\Gamma(1-\kappa_1)}\frac{1}{y_1{}^{\kappa+1}}dy_1$ and $\nu_2(dy_2) = \frac{\kappa_2}{\Gamma(1-\kappa_2)}\frac{1}{y_2{}^{\kappa_2+1}}dy_2$.

Based on Theorem 2, we can derive the Laplace transform of reliability function as

$$R^{LL}(u,w) = u^{-1}\{w+\langle b^*,u\rangle - \int_{R_+^2}(e^{-\langle u,y\rangle}-1)\nu(dy)\}^{-1}$$
$$= (u_1u_2)^{-1}\left\{w + (b_1^*u_1 + b_2^*u_2) - \int_{R_+^2}(e^{-(u_1y_1+u_2y_2)}-1)(1+\theta)\frac{\kappa_1\kappa_2}{y_1y_2}\right.$$
$$(\Gamma(1-\kappa_1)y_1^{\kappa 1})^{1+\theta}(\Gamma(1-\kappa_2)y_2^{\kappa 2})^{1+\theta}$$
$$\left.[(\Gamma(1-\kappa_1)y_1^{\kappa 1})^{\theta} + (\Gamma(1-\kappa_2)y_2^{\kappa 2})^{\theta}]^{-2-\frac{1}{\theta}}dy_1dy_2\right\}^{-1},$$

where w is the Laplace transform of time t. We choose a simple case that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, which means $\kappa_1 = \kappa_2 = 0.5$. When $\theta = 2$, the integral part of the Laplace transform of reliability function can be derived as

$$\int_{R_+^2}(e^{-(u_1y_1+u_2y_2)}-1)(1+\theta)\frac{\kappa_1\kappa_2}{y_1y_2}(\Gamma(1-\kappa_1)y_1^{\kappa 1})^{1+2}$$
$$(\Gamma(1-\kappa_2)y_2^{\kappa 2})^{1+2}[(\Gamma(1-\kappa_1)y_1^{\kappa 1})^2$$
$$+(\Gamma(1-\kappa_2)y_2^{\kappa 2})^2]^{-2-\frac{1}{2}}dy_1dy_2$$
$$= \frac{3}{4\sqrt{\pi}}\int_{R_+^2}(e^{-(u_1y_1+u_2y_2)}-1)(y_1+y_2)^{\frac{-5}{2}}dy_1dy_2$$
$$= -\frac{u_1^{3/2}-u_2^{3/2}}{u_1-u_2},$$

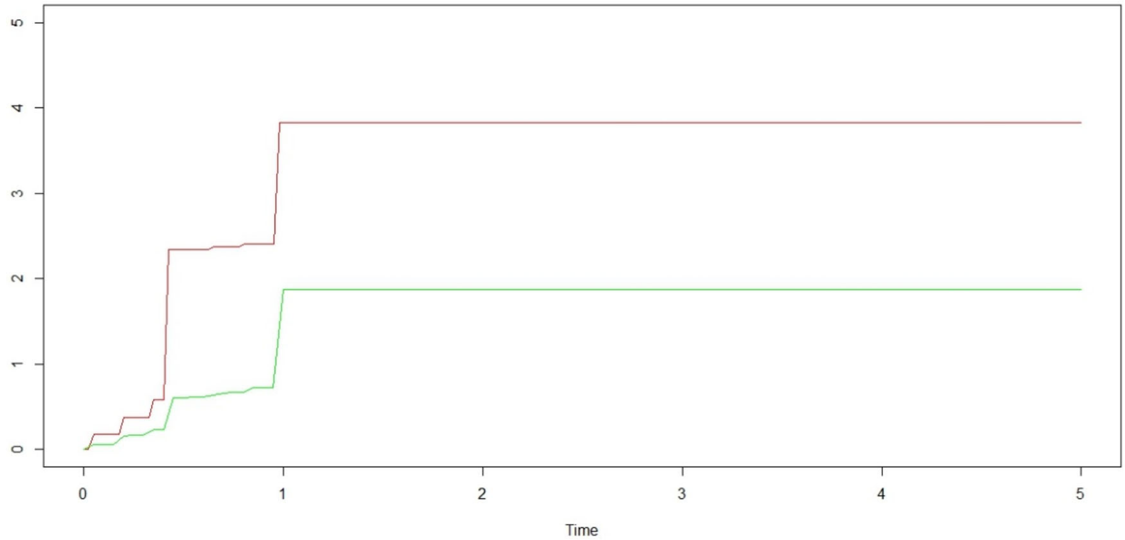

Figure 1. 2-dimensional Lévy subordinators ($\theta = 2$).

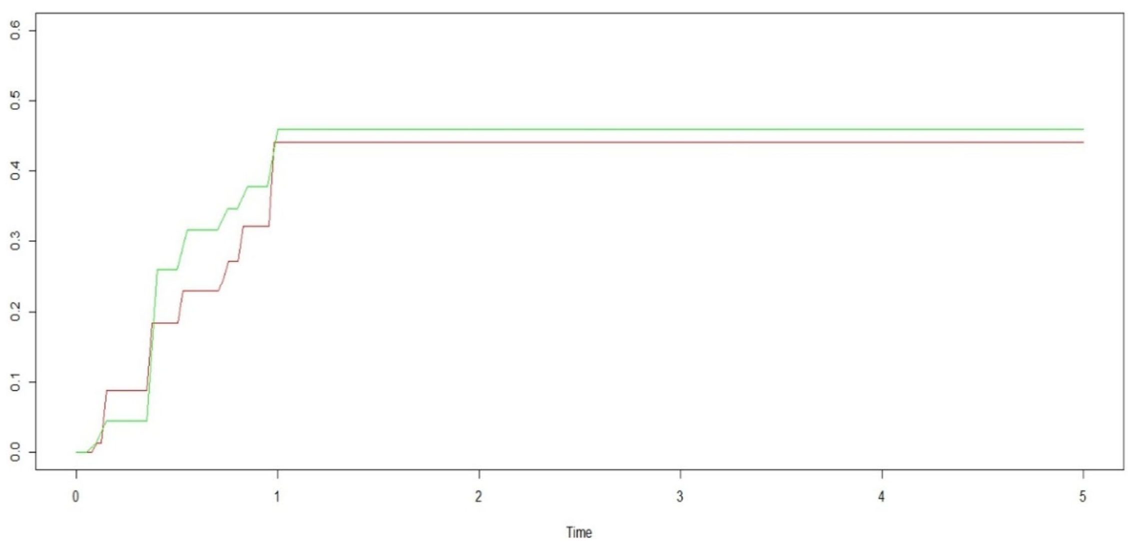

Figure 2. 2-dimensional Lévy subordinators ($\theta = 5$).

where u_1 and u_2 are Laplace transform of two failure thresholds x_1 and x_2.

The simulation results are shown in Figures 1–3. We choose $\theta = 2$, $\theta = 5$ and $\theta = 10$ to represent the relatively weak, medium and strong dependence relationships, respectively. These figures show that a larger dependence parameter θ indicates the stronger dependence between two degradation processes than a smaller dependence parameter does. In addition, the rank correlation coefficients are obtained as 0.333, 0.714, and 0.833 for $\theta = 2$, $\theta = 5$ and $\theta = 10$, respectively, by using the copulastats() function in Matlab.

Figure 4 shows that the first moment of lifetime increases as the thresholds increase on each dimension. For different sets of failure threshold values, Figure 5 illustrates that the Laplace transform of reliability function for 2-D Lévy subordinators decreases monotonously as w increases. When we fix the two failure thresholds, the reliability function for 2-D Lévy subordinators decreases as the time t increases in Figures 6 and 7, with different values of dependent parameters.

Three-dimensional Lévy subordinators

For a three-dimensional Lévy subordinator $X_S(t) = \{X_S{}^1(t),\ X_S{}^2(t),\ X_S{}^3(t)\}$, the Clayton Lévy copula is extended to be $C(u_1, u_2, u_3, \theta) = (u_1{}^{-\theta} + u_2{}^{-\theta} + u_3{}^{-\theta})^{\frac{-1}{\theta}}$. The three-dimensional Lévy measure is the third order derivative of the Lévy copula, which can be derived as

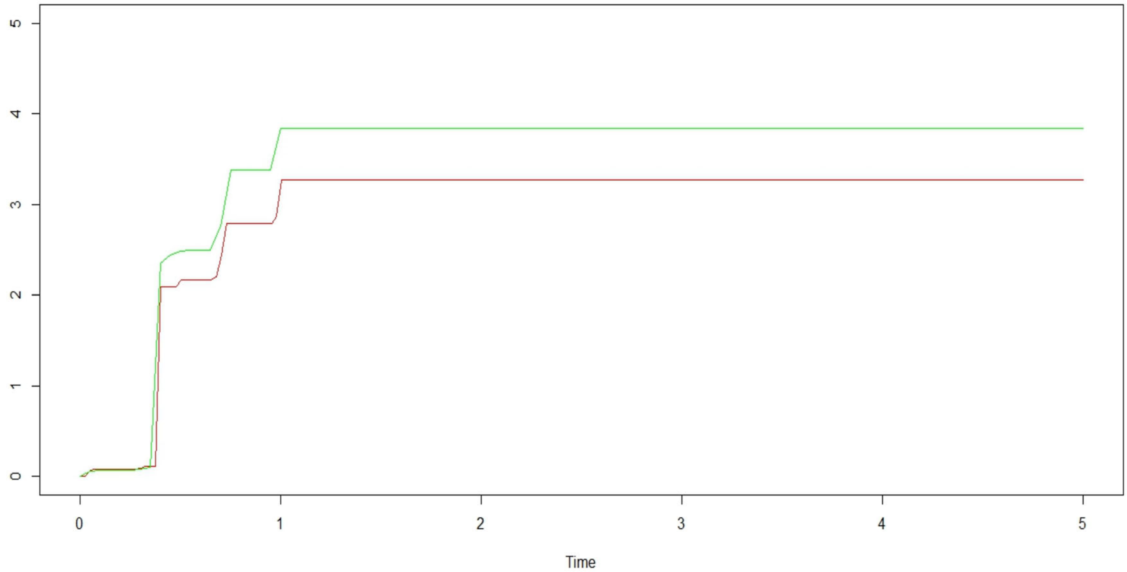

Figure 3. 2-dimensional Lévy subordinators ($\theta = 10$).

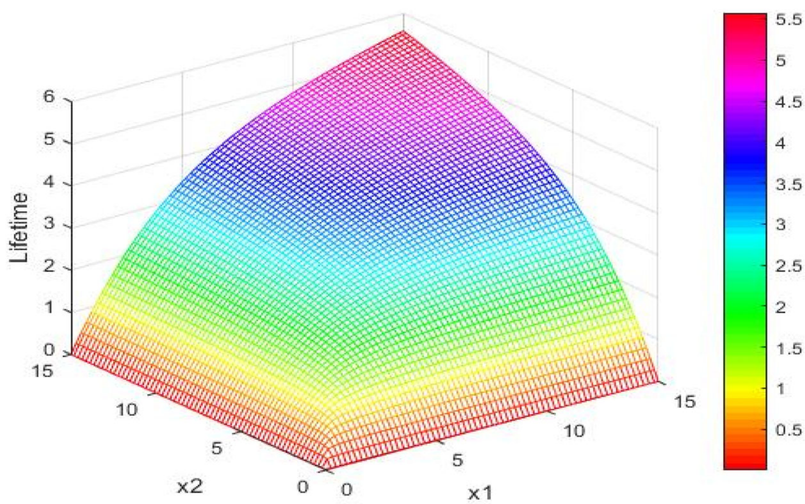

Figure 4. The first moment of lifetime for 2-D Lévy subordinators ($\theta = 2$).

$$
\begin{aligned}
v(y_1, y_2, y_3) &= \frac{\partial^3 C(z_1, z_2, z_3)}{\partial z_1 \partial z_2 \partial z_3}\Big|_{z_1=F_1(x_1),\, z_2=F_2(x_2),\, z_3=F_3(x_3)} v_1 v_2 v_3 \\
&= \frac{\partial^2 (z_1{}^{-\theta} + z_2{}^{-\theta} + z_3{}^{-\theta})^{\frac{-1}{\theta}-1} z_1{}^{-\theta-1}(-\theta)\left(-\frac{1}{\theta}\right)}{\partial z_2 \partial z_3} v_1 v_2 v_3 \\
&= \frac{\partial (z_1{}^{-\theta} + z_2{}^{-\theta} + z_3{}^{-\theta})^{\frac{-1}{\theta}-2} z_1{}^{-\theta-1} z_2{}^{-\theta-1}(-\theta)\left(-\frac{1+\theta}{\theta}\right)}{\partial z_3} v_1 v_2 v_3 \\
&= (1+\theta)(1+2\theta)\left[\int_{x_1}^{\infty} v_1 dy_1\right]^{-1-\theta}\left[\int_{x2}^{\infty} v_2 dy_2\right]^{-1-\theta}\left[\int_{x3}^{\infty} v_3 dy_3\right]^{-1-\theta}\left[\int_{x_1}^{\infty} v_1 dy_1{}^{-\theta} + \int_{x2}^{\infty} v_2 dy_2{}^{-\theta} + \int_{x3}^{\infty} v_3 dy_3{}^{-\theta}\right]^{-\frac{1}{\theta}-3} v_1 v_2 v_3.
\end{aligned}
$$

All marginal Lévy measures v_1, v_2 and v_3 are α-stable Lévy measures $v(dy) = \frac{\kappa}{\Gamma(1-\kappa)}\frac{1}{y^{\kappa+1}}dy$, $0 < \kappa < 1$. The Laplace transform of reliability function for three-dimensional Lévy subordinators can be derived based on Theorem 2:

$$R^{LL}(u,w) = u^{-1}\{w + \langle b^*, u\rangle - \int_{R_+^3}(e^{-\langle u,y\rangle} - 1)v(dy)\}^{-1}$$
$$= (u_1u_2u_3)^{-1}\left\{\begin{array}{l} w + (b_1^*u_1 + b_2^*u_2 + b_3^*u_3) \\ -\int_{R_+^3}(e^{-(u_1y_1+u_2y_2+u_3y_3)} - 1)(1+\theta)(1+2\theta)\frac{\kappa_1\kappa_2\kappa_3}{y_1y_2y_3} \\ (\Gamma(1-\kappa_1)y_1^{\kappa 1})^{\theta}(\Gamma(1-\kappa_2)y_2^{\kappa 2})^{\theta}(\Gamma(1-\kappa_3)y_3^{\kappa 3})^{\theta}[(\Gamma(1-\kappa_1)y_1^{\kappa 1})^{\theta} + (\Gamma(1-\kappa_2)y_2^{\kappa 2})^{\theta} + (\Gamma(1-\kappa_3)y_3^{\kappa 3})^{\theta}]^{-3-\frac{1}{\theta}}dy_1dy_2dy_3 \end{array}\right\}^{-1}.$$

$$R^{LL}(u,w) = (u_1u_2u_3)^{-1}$$
$$\left\{w + (b_1^*u_1 + b_2^*u_2 + b_3^*u_3) - \frac{u_3^{\frac{5}{2}} + \frac{-u_1u_2^{\frac{5}{2}} + u_1^{\frac{5}{2}}(u_2-u_3) + u_2^{\frac{5}{2}}u_3}{u_1-u_2}}{(u_1-u_3)(u_3-u_2)}\right\}^{-1}.$$

Then the Laplace transform of lifetime moments in the three-dimensional case is

$$M(T_X^n, x) = n!(u_1u_2u_3)^{-1}\{\langle b^*, u\rangle - \int_{R_+^3}(e^{-\langle u,y\rangle} - 1)v(dy)\}^{-n}$$
$$= n!(u_1u_2u_3)^{-1}\left\{\begin{array}{l} (b_1^*u_1 + b_2^*u_2 + b_3^*u_3) \\ -\int\limits_{R_+^3}(e^{-(u_1y_1+u_2y_2+u_3y_3)} - 1)(1+\theta)(1+2\theta)\dfrac{\kappa_1\kappa_2\kappa_3}{y_1y_2y_3}(\Gamma(1-\kappa_1)y_1^{\kappa 1})^{\theta}(\Gamma(1-\kappa_2)y_2^{\kappa 2})^{\theta}(\Gamma(1-\kappa_3)y_3^{\kappa 3})^{\theta} \\ \qquad [(\Gamma(1-\kappa_1)y_1^{\kappa 1})^{\theta} + (\Gamma(1-\kappa_2)y_2^{\kappa 2})^{\theta} + (\Gamma(1-\kappa_3)y_3^{\kappa 3})^{\theta}]^{-3-\frac{1}{\theta}}dy_1dy_2dy_3 \end{array}\right\}^{-n}.$$

The Laplace transform of lifetime moments is

$$M(T_X^n, x) = n!(u_1u_2u_3)^{-1}$$
$$\left\{(b_1^*u_1 + b_2^*u_2 + b_3^*u_3) - \frac{u_3^{\frac{5}{2}} + \frac{-u_1u_2^{\frac{5}{2}} + u_1^{\frac{5}{2}}(u_2-u_3) + u_2^{\frac{5}{2}}u_3}{u_1-u_2}}{(u_1-u_3)(u_3-u_2)}\right\}^{-n}.$$

We choose a simple case of $\kappa_1 = \kappa_2 = \kappa_3 = 0.5$ indicating $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\theta = 2$. The complicated integral parts can be obtained as

$$\int\limits_{R_+^3}(e^{-(u_1y_1+u_2y_2+u_3y_3)} - 1)(1+2)(1+2*2)\frac{\frac{1}{2}\frac{1}{2}\frac{1}{2}}{\pi^{\frac{3}{2}}y_1^{\frac{3}{2}}y_2^{\frac{3}{2}}y_3^{\frac{3}{2}}}$$
$$\left(\sqrt{\pi}y_1^{\frac{1}{2}}\right)^3\left(\sqrt{\pi}y_2^{\frac{1}{2}}\right)^3\left(\sqrt{\pi}y_3^{\frac{1}{2}}\right)^3$$
$$\left[\left(\sqrt{\pi}y_1^{\frac{1}{2}}\right)^2 + \left(\sqrt{\pi}y_2^{\frac{1}{2}}\right)^2 + \left(\sqrt{\pi}y_3^{\frac{1}{2}}\right)^2\right]^{-3-\frac{1}{2}}dy_1dy_2dy_3$$
$$= \frac{15}{8\sqrt{\pi}}\int\limits_{R_+^3}(e^{-(u_1y_1+u_2y_2+u_3y_3)} - 1)[y_1 + y_2 + y_3]^{\frac{-7}{2}}dy_1dy_2dy_3$$
$$= \frac{u_3^{\frac{5}{2}} + \frac{-u_1u_2^{\frac{5}{2}} + u_1^{\frac{5}{2}}(u_2-u_3) + u_2^{\frac{5}{2}}u_3}{u_1-u_2}}{(u_1-u_3)(u_3-u_2)}.$$

Therefore, the Laplace transform of the reliability function is

Similar to the two-dimensional case, we choose different values of dependence parameter to simulate the processes. The three-dimensional simulated degradation paths are illustrated in Figure 8. The Laplace transform of reliability function for three-dimensional Lévy subordinators when $\theta = 2$ is given in Figure 9 for different sets of failure threshold values, which shows that the reliability decreases monotonously as w increases. Figures 10 shows that the reliability function of a 3-D Lévy subordinator degradation process decreases with time t under three fixed thresholds.

Conclusions and discussion

In this research, we develop general multi-dimensional Lévy processes governed by Lévy copulas to describe multiple degradation processes by integrally capturing

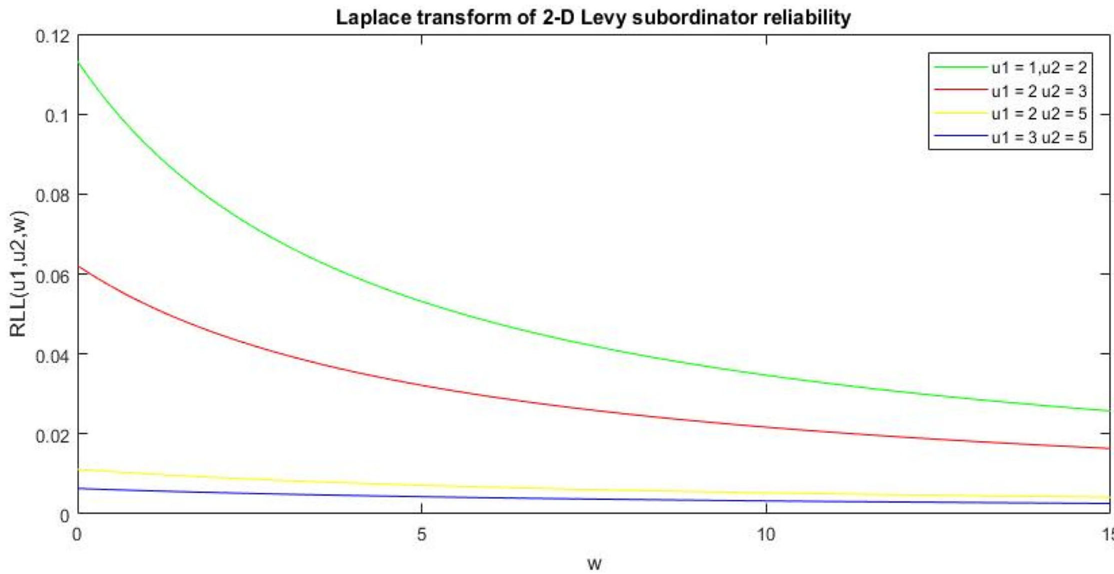

Figure 5. Laplace transform of reliability function for 2-D Lévy subordinators ($\theta = 2$).

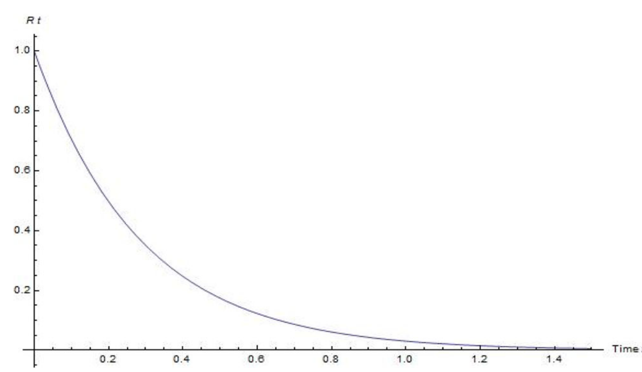

Figure 6. Reliability function for 2-D Lévy subordinators ($\theta = 2$) when u_1=1, u_2=1.

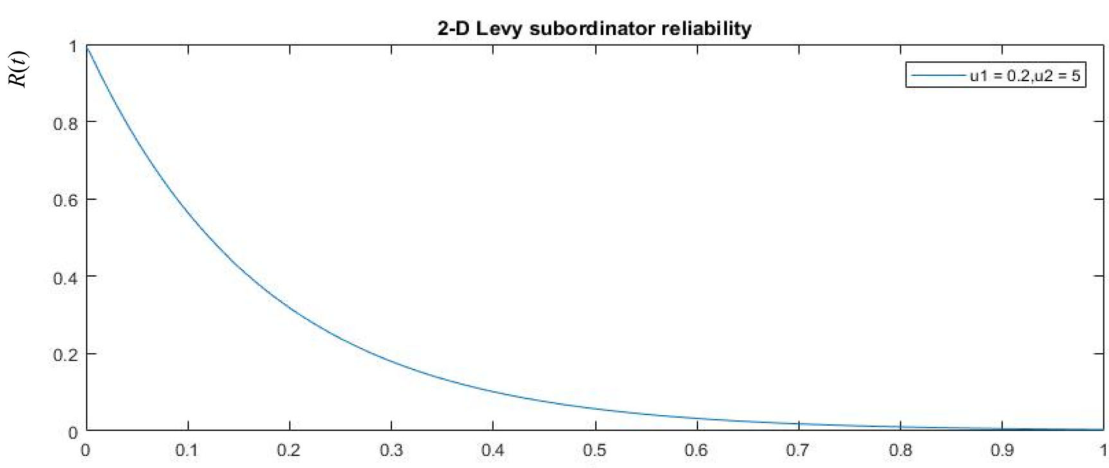

Figure 7. Reliability function for 2-D Lévy subordinators ($\theta = 5$) when u_1=0.2, u_2=5.

the jumps with uncertainties and the dependence among degradation processes in engineering systems. One of the most important advantages of using Lévy processes is that their jump parts represented by Lévy measures can model a great deal of jump mechanisms in degradation, which provides flexibility in fitting

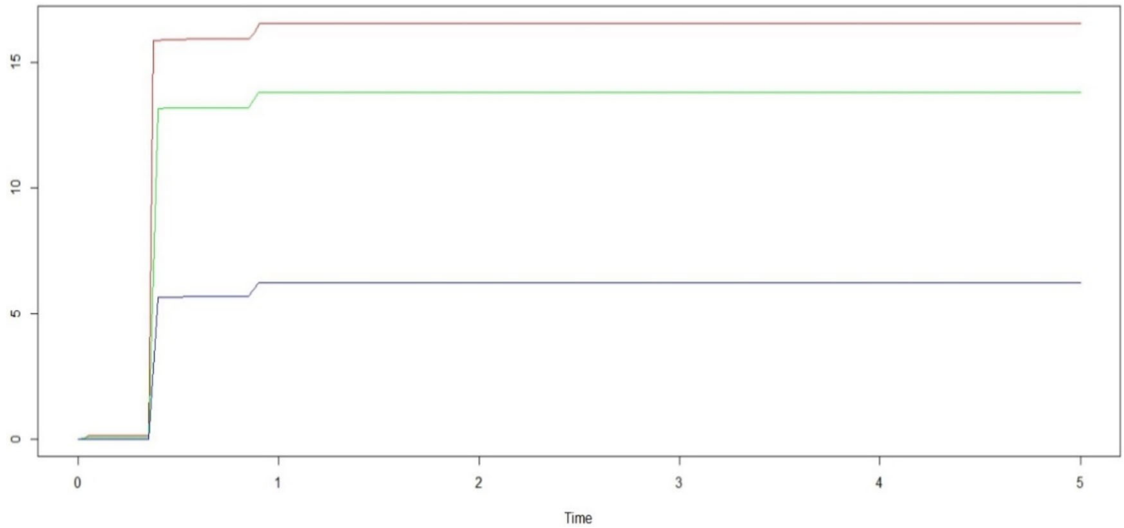

Figure 8. 3-dimensional Lévy subordinators ($\theta = 5$).

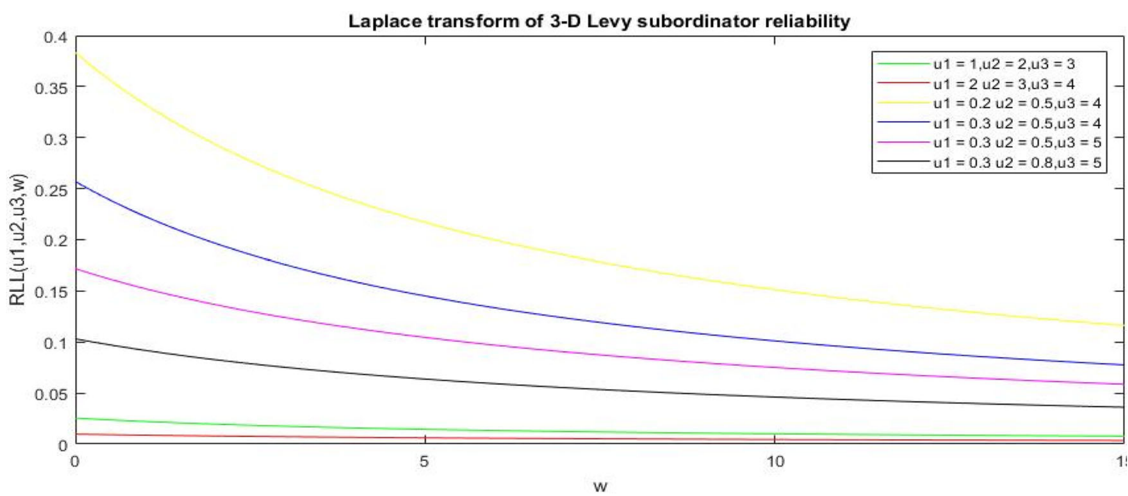

Figure 9. Laplace transform of reliability function for 3-D Lévy subordinators ($\theta = 2$).

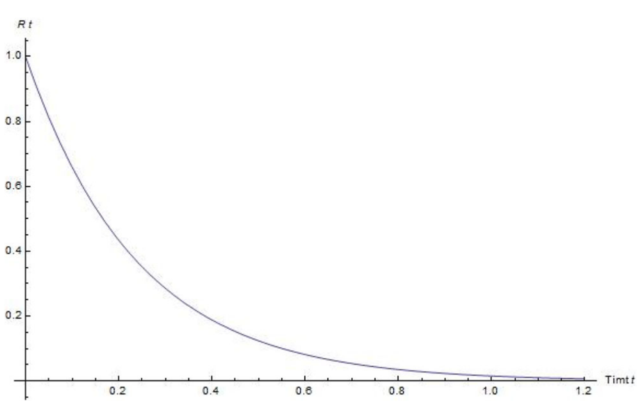

Figure 10. Reliability function for 3-D Lévy subordinators ($\theta = 2$) when u_1=0.2, $u_2 = 2.5$, $u_3 = 2$.

various degradation data series. The multi-dimensional degradation processes are modeled as a multi-dimensional Lévy subordinator and each marginal dimension is a one-dimensional Lévy subordinator. The lifetime characteristics including reliability function and lifetime moments are derived by multi-dimensional Lévy measures and Lévy copulas using Fokker-Planck-Equations. The results provide a framework to model the cumulative degradation and a guideline for enhancing the long-term operation of engineering systems.

A challenge in applying multi-dimensional Lévy subordinators is how to estimate parameters in degradation models and lifetime characteristics. Traditional maximum likelihood estimation and Bayesian estimation are not convenient for such general stochastic processes without closed-form distributions. To apply our models to real degradation datasets, the parametric estimation for subordinators in (Jongbloed and van der Meulen 2006) has been explored for the one-dimensional case (Shu, Feng, and

Liu 2019), where the cumulant M-estimation (CME) method is developed based on the characteristic function of Lévy subordinators. In multi-dimensional degradation processes, the dependence parameters coming from interactions between two or more dimensions need to be considered and estimated. The parameter estimation of multi-dimensional Lévy subordinators is a challenging problem that can be explored based on the one-dimensional CME method and Lévy copulas, which is a potential research direction in this field.

Funding

This paper is based upon work supported by USA National Science Foundation (NSF) under Grant no. 1728321 and Grant no. 1855408. The insightful comments from the editor and the anonymous reviewers are sincerely appreciated. Their constructive suggestions on the earlier versions have significantly improved this paper.

About the authors

Yu Shi is a PhD candidate in the Department of Industrial Engineering at the University of Houston. She received a bachelor's degree in mathematics from Shenzhen University, China and a master's degree in mathematics from Texas A&M University. Her research focuses on using stochastic processes for reliability analysis, and data analytics.

Qianmei Feng is a Professor and the Brij and Sunita Agrawal Faculty Fellow in the Department of Industrial Engineering at the University of Houston. She received a PhD degree in Industrial Engineering from the University of Washington, Seattle, in 2005. She has focused her research on the areas of system modeling, analysis, and optimization in quality and reliability engineering, with applications in evolving technologies (e.g., MEMS, biomedical implant devices), homeland security, and healthcare. Her research has been supported by the National Science Foundation, Department of Homeland Security, Texas Department of Transportation, and Texas Higher Education Coordinating Board. She served as the President for the Division of Quality Control & Reliability Engineering in the Institute of Industrial Engineers between 2009 and 2010. She is also a member of INFORMS, ASQ, and Alpha Pi Mu.

Yin Shu is currently a reliability scientist at Corning Incorporated. He obtained his PhD degree in Industrial Engineering from the University of Houston, Texas in 2016. His research interests are mainly focused on applied probability theory and statistics in Lévy-based stochastic processes, related mathematical methodology, and data analytics, with applications in reliability/safety engineering. He is a member of INFORMS and IIE.

Yisha Xiang is an Assistant Professor in the Department of Industrial, Manufacturing and Systems Engineering at Texas Tech University. She received her BS in Industrial Engineering from Nanjing University of Aeronautics & Astronautics, China, and MS and PhD in Industrial Engineering from the University of Arkansas. Her current research and teaching interests involve reliability modeling and optimization, risk analysis, and healthcare. She has published articles in refereed journals, such as *IIE Transactions*, *European Journal of Operational Research*, and *Computers and Industrial Engineering*. She is a member of IIE and INFORMS.

References

Aas, K., C. Czado, A. Frigessi, and H. Bakken. 2009. Pair-copula constructions of multiple dependence. *Insurance: Mathematics and Economics* 44 (2):182–98. doi:10.1016/j.insmatheco.2007.02.001.

Abdel-Hameed, M. 1984. Life distribution properties of devices subject to a Lévy wear process. *Mathematics of Operations Research* 9 (4):606–14. doi:10.1287/moor.9.4.606.

Barndorff-Nielsen, O. E., and A. M. Lindner. 2007. Lévy copulas: Dynamics and transforms of upsilon-type. *Scandinavian Journal of Statistics* 34 (2):298–316. doi:10.1111/j.1467-9469.2006.00527.x.

Bauerle, N., A. Blatter, and A. Muller. 2008. Dependence properties and comparison results for Lévy processes. *Mathematical Methods of Operations Research* 67 (1):161–86. doi:10.1007/s00186-007-0185-6.

Çinlar, E. 1977. Shock and wear models and markov additive processes. In *Theory and applications of reliability*, eds., I. N. Shimi and C. P. Tsokoseds, 193–214. New York: Academic Press.

Cont, R., and P. Tankov. 2004. *Financial modeling with jump processes*. Boca Raton, FL: Chapman & Hall/CRC.

Embrechts, P., A. Mcneil, and D. Straumann. 2002. Correlation and dependency in risk management: properties and pitfalls. *Risk Management: Value at Risk and Beyond*. 176–223.

Fang, G., R. Pan, and Y. Hong. 2020. Copula-based reliability analysis of degrading systems with dependent failures. *Reliability Engineering & System Safety* 193:106618. doi:10.1016/j.ress.2019.106618.

Hao, H., C. Su, and C. Li. 2015. LED lighting system reliability modeling and inference via random effects gamma process and copula function. *International Journal of Photoenergy* 2015:1–8.

Jaworski, P., F. Durante, W. Hardle, and T. Rychilik. 2009. *Copula theory and its applications*. New York: Springer-Verlag.

Joe, H. 1993. Parametric families of multivariate distributions with given margins. *Journal of Multivariate Analysis* 46 (2):262–82.

Jongbloed, G., and F. H. van der Meulen. 2006. Parametric estimation for subordinators and induced OU processes. *Scandinavian Journal of Statistics* 33 (4):825–47. doi:10.1111/j.1467-9469.2006.00498.x.

Li, D. 1999. On default correlation: A copula function approach. *AIG Asset Management*.

Liu, X., K. N. Al-Khalifa, E. A. Elsayed, D. W. Coit, and A. S. Hamouda. 2014. Criticality measures for components with multi-dimensional degradation. *IIE Transactions* 46 (10):987–98. doi:10.1080/0740817X.2013.851433.

Naifar, N. 2011. Modelling dependence structure with Archimedean Copulas and applications to the iTraxx CDS index. *Journal of Computational and Applied Mathematics* 235 (8):2459–66. doi:10.1016/j.cam.2010.10.047.

Nelsen, R. B. 2006. *An introduction to copulas*. 2nd ed. New York: Springer-Verlag.

Peng, W., Y.-F. Li, J. Mi, L. Yu, and H.-Z. Huang. 2016. Reliability of complex systems under dynamic conditions: A Bayesian multivariate degradation perspective. *Reliability Engineering & System Safety* 153:75–87. doi:10.1016/j.ress.2016.04.005.

Peng, W., Z. Ye, and N. Chen. 2019. Joint online RUL prediction for multivariate deteriorating systems. *IEEE Transactions on Industrial Informatics* 15 (5):2870–8. doi:10.1109/TII.2018.2869429.

Sari, J. K., M. J. Newby, A. C. Brombacher, and L. C. Tang. 2009. Bivariate constant stress degradation model: LED lighting system reliability estimation with two-stage modelling. *Quality and Reliability Engineering International* 25 (8):1067–84. doi:10.1002/qre.1022.

Shu, Y., Q. Feng, and D. W. Coit. 2015. Life distribution analysis based on Lévy subordinators for degradation with random jumps. *Naval Research Logistics (NRL)* 62 (6):483–92. doi:10.1002/nav.21642.

Shu, Y., Q. Feng, and H. Liu. 2019. Using degradation-with-jump measures to estimate life characteristics of lithium-ion battery. *Reliability Engineering & System Safety* 191:106515. doi:10.1016/j.ress.2019.106515.

Shu, Y., Q. Feng, E. Kao, and H. Liu. 2016. Lévy-driven Non-Gaussian Ornstein-Uhlenbeck processes for degradation-based reliability analysis. *IIE Transactions* 48 (11):993–1003. doi:10.1080/0740817X.2016.1172743.

Sklar, A. 1996. Random variables, distribution functions, and copulas - A personal look backward and forward. In *Distributions with Fixed Marginals and Related Topics*, eds., L. Ruschendorf, B. Schweizer, and M. D. Taylor, 1–14. Institute of Mathematical Statistics, Hayward, CA.

Soize, C. 1994. The Fokker-Planck equation for stochastic dynamical systems and its explicit steady state solutions. *Series on Advances in Mathematics for Applied Sciences* 17. 340.

Sun, F., L. Liu, X. Li, and H. Liao. 2016. Stochastic modeling and analysis of multiple nonlinear accelerated degradation processes through information fusion. *Sensors* 16 (8):1242. doi:10.3390/s16081242.

Sun, X., and J. Duan. 2012. Fokker-Planck equations for nonlinear dynamical systems driven by non-Gaussian levy processes. *Journal of Mathematical Physics* 53 (7):072701. doi:10.1063/1.4732102.

van Noortwijk, J. M. 2009. A survey of the application of gamma processes in maintenance. *Reliability Engineering & System Safety* 94 (1):2–21. doi:10.1016/j.ress.2007.03.019.

Wang, X., and D. Xu. 2010. An inverse Gaussian process model for degradation data. *Technometrics* 52 (2):188–97. doi:10.1198/TECH.2009.08197.

Wang, X., N. Balakrishnan, B. Guo, and P. Jiang. 2015. Residual life estimation based on bivariate non-stationary gamma degradation process. *Journal of Statistical Computation and Simulation* 85 (2):405–21.

Wang, Y. C., J. L. Wu, and Y. H. Lai. 2013. A revisit to the dependence structure between the stock and foreign exchange markets: A dependence-switching copula approach. *Journal of Banking & Finance* 37 (5):1706–19. doi:10.1016/j.jbankfin.2013.01.001.

Ye, Z., Y. Wang, K. L. Tsui, and M. Pecht. 2013. Degradation data analysis using wiener processes with measurement errors. *IEEE Transactions on Reliability* 62 (4):772–80.

Appendix: Preliminaries of *m-dimensional* Lévy processes and copulas

In this section, we introduce some fundamental concepts and related properties of Lévy processes and Lévy copulas, considered on R_+^m space.

Definition A1. ***(Cont and Tankovs 2004).*** *An m ($m \geq 1$)-dimensional Lévy process $X(t)$ is a cadlag stochastic process with $X^m(0) = 0$, and satisfies the following properties:*

1. *Independent increments: for a time sequence $t_0, \dots, t_n$, the increment random variables $X(t_0), \dots, X(t_n) - X(t_{n-1})$ are independent.*
2. *Stationary increments: the law of $X(t+h) - X(t)$ does not depend on t.*
3. *Stochastic continuity: $\forall \varepsilon > 0$, $\lim_{h \to 0} P(|X(t+h) - X(t)| \geq \varepsilon) = 0$.*

Based on the independent and stationary properties, we can generate a random walk $S_n(\Delta) = \sum_{k=0}^{n-1} \left(X((k+1)t) - X(kt) \right)$. When $n\Delta = t$, $X(t) = S_n(\Delta)$ can be represented as a sum of n *i.i.d.* parts. A *Lévy subordinator* $X_S(t)$ is a class of Lévy processes that takes values in $[0, \infty)$ with a non-decreasing path.

For a one-dimensional Lévy process $X(t)$, Lévy-Itô decomposition shows that it can be decomposed into four parts: (1) a drift term, bt, (2) a Brownian motion $B_A(t)$ with a covariance matrix A, (3) a jump part $\int_{|y| \geq 1} yJ(t, dy)$ that is a compound Poisson process, and (4) another jump part $\int_{|y|<1} y\left(J(t, dy) - \nu(t, dy)\right)$ that is the compensated version of Poisson process $\int_{0 \leq |y| \leq 1} yJ(t, dy)$. The last part can have finite or infinite activities, where $(J(t, dy) - \nu(t, dy))$ is the compensated Poisson random measure. The Lévy-Itô decomposition can be extended to multi-dimensional Lévy processes.

Lemma A1. ***(Multi-dimensional Lévy-Itô Decomposition).*** *If $X(t)$ is an m-dimensional Lévy process, then there are $a, b \in R^m$, a Brownian motion B_A with a covariance matrix A, and an independent Poisson random measure J on $(0,\infty) \times R^m$. For each $t \geq 0$,*

$$X(t) = bt + B_A(t) + \int_{|y|<1} y\left(J(t, dy) - \nu(t, dy)\right) + \int_{|y| \geq 1} yJ(t, dy),$$

where $\nu(t,dy)$ is an m-dimensional Lévy measure that satisfies $\nu(\{0\}) = 0$ and $\int_{R^m} \min\{1, |y|^2\} \nu(dy) < \infty$. When the covariance matrix $A = 0$ and the drift $b \geq 0$, Lévy process $X(t)$ has a non-decreasing path, i.e., a Lévy subordinator $X_S(t)$.

To analyze the dependence structure of a multi-dimensional Lévy subordinator $X_S(t) = \{X_S{}^1(t), \ldots, X_S{}^m(t)\}$ where m is the number of dimensions, the concepts of copulas and sub-copulas need to be introduced.

Definition A2. *(*Nelsen 2006*). An m-dimensional sub-copula is a function C' with the following properties:*

1. *Dom C'=$S_1 \times S_2 \times \cdots \ldots \times S_m$, where each S_k is a subset of I= [0, 1];*
2. *C' is grounded and m-increasing;*
3. *C' has one-dimensional margins C_k', $k = 1, 2, \ldots, m$, which satisfy C_k'(u) = u for all u in S_k .*

Definition A3. *(*Nelsen 2006*). An m-dimensional copula C is an m-dimensional sub-copula whose domain is I^m.*

For an m-dimensional copula C $(m \geq 3)$, every k dimensions in C is a k-dimensional sub-copula C' and there are $\binom{m}{k}$ k-dimensional sub-copulas for C in total, where $2 \leq k \leq m$.