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Maximally highly proximal flows

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Abstract. For G a Polish group, we consider G-flows which either contain a comeager orbit or have all orbits meager. We single out a class of flows, the *maximally highly proximal* (MHP) flows, for which this analysis is particularly nice. In the former case, we provide a complete structure theorem for flows containing comeager orbits, generalizing theorems of Melleray, Nguyen Van Thé, and Tsankov and of Ben Yaacov, Melleray, and Tsankov. In the latter, we show that any minimal MHP flow with all orbits meager has a metrizable factor with all orbits meager, thus 'reflecting' complicated dynamical behavior to metrizable flows. We then apply this to obtain a structure theorem for Polish groups whose universal minimal flow is distal.

Key words: topological dynamics, comeager orbits, highly proximal extensions
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1. *Introduction*

Let G be a Polish group. A *G-flow* is a compact Hausdorff space equipped with a continuous (right) G-action $X \times G \to X$. If X and Y are G-flows, a map $\varphi\colon X \to Y$ is a *G-map* if φ is continuous and respects the G-actions. A *subflow* of a G-flow X is any non-empty closed invariant subspace $Y \subseteq X$. We say X is *minimal* if the only subflow of X is X itself. Equivalently, X is minimal if for every $x \in X$, the orbit $x \cdot G \subseteq X$ is dense. Notice that if $\varphi\colon X \to Y$ is a G-map, then the image $\varphi[X] \subseteq Y$ is a subflow; if X is minimal, so is $\varphi[X]$, and if Y is minimal, then φ is surjective. We often call a surjective G-map a *factor*.

By a classical theorem of Ellis, there is a *universal minimal flow* $M(G)$; this is a minimal G-flow which admits a G-map onto any other minimal G-flow, and $M(G)$ is unique up to isomorphism. The study of $M(G)$ is useful because it captures information about all minimal G-flows. For instance, if $M(G)$ is metrizable, then every minimal G-flow is metrizable, and if $M(G)$ has a (necessarily unique) comeager orbit, then so does every minimal G-flow [**1**]. However, $M(G)$ is often very complicated; for example, if G is

locally compact, then $M(G)$ is never metrizable, and all of its orbits are meager. However, there are Polish groups G for which $M(G)$ is a singleton, and many others for which $M(G)$ is metrizable and has a concrete description. See [**9**] for several examples of these phenomena.

The starting point of this paper is the following fact, first proved by the author [**12**] in the case that G is non-Archimedean, and then by Ben Yaacov, Melleray, and Tsankov [**7**] for general Polish groups.

Fact 1.1. If G is a Polish group and $M(G)$ is metrizable, then $M(G)$ has a comeager orbit.

This fact, along with the structure theorem due to Melleray, Nguyen Van Thé, and Tsankov [**11**], provides a complete understanding of the structure of $M(G)$ when it is metrizable. However, the property of $M(G)$ having a comeager orbit remained less well understood. Indeed, it was only recently shown, by an example of Kwiatkowska [**10**], that the converse of Fact 1.1 does not hold.

The study of $M(G)$ is often undertaken by attempting to understand the *Samuel compactification* $\mathrm{Sa}(G)$, the Gelfand space of the bounded left uniformly continuous functions on G. The group G canonically embeds into $\mathrm{Sa}(G)$, and for any G-flow X and any $x \in X$, there is a unique G-map $\lambda_x : \mathrm{Sa}(G) \to X$ with $\lambda_x(1_G) = x$. In particular, any minimal subflow of $\mathrm{Sa}(G)$ is isomorphic to $M(G)$. The main technical tool introduced in [**7**] is to view $\mathrm{Sa}(G)$ as a *topometric space*, a topological space endowed with a possibly finer metric ∂ which interacts with the topology in nice ways. Letting ∂ denote this finer metric, the authors of [**7**] show that if $M \subseteq \mathrm{Sa}(G)$ is a compact metrizable subspace, then $\partial|_M$ is a compatible metric. When the metrizable $M \subseteq \mathrm{Sa}(G)$ is a minimal subflow, the properties of the metric $\partial|_M$ allow them to show that M has a comeager orbit. However, much remained unclear about this metric, especially when $M(G)$ is non-metrizable. Namely, if $M \subseteq \mathrm{Sa}(G)$ is a minimal subflow, can we define $\partial|_M$ just using the dynamics of M?

This paper singles out a class of flows, called *maximally highly proximal flows* (MHP), which all admit a canonical topometric structure. In particular, $M(G)$ and $\mathrm{Sa}(G)$ are both MHP, and the topometric on $M(G)$ agrees with the metric inherited by any minimal subflow of $\mathrm{Sa}(G)$. Using this topometric structure, we provide a structure theorem for MHP flows with a comeager orbit. Here, a *compatibility point* is a point in X where the topology and the metric coincide (see Definition 5.1).

THEOREM 5.5. *Let X be an MHP flow. The following are equivalent.*

(1) *X has a compatibility point with dense orbit.*
(2) *The set $Y \subseteq X$ of compatibility points is comeager, Polish, and contains a point with dense orbit.*
(3) *X has a comeager orbit.*
(4) *$X \cong \mathrm{Sa}(H\backslash G)$ for some closed subgroup $H \subseteq G$ (see §3.2.1).*

In Theorem 7.5, we generalize the main result of [**11**] by considering the case where $X \in \{M(G), \Pi(G), \Pi_s(G)\}$, where $\Pi(G)$ and $\Pi_s(G)$ are the *universal minimal proximal flow* and the *Furstenberg boundary*, respectively. In the first and third case, we show that the closed subgroup H appearing in item (4) is extremely amenable or amenable, respectively,

and in the second case we present a partial result towards showing that H is strongly amenable.

As an application of Theorem 5.5, we prove the following 'reflection' theorem, which shows that complicated dynamical behavior of the group G already appears in the realm of metrizable flows. Note that in minimal flows, all orbits are either meager or comeager.

THEOREM 8.1. *Let X be a minimal MHP flow all of whose orbits are meager. Then there is a factor $\varphi\colon X \to Y$ such that Y is metrizable and also has all orbits meager.*

This theorem was first suggested in [**7**], but in private communication with the authors, it was realized that the problem remained open.

As an application of Theorem 8.1, we give a complete characterization of when $M(G)$ is distal in Theorem 9.2 and Corollary 9.3. The theorem says that if $M(G)$ is distal, then $M(G)$ is metrizable. Then using results from [**11**], the corollary shows that any such G has a normal, extremely amenable subgroup H with $M(G) \cong H\backslash G$.

Notation. We will use some non-standard notation. The phrases 'non-empty open subset of', 'open neighborhood of', etc. occur often enough that we introduce some notation for this. If X is a topological space, then $A \subseteq_{\mathrm{op}} X$ will mean that A is a non-empty open subset of X. If $x \in X$, we write $x \in_{\mathrm{op}} A$ or $A \ni_{\mathrm{op}} x$ to mean that $A \subseteq X$ is an open neighborhood of x. Omitting the 'op' subscript does not mean that a given set is not open; it is just an easy way to introduce and/or emphasize open sets.

Other notation is mostly standard. We write $\omega = \{0, 1, 2, \ldots\}$, and we identify a non-negative integer with the set of its predecessors, that is, $n = \{0, \ldots, n-1\}$. If $f\colon X \to Y$ is a function and $K \subseteq X$, we set $f[K] := \{f(x) : x \in K\}$. All topological spaces we consider are Hausdorff.

2. *Topometric spaces*

This short section collects the background material on topometric spaces that we will need going forward. Most of the material here can be found in [**4**] or [**6**].

Definition 2.1. A *compact topometric space* is a triple (X, τ, ∂), where (X, τ) is a compact Hausdorff space and ∂ is a metric which is *τ-lower semi-continuous* (or τ-lsc), meaning that for every $c \geq 0$, the set $\{(p, q) \in X^2 : \partial(p, q) \leq c\}$ is $(\tau \times \tau)$-closed.

Note that the metric need not agree with the underlying topology. As a convention, when discussing a topometric space, topological vocabulary will refer to τ, while metric vocabulary will refer to ∂.

Fact 2.2. Let (X, τ, ∂) be a compact topometric space.
(1) The metric ∂ is finer than the topology.
(2) The metric ∂ is complete.

Remark. One can also define topometric spaces where the underlying topological space is not compact. One then includes item (1) above in the definition.

The following fact will be needed going forward.

Fact 2.3. (Ben Yaacov [**4**]) Let (X, τ, ∂) be a compact topometric space. Then if K, $L \subseteq X$ are closed with $\partial(K, L) > r$, then there is a continuous, 1-Lipschitz function $f\colon X \to [0, 1]$ with $f[K] = \{0\}$ and $f[L] = \{r\}$.

If (X, τ, ∂) is a compact topometric space, $K \subseteq X$, and $c > 0$, we define $K(c) := \{p \in X : \partial(p, K) < c\}$ and $K[c] := \{p \in X : \partial(p, K) \leq c\}$. If $K = \{p\}$ for some $p \in X$, we just write $p(c)$ or $p[c]$, respectively.

Definition 2.4. [**6**, Definition 1.25] A topometric space (X, τ, ∂) is called *adequate* if for every open $A \subseteq X$ and every $c > 0$, we have $A(c)$ open.

We will prove (see Theorem 4.8) that the topometric spaces we consider in this paper are all adequate.

3. *Maximally highly proximal flows*

Throughout this section, G will denote a fixed Polish group. We let d_G denote a compatible left-invariant metric of diameter 1, and for $c > 0$, we set $U_c := \{g \in G : d_G(1_G, g) < c\}$. We will frequently and without explicit mention make use of the inclusion $U_c U_\epsilon \subseteq U_{c+\epsilon}$.

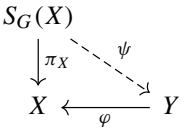
Definition 3.1. Let X be a G-flow. We say that X is *maximally highly proximal* if for every $A \subseteq_{\text{op}} X$, every $x \in \overline{A}$, and every $c > 0$, we have $x \in \text{int}(\overline{AU_c})$.

3.1. *Highly proximal extensions.* The name MHP comes from the notion of a *highly proximal* extension. If $\varphi\colon Y \to X$ is a surjective G-map, we define the *fiber image* of $B \subseteq_{\text{op}} Y$ to be $\varphi_{\text{fib}}(B) := \{x \in X : \varphi^{-1}(\{x\}) \subseteq B\}$. The set $\varphi_{\text{fib}}(B)$ is always open whenever $B \subseteq_{\text{op}} Y$, but possibly empty. We call φ *highly proximal* if $\varphi_{\text{fib}}(B) \neq \emptyset$ for every $B \subseteq_{\text{op}} Y$. The composition of highly proximal maps is also highly proximal. Also notice that if X is minimal and $\varphi\colon Y \to X$ is highly proximal, then Y is also minimal. More precisely, if X is any G-flow and $x \in X$ has dense orbit, then if $\varphi\colon Y \to X$ is any highly proximal extension, then any $y \in \varphi^{-1}(\{x\})$ also has dense orbit.

To motivate why this notion is named 'highly proximal', it is helpful to compare it to the notion of a proximal extension. A G-map $\varphi\colon Y \to X$ is called *proximal* if for any $y_0, y_1 \in Y$ with $\varphi(y_0) = \varphi(y_1)$, we can find a net g_i from G and $z \in Y$ with $\lim y_0 g_i = \lim y_1 g_i = z$. Now suppose that X is minimal and that $\varphi\colon Y \to X$ is highly proximal. Then φ is proximal. To see this, let $y_0, y_1 \in Y$ with $\varphi(y_0) = \varphi(y_1) = x$. Fix any $z \in Y$, and let $\{B_i : i \in I\}$ be a base of neighborhoods of z. For each B_i, we have $\varphi_{\text{fib}}(B_i) := A_i \neq \emptyset$. By minimality, let $g_i \in G$ be such that $xg_i \in A_i$. Then we see that $\lim yg_i = z$ for any $y \in \varphi^{-1}(\{x\})$, so in particular for y_0 and y_1. In fact, this is historically the definition of a highly proximal extension.

Fact 3.2. [**3**, p. 733] Let X be a minimal flow. Then the extension $\varphi\colon Y \to X$ is highly proximal if and only if for any $x \in X$, there exist a net $g_i \in G$ and a point $y \in Y$ with $\varphi^{-1}(\{xg_i\}) \to \{y\}$, that is, for any $B \ni_{\text{op}} y$, we eventually have $\varphi^{-1}(xg_i) \subseteq B_i$.

In the case where X is a minimal flow, Auslander and Glasner [**3**] prove the existence and uniqueness of a *universal highly proximal extension*; this is a highly proximal G-map $\pi_X\colon S_G(X) \to X$ such that for any other highly proximal $\varphi\colon Y \to X$, there is a G-map $\psi\colon S_G(X) \to Y$ with $\pi_X = \varphi \circ \psi$.



Such a ψ is necessarily also highly proximal.

The notion of a universal highly proximal extension was generalized to any G-flow in [**13**], where an explicit construction is given. We briefly review this construction here, referring to [**13**] for all proofs.

Definition 3.3. Fix a G-flow X, and write $\mathrm{op}(X) := \{A : A \subseteq_{\mathrm{op}} X\}$. A collection $p \subseteq \mathrm{op}(X)$ is called a *near ultrafilter* if the following statements hold.

(1) For every $k < \omega$, $A_0, \ldots, A_{k-1} \in p$, and $c > 0$, we have $\bigcap_{i<k} A_i U_c \neq \emptyset$. We call this property the *near finite intersection property* (NFIP).

(2) p is maximal with respect to satisfying item (1).

Let $S_G(X)$ denote the collection of near ultrafilters on $\mathrm{op}(X)$. For $A \subseteq_{\mathrm{op}} X$, we set $C_A = \{p \in S_G(X) : A \in p\}$ and $N_A = \{p \in S_G(X) : A \notin p\}$. We endow $S_G(X)$ with a compact Hausdorff topology given by the base $\{N_A := A \subseteq_{\mathrm{op}} X\}$. For $p \in S_G(X)$, a base of (not necessarily open) neighborhoods of p is given by $\{C_{AU_\epsilon} : A \in p, \epsilon > 0\}$. The group G acts on $S_G(X)$ in the obvious way, where $A \in pg$ if and only if $Ag^{-1} \in p$. We also have a canonical G-map $\pi_X \colon S_G(X) \to X$, where $\pi_X(p) = x$ if and only if for every $A \ni_{\mathrm{op}} x$, we have $A \in p$.

Fact 3.4. $\pi_X \colon S_G(X) \to X$ is the universal highly proximal extension of X.

In particular, the map $\pi_{S_G(X)} \colon S_G(S_G(X)) \to S_G(X)$ is an isomorphism. The construction of the space of near ultrafilters in fact works on any *G-space*, where the underlying space X need not be compact. While in this generality we do not get the map π_X, we will still refer to the universal highly proximal extension of the G-space X, and the construction will still be idempotent. A remark that will be useful later is that if $Y \subseteq X$ is a dense G-invariant subspace of a G-space X, then $S_G(X)$ and $S_G(Y)$ coincide.

PROPOSITION 3.5. *The G-flow X is MHP if and only if the universal highly proximal extension $\pi_X \colon S_G(X) \to X$ is an isomorphism.*

Proof. First let X be any G-flow. Fix $p \in S_G(X)$, and set $x = \pi_X(p)$. Then we must have $p \subseteq \mathcal{F}_x := \{A \subseteq_{\mathrm{op}} X : x \in \overline{A}\}$. To see why, if $x \notin \overline{A}$, we can find $B \ni_{\mathrm{op}} x$ and $c > 0$ with $AU_c \cap BU_c = \emptyset$. As $B \in p$ by definition of the map π_X, we cannot have $A \in p$.

Now suppose the G-flow X is MHP. Then for every $x \in X$, we have that $\mathcal{F}_x$ has the NFIP, so is a near ultrafilter. It follows that if $p \in S_G(X)$ with $\pi_X(p) = x$, then we in fact have $p = \mathcal{F}_x$. In particular, the map π_X is injective, hence an isomorphism.

Conversely, suppose X is not MHP. Find some $x \in X$, $B \subseteq_{\mathrm{op}} X$ with $x \in \overline{B}$, and $c > 0$ with $x \notin \mathrm{int}(\overline{BU_c})$. Setting $C = X \setminus \overline{BU_c}$, we have $x \in \overline{C}$. Notice that $BU_{c/2} \cap CU_{c/2} = \emptyset$, so B and C can never belong to the same near ultrafilter. Set $\mathcal{G}_x := \{A \subseteq_{\mathrm{op}} X : x \in A\}$. Let $p \in S_G(X)$ extend $\mathcal{G}_x \cup \{B\}$, and let $q \in S_G(X)$ extend $\mathcal{G}_x \cup \{C\}$. Then $p \neq q$ and $\pi_X(p) = \pi_X(q) = x$. □

3.2. *Examples of MHP flows.* We now collect some examples of MHP flows. Of course, the universal highly proximal extension of any G-space is an MHP flow, but it will be useful to have some explicit examples in mind.

3.2.1. *Samuel compactifications.* Let $H \subseteq G$ be a closed subgroup, and let $H\backslash G$ denote the right coset space. We equip $H\backslash G$ with the metric that it inherits from G, which we also denote by d_G. Explicitly, if $Hg \in H\backslash G$, the ball of radius $\epsilon > 0$ around Hg is given by HgU_ϵ. Then the *Samuel compactification* $\mathrm{Sa}(H\backslash G)$ is the Gelfand space of the bounded uniformly continuous functions on $H\backslash G$. It is a G-flow characterized by the property that for any G-flow Y containing a point $y \in Y$ with $y \cdot h = y_0$ for every $h \in H$, then there is a (necessarily unique) G-map $\varphi\colon \mathrm{Sa}(H\backslash G) \to Y$ with $\varphi(H) = y$. In the case $H = \{1_G\}$, we often write $yp := \varphi(p)$. We identify $H\backslash G$ with its image under the canonical embedding $i\colon H\backslash G \hookrightarrow \mathrm{Sa}(H\backslash G)$.

To see that $\mathrm{Sa}(H\backslash G)$ is MHP, suppose $\psi\colon X \to \mathrm{Sa}(H\backslash G)$ is highly proximal. Using the universal property of $\mathrm{Sa}(H\backslash G)$, it is enough to show that $\psi^{-1}(\{H\})$ is a singleton. First note that for any $x \in \psi^{-1}(\{H\})$ and any $A \ni_{\mathrm{op}} x$, we have $H \in \overline{\psi_{\mathrm{fib}}(A)}$. In particular, since $\psi_{\mathrm{fib}}(A)$ is open, we can for any $\epsilon > 0$ find $Hg \in (H\backslash G) \cap \psi_{\mathrm{fib}}(A)$ with $d_G(Hg, H) < \epsilon$. Now if $x \neq y \in X$ satisfies $\psi(x) = \psi(y) = H$, we can find $A \ni_{\mathrm{op}} x$, $B \ni_{\mathrm{op}} y$, and $\epsilon > 0$ with $\overline{AU_\epsilon} \cap \overline{BU_\epsilon} = \emptyset$. This implies that $\overline{\psi_{\mathrm{fib}}(A)U_\epsilon} \cap \overline{\psi_{\mathrm{fib}}(B)U_\epsilon} = \emptyset$, a contradiction as H is a member of this intersection.

In particular, by taking $H = \{1_G\}$, we see that $\mathrm{Sa}(G)$ is MHP. We also have that $M(G)$ is MHP. There are two ways of seeing this. One is that $S_G(M(G))$ is a minimal flow mapping onto $M(G)$, so by uniqueness of $M(G)$ we have that $\pi_{M(G)}\colon S_G(M(G)) \to M(G)$ is an isomorphism. The other way is to note that $M(G)$ is a retract of $\mathrm{Sa}(G)$ and observe that retracts of MHP flows are also MHP.

Also notice that since $H\backslash G$ is a dense G-invariant subspace of $\mathrm{Sa}(H\backslash G)$, we have, by the remark after Fact 3.4, that $\mathrm{Sa}(H\backslash G) \cong S_G(H\backslash G)$. When viewing $\mathrm{Sa}(H\backslash G)$ as a space of near ultrafilters, the following fact will be useful to keep in mind (see [**14**, Ch. 1]).

Fact 3.6. If X is a compact space and $f\colon H\backslash G \to X$ is a uniformly continuous function, then the unique continuous extension $f\colon \mathrm{Sa}(H\backslash G) \to X$ is defined by setting, for $p \in \mathrm{Sa}(H\backslash G)$ and $x \in X$, $f(p) = x$ if and only if $\{f^{-1}(U) : U \ni_{\mathrm{op}} x\} \subseteq p$. Given $p \in \mathrm{Sa}(H\backslash G)$, the existence of an $x \in X$ with this property is an easy consequence of compactness; the uniqueness of such an x requires the uniform continuity of f.

3.2.2. *Fraïssé expansion classes.* This example will not be needed in later sections and assumes some familiarity with Fraïssé theory and expansion classes (see [**9**] or [**12**]). Suppose L is a countable language and $G = \mathrm{Aut}(\mathbf{K})$ for some Fraïssé L-structure $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$ with underlying set ω. Let $\mathrm{Fin}(\mathbf{K})$ denote the collection of finite substructures of $\mathbf{K}$. Let $\mathcal{K}^*$ be a reasonable precompact expansion of $\mathcal{K}$ in a countable language $L^* \supseteq L$. Let X_{L^*} denote the space of L^*-structures on ω endowed with the logic topology. We can endow X_{L^*} with a continuous G-action, where for a structure $x \in X_{L^*}$, a relational symbol $R \in L^*$ of arity n, points $a_0, \ldots, a_{n-1} \in \omega$, and $g \in G$, we have

$$R^{x \cdot g}(a_0, \ldots, a_{n-1}) \Leftrightarrow R^x(ga_0, \ldots, ga_{n-1}).$$

The definition is similar for function and constant symbols. We then form the G-flow

$$X_{\mathcal{K}^*} = \{\mathbf{K}^* \in X_{L^*} : \mathbf{K}^*|_L = \mathbf{K} \text{ and } \mathbf{K}^*|_{\mathbf{A}} \in \mathcal{K}^* \text{ for every } \mathbf{A} \in \mathrm{Fin}(\mathbf{K})\}.$$

For $\mathbf{A} \in \mathrm{Fin}(\mathbf{K})$ and an expansion $\mathbf{A}^* \in \mathcal{K}^*$, a typical basic clopen neighborhood of $X_{\mathcal{K}^*}$ is given by

$$N_{\mathbf{A}^*} = \{\mathbf{K}^* \in X_{\mathcal{K}^*} : \mathbf{K}^*|_{\mathbf{A}} = \mathbf{A}^*\}.$$

PROPOSITION 3.7. *Suppose $\mathcal{K}^*$ has the amalgamation property (AP). Then $X_{\mathcal{K}^*}$ is MHP.*

Proof. For $\mathbf{A} \in \mathrm{Fin}(\mathbf{K})$, write $U_{\mathbf{A}} \subseteq G$ for the pointwise stabilizer of $\mathbf{A}$. Then $U_{\mathbf{A}} \subseteq G$ is a clopen subgroup and a typical basic open neighborhood of $1_G \in G$. Let $W \subseteq X_{\mathcal{K}^*}$ be open. It suffices to show that $\overline{WU_{\mathbf{A}}}$ is clopen. To that end, we will show that for any $\mathbf{B} \in \mathrm{Fin}(\mathbf{K})$ with $\mathbf{A} \subseteq \mathbf{B}$ and any expansion $\mathbf{B}^* \in \mathcal{K}^*$, we have $\overline{N_{\mathbf{B}^*}U_{\mathbf{A}}} = N_{\mathbf{A}^*}$, where $\mathbf{A}^*$ is the expansion of $\mathbf{A}$ inherited from $\mathbf{B}^*$. The left-to-right inclusion is clear. For the reverse, suppose $\mathbf{C} \in \mathrm{Fin}(\mathbf{K})$ is finite and $\mathbf{C}^*$ is an expansion so that $N_{\mathbf{C}^*} \subseteq N_{\mathbf{A}^*}$. By shrinking $N_{\mathbf{C}^*}$ if necessary, we may assume that $\mathbf{A}^* \subseteq \mathbf{C}^*$. Using the AP in $\mathcal{K}^*$, we can find $\mathbf{D} \in \mathrm{Fin}(\mathbf{K})$ and an expansion $\mathbf{D}^*$ so that $\mathbf{C}^* \subseteq \mathbf{D}^*$ and $f[\mathbf{B}^*] \subseteq \mathbf{D}^*$ for some $f \in \mathrm{Emb}(\mathbf{B}^*, \mathbf{D}^*)$ with $f|_{\mathbf{A}} = 1_{\mathbf{A}}$. If $g \in G$ satisfies $g|_{\mathbf{B}} = f$, then $N_{\mathbf{B}^*} \cdot g^{-1} \cap N_{\mathbf{C}^*} \supseteq N_{\mathbf{D}^*}$, so is non-empty as desired. □

We can provide a converse result as follows. Recall that a G-flow X is *topologically transitive* if for every $A, B \subseteq_{\mathrm{op}} X$, there is $g \in G$ with $Ag \cap B \neq \emptyset$.

PROPOSITION 3.8. *Suppose X is a metrizable MHP G-flow. Then there is an reasonable, precompact expansion class $\mathcal{K}^*$ with the AP so that $X \cong X_{\mathcal{K}^*}$. If X is also topologically transitive, then we can take the class $\mathcal{K}^*$ to be Fraïssé.*

Proof. Let $\mathbf{A} \in \mathrm{Fin}(\mathbf{K})$. Then if $W \subseteq X$ is open, the equality $\overline{W \cdot U_{\mathbf{A}}} \cdot U_{\mathbf{A}} = \overline{W \cdot U_{\mathbf{A}}}$ and MHP show that $\overline{W \cdot U_{\mathbf{A}}}$ is clopen. Call a clopen set $Y \subseteq X$ $U_{\mathbf{A}}$*-clopen* if $Y \cdot U_{\mathbf{A}} = Y$; the collection $\mathcal{B}(\mathbf{A})$ of $U_{\mathbf{A}}$-clopen sets forms an algebra.

Suppose $\mathcal{B}(\mathbf{A})$ were infinite. Then we could find $\{Y_n : n < \omega\}$ a collection of pairwise disjoint members of $\mathcal{B}(\mathbf{A})$. For $S \subseteq \omega$, write $Y_S = \bigcup_{n \in S} Y_n$. Then if $y \in \overline{Y_S}$, we have $y \in \mathrm{int}(\overline{Y_S \cdot U_{\mathbf{A}}}) = \mathrm{int}(\overline{Y_S})$, that is, the set $\overline{Y_S}$ is clopen. It follows that for $S, T \subseteq \omega$ disjoint, we have $\overline{Y_S} \cap \overline{Y_T} = \emptyset$. It follows that if $p_n \in Y_n$ for each $n < \omega$, then $\overline{\{p_n : n < \omega\}}$ is isomorphic to $\beta\mathbb{N}$, contradicting our assumption that X is metrizable. Hence $\mathcal{B}(\mathbf{A})$ is finite, hence atomic. Let $\mathrm{Atoms}(\mathbf{A}) \subseteq \mathcal{B}(\mathbf{A})$ denote the atoms.

For each $\mathbf{A} \in \mathrm{Fin}(\mathbf{K})$, we can view $\mathrm{Atoms}(\mathbf{A})$ as a set of 'expansions' of $\mathbf{A}$. Suppose $\mathbf{B} \in \mathrm{Fin}(\mathbf{K})$, $Z \in \mathrm{Atoms}(\mathbf{B})$, and $f : \mathbf{A} \to \mathbf{B}$ is an embedding. We need to determine which expansion of $\mathbf{A}$ is induced by f when we expand $\mathbf{B}$ using Z. We do this as follows: first find $g \in G$ with $g|_{\mathbf{A}} = f$. We will argue that Zg is contained in some $U_{\mathbf{A}}$-atom, and that this does not depend on the g we chose. So suppose W is $U_{\mathbf{A}}$-clopen. By choice of g, We have $g^{-1}U_{\mathbf{B}}g \subseteq U_{\mathbf{A}}$, so $Wg^{-1}U_{\mathbf{B}} = Wg^{-1}$. This shows that Wg^{-1} is $U_{\mathbf{B}}$-clopen. Therefore if $Zg \cap W \neq \emptyset$, then $Z \cap Wg^{-1} \neq \emptyset$, so $Z \subseteq Wg^{-1}$ and $Zg \subseteq W$. It follows that Zg is contained in some $U_{\mathbf{A}}$-atom, say Y. If $h \in G$ also satisfies $h|_{\mathbf{A}} = f$, then $g^{-1}h \in U_{\mathbf{A}}$, so $Zg(g^{-1}h) \subseteq Y$ as well. Therefore if $\mathbf{B}^Z$ is the corresponding expansion of $\mathbf{B}$, we declare that $\mathbf{A}^Y$ is the expansion that $\mathbf{A}$ inherits from $\mathbf{B}^Z$ along the map $f : \mathbf{A} \to \mathbf{B}$. All of this can

be coded by adding countably many new relational symbols to L, producing a language $L^* \supseteq L$ and a reasonable precompact expansion class $\mathcal{K}^*$ of $\mathcal{K}$.

For each $\mathbf{A} \in \mathrm{Fin}(\mathbf{K})$, the set $\mathrm{Atoms}(\mathbf{A})$ is a finite clopen partition of the space X. If $x \in X$, it follows that $x \in Y$ for exactly one $U_{\mathbf{A}}$-atom for each $\mathbf{A} \in \mathrm{Fin}(\mathbf{K})$, giving rise to a surjective G-map $\varphi\colon X \to X_{\mathcal{K}^*}$. If $x \neq y \in X$, then by continuity of the action, we can find $V \ni_{\mathrm{op}} x$, $W \ni_{\mathrm{op}} y$, and $\mathbf{A} \in \mathrm{Fin}(\mathbf{K})$ with $\overline{VU_{\mathbf{A}}} \cap \overline{WU_{\mathbf{A}}} = \emptyset$, showing that φ is injective, hence an isomorphism.

To show that this expansion class has the AP, suppose we have $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathrm{Fin}(\mathbf{K})$ with $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{A} \subseteq \mathbf{C}$. Let $Y_A, Y_B, Y_C \subseteq X$ be clopen atomic sets for $U_{\mathbf{A}}, U_{\mathbf{B}}, U_{\mathbf{C}}$, respectively, with $Y_B \subseteq Y_A$ and $Y_C \subseteq Y_A$. Since Y_A is a $U_{\mathbf{A}}$-atom, the action of $U_{\mathbf{A}}$ on Y_A is topologically transitive, so we can find $g \in U_{\mathbf{A}}$ with $Y_C g \cap Y_B \neq \emptyset$. We can then find some suitably large finite $\mathbf{D} \subseteq \mathbf{K}$ so that for some $U_{\mathbf{D}}$-atom Y_D we have $Y_D \subseteq Y_C g \cap Y_B$. By enlarging $\mathbf{D}$ more if needed, we can assume that $\mathbf{B} \subseteq \mathbf{D}$ and $g^{-1}[\mathbf{C}] \subseteq \mathbf{D}$. It follows that $i_{\mathbf{B}}\colon \mathbf{B}^{Y_B} \to \mathbf{D}^{Y_D}$ and $(g^{-1})|_{\mathbf{C}}\colon \mathbf{C}^{Y_C} \to \mathbf{D}^{Y_D}$ amalgamate the maps $i_{\mathbf{A}}\colon \mathbf{A}^{Y_A} \to \mathbf{B}^{Y_B}$ and $i_{\mathbf{A}}\colon \mathbf{A}^{Y_A} \to \mathbf{C}^{Y_C}$.

A similar argument shows that if X is topologically transitive, then the expansion $\mathcal{K}^*$ that we constructed above will have the joint embedding property (JEP) as well. □

In the case where X is topologically transitive, MHP, but not necessarily metrizable, two important cases emerge. Either for every finite $\mathbf{A} \subseteq \mathbf{K}$, the algebra of $U_{\mathbf{A}}$-clopen sets is atomic, or this fails for some $\mathbf{A}$; the equivalent conditions of Theorem 5.5 correspond to the first case.

4. *Topometrics on MHP flows*

For the rest of this section, fix an MHP flow (X, τ), where τ is the compact topology on X. Our goal is to endow X with a canonical topometric structure. This has been done in the case of $\mathrm{Sa}(G)$ in [**7**], where they use the definition below. Before stating the definition, we note that if $f\colon G \to [0, 1]$ is left-uniformly continuous, we can continuously extend it to $\mathrm{Sa}(G)$, and we will also use $f\colon \mathrm{Sa}(G) \to [0, 1]$ to denote this extension.

Definition 4.1. Given $p, q \in \mathrm{Sa}(G)$, we set

$$\partial(p, q) = \sup(|f(p) - f(q)| :\ f\colon G \to [0, 1] \text{ 1-Lipschitz}).$$

Notice that if $f\colon G \to [0, 1]$ is 1-Lipschitz and we continuously extend to $\mathrm{Sa}(G)$, then f has the following property, which we define more generally.

Definition 4.2. Let X be a G-flow. A function $f \in C(X, [0, 1])$ is called *orbit Lipschitz* if whenever $x \in X$ and $g \in G$, we have

$$|f(x) - f(xg)| \leq d_G(1_G, g).$$

We write $C_{\mathrm{OL}}(X, [0, 1])$ for the collection of orbit Lipschitz functions.

Eventually, we will show that the analogue of Definition 4.1 with 1-Lipschitz replaced by orbit Lipschitz provides the MHP flow X with a topometric structure. The problem is that *a priori*, we do not know whether X has any non-constant orbit Lipschitz functions. Therefore we start with an entirely different definition of the topometric structure, then use

Fact 2.3 to produce an ample supply of continuous Lipschitz functions, which will turn out to be precisely the orbit Lipschitz functions.

Definition 4.3. Given $x, y \in X$ and $c \geq 0$, we define $\partial(x, y) \leq c$ if and only if any of the following four equivalent items hold.

(1) Whenever $A \subseteq_{\text{op}} X$ with $x \in \overline{A}$ and $\epsilon > 0$, we have $y \in \text{int}(\overline{AU_{c+\epsilon}})$.
(2) Whenever $A \subseteq_{\text{op}} X$ with $x \in \overline{A}$ and $\epsilon > 0$, we have $y \in \overline{AU_{c+\epsilon}}$.
(3) Whenever $A \ni_{\text{op}} x$ and $\epsilon > 0$, we have $y \in \text{int}(\overline{AU_{c+\epsilon}})$.
(4) Whenever $A \ni_{\text{op}} x$ and $\epsilon > 0$, we have $y \in \overline{AU_{c+\epsilon}}$.

Remark. The directions (1) $\Rightarrow$ (2) $\Rightarrow$ (4) as well as (1) $\Rightarrow$ (3) $\Rightarrow$ (4) are clear. Suppose (4) holds, and let $A \subseteq_{\text{op}} X$ with $x \in \overline{A}$. Also fix $\epsilon > 0$. Then as X is MHP, we have $x \in \text{int}(\overline{AU_\epsilon})$. By (4), we have $y \in \overline{\text{int}(\overline{AU_\epsilon})U_{c+\epsilon}} \subseteq \overline{AU_{c+2\epsilon}}$. Using MHP once more, we obtain $y \in \text{int}(\overline{AU_{c+3\epsilon}})$, showing that (1) holds.

PROPOSITION 4.4. *The function ∂ from Definition 4.3 is a topometric on X.*

Proof. Suppose $x, y \in X$ have $\partial(x, y) = 0$. If $A \ni_{\text{op}} x$, we can find $B \ni_{\text{op}} x$ and $\epsilon > 0$ with $\overline{BU_\epsilon} \subseteq A$. So in particular $y \in A$, so $x = y$.

Suppose $\partial(x, y) \leq c$ for some $c \geq 0$, toward showing that $\partial(y, x) \leq c$. Let $B \ni_{\text{op}} y$ and $\epsilon > 0$. Notice that if $A \ni_{\text{op}} x$, then $AU_{c+\epsilon} \cap B \neq \emptyset$. So also $A \cap BU_{c+\epsilon} \neq \emptyset$. It follows that $x \in \overline{BU_{c+\epsilon}}$.

Now suppose $\partial(x, y) \leq c$ and $\partial(y, z) \leq d$. Fix $A \ni_{\text{op}} x$ and $\epsilon > 0$. Then $AU_{c+\epsilon}$ is open with $y \in \overline{AU_{c+\epsilon}}$. We then have $z \in \overline{AU_{c+d+2\epsilon}}$, showing that $\partial(x, z) \leq c + d$ as desired.

Having shown that ∂ is a metric on X, we now show that it is τ-lsc. Fix $c \geq 0$, and let $x_i \to x$ and $y_i \to y$ be nets with $\partial(x_i, y_i) \leq c$. Let $A \ni_{\text{op}} x$, and fix $\epsilon > 0$. Then for a tail of x_i, we also have $x_i \in A$, implying that $y_i \in \overline{AU_{c+\epsilon}}$. So $y \in \overline{AU_{c+\epsilon}}$, and by MHP, $y \in \text{int}(\overline{AU_{c+2\epsilon}})$. It follows that $\partial(x, y) \leq c$. □

Remark. If G is a discrete group and X is an MHP G-flow, then ∂ is just the discrete metric on X. If G is locally compact and $c \geq 0$ is small enough so that $U_{c+\epsilon} \subseteq G$ is precompact for some $\epsilon > 0$, then, given an MHP G-flow X and $x, y \in X$, we have $\partial(x, y) \leq c$ if and only if there is $g \in G$ with $d(g, 1_G) \leq c$ and $xg = y$. Hence topometric structures on MHP flows are most interesting when G is not locally compact.

Remark. Suppose $H \subseteq G$ is a closed subgroup, and form $\text{Sa}(H\backslash G)$. On the orbit $H\backslash G \subseteq \text{Sa}(H\backslash G)$, the metric ∂ coincides with the metric d. In particular, this is true for $G \subseteq \text{Sa}(G)$. This will be easiest to see by using Corollary 4.7.

Remark. Suppose $\mathbf{K} = \text{Flim}(\mathcal{K})$ is a Fraïssé structure with $G = \text{Aut}(\mathbf{K})$. Write $\mathbf{K} = \bigcup_{n \geq 1} \mathbf{A}_n$ as an increasing union of finite structures. A compatible left-invariant metric d on G is given by $d(g, h) \leq 1/n$ if and only if $g|_{\mathbf{A}_n} = h|_{\mathbf{A}_n}$. Write $V_n = \{g \in G : g|_{\mathbf{A}_n} = \text{id}_{\mathbf{A}_n}\}$, and notice that for any suitably small $\epsilon > 0$, we have $V_n = U_{1/n+\epsilon}$.

Now suppose X is an MHP G-flow. As in the discussion before Proposition 3.8, let $\mathcal{B}_n$ be the Boolean algebra of V_n-clopen subsets of X. As we make no metrizability assumption here, $\mathcal{B}_n$ may be infinite. However, if $\mathcal{Y} \subseteq \mathcal{B}$ and we set $Y = \bigcup \mathcal{Y}$, we see that for $y \in \overline{Y}$, we have $y \in \text{int}(\overline{Y \cdot V_n}) = \text{int}(\overline{Y})$. In particular, $\overline{Y} \in \mathcal{B}_n$. Setting $\bigvee \mathcal{Y} = \overline{Y}$, we see that $\mathcal{B}_n$ is

a complete Boolean algebra. Let $X_n = \mathrm{St}(\mathcal{B}_n)$ be the Stone space. Then $X \cong \varprojlim X_n$, and, given $x = (x_n)_n$ and $y = (y_n)_n$ in X, we have that $\partial(x, y) \leq 1/n$ if and only if $x_n = y_n$. To see this, first suppose $x_n \neq y_n$, and find some $A \in \mathcal{B}_n$ with $x \in A$ and $y \notin A$. But since $A = AU_{1/n+\epsilon} = \overline{AU_{1/n+\epsilon}}$, we have $\partial(x, y) > 1/n$ by item (4) of Definition 4.3. In the other direction, suppose $x_n = y_n$. Then if $A \ni_{\mathrm{op}} x$, we have $\mathrm{int}(\overline{AU_{1/n+\epsilon}}) \in \mathcal{B}_n$, hence $y \in \mathrm{int}(\overline{AU_{1/n+\epsilon}})$. Therefore $\partial(x, y) \leq 1/n$ by item (3) of Definition 4.3.

We next investigate how this topometric structure interacts with the G-flow structure. Not only is this a canonical topometric to place on an MHP flow X, but it will also behave well when comparing different MHP flows. When discussing multiple MHP flows X, Y, etc., we write ∂_X, ∂_Y, etc. to refer to the topometric structure on each flow.

PROPOSITION 4.5. *Let X and Y be MHP flows endowed with the topometric structure from Definition 4.3.*

(1) *If $x \in X$ and $g \in G$, then $\partial(x, xg) \leq d_G(1_G, g)$.*
(2) *For each $g \in G$, the map $\rho_g \colon (X, \partial) \to (X, \partial)$ given by $\rho_g(x) = xg$ is uniformly continuous.*
(3) *If $\varphi \colon X \to Y$ is a G-map, then φ is metrically non-expansive, that is, for any $x, y \in X$, we have $\partial_Y(\varphi(x), \varphi(y)) \leq \partial_X(x, y)$.*

Proof. For item (1), write $c = d_G(1_G, g)$. Then for any $\epsilon > 0$, we have $g \in U_{c+\epsilon}$. Hence if $A \ni_{\mathrm{op}} x$, we have $xg \in AU_{c+\epsilon}$.

For item (2), fix $c > 0$. Find $d > 0$ so that $g^{-1}U_d g \subseteq U_c$. Now suppose $x, y \in X$ satisfy $\partial(x, y) < d$. Let $A \ni_{\mathrm{op}} xg$, and fix $\epsilon > 0$. Then $Ag^{-1} \ni_{\mathrm{op}} x$, so $y \in \overline{Ag^{-1}U_d}$. It follows that $yg \in \overline{AU_c}$, so $\partial(x, y) \leq c$.

For item (3), write $c = \partial_X(x, y)$, and let $B \ni_{\mathrm{op}} \varphi(x)$. Then $x \in \varphi^{-1}(B)$, so we have $y \in \overline{\varphi^{-1}(B)U_{c+\epsilon}} = \overline{\varphi^{-1}(BU_{c+\epsilon})} \subseteq \varphi^{-1}(\overline{BU_{c+\epsilon}})$ for any $\epsilon > 0$. So $\varphi(y) \in \overline{BU_{c+\epsilon}}$ as desired. □

Remark. Notice that item (3) shows that if $M \subseteq \mathrm{Sa}(G)$ is a minimal subflow, then the topometric structure computed internally in M is the same as the topometric structure inherited from $\mathrm{Sa}(G)$. This is because M is a retract of $\mathrm{Sa}(G)$.

Denote by $C_L(X, [0, 1])$ the collection of continuous, 1-Lipschitz functions from X to $[0, 1]$. The next proposition, along with Fact 2.3, will give us Corollary 4.7, the analogue of Definition 4.1 for any MHP flow.

PROPOSITION 4.6. $C_L(X, [0, 1]) = C_{\mathrm{OL}}(X, [0, 1])$.

Proof. First suppose $f \in C_L(X, [0, 1])$. Then since for any $p \in X$ and $g \in G$, we have $\partial(p, pg) \leq d(1_G, g)$, we see that $f \in C_{\mathrm{OL}}(X, [0, 1])$.

Now suppose $f \in C_{\mathrm{OL}}(X, [0, 1])$, and fix $p, q \in X$. Suppose $\partial(p, q) \leq c$, and let $\epsilon > 0$. Find $A \ni_{\mathrm{op}} p$ so that $|f(p') - f(p)| < \epsilon$ for $p' \in A$. Then $q \in \overline{AU_{c+\epsilon}}$, so find $p_i \in A$ and $g_i \in U_{c+\epsilon}$ with $p_i g_i \to q$. As f is orbit Lipschitz, we have $|f(p_i g_i) - f(p)| < c + 2\epsilon$. As $\epsilon > 0$ is arbitrary, we have $|f(q) - f(p)| \leq c$ as desired. □

COROLLARY 4.7. *Let $x, y \in X$. Then $\partial(x, y) = \sup\{|f(x) - f(y)| : f \in C_{\mathrm{OL}}(X, [0, 1])\}$.*

We end this section by proving that the topometric space (X, τ, ∂) is adequate. For the proof, it will be easier to work with closed sets rather than open sets. Given $K \subseteq X$ and $c > 0$, we write $K(-c) := X \setminus ((X \setminus K)(c)) = \{x \in X : x(c) \subseteq K\}$. So a topometric space (X, τ, ∂) is adequate if for every closed $K \subseteq X$ and every $c > 0$, we have $K(-c)$ closed.

THEOREM 4.8. *The topometric space (X, τ, ∂) is adequate.*

Proof. Fix $K \subseteq X$ closed. We show that the set $K(-c)$ is also closed. Write $K = \bigcap_i K_i$ with each K_i a regular closed set. Then $K(-c) = \bigcap_i K_i(-c)$. So it suffices to prove the theorem in the case where K is regular closed (we will only need this at the very end). For such K, we will show that

$$K(-c) = \bigcap_{\substack{\epsilon>0 \\ r<c}} X \setminus \operatorname{int}\Big(\overline{(X \setminus \overline{KU_\epsilon})U_r}\Big).$$

Suppose $p \in X$ is not in the left-hand side. Then there is $q \in X \setminus K$ with $\partial(p, q) < c$. Given r with $\partial(p, q) < r < c$, then for every $A \ni_{\mathrm{op}} q$, we have $p \in \operatorname{int}(\overline{AU_r})$. Now for some suitably small $\epsilon > 0$, we have $q \in X \setminus \overline{KU_\epsilon}$. Taking $A = X \setminus \overline{KU_\epsilon}$, we see that p is not in the right-hand side.

Now suppose $p \in X$ is not in the right-hand side as witnessed by $\epsilon > 0$ and $r < c$. In particular, we have $p \in \overline{(X \setminus \overline{KU_\epsilon})U_r}$. Let $A \ni_{\mathrm{op}} p$. Then $A \cap (X \setminus \overline{KU_\epsilon})U_r \neq \emptyset$. It follows that $AU_r \cap (X \setminus \overline{KU_\epsilon}) \neq \emptyset$. Therefore we have

$$\Big(\overline{X \setminus \overline{KU_\epsilon}}\Big) \cap \Big(\bigcap \{\overline{AU_r} : A \ni_{\mathrm{op}} p\}\Big) \neq \emptyset.$$

Fix some q from this set. It follows that $\partial(p, q) \leq r < c$. To see that $q \notin K$, notice that for any $x \in K$, we have $x \in \overline{\operatorname{int}(K)}$, so we have $x \in \operatorname{int}(\overline{\operatorname{int}(K)U_\epsilon}) = \operatorname{int}(\overline{KU_\epsilon})$. □

5. *Comeager orbits in MHP flows*

We continue with most of the notation of the previous section. In particular, G is a Polish group, and (X, τ, ∂) is an MHP G-flow endowed with the topometric structure from Definition 4.3. In this section, we undertake a deeper study of the interaction between the topology τ and the metric ∂, connecting this to various properties that the G-flow X might enjoy. The main theorem is Theorem 5.5, which gives a complete characterization of when an MHP flow has a comeager orbit.

Definition 5.1. Let $p \in X$. We say that ∂ is *compatible* at p or that p is a *compatiblity point* if for every $c > 0$, we have $p \in \operatorname{int}(p(c))$.

Compatibility points are precisely the points in X where the topologies given by τ and ∂ coincide. This is a notion which has been studied in the context of continuous logic, especially with regard to type spaces and the omitting types theorem (see [**5**, Ch. 12]). We can now generalize one of the key theorems from [**7**]. We will repeatedly use the fact that if $A, B \subseteq_{\mathrm{op}} X$ with $A \cap B = \emptyset$, then $\operatorname{int}(\overline{A}) \cap \operatorname{int}(\overline{B}) = \emptyset$.

LEMMA 5.2. *Suppose $x, y \in X$ satisfy $\partial(x, y) > 2c$. Then there are $A \ni_{\mathrm{op}} x$ and $B \ni_{\mathrm{op}} y$ with $\overline{AU_c} \cap \overline{BU_c} = \emptyset$.*

Proof. We can find $A \ni_{\text{op}} x$ and $\epsilon > 0$ with $y \notin \overline{AU_{2c+2\epsilon}}$. Setting $B = X \setminus \overline{AU_{2c+2\epsilon}}$, we have that $\overline{AU_c} \cap \overline{BU_c} = \emptyset$ as desired. Indeed if $x \in \overline{AU_c} \cap \overline{BU_c}$, then by MHP $x \in \text{int}(\overline{AU_{c+\epsilon}}) \cap \text{int}(\overline{BU_{c+\epsilon}})$, contradicting that $AU_{c+\epsilon} \cap BU_{c+\epsilon} = \emptyset$. □

THEOREM 5.3. *(X, τ) is metrizable if and only if ∂ is a compatible metric for τ, that is, if and only if ∂ is compatible at every point in X. Furthermore, if (X, τ) is not metrizable, then X embeds a copy of $\beta\omega$, the space of ultrafilters on ω.*

Proof. One direction is clear, so suppose ∂ generates a strictly finer topology than τ. In particular, (X, ∂) is not compact, so find $c > 0$ and an infinite $Y \subseteq X$ with $\partial(x, y) > 2c$ for any $x \neq y \in Y$.

We will inductively define infinite $Y_n \subseteq Y$, $x_n \in Y_n$, and $A_n \ni_{\text{op}} x_n$ for each $n < \omega$. We will ensure that the following all hold.

(1) $Y_{n+1} \subseteq Y_n$ for every $n < \omega$.
(2) $Y_n \cap \overline{A_k U_c} = \emptyset$ for every $k < n < \omega$.
(3) $A_n \cap A_k U_c = \emptyset$ for each $k < n < \omega$.

Set $Y_0 = Y$. Suppose $Y_0, \ldots, Y_n$, $x_0, \ldots x_{n-1}$, and $A_0 \ldots A_{n-1}$ have been chosen. Pick $x \neq y \in Y_n$, and use Lemma 5.2 to find $A \ni_{\text{op}} x$ and $B \ni_{\text{op}} y$ with $\overline{AU_c} \cap \overline{BU_c} = \emptyset$. We also demand, by shrinking A and B if needed, that $A \cap A_k U_c = \emptyset$ and $B \cap A_k U_c = \emptyset$ for every $k < n$; this is possible by item (2). Now at least one of $Y_n \setminus \overline{AU_c}$ and $Y_n \setminus \overline{BU_c}$ is infinite—without loss of generality the former. Set $Y_{n+1} = Y_n \setminus \overline{AU_c}$, $x_n = x$, and $A_n = A$.

Having completed the inductive construction, define $\varphi \colon \beta\omega \to X$ to be the continuous extension of the map $\varphi(n) = x_n$. We show that φ is injective. If $S \subseteq \omega$, set $A_S = \bigcup_{n \in S} A_n$. It is enough to show that if $S, T \subseteq \omega$ with $S \cap T = \emptyset$, then $\overline{A_S} \cap \overline{A_T} = \emptyset$. To see why this is, note that $\overline{A_S} \subseteq \text{int}(\overline{A_S U_{c/2}})$, likewise for A_T, and that $A_S U_{c/2} \cap A_T U_{c/2} = \emptyset$. □

Next we investigate what happens when some, but not all, points in X are compatibility points. We remind the reader that the topometric space (X, τ, ∂) was proven in Theorem 4.8 to be adequate.

LEMMA 5.4.

(1) *Let $Y \subseteq X$ denote the set of compatibility points. Then Y is G-invariant, ∂-closed, and topologically G_δ.*
(2) *Suppose $x \in X$ is not a compatibility point. Then there is $c > 0$ such that* $\text{int}(x(c)) = \emptyset$.

Proof. (1) That Y is G-invariant follows from item (2) of Proposition 4.5. To show Y is ∂-closed, let $y_n \xrightarrow{\partial} y$, and fix $c > 0$. Then for some $n < \omega$, we have $y_n(c/2) \subseteq y(c)$. By assumption, $y_n \in \text{int}(y_n(c/2))$, so in particular, we have $\text{int}(y(c)) \neq \emptyset$. Using adequacy, we have $y \in (\text{int}(y(c)))(c) \subseteq \text{int}(y(2c))$. Lastly, to show that Y is G_δ, let $Y_c = \bigcup_{y \in Y} \text{int}(y(c))$. Then $Y_c \subseteq X$ is open with $Y = \bigcap_{c>0} Y_c$.

(2) Suppose $x \in X$ is a point with $\text{int}(x(c)) \neq \emptyset$ for every $c > 0$. Then x is a compatibility point, as by adequacy, we have $x \in (\text{int}(x(c)))(c) \subseteq \text{int}(x(2c))$. □

THEOREM 5.5. *The following are equivalent.*

(1) *X has a compatibility point with dense orbit.*

(2) *The set $Y \subseteq X$ of compatibility points is comeager, Polish, and G acts on Y topologically transitively.*
(3) *X has a comeager orbit.*
(4) *$X \cong \mathrm{Sa}(H\backslash G)$ for some closed subgroup $H \subseteq G$.*

Proof. (1) $\Rightarrow$ (2) Letting $Y \subseteq X$ denote the set of compatibility points, item (1) of Lemma 5.4 shows us that Y is G-invariant, ∂-closed, and G_δ. By (1), $Y \subseteq X$ is dense. As (Y, τ) and (Y, ∂) are homeomorphic, we see that (Y, τ) is separable and that ∂ is a compatible complete metric, hence Y is Polish. As Y contains a dense orbit, the action of G on Y is topologically transitive.

(2) $\Rightarrow$ (3) It is enough to show that Y has a comeager orbit. We mostly follow the proof from [**7**], with a few differences to adapt to our more general setting. Using a criterion due to Rosendal (see [**7**] for a proof of the criterion), we need to show that for every $\epsilon > 0$ and every $A \subseteq_{\mathrm{op}} Y$, there is $B \subseteq_{\mathrm{op}} A$ such that the local action of U_ϵ on B is topologically transitive. To that end, let $B \subseteq_{\mathrm{op}} Y$ be any open set of ∂-diameter less than ϵ; any $A \subseteq_{\mathrm{op}} Y$ will contain such a B since (Y, ∂) and (Y, τ) are homeomorphic. Fix $C_0, C_1 \subseteq_{\mathrm{op}} B$. If $p_0 \in C_0$ and $p_1 \in C_1$, then $\partial(p_0, p_1) < \epsilon$. So $p_1 \in \overline{C_0 U_\epsilon}$. In particular, $C_0 U_\epsilon \cap C_1 \neq \emptyset$ as desired.

(3) $\Rightarrow$ (4). Let $Z \subseteq X$ denote the comeager orbit, and pick $p \in Z$. Let $H = \mathrm{Stab}(p)$. By the Effros theorem, we have that $Z \cong H\backslash G$ as G-spaces. So also $S_G(Z) \cong S_G(H\backslash G) \cong \mathrm{Sa}(H\backslash G)$. But since $Z \subseteq X$ is dense, we have $S_G(Z) \cong S_G(X) \cong X$.

(4) $\Rightarrow$ (1) We will make use of Fact 3.6. For each $\epsilon > 0$, we have that $C_{HU_\epsilon} \subseteq \mathrm{Sa}(H\backslash G)$ is a neighborhood of H. Let $f\colon \mathrm{Sa}(H\backslash G) \to [0, 1]$ be continuous and orbit Lipschitz. In particular, $f|_{H\backslash G}$ is 1-Lipschitz. So if $p \in C_{HU\epsilon}$, we have $|f(p) - f(H)| \leq \epsilon$. Therefore $C_{HU_\epsilon} \subseteq H(2\epsilon)$, so H is a compatibility point in $\mathrm{Sa}(H\backslash G)$. □

6. *More on Samuel compactifications*

Given item (4) in Theorem 5.5, let us spend some time developing a more detailed understanding of the topometric G-space $\mathrm{Sa}(H\backslash G)$, which we continue to view as a space of near ultrafilters. We first consider the left completion $\widehat{H\backslash G}$. Notice that if $f\colon H\backslash G \to X$ is a uniformly continuous function with X a complete uniform space, then f continuously extends to $\widehat{H\backslash G}$. In particular, by considering the inclusion $i\colon H\backslash G \hookrightarrow \mathrm{Sa}(H\backslash G)$, we obtain a continuous map from $\widehat{H\backslash G}$ to $\mathrm{Sa}(H\backslash G)$. This map turns out to be an embedding, and we will identify $\widehat{H\backslash G}$ with its image in $\mathrm{Sa}(H\backslash G)$. We have the following fact (see [**14**, Ch. 1.2]).

Fact 6.1. Given $p \in \mathrm{Sa}(H\backslash G)$, we have $p \in \widehat{H\backslash G}$ if and only if for every $\epsilon > 0$, there is $A \subseteq_{\mathrm{op}} H\backslash G$ of diameter less than ϵ, with $A \in p$.

From the proof of Theorem 5.5, we know that $H \in \mathrm{Sa}(H\backslash G)$ is a compatibility point. As H has dense orbit in $\mathrm{Sa}(H\backslash G)$, and since the topology and the metric coincide on the set of compatibility points, we see that H has a ∂-dense orbit in the set of compatibility points. Since ∂ and d coincide on $H\backslash G$, we obtain the following proposition.

PROPOSITION 6.2. *In $\mathrm{Sa}(H\backslash G)$, the set of compatibility points is precisely $\widehat{H\backslash G}$.*

In particular, by Theorem 5.5, we have that $\widehat{H\backslash G} \subseteq \mathrm{Sa}(H\backslash G)$ is comeager. As $H\backslash G \subseteq \widehat{H\backslash G}$ is comeager, we obtain the following result.

PROPOSITION 6.3. *In* $\mathrm{Sa}(H\backslash G)$*, the orbit* $H\backslash G \subseteq \mathrm{Sa}(H\backslash G)$ *is comeager.*

Remark. This proposition is really a statement about topology rather than dynamics. Whenever (X, d) is a Polish metric space and $S(X)$ is the Samuel compactification of X with its metric uniformity, then $X \subseteq S(X)$ is comeager.

We now take some time to understand the canonical G-map $\pi : \mathrm{Sa}(G) \to \mathrm{Sa}(H\backslash G)$. To do this, we first need to understand how near ultrafilters on H interact with those on G. Let $p \in \mathrm{Sa}(H)$. Then if $A \in p$ and $\epsilon > 0$, we have $AU_\epsilon \subseteq_{\mathrm{op}} G$, and the collection $\{AU_\epsilon : A \in p, \epsilon > 0\}$ extends to a unique near ultrafilter in $\mathrm{Sa}(G)$. This gives rise to an embedding $i : \mathrm{Sa}(H) \hookrightarrow \mathrm{Sa}(G)$. More explicitly, given $p \in G$, we set

$$i(p) = \{B \subseteq_{\mathrm{op}} G : B \cap AU_\epsilon \neq \emptyset \text{ for every } A \in p, \epsilon > 0\}.$$

Now given $p \in \mathrm{Sa}(G)$, we have $p \in i[\mathrm{Sa}(H)]$ if and only if $HU_\epsilon \in p$ for every $\epsilon > 0$. One direction is clear. For the other, if $HU_\epsilon \in p$ for every $\epsilon > 0$, it follows that for every $A \in p$ and $\epsilon > 0$, we have $AU_\epsilon \cap H \neq \emptyset$, and the collection

$$\{B \subseteq_{\mathrm{op}} H : B \cap AU_\epsilon \neq \emptyset \text{ for every } A \in p, \epsilon > 0\}$$

is a near ultrafilter q on H satisfying $i(q) = p$.

From here on out, we will identify $\mathrm{Sa}(H)$ as a subspace of $\mathrm{Sa}(G)$ and suppress the embedding i. We now consider the quotient $\pi : G \to H\backslash G$ and extend it continuously to the respective Samuel compactifications. Given $p \in \mathrm{Sa}(G)$ and $q \in \mathrm{Sa}(H\backslash G)$, we have by Fact 3.6 that $\pi(p) = q$ if and only if $\pi^{-1}(AU_\epsilon) \in p$ for every $A \in q$ and $\epsilon > 0$. In particular, $\pi(p) = H$ if and only if $HU_\epsilon \in p$ for every $\epsilon > 0$. We obtain the following proposition.

PROPOSITION 6.4. *With* $\pi : \mathrm{Sa}(G) \to \mathrm{Sa}(H\backslash G)$ *the canonical map, we have* $\pi^{-1}(\{H\}) = \mathrm{Sa}(H)$.

In the next section we will be particularly interested in *minimal* MHP flows. Recall that $S \subseteq G$ is called *syndetic* if there is a finite set $F \subseteq G$ with $SF = G$. We have the following folklore fact.

Fact 6.5. [**2**, Ch. 1, Lemma 6] Suppose X is a G-flow and $x \in X$. Then $x \in X$ belongs to a minimal subflow if and only if for every $A \ni_{\mathrm{op}} X$, the set $\{g \in G : xg \in A\}$ is syndetic.

The following simple proposition gives a combinatorial characterization of when $\mathrm{Sa}(H\backslash G)$ is minimal.

PROPOSITION 6.6. *Let* $H \subseteq G$ *be a closed subgroup. Then the following are equivalent.*

(1) $\mathrm{Sa}(H\backslash G)$ *is minimal.*

(2) *For every* $\epsilon > 0$*, the set* $HU_\epsilon \subseteq G$ *is syndetic.*

Remark. Compare this to the notion of co-precompactness, where $H \subseteq G$ is *co-precompact* if $\mathrm{Sa}(H\backslash G) \cong \widehat{H\backslash G}$, the left completion of $H\backslash G$. This occurs if and only if for every $\epsilon > 0$, there is a finite $F \subseteq G$ with $HFU_\epsilon = G$.

Proof. First assume $\mathrm{Sa}(H\backslash G)$ is minimal. Since $H \in \mathrm{Sa}(H\backslash G)$ is a compatibility point, we have that $HU_\epsilon \subseteq H\backslash G$ is relatively open. Item (2) then follows from minimality.

Conversely, assume item (2) holds. It follows that in $\mathrm{Sa}(H\backslash G)$, the return times of H to any open neighborhood of H are syndetic. Then by Fact 6.5, $H \in \mathrm{Sa}(H\backslash G)$ belongs to a minimal subflow, and the orbit of H is dense in $\mathrm{Sa}(H\backslash G)$. □

Also in the next section, we will need to consider two closed subgroups H, $H' \subseteq G$ and understand when a G-map $\varphi\colon \mathrm{Sa}(H\backslash G) \to \mathrm{Sa}(H'\backslash G)$ can exist.

PROPOSITION 6.7. *Suppose H, $H' \subseteq G$ are closed subgroups with both $\mathrm{Sa}(H\backslash G)$ and $\mathrm{Sa}(H'\backslash G)$ minimal. Then there is a G-map $\varphi\colon \mathrm{Sa}(H\backslash G) \to \mathrm{Sa}(H'\backslash G)$ if and only if there is $g \in G$ with $H \subseteq g^{-1}H'g$.*

Proof. For the forward direction, let φ be a G-map as above. By [**1**, Proposition 14.1], we know that φ must preserve the comeager orbit. In particular, $\varphi(H) = H'g$ for some $g \in G$. It follows that for every $h \in H$, we have $H'gh = H'g$, that is, that $H \subseteq g^{-1}H'g$.

For the reverse direction, if $H \subseteq g^{-1}Hg$ for some $g \in G$, it follows that H stabilizes the point $H'g \in \mathrm{Sa}(H'\backslash G)$. Then the existence of a G-map φ as above follows from the universal property of $\mathrm{Sa}(H\backslash G)$. □

7. *Canonical minimal flows*

In this section, we consider the universal minimal flow as well as two other 'canonical' minimal flows in the context of Theorem 5.5. These other special flows both deal with the notion of *proximality*.

Definition 7.1. Fix a G-flow X.

(1) We say that X is *proximal* if for any x, $y \in X$, there exist a net $g_i \in G$ and $z \in X$ with $xg_i \to z$ and $yg_i \to z$. Equivalently, there is $p \in \mathrm{Sa}(G)$ with $xp = yp$.

(2) Let $P(X)$ denote the compact space of probability measures on X endowed with the weak*-topology. Then $P(X)$ is also a G-flow. We say that X is *strongly proximal* if $P(X)$ is proximal. Equivalently, X is strongly proximal if and only if X is proximal and for any $\mu \in P(X)$, there is a net g_i from G with $\mu g_i \to \delta_x$ for some $x \in X$, where δ_x denotes the Dirac measure supported at x.

In [**8**], it is shown that there exist a *universal minimal proximal flow*, denoted $\Pi(G)$, and a *universal minimal strongly proximal flow*, denoted $\Pi_s(G)$ and often called the *Furstenberg boundary*. Here, if P is a property of flows, a *universal minimal P flow* is a minimal flow with property P which admits a G-map onto any other minimal flow with property P. Both are unique up to isomorphism.

LEMMA 7.2. *If X is a minimal, proximal G-flow, then the only G-map from X to X is the identity.*

Proof. Suppose $\varphi\colon X \to X$ is a G-map. If there is $x \in X$ with $\varphi(x) = x$, then also $\varphi(xp) = xp$ for every $p \in \mathrm{Sa}(G)$. As X is minimal, this implies that φ is the identity map. Now suppose $\varphi \neq \mathrm{id}_X$. Fix $x \in X$, and find $p \in \mathrm{Sa}(G)$ with $xp = \varphi(x)p$. But as $\varphi(x)p = \varphi(xp)$, this is a contradiction since φ has no fixed points. □

LEMMA 7.3. *The flows $\Pi(G)$ and $\Pi_s(G)$ are both MHP.*

Proof. Suppose $\varphi: X \to \Pi(G)$ is a non-trivial highly proximal G-map. Then it follows that X is also minimal and proximal, so let $\psi: \Pi(G) \to X$ be a G-map. It follows that $\psi \circ \varphi: X \to X$ is a non-trivial G-map, contradicting Lemma 7.2.

To show that $\Pi_s(G)$ is MHP, suppose $\varphi: X \to \Pi_s(G)$ is a non-trivial highly proximal G-map. As a highly proximal extension of a minimal proximal flow, X is proximal. Now suppose $\mu \in P(X)$. We can find a net $g_i \in G$ such that $\varphi_* \mu g_i \to \delta_p$ for some $p \in \Pi_s(G)$. We may assume that $\mu g_i \to \nu$ for some $\nu \in P(X)$ supported on $\varphi^{-1}(\{p\})$. Then since φ is highly proximal and X is minimal, we can use Fact 3.2 and find another net $h_j \in G$ so that $\varphi^{-1}(\{p\})h_j$ shrinks down to some point $x \in X$. Hence $\nu h_j \to \delta_x$, showing that X is strongly proximal. Now a similar argument to the proximal case shows that φ must be an isomorphism. □

We can use $M(G)$ to create a particularly nice representation of $\Pi_s(G)$. Form the G-flow $P(M(G))$, and let $A \subseteq P(M(G))$ be a minimal *affine* subflow of $P(M(G))$, that is, a subflow which is closed under convex combinations and minimal with this property. Then A is strongly proximal, and $\overline{ex(A)}$, the closure of the extreme points of A, is the unique minimal subflow of A. We then obtain $\overline{ex(A)} \cong \Pi_s(G)$. More details can be found in [**8**, Ch. 3].

From this characterization of $\Pi_s(G)$, it follows that a topological group G is amenable if and only if G admits no non-trivial minimal strongly proximal actions. As for proximal actions, we call G *strongly amenable* if G admits no non-trivial minimal proximal actions. In particular, every strongly amenable group is amenable.

LEMMA 7.4. *Let X be a proximal G-flow, and let $H \subseteq G$ be a closed subgroup with $\mathrm{Sa}(H\backslash G)$ minimal. Then H acts proximally on X.*

Proof. Let $x, y \in X$. As X is a proximal G-flow, find $p \in \mathrm{Sa}(G)$ with $xp = yp$. Since $\mathrm{Sa}(H\backslash G)$ is minimal, we can find $q \in \mathrm{Sa}(G)$ with $pq \in \mathrm{Sa}(H)$. Then $xpq = ypq$, showing that H acts proximally on X. □

The following theorem provides a generalization of [**11**, Theorem 1.2].

THEOREM 7.5. *Fix a minimal MHP flow X with a comeager orbit.*

(1) *$X \cong M(G)$ if and only if $X \cong \mathrm{Sa}(H\backslash G)$ for some extremely amenable closed subgroup $H \subseteq G$.*
(2) *$X \cong \Pi_s(G)$ if and only if $X \cong \mathrm{Sa}(H\backslash G)$ for some maximal amenable subgroup $H \subseteq G$.*
(3) *If $X \cong \mathrm{Sa}(H\backslash G)$ for some strongly amenable closed subgroup $H \subseteq G$ and X is proximal, then $X \cong \Pi(G)$.*

Proof. (1) First assume $X \cong M(G)$, and let $H \subseteq G$ be the closed subgroup given by item (4) of Theorem 5.5. Fix a minimal subflow $M \subseteq \mathrm{Sa}(G)$, and consider the canonical map $\pi: \mathrm{Sa}(G) \to \mathrm{Sa}(H\backslash G)$. Then π is surjective, and $\pi|_M$ is an isomorphism. Since by Proposition 6.4 we have $\pi^{-1}(\{H\}) = \mathrm{Sa}(H)$, it follows that $M \cap \mathrm{Sa}(H)$ is a singleton and

an H-flow. As any minimal subflow of $\mathrm{Sa}(H)$ is isomorphic to $M(H)$, we see that H is extremely amenable.

Conversely, suppose $H \subseteq G$ is an extremely amenable closed subgroup of G with $\mathrm{Sa}(H\backslash G)$ minimal. Then $M(G)$ must have an H-fixed point. It follows that there is a G-map $\varphi\colon \mathrm{Sa}(H\backslash G) \to M(G)$. As we assumed that $\mathrm{Sa}(H\backslash G)$ was minimal, it follows that φ is an isomorphism.

(2) We break the argument into the following parts.

- If $X \cong \Pi_s(G)$, then $X \cong \mathrm{Sa}(H\backslash G)$ with $H \subseteq G$ a closed amenable subgroup.
- If $X \cong \mathrm{Sa}(H'\backslash G)$ with $H' \subseteq G$ a closed amenable subgroup, then X maps onto any strongly proximal flow.

From these two items the theorem follows, since if $H \subsetneq H'$ are both closed amenable subgroups of G, then by Proposition 6.7, we have a non-trivial factor map $\mathrm{Sa}(H\backslash G) \to \mathrm{Sa}(H'\backslash G)$. If we had $\Pi_s(G) \cong \mathrm{Sa}(H\backslash G)$, then the second item would allow us to build a non-trivial G-map from $\Pi_s(G)$ to itself, contradicting Lemma 7.2. Conversely, if $X \cong \mathrm{Sa}(H'\backslash G)$ for $H' \subseteq G$ a maximal amenable subgroup, then, using the second item, we obtain a map $\mathrm{Sa}(H'\backslash G) \to \Pi_s(G)$. By the first item, we have $\Pi_s(G) \cong \mathrm{Sa}(H\backslash G)$ for some closed amenable subgroup $H \subseteq G$. By Proposition 6.7 we must have $H \subseteq g^{-1}H'g$, so in fact $H = g^{-1}H'g$ as H was assumed maximal. It follows that $\mathrm{Sa}(H\backslash G) \cong \mathrm{Sa}(H'\backslash G) = \Pi_s(G)$.

To prove the first item, suppose $X \cong \Pi_s(G) \cong \mathrm{Sa}(H\backslash G)$. Let $M \subseteq \mathrm{Sa}(G)$ be a minimal subflow, and let $A \subseteq P(M)$ be a minimal affine subflow. Then $X \cong \overline{ex(A)}$, the unique minimal subflow of A. Now letting $\pi : \mathrm{Sa}(G) \to \mathrm{Sa}(H\backslash G)$ be the canonical map, we have the affine extension $\pi_*\colon P(\mathrm{Sa}(G)) \to P(\mathrm{Sa}(H\backslash G))$ to the spaces of measures. Identifying each $p \in \mathrm{Sa}(H\backslash G)$ with the Dirac measure δ_p, we have that $\mathrm{Sa}(H\backslash G)$ is the unique minimal subflow of $P(\mathrm{Sa}(H\backslash G))$. It follows that $\pi_*|_{\overline{ex(A)}}\colon \overline{ex(A)} \to \mathrm{Sa}(H\backslash G)$ is an isomorphism. However, we also have $\pi_*^{-1}(\{H\}) = P(\mathrm{Sa}(H))$, so $P(\mathrm{Sa}(H)) \cap \overline{ex(A)}$ is a singleton and an H-flow, that is, an H-invariant measure on $\mathrm{Sa}(H)$. Hence H is amenable.

To prove the second item, we assume $X \cong \mathrm{Sa}(H'\backslash G)$ with $H' \subseteq G$ a closed amenable subgroup. On $P(\Pi_s(G))$, H' acts proximally by Lemma 7.4, hence H' acts strongly proximally on $\Pi_s(G)$. Since H' is amenable, it follows that $\Pi_s(G)$ has an H'-fixed point, so there is a G-map from $\mathrm{Sa}(H'\backslash G)$ to $\Pi_s(G)$.

(3) We assume that $X \cong \mathrm{Sa}(H\backslash G)$ is proximal and that $H \subseteq G$ is strongly amenable. By Lemma 7.4, H acts proximally on $\Pi(G)$. As H is strongly amenable, $\Pi(G)$ has an H-fixed point, so there is a G-map from $\mathrm{Sa}(H\backslash G)$ to $\Pi(G)$. As $\mathrm{Sa}(H\backslash G)$ was assumed proximal, we have $\mathrm{Sa}(H\backslash G) \cong \Pi(G)$. □

Remark. When considering the Furstenberg boundary or the universal minimal proximal flow of locally compact groups, we note that if $\mathrm{Sa}(H\backslash G)$ is minimal, then in fact $\mathrm{Sa}(H\backslash G) = H\backslash G$, that is, that H is a cocompact subgroup of G. This is because $H\backslash G \subseteq \mathrm{Sa}(H\backslash G)$ is comeager, but also F_σ, being an orbit of a locally compact group action. So $\mathrm{Sa}(H\backslash G) \setminus (H\backslash G)$ is G_δ, and if it were non-empty, then by minimality it would be dense, a contradiction.

The following question addresses whether item (3) in Theorem 7.5 can be strengthened to have the same form as items (1) and (2).

Question 7.6. Suppose $\Pi(G) \cong \mathrm{Sa}(H\backslash G)$ for some closed subgroup $H \subseteq G$. Then must H be strongly amenable?

8. *Reflecting meager orbits*

The main theorem of this section is the following 'reflection' theorem.

THEOREM 8.1. *Let X be a minimal MHP flow all of whose orbits are meager. Then there is a factor $\varphi\colon X \to Y$ such that Y is metrizable and also has all orbits meager.*

Therefore in addition to the notation of the previous sections, we assume that X is minimal and does not have a comeager orbit.

The metrizable factor of X that we produce will be a space of uniformly continuous functions from G (with its left-invariant metric uniformity) to a compact metric space. If Y is a compact metric space, then Y^G is a compact space when endowed with the product topology. The group G acts on Y^G by shift, where for $y \in Y^G$ and $g, h \in G$, we have $y \cdot g(h) = y(gh)$. Now suppose $y \in Y^G$ is uniformly continuous. Then $\overline{y \cdot G}$ is a uniformly equi-continuous family, and furthermore, the space $\overline{y \cdot G}$ is metrizable. To see why the last claim is true, note that pointwise convergence of a net of uniformly equi-continuous functions is determined by pointwise convergence on some countable dense subset of G.

In order to obtain factors of X, we use functions which arise from X in the following way. Suppose $f\colon X \to Y$ is continuous, and fix $x \in X$. Then we obtain a uniformly continuous function $f_x\colon G \to Y$ via $f_x(g) = f(xg)$. Then notice that $f_x \cdot g = f_{xg}$, and if $x_i \to y$, then $f_{x_i} \to f_y$. It follows that the map $x \to f_x$ is a surjective G-map of X onto $\overline{f_x \cdot G}$.

We now turn to the proof of the theorem. Our first task is to provide a 'global' version of item (2) from Lemma 5.4. This does not require minimality.

LEMMA 8.2. *Suppose Z is an MHP flow with no comeager orbit. Then there exist some $c > 0$ and $A \subseteq_{\mathrm{op}} Z$ with $x[c]$ nowhere dense for every $x \in A$.*

Proof. Notice by the lower semi-continuity of ∂ that $x[c]$ is closed for every $c > 0$. Suppose toward a contradiction that for every $c > 0$, the set $D_c := \{x \in Z : \mathrm{int}(x[c]) \neq \emptyset\}$ is dense. Using adequacy, we see that for every $c > 0$, we have $D_{c/3} \subseteq E_c := \{x \in Z : x \in \mathrm{int}(x[c])\}$, so E_c is also dense. Then ∂ is compatible at any point in the comeager set $\bigcap_{c>0} \bigcup_{x \in E_c} \mathrm{int}(x[c])$. Theorem 5.5 then shows that Z has a comeager orbit, contradicting our assumption. □

Fix $c > 0$ and $A \subseteq_{\mathrm{op}} X$ as given by Lemma 8.2. Fix $D \subseteq X$ a countable dense set, and write $[D]^2 = \{\{p_i, q_i\} : i < \omega\}$. Keeping in mind Corollary 4.7, find $\gamma_i \in C_{\mathrm{OL}}(X, [0, 1])$ with $|\gamma_i(p_i) - \gamma_i(q_i)| > \partial(p_i, q_i)/2$. Let $\gamma\colon X \to [0, 1]^\omega$ be the concatenation of the γ_i. It will be helpful to view $[0, 1]^\omega$ as a topometric space whose metric is given by the uniform distance d_u.

LEMMA 8.3. *Let $B \subseteq [0, 1]^\omega$ be a closed d_u-ball of radius $c/4$. Then $\gamma^{-1}(B) \cap A \subseteq X$ is nowhere dense.*

Proof. As $\gamma^{-1}(B) \cap A$ is relatively closed in A, we show that it has empty interior. Let $W \subseteq A$ be non-empty open. Pick $p \in W \cap D$. Then $p[c]$ is a closed, nowhere dense set, so find $q \in (W \setminus p[c]) \cap D$. Then $p, q \in W$ with $\partial(p, q) > c$. Suppose that $\{p, q\} = \{p_k, q_k\}$. Then $|\gamma_k(p) - \gamma_k(q)| > c/2$. In particular, $d_u(\gamma(p), \gamma(q)) > c/2$, so $W \not\subseteq \gamma^{-1}(B)$. □

LEMMA 8.4. *Suppose Z is a minimal G-flow, $x \in Z$, and $S \subseteq G$ is syndetic. Then $x \cdot S \subseteq Z$ is somewhere dense.*

Proof. Since $S \subseteq G$ is syndetic, find $g_0, \ldots, g_{k-1} \in G$ with $\bigcup_{i<k} Sg_i = G$. Then $\bigcup_{i<k}(x \cdot S) \cdot g_i = x \cdot G \subseteq Z$ is dense, so $x \cdot Sg_i$ is somewhere dense for some $i < k$. Then by translating, $x \cdot S$ is somewhere dense as well. □

Now let $\alpha\colon X \to [0, 1]$ be a continuous function with $\alpha^{-1}(\{1\}) \neq \emptyset$ and $\alpha[X \setminus A] = \{0\}$. Form the function $\theta = \alpha \times \gamma\colon X \to [0, 1] \times [0, 1]^\omega$. Pick $p \in X$, and then form $\theta_p\colon G \to [0, 1] \times [0, 1]^\omega$.

We will show that $\overline{\theta_p \cdot G}$ has all orbits meager. Towards a contradiction, suppose $\theta_q \in \overline{\theta_p \cdot G}$ belonged to a comeager orbit; as $\{q \in X : \alpha(q) > 3/4\} \subseteq X$ is open, we may assume that q belongs to this set. Let $r > 0$ be small enough so that both $r < c/4$ and for $g \in U_r$, we have $\alpha(qg) > 1/2$. By the Effros theorem, $\theta_q \cdot U$ is a relatively open subset of $\theta_q \cdot G$. By Fact 6.5, it follows that $S := \{g \in G : \theta_q \cdot g \in \theta_q \cdot U\}$ is syndetic, so by Lemma 8.4, $q \cdot S \subseteq X$ is somewhere dense. Furthermore, for $g \in G$, we have

$$\begin{aligned} \theta_q \cdot g(1_G) &= \theta(qg) \\ &= (\alpha(qg), \gamma(qg)). \end{aligned}$$

It follows that for $g \in S$, there is $h \in U$ with $\alpha(qg) = \alpha(qh) > 1/2$. Hence $q \cdot S \subseteq A$. However, by item (1) of Proposition 4.5, $\gamma[q \cdot S] = \gamma[q \cdot U]$ lies in a d_u-ball of radius $c/4$, contradicting Lemma 8.3.

Question 8.5. Theorem 8.1 shows that for *any* non-metrizable minimal G-flow X all of whose orbits are meager, we have a factor $\varphi\colon S_G(X) \to Y$ where Y is metrizable and has all orbits meager. Is it necessary to pass to the universal highly proximal extension? More precisely, is there an example of a Polish group G and a minimal G-flow X with all meager orbits, but all of whose metrizable factors have a comeager orbit?

9. *Distal universal minimal flows*

As an application of Theorem 8.1, we prove Theorem 9.2, a characterization of when a Polish group G has distal universal minimal flow.

Definition 9.1. A G-flow X is called *distal* if for any pair of points $x \neq y \in X$ and any net g_i from G with $xg_i \to z \in X$, we have $yg_i \not\to z$.

THEOREM 9.2. *Let G be a Polish group, and assume that $M(G)$ is distal. Then $M(G)$ is metrizable.*

In [**11**], the authors consider Polish groups which are *strongly amenable*, groups which admit no non-trivial minimal proximal flows. They prove that if G is strongly amenable

and $M(G)$ is metrizable, then G has a closed, normal, extremely amenable subgroup H with G/H compact and $M(G) \cong G/H$. As any group G with $M(G)$ distal is also strongly amenable, we obtain the following corollary.

COROLLARY 9.3. *Let G be a Polish group with $M(G)$ distal. Then G has a closed, normal, extremely amenable subgroup H with G/H compact and $M(G) \cong G/H$.*

We briefly review some facts about enveloping semi-groups and distal flows; see [**2**] for more detail. To any G-flow X, we can associate to it the *enveloping semi-group* $E(X)$. Given $g \in G$, form the function $\rho_g \colon X \to X$ given by $\rho_g(x) = xg$. Then $E(X)$ is the closure of the set $\{\rho_g : g \in G\}$ in the compact space X^X. Each $f \in E(X)$ is a function, and because we take our G-flows to be right actions, it will be more convenient to write function application and composition on the right, that is, for $x \in X$ and $f \in E(X)$, we write xf instead of $f(x)$. Then $E(X)$ becomes a compact left-topological semi-group, in particular a G-flow, where $f \cdot g = \rho_g \circ f$. For any $x \in X$, the map $\lambda_x \colon E(X) \to X$ given by $\lambda_x(f) = xf$ is a G-map.

When X is distal, $E(X)$ is a group. Furthermore, if X is also minimal, then $E(X)$ is a minimal distal system. If $f \in E(X)$, then the left multiplication map λ_f is a G-flow automorphism. In particular, if $M(G)$ is distal, then $E(M(G)) \cong M(G)$, and for any $p, q \in M(G)$, there is a G-flow automorphism φ with $\varphi(p) = q$.

In the proof of Theorem 9.2, we will need the following simple proposition.

PROPOSITION 9.4. [**2**, Corollary 7(c)] *Let Y be a distal flow, and let $\varphi \colon Y \to X$ be a factor. Then X is also distal*

We will also need to recall the main result of [**13**].

Fact 9.5. [**13**, Corollary 3.3] If X is a minimal, metrizable flow with all orbits meager, then the universal highly proximal extension $S_G(X)$ is non-metrizable. In particular, the map $\pi_X \colon S_G(X) \to X$ is a non-trivial highly proximal extension.

We can now complete the proof of Theorem 9.2. Toward a contradiction, suppose $M(G)$ is distal, but not metrizable. Then by Theorem 5.3, we have $|M(G)| = 2^{\mathfrak{c}}$, so in particular $M(G)$ contains more than one orbit. As there is a G-flow automorphism bringing any one orbit to any other, we see that $M(G)$ contains all meager orbits. By Theorem 8.1, let X be a minimal metrizable flow with all meager orbits. Then by Fact 9.5, $\pi_X \colon S_G(X) \to X$ is a non-trivial highly proximal extension of minimal flows, which implies that $S_G(X)$ is not distal. Now let $\varphi \colon M(G) \to S_G(X)$ be a G-map. By Proposition 9.4, we must also have $M(G)$ not distal, completing our contradiction.

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