PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 148, Number 3, March 2020, Pages 1161–1168
https://doi.org/10.1090/proc/14765
Article electronically published on September 20, 2019

A NOTE ON MINIMAL MODELS FOR PMP ACTIONS

ANDY ZUCKER

(Communicated by Nimish Shah)

ABSTRACT. Given a countable group G, we say that a metrizable flow Y is *model-universal* if by considering the various invariant measures on Y, we can recover every free measure-preserving G-system up to isomorphism. Weiss in[*Dynamical systems and group actions*, American Mathematical Society, Providence, RI, 2012, pp. 249–264] constructs a minimal model-universal flow. In this note, we provide a new, streamlined construction, allowing us to show that a minimal model-universal flow is far from unique.

In this paper, we consider actions of an infinite countable group G on a standard Borel probability space (X, μ) by Borel, measure-preserving bijections. When an action $a\colon G \times X \to X$ is understood, we will suppress the action notation, and given $g \in G$ and $x \in X$ just write gx or $g \cdot x$ for $a(g, x)$. We will refer to (X, μ) as a *G-system.* A G-system is *free* if for μ-almost every $x \in X$, we have $G_x = \{1_G\}$, where $G_x := \{g \in G : gx = x\}$ is the *stabilizer* of $x \in X$. By passing to a subset of measure 1, we will often implicitly assume that every point in a free G-system has trivial stabilizer. If (X, μ) and (Y, ν) are two G-systems, we say that (Y, ν) is a *factor* of (X, μ) if there is a Borel $X' \subseteq X$ with $\mu(X') = 1$ and a Borel G-equivariant map $f\colon X' \to Y$ with $\nu = f^*\mu$. If we can find such an f that is also injective, then we call (X, μ) and (Y, ν) *isomorphic G-systems.*

A *G-flow* is an action of G by homeomorphisms on a compact Hausdorff space. We similarly suppress the action notation. Given a G-system (X, μ), a *model* for (X, μ) is a compact metric G-flow Y and an invariant Borel probability measure ν so that (X, μ) and (Y, ν) are isomorphic G-systems. We will be most interested in *minimal* G-flows, those G-flows in which every orbit is dense. Notice that any minimal model of a free G-system must be *essentially free*, where a G-flow Y is essentially free if for each $g \in G \setminus \{1_G\}$, the set $\{y \in Y : gy = y\}$ is nowhere dense.

We say that a metrizable G-flow Y is *model-universal* if by considering the various invariant measures ν on Y, the G-systems (Y, ν) recover every (standard) free G-system up to isomorphism. In [5], Weiss constructs for every countable group G a minimal model-universal flow. It is natural to ask in what sense a minimal model-universal flow must be unique. Here, we prove a strong negative result. Given a family $\{Y_i : i \in I\}$ of minimal G-flows, we say that $\{Y_i : i \in I\}$ is *mutually disjoint* if the product $\prod_{i\in I} Y_i$ is minimal. In particular, this implies that the Y_i are pairwise non-isomorphic G-flows.

Received by the editors March 19, 2019, and, in revised form, July 11, 2019.
2010 *Mathematics Subject Classification.* Primary 37B05; Secondary 28D15.
The author was supported by NSF Grant no. DMS 1803489.

Theorem 1. *For any countable group G, there is a mutually disjoint family $\{Y_i : i < \mathfrak{c}\}$ of minimal model-universal flows.*

Let us call a G-flow Y *weakly model-universal* if for every free G-system (X, μ), there is an invariant measure ν on Y so that (Y, ν) is a factor of (X, μ). In [5], Weiss first constructs a minimal, essentially free, weakly model-universal flow, then proves that any flow with these properties admits an almost one-to-one extension which is model-universal. We instead build our model-universal flows in one step.

A recent result of Elek in [1] shows the existence of a *free* minimal model-universal flow. Recall that a G-flow Y is free when for any $y \in Y$ and any $g \in G \setminus \{1_G\}$, we have $gy \neq y$. In the last section of this paper, we show how one can deduce this result using rather soft arguments.

Theorem 2. *Let Y be a minimal, model-universal, Cantor flow. Then there is an almost one-to-one extension $\pi\colon Z \to Y$ so that Z is free, minimal, and model-universal.*

As almost one-to-one extensions always preserve minimality and disjointness, we can strengthen Theorem 1 as follows.

Theorem 3. *For any countable group G, there is a mutually disjoint family $\{Y_i : i < \mathfrak{c}\}$ of free, minimal, model-universal flows.*

1. Basic examples of model-universal flows

We briefly collect a few simple examples which will be important in what follows. Let K be a compact space. Then K^G is a G-flow with the right shift action, where given $g, h \in G$ and $s \in K^G$, we have $g \cdot s(h) = s(hg)$. Mostly we take $K = 2^n$ or 2^ω.

Proposition 4. *The flow $(2^\omega)^G$ is model-universal.*

Proof. Let (X, μ) be a free G-system, and fix $\varphi\colon X \to 2^\omega$ a Borel bijection. Now define $\psi\colon X \to (2^\omega)^G$ via $\psi(x)(g) = \varphi(g \cdot x)$. Then ψ is injective, and $(X, \mu) \cong ((2^\omega)^G, \psi^*\mu)$. □

A *subshift* of K^G is a closed, G-invariant subspace. The following family of subshifts of 2^G will be an important source of weakly model-universal flows. Let $Q \subseteq G$ be a finite symmetric set. We say that $S \subseteq G$ is *Q-spaced* if whenever $g, h \in S$ with $g \neq h$, then $Qg \cap Qh = \emptyset$. We say that S is *Q-syndetic* if we have $\bigcup_{g \in Q} gS = \bigcup_{g \in S} Qg = G$. Notice that maximal Q-spaced sets exist and are Q^2-syndetic. Conversely, any Q^2-syndetic Q-spaced set is a maximal Q-spaced set. We define

$$Y_Q = \{s \in 2^G : s^{-1}(\{1\}) \text{ is a maximal } Q\text{-spaced set}\}.$$

Proposition 5. *The flow Y_Q is weakly model-universal.*

Remark. This proposition is also one of the key ingredients used by Weiss (see [5, Lemma 2.2]).

Proof. Let (X, μ) be a free G-system. By freeness, we can find for every Borel $B \subseteq X$ with $\mu(B) > 0$ a Borel subset $A \subseteq B$ with $\mu(A) > 0$ and with $gA \cap A = \emptyset$ for any $g \in Q^2$. Let us call a Borel set A with this property a *Q^2-disjoint* set. Now if $\bigcup_{g \in Q^2} gA$ doesn't have full measure, we can find a Q^2-disjoint Borel set $A' \subseteq X$

with $\mu(A') > 0$ and $gA \cap A' = \emptyset$ for every $g \in Q^2$. As Q is assumed symmetric, it follows that $A \cup A'$ is also Q^2-disjoint.

Thus using a measure exhaustion argument, we can find $A \subseteq X$ a Q^2-disjoint Borel set so that $\mu\left(\bigcup_{g\in Q^2} gA\right) = 1$. We now let $\varphi\colon X \to 2^G$ be the map given by $\varphi(x)(g) = 1$ iff $gx \in A$. Then for almost every $x \in X$, $\varphi(x)^{-1}(\{1\})$ is both Q^2-syndetic and Q-spaced, so a maximal Q-spaced set. It follows that Y_Q contains the closed support of $\varphi^*\mu$, so $(Y_Q, \varphi^*\mu)$ is a factor of (X, μ). □

We end the section by noting a simple closure property of (weakly) model-universal flows.

Proposition 6. *Let Y_n be weakly model-universal G-flows. Then $Y := \prod_n Y_n$ is weakly model-universal. If at least one of the Y_n is model-universal, then so is Y.*

Proof. Let (X, μ) be a free G-system, and for each $n < \omega$, let $\varphi_n\colon X_n \to Y_n$ be a Borel, G-equivariant map, where $X_n \subseteq X$ satisfies $\mu(X_n) = 1$. Set $X' = \bigcap_n X_n$. Then $\mu(X') = 1$, and the map $\varphi\colon X' \to \prod_n Y_n$ given by $\varphi(x) = (\varphi_n(x))_{n<\omega}$ is Borel and G-equivariant. If for some $n < \omega$, the map φ_n is injective, then φ will also be injective. □

2. Strongly irreducible subshifts

The key technical tool we use here is the notion of a *strongly irreducible subshift.* First, we introduce some general terminology. Write $\mathrm{Fin}(G)$ for the collection of finite subsets of G. Given $S_1, S_2 \subseteq G$ and $D \in \mathrm{Fin}(G)$, we say that S_1 and S_2 are *D-apart* if $DS_1 \cap DS_2 = \emptyset$. Let A be a finite set. If $Y \subseteq A^G$ is a subshift and $F \in \mathrm{Fin}(G)$, we define the *F-patterns* of Y to be the set $S_F(Y) := \{s|_F : s \in Y\} \subseteq A^F$. Given $\alpha \in S_F(Y)$, we define the basic clopen neighborhood $N_Y(\alpha) := \{y \in Y : y|_F = \alpha\}$. If $F \in \mathrm{Fin}(G)$, $S \subseteq G$, $\alpha \in A^F$, and $\beta \in A^S$, we say that α *appears in* β if there is $g \in G$ with $Fg \subseteq S$ and $\beta(fg) = \alpha(f)$ for each $f \in F$. We say in this case that α *appears at* $g \in G$.

We say that Y is *strongly irreducible* if there is $D \in \mathrm{Fin}(G)$ so that for any $F_0, F_1 \in \mathrm{Fin}(G)$ which are D-apart and any $\alpha_i \in S_{F_i}(Y)$, there is $y \in Y$ with $y|_{F_i} = \alpha_i$. We sometimes say that Y is D-irreducible. We will frequently use the following facts about strongly irreducible subshifts. Here A and B are finite sets.

(1) If $Y \subseteq A^G$ is D_Y-irreducible and $Z \subseteq B^G$ is D_Z-irreducible, then $Y \times Z \subseteq (A \times B)^G$ is $(D_Y \cup D_Z)$-irreducible.
(2) Suppose $Y \subseteq A^G$ is D-irreducible and $\varphi\colon Y \to B^G$ is continuous and G-equivariant. By continuity, there is $F \in \mathrm{Fin}(G)$ so that $\varphi(y)(1_G)$ depends only on $y|_F$. Then $Z := \varphi[Y]$ is DF-irreducible.

We will also need a method of making explicit choices of patterns in $S_F(Y)$. To that end, suppose that A is linearly ordered, and enumerate the group G in some fashion. This allows us to order $S_F(Y)$ lexicographically. We will use this ordering in the following two ways. Fix $Y \subseteq A^G$ a D-irreducible subshift.

(1) If $F_0, ..., F_{n-1} \in \mathrm{Fin}(G)$ are pairwise D-apart, $\alpha_i \in S_{F_i}(Y)$, and $E \in \mathrm{Fin}(G)$ contains each F_i, then we let $\mathrm{Conf}_Y(\alpha_0, ..., \alpha_{n-1}, E) \in S_E(Y)$ be the lexicographically least E-pattern β satisfying $\beta|_{F_i} = \alpha_i$.
(2) Every strongly irreducible subshift is topologically transitive. In particular, fix $F \in \mathrm{Fin}(G)$. Then for any $E \in \mathrm{Fin}(G)$ containing at least $|S_F(Y)|$ many disjoint right translates of DF, there is $\beta \in S_E(Y)$ so that every $\alpha \in S_F(Y)$

appears in β. We let $\mathrm{Trans}_Y(F,E)$ be the lexicographically least E-pattern with this property.

Most of the time, we take $A = 2^n$ for some $n < \omega$, and we take the lexicographic ordering on 2^n as the ordering on A.

3. The operator Φ

A subset $S \subseteq G$ is called *syndetic* if S is Q-syndetic for some $Q \in \mathrm{Fin}(G)$. Given $F \in \mathrm{Fin}(G)$ and $Y \subseteq A^G$ a subshift, we say that Y is *F-minimal* if for every $y \in Y$, every $\alpha \in S_F(Y)$ appears in y. Equivalently, for every $y \in Y$, every $\alpha \in S_F(Y)$ appears syndetically often. The following observation will be useful; suppose $Y \subseteq A^G$ is F-minimal and that every $\alpha \in S_F(Y)$ appears E-syndetically for some $E \in \mathrm{Fin}(G)$. Then every $\alpha \in S_F(Y)$ appears in every $\beta \in S_{FE}(Y)$.

The following is our main method of producing strongly irreducible, F-minimal flows. First, recalling the flow Y_Q from section 1, we note that Y_Q is Q^3-irreducible. Now let $Y \subseteq A^G$ be D-irreducible. Let $E \in \mathrm{Fin}(G)$ be symmetric, contain D, and be large enough to contain at least $|S_F(Y)| \leq |A|^{|F|}$ many disjoint right translates of DF. Let $C \in \mathrm{Fin}(G)$ be symmetric with $E^5 \subseteq C$. We define a continuous, G-equivariant map $\varphi\langle Y,F,E,C\rangle = \varphi \colon Y \times Y_C \to A^G$ as follows. Suppose $(y,s) \in Y \times Y_C$, and write $z = \varphi(y,s)$. Let $g \in G$.

- If $g = kh$, where $s(h) = 1$ and $k \in E$, set $z(g) = \mathrm{Trans}_Y(F,E)(k)$.
- If there are not $k \in E^3$ and $h \in G$ with $s(h) = 1$ and $g = kh$, set $z(g) = y(g)$.
- If $g = kh$, where $s(h) = 1$ and $k \in E^3 \setminus E$, set

$$z(g) = \mathrm{Conf}_Y(\mathrm{Trans}_Y(F,E),\, (h\cdot y)|_{E^5\setminus E^3},\, E^5)(k).$$

The idea behind this definition is to reprint y most of the time, using s to tell us where to overwrite with the pattern $\mathrm{Trans}_Y(F,E)$, and using strong irreducibility to blend everything together. This construction is a slight modification of a construction in [2]; see their Figure 3 for a good illustration.

It is routine to verify that φ as defined is continuous and G-equivariant. Denote by $\Phi(Y,F,E,C)$ the image of $\varphi = \varphi\langle Y,F,E,C\rangle$. Then $\Phi(Y,F,E,C)$ is C^5-irreducible.

Lemma 7. *We have $S_F(Y) = S_F(\Phi(Y,F,E,C))$.*

Proof. The $\subseteq$ direction is clear. For the $\supseteq$ direction, suppose $z \in \Phi(Y,F,E,C)$ with $z = \varphi(y,s)$. It is enough to show that $z|_F \in S_F(Y)$. If there is $h \in G$ with $s(h) = 1$ and $F \cap E^3h \neq \emptyset$, then $F \subseteq E^5h$, so we have

$$z|_F = \mathrm{Conf}_Y(\mathrm{Trans}_Y(F,E),(h\cdot y)|_{E^5\setminus E^3},E^5)|_F.$$

If there is no such $h \in G$, then we have $z|_F = y|_F$. □

For any $z \in \Phi(Y,F,E,C)$, the E-pattern $\mathrm{Trans}_Y(F,E)$ appears in z, so in particular every pattern in $S_F(Y)$ appears in z. Hence $\Phi(Y,F,E,C)$ is F-minimal. Indeed, every F-pattern appears C^3-syndetically, since maximal C-spaced sets are C^2-syndetic. So every pattern in $S_F(Y)$ appears in every pattern in $S_{C^4}(\Phi(Y,F,E,C))$.

4. A tree of subshifts

We now use the operator Φ to produce a tree of strongly irreducible flows. We will construct for each $s \in 2^{<\omega}$ a strongly irreducible flow $X_s \subseteq (2^{|s|})^G$ by induction.

This tree will be controlled by rapidly increasing sequences $\{D_k : k < \omega\}$, $\{E_k : k < \omega\}$, and $\{F_k : k < \omega\}$ of finite symmetric subsets of G. We will continue to add assumptions about how rapid this needs to be, but for now, we assume that

- $\bigcup_n D_n = \bigcup_n E_n = \bigcup_n F_n = G$.
- E_n contains at least $2^{|D_n|(n+1)}$-many pairwise disjoint translates of D_n^2.
- $F_n \supseteq E_n^5$.
- $D_{n+1} \supseteq F_n^5$.

Let $X_\emptyset$ be the trivial flow. If $s \in 2^{<\omega}$ and X_s is defined, and $t = s^\frown 0$, then we set $X_t = X_s \times 2^G$. Suppose we are given $k < \omega$, $s \in 2^k$, and $t = s^\frown 1 \in 2^{k+1}$. Then we set $X_t = \Phi(X_s \times 2^G, D_k, E_k, F_k)$.

In order to discuss the key properties of this construction, we think of $(2^n)^G$ as embedded into $(2^\omega)^G$ by adding zeros to the end. In this way, we can refer to the *$(n \times F)$-patterns* of a subflow $Y \subseteq (2^N)^G \cong 2^{N\times G}$, the set $S_{n\times F}(Y) := \{y|_{n\times F} : y \in Y\}$, whenever $N \geq n$.

(1) Each X_s is $D_{|s|}$-irreducible.
(2) For any $s \sqsubseteq t \in 2^{<\omega}$ with $|s| = n$, we have $S_{n\times D_n}(X_s) = S_{n\times D_n}(X_t)$.
(3) Suppose $s \in 2^{<\omega}$ is such that $|s| > n$ and $s(n) = 1$. Then every pattern in $S_{(n+1)\times D_n}(X_s)$ appears in every pattern in $S_{(n+1)\times D_{n+1}}(X_s)$.
(4) Suppose $s \in 2^n$. Then $S_{(n+1)\times D_{n+1}}(X_{s^\frown 0}) \neq S_{(n+1)\times D_{n+1}}(X_{s^\frown 1})$. This is because the conclusion of item (3) is true for $X_{s^\frown 1}$ and false for $X_{s^\frown 0} = X_s \times 2^G$.

We can now consider taking limits along the branches. It follows from item (2) above that for any $\alpha \in 2^\omega$, the flow $X_\alpha \subseteq (2^\omega)^G$ is well defined. We can think of X_α as a point in the space $K((2^\omega)^G)$ of compact subsets of $(2^\omega)^G$. The subshifts form a closed subspace, and given subshifts $\{Z_n : n < \omega\} \subseteq K((2^\omega)^G)$ and $Z \in K((2^\omega)^G)$, we have $Z_n \to Z$ iff for each finite $F \subseteq G$ and $k < \omega$, we eventually have $S_{k\times F}(Z_n) = S_{k\times F}(Z)$. With this topology, the map $\Theta\colon 2^\omega \to K((2^\omega)^G)$ given by $\Theta(\alpha) = X_\alpha$ is continuous. Item (4) shows that Θ is injective. Whenever $\alpha \in 2^\omega$ has $\alpha^{-1}(\{1\})$ infinite, then item (3) implies that X_α is a minimal flow.

Proposition 8. *For any $\alpha \in 2^\omega$ with $\alpha^{-1}(\{0\})$ and $\alpha^{-1}(\{1\})$ infinite, the flow X_α is a minimal, model-universal flow.*

Proof. Having already discussed minimality, we focus on model-universality. Write $T = \alpha^{-1}(\{1\})$, and form the flow $Y_\alpha := (2^G)^\omega \times \prod_{n\in T} Y_{F_n}$. Then Y_α is model-universal. We have a continuous G-map $\psi_\alpha\colon Y_\alpha \to \prod_n X_{\alpha|_n}$ given inductively as follows. First let $f_\omega\colon \omega \to (\omega \setminus T)$ and $f_T\colon T \to T$ be infinite-to-one surjections. Let $y \in Y_\alpha$, and write $y = \{(y_n)_{n<\omega}, (s_n)_{n\in T}\}$ with $y_n \in 2^G$ and $s_n \in Y_{F_n}$. Then we write $\psi_\alpha(y) = (\psi_\alpha(y)_n)_{n<\omega}$ with each $\psi_\alpha(y)_n \in X_{\alpha|_n}$. We let $\psi_\alpha(y)_0$ be the unique member of the trivial flow $X_\emptyset$. If $\psi_\alpha(y)_n$ has been defined and $n \notin T$, then $\psi_\alpha(y)_{n+1} = (\psi_\alpha(y)_n, y_{f_\omega(n)})$. If $n \in T$, then $\psi_\alpha(y)_{n+1} = \varphi_n((\psi_\alpha(y)_n, s_{f_T(n)}), s_n)$, where $\varphi_n = \varphi\langle X|_{\alpha|_n} \times 2^G, D_n, E_n, F_n\rangle$.

Notice that if the sequence $(\psi_\alpha(y)_n)_{n<\omega}$ converges to some $x \in (2^\omega)^G$, then $x \in X_\alpha$. Let $Y'_\alpha \subseteq Y_\alpha$ be the subset of those y for which $\psi_\alpha(y)_n$ is convergent. Then the map $\eta\colon Y'_\alpha \to X_\alpha$ with $\eta(y) = \lim_n \psi_\alpha(y)_n$ is Borel. It suffices to show that if the D_n grow rapidly enough, then Y'_α has measure 1 for any G-invariant measure on Y_α. To that end, fix $y = ((y_n)_{n<\omega}, (s_n)_{n\in T})$, and consider some $g \in G$. A sufficient condition for the sequence $\psi_\alpha(y)_n(g)$ to be convergent is that for a tail of $n \in T$, we have $s_n(h) = 0$ whenever $h \in E_n^3 g$. This condition ensures that for suitably large $n \in T$, we have $\psi_\alpha(y)_{n+1}(g) = (\psi_\alpha(y)_n(g), s_{f_T(n)}(g))$. Define $Y''_\alpha \subseteq Y'_\alpha$ to be those y for which on a tail of $n \in T$, we have $s_n(g) = 0$ for any $g \in E_n^4$. Notice that Y''_α is also Borel and G-invariant.

Fix ν an invariant measure on Y_{F_n}. Then letting $U = \{s \in Y_{F_n} : s(1_G) = 1\}$, we have $\nu(U) \leq 1/|F_n|$. This is because $g \cdot U = \{s \in Y_{F_n} : s(g^{-1}) = 1\}$, so by definition of the subshift Y_{F_n}, we have that the collection $\{g \cdot U : g \in F_n\}$ is pairwise disjoint. Then by invariance and a union bound, we have $\nu(\{s \in Y_{F_n} : s(g) = 1 \text{ for some } g \in E_n^4\}) \leq |E_n^4|/|F_n|$. We now add our last assumption to the growth of the D_n.

- $|E_n^4|/|F_n| < 1/2^n$.

From this assumption, it follows from the Borel-Cantelli lemma that for any invariant measure μ on Y_α that $\mu(Y''_\alpha) = 1$.

Furthermore, we claim that η is injective on Y''_α. To see this, suppose that $y \neq y' \in Y''_\alpha$, with $y = \{(y_n)_{n<\omega}, (s_n)_{n\in T}\}$ and $y' = \{(y'_n)_{n<\omega}, (s'_n)_{n\in T}\}$. First suppose that $y_n(g) \neq y'_n(g)$ for some $n < \omega$ and $g \in G$. Then for some large enough $N < \omega$ and any $k, \ell \geq N$, we have $\psi_\alpha(y)_k(g) = \psi_\alpha(y)_\ell(g)$, and the same for y'. Now pick some suitably large $k \in \omega \setminus T$ with $f_\omega(k) = n$. Then $\psi_\alpha(y)_{k+1}(g) = \psi_\alpha(y)_k(g) \times y_n(g)$, and similarly for y'. It follows that $\eta(y) \neq \eta(y')$. In the case that $s_n(g) \neq s'_n(g)$ for some $n \in T$, the argument is almost the same. For a suitably large $k \in T$ with $f_T(k) = n$, we use the assumption that y and y' are in Y''_α to see that $\psi_\alpha(y)_{k+1}(g) = \psi_\alpha(y)_k(g) \times s_n(g)$, and similarly for y'. Once more, we have $\eta(y) \neq \eta(y')$. ☐

To prove Theorem 1, we need to recall some results from [3] (in particular, see Corollary 6.8). There, it is shown that every minimal flow is disjoint from every strongly irreducible subshift. From this, it follows that every minimal flow is disjoint from any X_α where α has a tail of zeros. Since disjointness is a G_δ condition ([3, Proposition 6.4]), it follows that every minimal flow is disjoint from X_α for comeagerly many $\alpha \in 2^\omega$. We are now in a position to apply Mycielski's theorem (see [4, 19.1]) to find our mutually disjoint family $\{X_{\alpha_i} : i < \mathfrak{c}\}$ of minimal, model-universal shifts.

5. From essentially free to free

Recall that if Y is a minimal metrizable flow, then an extension $\pi\colon Z \to Y$ is called *almost one-to-one* if the set $\{z \in Z : |\pi^{-1}(\{\pi(z)\})| = 1\}$ is comeager. Notice that Z must also be minimal. To see this, let $z \in Z$ and $V \subseteq Z$ be non-empty open. Then find $z' \in V$ with $|\pi^{-1}(\{\pi(z')\})| = 1$. We can find a net $g_i \in G$ with $g_i \cdot \pi(z) \to \pi(z')$. It follows that $g_i \cdot z \to z'$. In particular, the orbit of z meets V.

One method of producing almost one-to-one extensions of a given minimal G-flow is to consider $\mathrm{Reg}(Y)$, the Boolean algebra of regular open subsets of Y. Recall that $A \subseteq Y$ is *regular open* if $\mathrm{Int}(\overline{A}) = A$. We remind the reader that in this Boolean algebra, we have $A^c = Y \setminus \overline{A}$, $A \vee B = \mathrm{Int}(\overline{A \cup B})$, and $A \wedge B = A \cap B$. If $\mathcal{B} \subseteq \mathrm{Reg}(Y)$ is a subalgebra, then $\mathrm{St}(\mathcal{B})$, the space of ultrafilters on $\mathcal{B}$, is a compact, zero-dimensional space whose basic clopen neighborhood has the form $\{p \in \mathrm{St}(\mathcal{B}) : A \in p\}$, where $A \in \mathcal{B}$. If $\mathcal{B}$ is also G-invariant, then $\mathrm{St}(\mathcal{B})$ is a G-flow. If $\mathcal{B}$ is countable, then $\mathrm{St}(\mathcal{B})$ is homeomorphic to Cantor space. Now suppose that $\mathcal{B}$ contains a basis for the topology on Y. Then we have a G-map $\pi\colon \mathrm{St}(\mathcal{B}) \to Y$ given by $\pi(p) = y$ iff every $A \in \mathcal{B}$ with $A \ni y$ satisfies $A \in p$. Furthermore, the map π is *pseudo-open*, meaning that images of open sets have non-empty interior.

For $y \in Y$, we have $|\pi^{-1}(\{y\})| = 1$ iff for every $A \in \mathcal{B}$, we have $y \in A$ or $y \in Y \setminus \overline{A}$. So when $\mathcal{B}$ is countable, the set $\{y \in Y : |\pi^{-1}(y)| = 1\}$ is comeager. Since π is pseudo-open, it follows that $\{z \in Z : |\pi^{-1}(\pi(z))| = 1\}$ is also comeager.

In general, an almost one-to-one extension can have very different measure-theoretic behavior than the base flow. Indeed, this fact is heavily exploited in [5]. For us however, we will seek to build almost one-to-one extensions which preserve the measure-theoretic properties of the base flow. For the remainder of the section, fix Y a minimal, model-universal flow whose underlying space is a Cantor set. Recall that this implies that Y is essentially free. We will call an invariant measure μ on Y *free* if for every $g \in G$, we have $\mu(Y_g) = 0$, where $Y_g = \{y \in Y : gy = y\}$.

Definition 9. Given $A \subseteq Y$, we call A *strongly regular open* if A is regular open and for every free invariant measure μ, we have $\mu(A) + \mu(Y \setminus \overline{A}) = 1$. Denote by $\mathrm{SReg}(Y)$ the collection of strongly regular open sets.

Proposition 10. *$\mathrm{SReg}(Y)$ is a G-invariant subalgebra of $\mathrm{Reg}(Y)$.*

Proof. Clearly $\mathrm{SReg}(Y)$ is G-invariant and closed under complements, so it is enough to check closure under intersection. Given $A, B \in \mathrm{SReg}(Y)$, we have

$$\begin{aligned}\overline{(A \cap B)} \setminus (A \cap B) &= \overline{(A \cap B)} \setminus A \cup \overline{(A \cap B)} \setminus B \\ &\subseteq (\overline{A} \setminus A) \cup (\overline{B} \setminus B).\end{aligned}$$

Since A and B are both strongly regular open, the last entry must have measure zero for any free invariant measure μ. ☐

Of course, we have yet to prove the existence of any interesting strongly regular open sets. We do this in the next lemma.

Lemma 11. *For every $g \in G \setminus \{1_G\}$, there is a partition of $Y \setminus Y_g$ into three relatively clopen pieces A_g, B_g, and C_g with the property that $gA_g \cap A_g = \emptyset$, and likewise for B_g and C_g. In particular, A_g, B_g, and C_g are all strongly regular open sets.*

Proof. Write $Y \setminus Y_g = \bigcup_n U_n$ with each U_n compact open. We may assume that the U_n are pairwise disjoint, and by further partitioning each U_n into finitely many clopen pieces if needed, we may assume that $gU_n \cap U_n = \emptyset$ for each $n < \omega$. We will inductively partition $V_n := \bigcup_{k<n} U_n$ into pieces A_n, B_n, and C_n with the property that $A_N \cap V_n = A_n$ for $N \geq n$, likewise for B_N and C_N. We then set $A_g = \bigcup_n A_n$, and likewise for B_g and C_g.

We set $A_0 = B_0 = C_0 = \emptyset$. Assume A_k, B_k, and C_k have been defined for some $k < \omega$. We will form clopen sets A_k', B_k', and C_k' so that $U_k = A_k' \cup B_k' \cup C_k'$. Partition U_k into finitely many clopen sets $\{W_j : j < m\}$ with the property that for each $j < m$ and for each $h \in \{g^{-1}, g\}$, we either have $hW_j \subseteq A_k$, $hW_j \subseteq B_k$, $hW_j \subseteq C_k$, or $hW_j \cap (A_k \cup B_k \cup C_k) = \emptyset$. Add each W_j to the set A_k', B_k', or C_k' in such a way so that if $hW_j \subseteq A_k$ for some h as above, then W_j is not added to A_k', and likewise for B_k' and C_k'. We then set $A_{k+1} = A_k \cup A_k'$, and likewise for B_{k+1} and C_{k+1}.

Notice that for each $n < \omega$, we have $gA_n \cap A_n = \emptyset$, and likewise for B_n and C_n. Hence A_g will also satisfy $gA_g \cap A_g = \emptyset$ as desired, and likewise for B_g and C_g. ☐

The last lemma we will need shows that metrizable, almost one-to-one extensions of Y using strongly regular open sets preserve the measure-theoretic properties of Y.

Lemma 12. *Let $\mathcal{B}$ be a countable G-invariant subalgebra of* $\mathrm{SReg}(Y)$ *extending the clopen algebra of Y. Let $Z = \mathrm{St}(\mathcal{B})$, and let $\pi\colon Z \to Y$ be the associated almost one-to-one extension. Then for any free invariant measure μ on Y, we have $\mu(\{y : |\pi^{-1}(\{y\})| = 1\}) = 1$.*

Proof. By the discussion at the beginning of the section, we have

$$\{y \in Y : |\pi^{-1}(\{y\})| = 1\} = \bigcap_{A\in\mathcal{B}} A \cup (Y \setminus \overline{A}).$$

Since $\mathcal{B}$ is a countable collection of strongly regular open sets, this set must have measure 1 for any free μ. $\square$

Proof of Theorem 2. Let $\mathcal{B} \subseteq \mathrm{SReg}(Y)$ be a countable, G-invariant subalgebra containing all of the sets A_g, B_g, C_g from Lemma 11. Then $\mathrm{St}(\mathcal{B})$ will be the desired flow. To see that $\mathrm{St}(\mathcal{B})$ is free, let $p \in \mathrm{St}(\mathcal{B})$ and $g \in G \setminus \{1_G\}$. Then p contains one of A_g, B_g, or C_g, WLOG say $A_g \in p$. Then since $gA_g \cap A_g = \emptyset$, we must have $gp \neq p$. To see that $\mathrm{St}(\mathcal{B})$ is model-universal, we note that on the set $Y_0 := \{y \in Y : |\pi^{-1}(\{y\})| = 1\}$, the map $\pi^{-1}\colon Y_0 \to Z$ is well defined. By Lemma 12, this set has measure 1 for all free invariant measures on Y. $\square$

Acknowledgments

The author would like to thank Benjamin Weiss for many helpful comments on an earlier draft, as well as the anonymous referee for suggesting many improvements.

References

[1] G. Elek, *Free minimal actions of countable groups with invariant probability measures*, preprint, https://arxiv.org/abs/1805.11149.

[2] J. Frisch, O. Tamuz, and P. Vahidi-Ferdowsi, *Strong amenability and the infinite conjugacy class property*, Invent. Math., to appear.

[3] E. Glasner, T. Tsankov, B. Weiss, and A. Zucker, *Bernoulli disjointness*, submitted, https://arxiv.org/abs/1901.03406.

[4] Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR1321597

[5] Benjamin Weiss, *Minimal models for free actions*, Dynamical systems and group actions, Contemp. Math., vol. 567, Amer. Math. Soc., Providence, RI, 2012, pp. 249–264, DOI 10.1090/conm/567/11253. MR2931921

Institut de Mathématiques de Jussieu - PRG, Université Paris Diderot, Paris, France 75013

Email address: `andrew.zucker@imj-prg.fr`