ON THE KINETIC WAVE TURBULENCE DESCRIPTION FOR NLS

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Abstract. The purpose of this note is two-fold: A) We give a brief introduction into the problem of rigorously justifying the fundamental equations of wave turbulence theory (the theory of nonequilibrium statistical mechanics of nonlinear waves), and B) we describe a recent work of the authors in which they obtain the so-called *wave kinetic equation*, predicted in wave turbulence theory, for the nonlinear Schrödinger equation on short but nontrivial time scales.

1. Introduction

1.1. **What is wave turbulence.** Wave turbulence can be succinctly defined as the theory of non-equilibrium statistical mechanics for nonlinear wave and dispersive systems. Untangling this definition serves to illustrate the rich set of problems addressed by this theory: Statistical mechanics aims at understanding the long-time behavior of systems with many (possibly infinite) degrees of freedom, typically by tracking the effective dynamics of some macroscopic quantities obtained by averaging out some of available (microscopic) degrees of freedom. Equilibrium statistical mechanics often addresses questions pertaining to isolated systems (no forcing or dissipation), with a focus on *equilibrium steady states* or invariant measures. On the other hand, non-equilibrium statistical mechanics is interested in transient phenomena for isolated systems, like *How does the system relax to its steady state or invariant measure?*, as well as steady states of systems that are not isolated, like those affected by external forcing and/or dissipation.

The statistical mechanics of particle systems is one of the great triumphs of physics in the nineteenth and twentieth century. It has developed into a highly successful theory whose ideas had a profound impact both on mathematics, as well as other fields of physics (like quantum mechanics). The most relevant aspect of this theory for us here is its non-equilibrium version as presented by the Boltzmann theory. The fundamental outcome of that theory is a kinetic equation governing the particle distribution function $f(t, x, v)$, which describes the effective dynamics of the density of particles at space point x, with velocity v at time t. This is given by Boltzmann's celebrated equation, which has the form

$$\partial_t f + v \cdot \nabla_x f = C(f, f) \tag{1.1}$$

where $C(f, f)$ is a nonlinear collision term.

The fundamental mathematical question here is to justify this passage from the reversible (in time) microscopic dynamics of $N-$ particles to the irreversible dynamics of the effective quantity $f(t, x, v)$. This was first done rigorously in Lanford [18], and later clarified in [15], which allows to approximate the microscopic density of the $N-$particles with the solution of the above Boltzmann equation in the so-called Boltzmann-Grad limit.

Being as fundamental as particle systems, physicists proposed a parallel kinetic theory for wave systems starting with the work of Peierls [24] in his investigations of solid state physics, as well as the work of Hasselmann [16, 17] on water waves. In both investigations, a kinetic equation analogous to (1.1) was derived for the corresponding wave system. The subject was later tremendously invigorated by Zakharov and his collaborators [26] after the discovery of special power-type stationary solutions for such kinetic equations, which provided wave-analogs of Kolmogorov's spectra of hydrodynamic turbulence. This is one reason why this kinetic theory of non-equilibrium statistical mechanics of waves came to be called "wave turbulence" (or weak turbulence in some of the literature). These so-called *Kolmogorov-Zakharov spectra* predict steady states of the corresponding microscopic wave system (possibly with forcing and dissipation at well-separated extreme scales), where the energy *cascades* at a constant flux through the (intermediate) frequency scales.

We won't go much into details of these spectra and their implications on the statistical physics of the dispersive system understudy; for this, we refer to [22, 23] for recent reviews. One should mention that the wave kinetic equation is of utmost importance in several areas of physics and engineering. For instance, in oceanography, the relevant kinetic equation is integrated on a daily basis to perform ocean forecasting.

1.2. **The wave kinetic equation.** As mentioned above, the fundamental mathematical question here is to provide a rigorous derivation of the wave kinetic equation from the dispersive PDE describing the microscopic systems. This means to justify the approximation of the relevant averaged quantity of the microscopic system by the solution of the kinetic equation assuming well-prepared initial data. To illustrate the problematic concretely, let us consider the nonlinear Schrödinger equation as our microscopic dispersive model. A natural starting setting is to consider this equation on a large box of size L with periodic boundary conditions, which we denote by $\mathbb{T}^d_L$. The equation is given by

$$\begin{cases} i\partial_t v - \frac{1}{2\pi}\Delta_\beta v = -|v|^2 v, \quad x \in \mathbb{T}^d_L = [0,L]^d, \\ v(0,x) = v_0(x). \end{cases}$$

In addition to the parameter L, another parameter that plays a central role in the analysis, is the characteristic size of the initial data. To track this parameter, we adopt the ansatz $v = \lambda u$, and think of u to be uniformly bounded in some relevant norm. The equation satisfied by u is then given by

$$\begin{cases} i\partial_t u - \frac{1}{2\pi}\Delta_\beta u = -\lambda^2|u|^2 u, \quad x \in \mathbb{T}^d_L = [0,L]^d, \\ u(0,x) = u_0(x). \end{cases} \tag{NLS}$$

where

$$\Delta_\beta := \sum_{i=1}^d \beta_i \partial_i^2,$$

and $\beta := (\beta_1, \ldots, \beta_d) \in [1,2]^d$. The case when $\beta_i = 1$ for all $1 \le i \le d$ corresponds to the rational torus, whereas the case when β_i are rationally independent is equivalent to working with the usual nonlinear Schrödinger equation on the irrational torus. This will be the setting where our result holds. We will denote by $\mathbb{Z}^d_L := \frac{1}{L}\mathbb{Z}^d$, the Fourier dual space of $\mathbb{T}^d_L$.

Typically in this theory, the initial data are randomly distributed in an appropriate fashion. For us, we take random initial data of the form

$$u_0(x) = \frac{1}{L^d}\sum_{k\in\mathbb{Z}^d_L} \widehat{u}_0(k)e^{2\pi i k\cdot x}; \qquad \widehat{u}_0(k) = \sqrt{\phi(k)}e^{2\pi i\vartheta_k(\omega)}, \tag{1.2}$$

for some nice (say Schwartz) deterministic function $\phi : \mathbb{R}^d \to [0,\infty)$. The phases $\vartheta_k(\omega)$ are independent random variables, uniformly distributed on $[0,1]$. Notice that with this normalization of the Fourier transform is chosen so that

$$\|u_0\|_{L^2} \sim 1,$$

and we should remark here that different normalizations of the Fourier transform would yield different definitions of the kinetic time scale mentioned below[1], however they are all of course mathematically equivalent.

Taking the Fourier transform of the equation, and recalling that the Fourier dual of $\mathbb{T}^d_L$ is $Z^D_L := (\mathbb{Z}/L)^d$, we obtain the following equation

$$i\partial_t \widehat{u}_k + 2\pi Q(k)\widehat{u}_k = \frac{i\lambda^2}{L^{2d}} \sum_{k_1-k_2+k_3=k} \widehat{u}_{k_1}\overline{\widehat{u}_{k_2}}\widehat{u}_{k_3}, \qquad k\in Z^d_L, \quad Q(k) := \sum_{i=1}^{d} \beta_i (k_i)^2.$$

Filtering by the linear oscillations, one sets the ansatz $a_k(t) = u_k(t)e^{itQ(k)t}$, to obtain the following integral equation for $a_k(t)$

$$a_k(t) = a^0_k + \frac{i\lambda^2}{L^{2d}} \int_0^t \sum_{\substack{(k_1,k_2,k_3)\in(\mathbb{Z}^d_L)^3 \\ k-k_1+k_2-k_3=0}} a_{k_1}\overline{a_{k_2}}a_{k_3} e^{-2\pi i s \Omega(k,k_1,k_2,k_3)}\, ds \tag{1.3}$$

The main conjecture of wave turbulence theory is that as $L\to\infty$ (big box limit) and $\frac{\lambda^2}{L^d}\to 0$ (weakly nonlinear limit), the mass density

$$\rho^L_k(t) = \mathbb{E}|a_k(t)|^2$$

converges to the solution of a kinetic equation. More precisely, it is conjectured that, as $L\to\infty$ and $\frac{\lambda^2}{L^d}\to 0$, $\rho^L_k(t)\sim\rho(t,k)$ as $t\to\infty$, where $\rho:\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}_+$ satisfies the wave kinetic equation

$$\begin{cases} \partial_t\rho = \frac{1}{\tau}\mathcal{T}(\rho), & \text{where} \quad \tau\sim\left(\frac{L^d}{\lambda^2}\right)^2, \\ \rho(0,k) = \phi(k). \end{cases} \tag{WKE}$$

and furthermore

$$\mathcal{T}(\rho)(k) = \iiint_{(\mathbb{R}^d)^r} \delta(\Sigma)\delta(\Omega)\rho(k)\prod_{i=1}^{3}\rho(k_i)\left[\frac{1}{\rho(k)} + \sum_{i=1}^{3}\frac{(-1)^i}{\rho(k_i)}\right] dk_1\dots dk_3$$

with

$$\begin{cases} \Sigma = \Sigma(k,k_1,\dots,k_3) = k + \sum_{i=1}^{3}(-1)^i k_i \\ \Omega = \Omega(k,k_1,\dots,k_3) = Q(k) + \sum_{i=1}^{3}(-1)^i Q(k_i). \end{cases}$$

Several partial or heuristic derivations have been put forward for equations similar to (WKE) [1, 2, 3, 12, 10, 13, 19, 21, 25]. However, to the best of our knowledge, there is no rigorous mathematical statement on the derivation of (WKE) from random initial data. The closest attempt in this direction is due to Lukkarinen and Spohn [20], who studied behavior of correlations for the Gibbs invariant measure of the discrete nonlinear Schrödinger equation. We remark that this invariant measure corresponds to the stationary solution $\rho(k) = \frac{1}{a+b|k|^2}$ of (WKE).

[1]For instance, a common normalization in the physics literature is to have $f(x) = \frac{1}{L^{d/2}}\sum_{k\in\mathbb{Z}^d_L}\widehat{f}(k)e^{2\pi i k\cdot x}$, which would make the kinetic time scale $\tau\sim\lambda^{-4}$ instead of $L^{2d}\lambda^{-4}$.

2. The wave kinetic equation approximation

2.1. **Formal derivation of the kinetic equation.** The starting point is the Fourier space formulation of equation (NLS) given in (1.3).

Step 1: expanding in the data. Noting the symmetry in (1.3) in the variables k_1 and k_3, we obtain after integrating by parts twice,

$$
\begin{aligned}
a_k(t) = & a_k^0 && \text{(2.1a)}\\
& + \frac{\lambda^2}{L^{2d}} \sum_{k-k_1+k_2-k_3=0} a^0_{k_1} \overline{a^0_{k_2}} a^0_{k_3} \frac{1-e^{-2\pi i t \Omega(k,k_1,k_2,k_3)}}{2\pi\Omega(k,k_1,k_2,k_3)} && \text{(2.1b)}\\
& + 2\frac{\lambda^4}{L^{4d}} \sum_{\substack{k-k_1+k_2-k_3=0\\ k_1-k_4+k_5-k_6=0}} a^0_{k_4} \overline{a^0_{k_5}} a^0_{k_6} \overline{a^0_{k_2}} a^0_{k_3} \frac{1}{2\pi\Omega(k,k_1,k_2,k_3)} \\
& \left[\frac{e^{-2\pi i t \Omega(k,k_4,k_5,k_6,k_2,k_3)}-1}{2\pi\Omega(k,k_4,k_5,k_6,k_2,k_3)} - \frac{e^{-2\pi i t \Omega(k_1,k_4,k_5,k_6)}-1}{2\pi\Omega(k_1,k_4,k_5,k_6)} \right] && \text{(2.1c)}\\
& + \frac{\lambda^4}{L^{4d}} \sum_{\substack{k-k_1+k_2-k_3=0\\ k_2-k_4+k_5-k_6=0}} a^0_{k_1} \overline{a^0_{k_4}} a^0_{k_5} \overline{a^0_{k_6}} a^0_{k_3} \frac{1}{2\pi\Omega(k,k_1,k_2,k_3)} \\
& \left[\frac{e^{-2\pi i t \Omega(k,k_1,k_4,k_5,k_6,k_3)}-1}{2\pi\Omega(k,k_1,k_4,k_5,k_6,k_3)} - \frac{e^{-2\pi i t \Omega(k_2,k_4,k_5,k_6)}-1}{2\pi\Omega(k_2,k_4,k_5,k_6)} \right] && \text{(2.1d)}\\
& + \{\text{higher order terms}\}. && \text{(2.1e)}
\end{aligned}
$$

where we denoted $\Omega(k,k_1,k_2,k_3,k_4,k_5) = Q(k)-Q(k_1)+Q(k_2)-Q(k_3)+Q(k_4)-Q(k_5)$; we also used the convention that, if $a=0$, $\frac{e^{2\pi i t a}-1}{2\pi a} = it$, while, if $a=b=0$, $\frac{1}{2\pi a}\left(\frac{e^{2\pi i t(a+b)}-1}{2\pi(a+b)} - \frac{e^{2\pi i t a}-1}{2\pi a}\right) = -\frac{1}{2}t^2$.

Step 2: Expectation pairing. We now compute $\mathbb{E}|a_k|^2$, where the expectation $\mathbb{E}$ is understood with respect to the random phases, and we use

$$
\mathbb{E}(a^0_{k_1}\dots a^0_{k_s}\overline{a^0_{\ell_1}\dots a^0_{\ell_s}}) = \begin{cases} \phi_{k_1}\dots\phi_{k_s} & \text{if there exists a permutation } \nu \text{ such that } k_{\nu(i)} = \ell_i \\ 0 & \text{otherwise.} \end{cases}
$$

(for $k \in \mathbb{Z}^d_L$, we write $\phi_k = \phi(k)$). Computing $\mathbb{E}\left(|a_k|^2\right)$, we see that there are no terms of order λ^2. Terms of order λ^4 can be obtained in different ways: either by pairing the term of order λ^2, namely (2.1b), with its conjugate, or by pairing one of the terms of order λ^4, (2.1c) or (2.1d), with the term of order 1, namely a^0_k. Overall, this leads to

$$
\begin{aligned}
\mathbb{E}|a_k|^2(t) = \phi_k + \frac{2\lambda^4}{L^{4d}} \sum_{k-k_1+k_2-k_3=0} \phi_k\phi_{k_1}\phi_{k_2}\phi_{k_3} \left[\frac{1}{\phi_k} - \frac{1}{\phi_{k_1}} + \frac{1}{\phi_{k_2}} - \frac{1}{\phi_{k_3}}\right] \left|\frac{\sin(t\pi\Omega(k,k_1,k_2,k_3))}{\pi\Omega(k,k_1,k_2,k_3)}\right|^2 \\
+ \{\text{higher order terms}\} + \{\text{degenerate cases}\}, \qquad \text{(2.2)}
\end{aligned}
$$

where degenerate cases occur for instance if k, k_1, k_2, k_3 are not distinct[2]. The details of the computation are as follows:

(1) Consider first $\mathbb{E}|(2.1b)|^2 = \mathbb{E}(2.1b)\overline{(2.1b)}$, and denote k_1, k_2, k_3 the indices in (2.1b) and k_1', k_2', k_3' the indices in $\overline{(2.1b)}$. There are two possibilities:

[2]Degenerate cases, like higher order terms, have smaller order of magnitude, on the time scales we consider as will be illustrated in Section ??.

- $\{k_1, k_3\} = \{k_1', k_3'\}$, in which case $k_2 = k_2'$, and $\Omega(k, k_1, k_2, k_3) = \Omega(k, k_1', k_2', k_3')$.
- ($k_2 = k_1$ or k_3) and ($k_2' = k_1'$ or k_3'), in which case $\Omega(k, k_1, k_2, k_3) = \Omega(k, k_1', k_2', k_3') = 0$.

Overall, we find, neglecting degenerate cases (which occur for instance if k, k_1, k_2, k_3 are not distinct),

$$\mathbb{E}|(2.1b)|^2 = \frac{2\lambda^4}{L^{4d}} \sum_{k-k_1+k_2-k_3=0} \phi_{k_1}\phi_{k_2}\phi_{k_3} \left| \frac{\sin(\pi t \Omega(k, k_1, k_2, k_3))}{\pi \Omega(k, k_1, k_2, k_3)} \right|^2 + \frac{4\lambda^4}{L^{4d}} t^2 \sum_{k_1, k_3} \phi_k \phi_{k_1} \phi_{k_2}.$$

(2) Consider next the pairing of a_k^0 with (2.1c), which contributes $2\mathbb{E}\mathfrak{Re}\left[(2.1c)\overline{a_k^0}\right]$. The possible pairings are

- $\{k, k_2\} = \{k_4, k_6\}$, implying $k_3 = k_5$, and leading to $\Omega(k_1, k_4, k_5, k_6) = -\Omega(k, k_1, k_2, k_3)$, and $\Omega(k, k_4, k_5, k_6, k_2, k_1) = 0$.
- ($k_3 = k_2$ or k) and ($k_5 = k_4$ or k_6) in which case $\Omega(k, k_1, k_2, k_3) = \Omega(k_1, k_4, k_5, k_6) = 0$.

This gives, neglecting degenerate cases,

$$\begin{aligned}
&2\mathbb{E}\mathfrak{Re}\left[\overline{a_k^0}(2.1c)\right] = \frac{8\lambda^4}{L^{4d}} \times \\
&\sum_{k-k_1+k_2-k_3=0} \phi_k\phi_{k_2}\phi_{k_3} \mathfrak{Re}\left[\frac{e^{-2\pi i t\Omega(k,k_1,k_2,k_3)} - 1}{4\pi^2 \Omega(k, k_1, k_2, k_3)^2}\right] - \frac{8\lambda^4}{L^{4d}} t^2 \sum_{k_1, k_3} \phi_k \phi_{k_2} \phi_{k_3} \\
&= -\frac{2\lambda^4}{L^{4d}} \sum_{k-k_1+k_2-k_3=0} \phi_k \phi_{k_1} \phi_{k_2} \phi_k \left[\frac{1}{\phi_{k_1}} + \frac{1}{\phi_{k_3}}\right] \left| \frac{\sin(\pi t \Omega(k, k_1, k_2, k_3))}{\pi \Omega(k, k_1, k_2, k_3)} \right|^2 - \frac{8\lambda^4}{L^{4d}} t^2 \sum_{k_1, k_3} \phi_k \phi_{k_2} \phi_{k_3},
\end{aligned}$$

where we used in the last line the symmetry between the variables k_1 and k_3, as well as the identity $\mathfrak{Re}(e^{iy} - 1) = -2|\sin(y/2)|^2$, for $y \in \mathbb{R}$.

(3) Finally, the pairing of a_k^0 with (2.1d) can be discussed similarly, to yield

$$2\mathbb{E}\mathfrak{Re}\left[\overline{a_k^0}(2.1d)\right] = \frac{2\lambda^4}{L^{4d}} \sum_{k-k_1+k_2-k_3=0} \phi_k \phi_{k_1} \phi_{k_3} \left| \frac{\sin(\pi t \Omega(k, k_1, k_2, k_3))}{\pi \Omega(k, k_1, k_2, k_3)} \right|^2 + \frac{4\lambda^4}{L^{4d}} t^2 \sum_{k_1, k_3} \phi_k \phi_{k_2} \phi_{k_3},$$

Summing the above expressions for $\mathbb{E}|(2.1b)|^2$, $2\mathbb{E}\mathfrak{Re}\left[\overline{a_k^0}(2.1c)\right]$ and $2\mathbb{E}\mathfrak{Re}\left[\overline{a_k^0}(2.1d)\right]$ gives (2.2).

Step 3: the big box limit $L \to \infty$. This is the part of the argument that often relies on number theory. We would like to replace the above sum by an integral similar to that on the R. H. S. of the (WKE). This is not obvious since the values of Ω may not be equidistributed at the scale $1/t$ which appears in the "cutoff" function $\left| \frac{\sin(\pi t \Omega(k,k_1,k_2,k_3))}{\pi \Omega(k,k_1,k_2,k_3)} \right|^2$. Assuming that $\Omega(k, k_1, k_2, k_3)$ is equidistributed at this scale, one obtains that as $L \to \infty$,

$$\begin{aligned}
&\sum_{k-k_1+k_2-k_3=0} \phi_k \phi_{k_1} \phi_{k_2} \phi_{k_3} \left[\frac{1}{\phi_k} - \frac{1}{\phi_{k_1}} + \frac{1}{\phi_{k_2}} - \frac{1}{\phi_{k_3}}\right] \left| \frac{\sin(\pi t \Omega(k, k_1, k_2, k_3))}{\pi \Omega(k, k_1, k_2, k_3)} \right|^2 \sim \\
&L^{2d} \int \delta(\Sigma) \phi(k) \phi(k_1) \phi(k_2) \phi(k_3) \left[\frac{1}{\phi(k)} - \frac{1}{\phi(k_1)} + \frac{1}{\phi(k_2)} - \frac{1}{\phi(k_3)}\right] \left| \frac{\sin(\pi t \Omega(k, k_1, k_2, k_3))}{\pi \Omega(k, k_1, k_2, k_3)} \right|^2 dk_1\, dk_2\, dk_3.
\end{aligned}$$

The proof of this fact will be discussed briefly in Section 4.

Step 4: the large time limit $t \to \infty$ Observe that[3] $\int \frac{(\sin x)^2}{x^2}\,dx = \pi^2$, so that, in the sense of distributions,

$$\left|\frac{\sin(\pi t\Omega)}{\pi\Omega}\right|^2 \sim t\delta(\Omega) \qquad \text{as } t \to \infty.$$

Therefore, as $t \to \infty$,

$$\sum_{k-k_1+k_2-k_3=0} \phi_k\phi_{k_1}\phi_{k_2}\phi_n \left[\frac{1}{\phi_k} - \frac{1}{\phi_{k_1}} + \frac{1}{\phi_{k_2}} - \frac{1}{\phi_{k_3}}\right] \left|\frac{\sin(\pi t\Omega(k,k_1,k_2,k_3))}{\pi\Omega(k,k_1,k_2,k_3)}\right|^2$$
$$\sim tL^{2d}\int \delta(\Sigma)\delta(\Omega)\phi(k)\phi(k_1)\phi(k_2)\phi(k_3)\left[\frac{1}{\phi(k)} - \frac{1}{\phi(k_1)} + \frac{1}{\phi(k_2)} - \frac{1}{\phi(k_3)}\right] dk_1\,dk_2\,dk_3$$
$$= tL^{2d}\mathcal{T}(\phi,\phi,\phi).$$

Conclusion Overall, we find, assuming that the above limits are justified

$$\mathbb{E}|a_k|^2(t) = \phi_k + 2\frac{\lambda^4}{L^{2d}}t\mathcal{T}(\phi,\phi,\phi) + \{\text{lower order terms}\}. \tag{2.3}$$

This suggests that the actual time scale of the problem is

$$\tau = \frac{L^{2d}}{2\lambda^4},$$

and that, setting $s = \frac{t}{\tau}$, the governing equation should read

$$\partial_s\phi = \mathcal{T}(\phi,\phi,\phi) \tag{2.4}$$

In addition to the number theoretic component of Step 3 above, making such a heuristic derivation rigorous hinges on making sure that all the remainder terms are indeed lower order compared to the second term on the R. H. S. of (2.3). This will be the subject of the discussion in Section 3.

2.2. **Statement of the result.** We now state a rough version of the main result in [6] in dimension $d = 3$. For other dimensions, and a more general result involving a larger range of the parameter λ, we refer the reader to [6].

Theorem 2.1. *Consider the cubic (NLS) on the three-dimensional torus $\mathbb{T}^3_L$. Assume that the initial data are chosen randomly as in (1.2) with $\phi \in \mathscr{S}(\mathbb{R}^d)$. There exists $\delta > 0$ such that the following holds for L sufficiently large and $\lambda \leq L^{\frac{9}{53}}$:*

$$\mathbb{E}|a_k(t)|^2 = \phi_k + \frac{t}{\tau}\mathcal{T}(\phi)(k) + O_{\ell^\infty}\left(L^{-\delta}\frac{t}{\tau}\right), \quad L^\delta \leq t \leq T, \tag{2.5}$$

where $\tau = \frac{1}{2}\left(\frac{L^d}{\lambda^2}\right)^2$ and $T \sim \min(L^{d-\delta}, \frac{L^{2.65}}{\lambda^2})$.

A few remarks about this result are in order:

- Notice that this is the same as (2.3), except that the time interval $[0,T]$ where this theorem holds is shorter than the kinetic time scale. Such a short time interval only allows the kinetic equation to affect a small change to the initial distribution ϕ_k. An $O(1)$ change to ϕ_k would require having $T \sim \tau$.

[3]This follows from Plancherel's theorem, and the fact that the Fourier transform of $\frac{1}{\pi}\frac{\sin x}{x}$ is the characteristic function of $[-\frac{1}{2\pi}, \frac{1}{2\pi}]$.

- The first upper bound on T given by $L^{d-\delta}$ is imposed so that the contribution of exact resonances (corresponding to $\Omega = 0$) is an error compared to the main term (which is an asymptotic for the so-called *quasi-resonances* corresponding to $\Omega \lesssim T^{-1}$). This restriction is essentially sharp (up to the L^δ factor).
- The other upper bound $T \leq \frac{L^{2.65}}{\lambda^2}$ is not optimal (the optimal being τ up to L^δ losses) can be motivated as follows. Roughly speaking, for a random field that is normalized to 1 in $L^2(\mathbb{T}^d_L)$, its L^∞ norm can be heuristically bounded on average by $L^{-d/2}$. Therefore, regarding the nonlinearity $\lambda^2|u|^2u$ as a nonlinear potential Vu with $V = \lambda^2|u|^2$ and $\|V\|_{L^\infty} \lesssim \lambda^2 L^d$, one would hope to control the solution as needed[4] on an interval $[0, T]$ provided that $T\lambda^2 L^d \ll 1$, which amounts to $T \leq \sqrt{\tau} \sim \frac{L^3}{\lambda^2}$ if $d = 3$. The fact that the result in Theorem 2.1 falls a bit short of this is a technical which we explain briefly in Section 3.2.

3. Feynman Diagram Expansion

One main component of the proof of the above result is the expansion of the solution as an infinite power series in terms of the initial data. The radius of convergence of this power series is what exactly dictates the restriction on the time interval $[0, T]$ in the statement of Theorem 2.1. Let us start by explaining this power series expansion, and how it can be organized in diagrams (ternary trees), that are sometimes called Feynman Diagrams. Similar expansions appeared in the work of Christ [9], but we rely more on the notation of n Lukkarinen-Spohn [20], Section 3.

3.1. **Expansion of the solution in the data.** Let us start by writing the equation satisfied by $a_k(t)$ in (1.3) as

$$a_k(t) = a_k^0 + \frac{i\lambda^2}{L^{2d}} \int_0^t \mathscr{P}_3(a)(s) e^{-2\pi i s \Omega}\, ds.$$

where the subscript in $\mathscr{P}_3$ indicates that it is a monomial of degree 3. The expansion can be obtained by integrating by parts on the oscillating factor $e^{-2\pi i s\Omega}$. Doing one integration by parts gives

$$a_k(t) = a_k^0 + \frac{i\lambda^2}{L^{2d}} \mathscr{P}_3(a)(0) F_0^t + \frac{i\lambda^2}{L^{2d}} \int_0^t \dot{\mathscr{P}}_3(a)(s) F_s^t\, ds, \quad F_s^t := \int_s^t e^{-2\pi i \tau \Omega} d\tau\, .$$

Using the equation for a, we see that $\dot{\mathscr{P}}_3(a)$ consists of three monomials of degree 5, and if we denote on of them by $\mathscr{P}_5$, then the integral term consists of three integrals of the type,

$$\left(\frac{i\lambda^2}{L^{2d}}\right)^2 \int_0^t \mathscr{P}_5(a)(s) e^{-2\pi i s\Omega} F_s^t\, ds.$$

Another integration by parts gives the quintic expansion, which has three terms of the form

$$\left(\frac{i\lambda^2}{L^{2d}}\right)^2 \mathscr{P}_5(a)(0) G_0^t + \left(\frac{i\lambda^2}{L^{2d}}\right)^2 \int_0^t \dot{\mathscr{P}}_5(a)(s) G_s^t\, ds, \quad G_s^t = \int_s^t e^{-2\pi i \tau\Omega} F_\tau^t\, d\tau\, .$$

[4] As we shall see, this means proving convergence of the Feynman diagram expansion discussed in the following section.

Integrating by parts N times, we obtain the expansion to order N given by:

$$a_k(t) = \sum_{n=0}^{N} \mathcal{J}_n(t,k)(\boldsymbol{a}^{(0)}) + R_{N+1}(t,k)(\boldsymbol{a}^{(t)}), \tag{3.1}$$

where $\mathcal{J}_n = \sum_{\boldsymbol{\ell}} \mathcal{J}_{n,\boldsymbol{\ell}}$, and each $\mathcal{J}_{n,\boldsymbol{\ell}}$ is a monomial of degree $2n+1$ generated by the n^{th} integration by parts. The index $\boldsymbol{\ell}$ is a vector whose entries keep track of the history of how the monomial $\mathcal{J}_{n,\boldsymbol{\ell}}$ was generated. R_{N+1} is the remaining time integral.

Each $\mathcal{J}_{n,\boldsymbol{\ell}}$ can be represented by a tree similar to Figure 1 below. which we now explain.

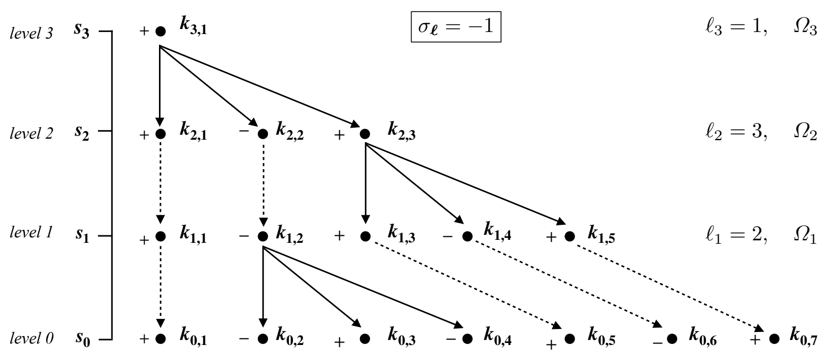

FIGURE 1. tree of depth 3.

The tree corresponding to $\mathcal{J}_{n,\boldsymbol{\ell}}$, is given as follows.

- There are $n+1$ levels in the tree, the bottom level is the 0^{th} level. Descending from the top to the bottom, each level is generated from the previous level by an integration by part step. Thus level j represent the terms present after $n-j$ integration by parts.
- $k_{j,m}$ denote the wave numbers present in level j, and therefore $1 \le m \le 2(n-j)+1$.
- $k_{j,m}$ has a parity σ_m due to complex conjugation. For m odd or even, $\sigma_m = +1$ or $\sigma_m = -1$ respectively.

$$a_{k_{j,m},\sigma_m} = \begin{cases} a_{k_{j,m}} & \text{if } \sigma_m = +1 \\ \overline{a_{k_{j,m}}} & \text{if } \sigma_m = -1 \end{cases}$$

- For each level j, we associate a number ℓ_j, which signals out the wave number k_{j,ℓ_j} which has 3 branches. This is the wave number of the a (or $\bar{a}$) that was differentiated by the j^{th} integration by parts. The index vector $\boldsymbol{\ell}$, keeps track of the integration by parts history in the tree for $\mathcal{J}_{n,\ell}$. The entries ℓ_j, $1 \le j \le n$, are given by

$$\boldsymbol{\ell} = (\ell_1, \ldots, \ell_n) \in \{1, \ldots, 2n-1\} \times \{1, \ldots, 2n-3\} \times \cdots \times \{1,2,3\} \times \{1\}.$$

- The tree has a signature $\sigma_{\boldsymbol{\ell}} = \prod_{j=1}^{n} (-1)^{\ell_j + 1}$.

- *Transition rules.* To go from level j to level $j-1$, the wave numbers are related as follows

$$\begin{cases} k_{j,m} = k_{j-1,m} & \text{for } m < \ell_j \\ k_{j,m} = k_{j-1,m+2} & \text{for } \ell_j < m \\ k_{j,\ell_j} = k_{j-1,\ell_j} - k_{j-1,\ell_j+1} + k_{j-1,\ell_j+2} \end{cases} \tag{3.2}$$

  Note that for any j, $\sum_{m=1}^{2(n-j)+1} (-1)^{m+1} k_{j,m} = k_{n,1} = k$. The wave numbers at level 0, i.e., those present in $\mathcal{J}_{n,\boldsymbol{\ell}}$, are labeled

$$\boldsymbol{k} = (k_{0,1}, \dots, k_{0,2n+1}) \in (\mathbb{Z}_L^d)^{2n+1},$$

- At each level j, the derivative of the element with wave number k_{j,ℓ_j} (due to the integration by parts), generates a oscillatory term with frequency

$$\Omega_j(\boldsymbol{k}) = (-1)^{\ell_j+1} \left(Q(k_{j,\ell_j}) - Q(k_{j-1,\ell_j}) + Q(k_{j-1,\ell_j+1}) - Q(k_{j-1,\ell_j+2}) \right),$$

- Integration by parts variables, $\boldsymbol{s} = (s_0, \dots, s_n) \in (\mathbb{R}^+)^{n+1}$; $t_j(\boldsymbol{s}) = \sum\limits_{k=0}^{j-1} s_k$, $1 \le j \le n$. This choice of variables can be explained as follows. Repeated integration by parts generates terms of the form

$$\int\limits_0^t g_0(s_0) \int\limits_{s_1}^t g_1(s_1) \dots \int\limits_{s_{n-2}}^t g_{n-1}(s_{n-1}) = \int\limits_0^t g_0(s_0) \int\limits_0^{t-s_0} g_1(s_0+s_1) \dots \int\limits_0^{t-s_0-\dots-s_{n-2}} g_{n-1}(s_0+\dots+s_{n-1})$$

  which can be written as

$$\int_{\mathbb{R}_+^{n+1}} g_0(s_0) g_1(s_0+s_1) \dots g_{n-1}(s_0+\dots+s_{n-1}) \delta(t - \sum_{l=0}^n s_l)$$

With this notation at hand,

$$\mathcal{J}_0 = a_k^0, \quad \mathcal{J}_1 = \mathcal{J}_{1,1} = (2.1b), \quad \mathcal{J}_2 = \mathcal{J}_{2,(1,1)} + \mathcal{J}_{2,(2,1)} + \mathcal{J}_{2,(3,1)},$$
$$\mathcal{J}_{2,(2,1)} = (2.1d), \quad \mathcal{J}_{2,(1,1)} = \mathcal{J}_{2,(3,1)} = \frac{1}{2}(2.1c),$$

and Figure 1 represents $\mathcal{J}_{3,(2,3,1)}$. The general formula for $\mathcal{J}_{n,\boldsymbol{\ell}}$ is given by

$$\mathcal{J}_{n,\boldsymbol{\ell}}(t,\boldsymbol{k}) = \left(\frac{i\lambda^2}{L^{2d}}\right)^n \sigma_{\boldsymbol{\ell}} \sum_{\boldsymbol{k}\in(\mathbb{Z}_L^d)^{2n+1}} \delta^k_{k_{n,1}} \prod_{j=1}^{2n+1} a^0_{k_{0,j},\sigma_j} \int_{(\mathbb{R}^+)^{n+1}} \prod_{m=1}^n e^{-2\pi i t_m(s)\Omega_m(\boldsymbol{k})} \delta\left(t - \sum_0^n s_i\right) d\boldsymbol{s} \tag{3.3}$$

Here and throughout the manuscript we write

$$\delta^k_j = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases}$$

while $\delta(\cdot)$ is the Dirac delta.

Finally, we write $R_n(t,k)(\boldsymbol{a}) = \sum_{\boldsymbol{\ell}} \int_0^t R_{n,\boldsymbol{\ell}}(t,s_0;k)(\boldsymbol{a}^{(s_0)})ds_0$, where

$$R_{n,\boldsymbol{\ell}}(t,s_0;k)(\boldsymbol{b}) = \left(\frac{i\lambda^2}{L^{2d}}\right)^n \sigma_{\boldsymbol{\ell}} \sum_{\boldsymbol{k}\in(\mathbb{Z}_L^d)^{2n+1}} \delta^k_{k_{n,1}} \prod_{j=1}^{2n+1} b_{k_{0,j},\sigma_j} \int_{(\mathbb{R}^+)^n} \prod_{j=1}^n e^{-2\pi i t_j(s)\Omega_j(\boldsymbol{k})} \delta\left(t - s_0 - \sum_1^n s_i\right) d\boldsymbol{s}. \quad (3.4)$$

3.2. **Convergence and estimates.** The first question to address for such a power series expansion is its convergence. For this we rely on an iteration scheme that takes advantage of two main facts:

A) Improved Strichartz estimates on irrational tori: These were derived in [11] and yield improvements on longer time intervals than those obtained by iterating the time-1 Strichartz estimates (the latter being optimal on the rational torus). Rather unfortunately, the crucial Strichartz estimate for us, which is the $L^4_{t,x}$ estimate is suboptimal in [11], which is the why our time T in Theorem 2.1 does not quite reach L^3/λ^2.

B) Improved integrability due to randomization of the initial data, namely that random initial data allow for better space-time estimates to be propagated for the solution. This fact is by-now classical, and has appeared in the past few years in many works on dispersive equations with random data (See for example [7] and references therein).

Now that we have a convergent series for $a_k(t)$ of Feynman diagrams, we can start computing as we did in Section 2.1 to obtain:

$$\begin{aligned}
\mathbb{E}|a_k(t)|^2 &= \sum_{n,n'\geq 0} \sum_{\ell,\ell'} \mathbb{E}(\mathcal{J}_{n,\boldsymbol{\ell}}(t,k)\overline{\mathcal{J}_{n',\boldsymbol{\ell}'}(t,k)}) \\
&= \phi_k + \frac{2\lambda^4}{L^{4d}} \sum_{k-k_1+k_2-k_3=0} \phi_k\phi_{k_1}\phi_{k_2}\phi_{k_3} \left[\frac{1}{\phi_k} - \frac{1}{\phi_{k_1}} + \frac{1}{\phi_{k_2}} - \frac{1}{\phi_{k_3}}\right] \left|\frac{\sin(t\pi\Omega(k,k_1,k_2,k_3))}{\pi\Omega(k,k_1,k_2,k_3)}\right|^2 \\
&\quad + \sum_{n+n'\geq 3} \sum_{\ell,\ell'} \mathbb{E}(\mathcal{J}_{n,\boldsymbol{\ell}}(t,k)\overline{\mathcal{J}_{n',\boldsymbol{\ell}'}(t,k)})
\end{aligned}$$

This leads us to the key estimate on the diagram interactions which is contained in the following proposition:

Proposition 3.1. *If $t < L^{d-\epsilon_0}$, then*

$$\left|\sum_{n+n'=S} \sum_{\ell,\ell'} \mathbb{E}(\mathcal{J}_{n,\boldsymbol{\ell}}(t,k)\overline{\mathcal{J}_{n',\boldsymbol{\ell}'}(t,k)})\right| \lesssim_S (\log t)^2 \left(\frac{t}{\sqrt{\tau}}\right)^S \frac{1}{t}. \quad (3.5)$$

Remark 3.2. *The trivial estimate would be that*

$$\left|\sum_{n+n'=S} \sum_{\ell,\ell'} \mathbb{E}(\mathcal{J}_{n,\boldsymbol{\ell}}(t,k)\overline{\mathcal{J}_{n',\boldsymbol{\ell}'}(t,k)})\right| \lesssim \left(\frac{t}{\sqrt{\tau}}\right)^S.$$

Indeed, $\mathcal{J}_{n,\boldsymbol{\ell}}\mathcal{J}_{n',\boldsymbol{\ell}'}$ comes with a prefactor $\left(\frac{\lambda^2}{L^{2d}}\right)^{n+n'}$; the size of the domains where the time integration takes place is $O(t^{n+n'})$; and the summation over $\boldsymbol{k}$ and $\boldsymbol{k'}$ is over $2d(n+n'+1)$ dimensions, *half of which are canceled by the pairing, out of which d further dimensions are canceled by the requirement that $k_{n,1} = k$. Overall, this gives a bound $\left(\frac{\lambda^2}{L^{2d}}\right)^{n+n'} \times t^{n+n'} \times L^{d(n+n')} = \left(\frac{t}{\sqrt{\tau}}\right)^{n+n'}$.*

Having such an estimate in hand, allows one to prove that the contribution of the higher order trees is an error term on any time interval shorter than $\sqrt{\tau}$.

4. Number theoretic results

Finally, we discuss how one can obtain an asymptotic for the second term on the R. H. S. of (2.2) that would give the interaction kernel appearing on the R. H. S. of (WKE) as stated in Theorem 2.1. Specifically one needs to derive an asymptotic formula for

$$\sum_{\substack{k_i\in\mathbb{Z}^d_L\\ k-k_1+k_2-k_3=0}} \phi_k\phi_{k_1}\phi_{k_2}\phi_{k_3}\left[\frac{1}{\phi_k}-\frac{1}{\phi_{k_1}}+\frac{1}{\phi_{k_2}}-\frac{1}{\phi_{k_3}}\right]\left|\frac{\sin(\pi t\Omega(k,k_1,k_2,k_3))}{\pi\Omega(k,k_1,k_2,k_3)}\right|^2 \tag{4.1}$$

for t and L large. Note that for t finite, it is easy to show that for L large the above sum is asymptotic to,

$$L^{2d}\int\delta(\Sigma)\phi_k\phi_{k_1}\phi_{k_2}\phi_{k_3}\left[\frac{1}{\phi_k}-\frac{1}{\phi_{k_1}}+\frac{1}{\phi_{k_2}}-\frac{1}{\phi_{k_3}}\right]\left|\frac{\sin(\pi t\Omega(k,k_1,k_2,k_3))}{\pi\Omega(k,k_1,k_2,k_3)}\right|^2 dk_1dk_2\,dk_3\,,$$

which is nothing but the Riemann sum formula. However for our problem we need to derive such formula in the regime where $\mu := tL^{-2} = O(L^a)$ for some positive power a. In such a regime, the function $(\sin(\mu x)/x)^2$ behaves like μDirac $\delta(x)$. In this situation one needs some deep results from analytic number theory, namely a recent result on pair correlations of generic quadratic forms by Bourgain [4].

Although asymptotic formulas are hard to prove, it is relatively easy to prove sharp upper bounds on such sums if one is willing to allow a loss of a small power of L, i.e., L^ϵ, which we will illustrate below.

Let us consider[5]

$$Q(n) = \sum_{i=1}^{d}\beta_i n_i^2, \quad n=(n_1,\ldots,n_d), \qquad Q(p,q) := Q(p)-Q(q), \tag{4.2}$$

for generic $\beta = (\beta_1,\ldots,\beta_d)\in[1,2]^d$, and try to find a sharp upper bound for the lattice points in the region,

$$R_{\mathbb{Z}} \overset{def}{=} \{(p,q)\in\mathbb{Z}^{2d}\cap[0,L]^{2d} \mid Q(p,q)\in[a,b], p\neq q\},$$

that is bound $\#R_{\mathbb{Z}} = \sum_{(p,q)\in R_{\mathbb{Z}}} 1$. The number of such lattice points in intimately related to the asymptotic of the sum in (4.1) (by replacing the ϕ_k by characteristic functions of the unit ball and the cutoff function $(\sin(\mu x)/x)^2$ by $\mathbf{1}_{[a,b]}$).

First we show that for linear forms, $\ell(n) = \beta\cdot n$, the number of lattice points such that $a \le \ell(n) \le b$, are bounded by

$$\#\{n\in\mathbb{Z}^d\cap[-M,M]^d \mid a\le\beta\cdot n\le b\} = \sum_{\substack{a\le\beta\cdot n\le b\\ |n|\le M}} 1 \lesssim M^{(d-1)^+}(b-a)+1 \tag{4.3}$$

[5]The quadratic form Ω can be transformed to $Q(p,q)$, see [6]

This fact is easy to verify using the genericity of $\beta = (\beta_1, \ldots, \beta_d)$ which implies a lower bound for ℓ whenever $0 < |n| \leq M$,

$$|\beta \cdot n| \gtrsim \frac{1}{M^{(d-1)^+}},$$

(see for example [8], Chapter VII), and the linearity of $\ell(n)$ which implies an upper bound on the distence between two lattice point in $R_{\mathbb{Z}}$, i.e., for arbitrary $n^{(1)} \neq n^{(2)} \in \mathbb{Z}^d$ satisfying $a \leq \beta \cdot n^{(i)} \leq b$ and $0 < \left|n^{(i)}\right| \leq M$,

$$\frac{1}{M^{(d-1)^+}} \lesssim \left|\beta \cdot (n^{(1)} - n^{(2)})\right| \leq b - a\,.$$

Consequently by the pigeonhole principle we obtain (4.3).

Using the fact that $a \leq Q(p,q) \leq b \iff a \leq \ell(k) \leq b$, where $k_i = (p_i - q_i)(p_i + q_i)$, and the divisor bound $d(k_i) \lesssim_\epsilon k_i^\epsilon$, we obtain

$$\# R_{\mathbb{Z}} \lesssim L^{2(d-1)^+}(b-a) + L^{(d-1)^+} \tag{4.4}$$

This upper bound allows us to derive the asymptotic formula for $\# R_{\mathbb{Z}}$ on a coarser scale, e.g. $b - a = L^{\frac{4}{3}}$. Note hat this is still better then the trivial Riemann sum scale of $b - a = O(L^2)$.

Proposition 4.1. *Fix $\delta > 0$ sufficiently small, then if $L^{1+4\delta} \leq b - a \leq L^{2-\delta}$, we have the asymptotic formula*

$$\# \left\{(p,q) \in \mathbb{Z}^d \cap [0,L]^{2d} \;\middle|\; Q(p,q) \in [a,b]\right\} = L^{2(d-1)}(b-a) \iint\limits_{\mathbb{R}^{2d}} \mathbb{1}_{[0,1]^{2d}}(x,y)\delta_{dirac}(Q(x,y))\,dxdy$$
$$+ O\left(L^{2(d-1)-\delta}(b-a)\right)\,.$$

The proof of this proposition is a consequence of (4.4) and Poisson summation formula.

To improve on this result to allow $b - a = O(L^{(d-1)^-})$ one needs some deep results from analytic number theory. In particular, we can generalize the resut of Bourgain in [4] to prove,

Theorem 4.2 (Equidistribution)**.** *Fix $\epsilon > 0$ and let $\delta > 0$ be sufficiently small. Then for generic $\beta \in [1,2]^d$, we have that for any smooth function $W : \mathbb{R}^{2d} \to \mathbb{R}$, compactly supported in a ball of radius L^δ, the following holds,*

$$\sum_{\substack{(p,q) \in \mathbb{Z}^{2d}_+ \\ p \neq q}} W\left(\frac{p}{L}, \frac{q}{L}\right) g(\mu Q(p,q)) = L^{2d} \iint\limits_{\mathbb{R}^{2d}} W(x,y) g(L^2 \mu Q(x,y))\,dxdy + O\left(\frac{L^{2(d-1)-\delta}}{\mu}\right)$$

where $0 < \mu \leq L^{d-1-\epsilon}$.

This allows us to obtain the following asymptotic formula:

Theorem 4.3. *For any $\epsilon > 0$, there exists a sufficiently small δ such that if $0 < t \leq L^{d-\epsilon}$, then,*

$$\sum_{\substack{k_i \in \mathbb{Z}^d_L \\ k - k_1 + k_2 - k_3 = 0}} \phi_k \phi_{k_1} \phi_{k_2} \phi_{k_3} \left[\frac{1}{\phi_k} - \frac{1}{\phi_{k_1}} + \frac{1}{\phi_{k_2}} - \frac{1}{\phi_{k_3}}\right] \left|\frac{\sin(\pi t \mathit{\Omega}(k, k_1, k_2, k_3))}{\pi \mathit{\Omega}(k, k_1, k_2, k_3)}\right|^2 =$$

$$t L^{2(d-1)} \int \phi_k \phi_{k_1} \phi_{k_2} \phi_{k_3} \left[\frac{1}{\phi_k} - \frac{1}{\phi_{k_1}} + \frac{1}{\phi_{k_2}} - \frac{1}{\phi_{k_3}}\right] \delta(\mathit{\Sigma}) \delta(\mathit{\Omega}) dk_1 dk_2\,dk_3 + O\left(t L^{2(d-1)-\delta}\right)\,.$$

This proves the validity of the asymptotic formula for the needed long time interval.

References

[1] D. Benedetto, F. Castella, R. Esposito, and M. Pulvirenti. Some considerations on the derivation of the nonlinear quantum Boltzmann equation. *J. Statist. Phys.*, 116(1-4):381–410, 2004.

[2] D. Benedetto, F. Castella, R. Esposito, and M. Pulvirenti. Some considerations on the derivation of the nonlinear quantum Boltzmann equation. II. The low density regime. *J. Stat. Phys.*, 124(2-4):951–996, 2006.

[3] D. Benedetto, F. Castella, R. Esposito, and M. Pulvirenti. From the N-body Schrödinger equation to the quantum Boltzmann equation: a term-by-term convergence result in the weak coupling regime. *Comm. Math. Phys.*, 277(1):1–44, 2008.

[4] J. Bourgain. On pair correlation for generic diagonal forms. *arXiv e-prints*, June 2016.

[5] T. Buckmaster, P. Germain, Z. Hani, and J. Shatah. Effective dynamics of the nonlinear Schrödinger equation on large domains. *Comm. Pure Appl. Math.*, 71(7):1407–1460, 2018.

[6] T. Buckmaster, P. Germain, Z. Hani, and J. Shatah. Onset of the wave turbulence description of the longtime behavior of the nonlinear Schrödinger equation Preprint.

[7] N. Burq and N. Tzvetkov. Random data Cauchy theory for supercritical wave equations. I. Local theory. *Invent. Math.*, 173(3):449–475, 2008.

[8] J. W. S. Cassels. *An introduction to Diophantine approximation*. Hafner Publishing Co., New York, 1972. Facsimile reprint of the 1957 edition, Cambridge Tracts in Mathematics and Mathematical Physics, No. 45.

[9] M. Christ. Power series solution of a nonlinear Schrdinger equation *Mathematical aspects of nonlinear dispersive equations*, 131–155 *Ann. of Math. Stud.*, 163, Princeton Univ. Press, Princeton, NJ, 2007.

[10] A. de Suzzoni and N. Tzvetkov. On the propagation of weakly nonlinear random dispersive waves. *Arch. Ration. Mech. Anal.*, 212(3):849–874, 2014.

[11] Y. Deng, P. Germain, and L. Guth. Strichartz estimates for the Schrödinger equation on irrational tori. *J. Funct. Anal.*, 273(9):2846–2869, 2017.

[12] L. Erdős, M. Salmhofer, and H. Yau. On the quantum Boltzmann equation. *J. Statist. Phys.*, 116(1-4):367–380, 2004.

[13] E. Faou. Linearized wave turbulence convergence results for three-wave systems. *arXiv e-prints*, May 2018.

[14] E. Faou, P. Germain, and H. Hani. The weakly nonlinear large-box limit of the 2D cubic nonlinear Schrödinger equation. *Journal of the American Mathematical Society*, 2015.

[15] I. Gallagher, L. Saint-Raymond, and B. Texier. *From Newton to Boltzmann: hard spheres and short-range potentials*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2013.

[16] K. Hasselmann. On the non-linear energy transfer in a gravity-wave spectrum Part 1. General theory. *Journal of Fluid Mechanics*, 12(04):481–500, Apr. 1962.

[17] K. Hasselmann. On the non-linear energy transfer in a gravity wave spectrum. II. Conservation theorems; wave-particle analogy; irreversibility. *J. Fluid Mech.*, 15:273–281, 1963.

[18] O. E. Lanford, III. *Time evolution of large classical systems*. Springer, Berlin, 1975.

[19] J. Lukkarinen and H. Spohn. Not to normal order—notes on the kinetic limit for weakly interacting quantum fluids. *Journal of Statistical Physics*, 134(5):1133–1172, Mar 2009.

[20] J. Lukkarinen and H. Spohn. Weakly nonlinear Schrödinger equation with random initial data. *Invent. Math.*, 183(1):79–188, 2011.

[21] A. J. Majda, D. W. McLaughlin, and E. G. Tabak. A one-dimensional model for dispersive wave turbulence. *J. Nonlinear Sci.*, 7(1):9–44, 1997.

[22] S. Nazarenko. *Wave turbulence*, volume 825 of *Lecture Notes in Physics*. Springer, Heidelberg, 2011.

[23] A. C. Newell and B. Rumpf. Wave turbulence: a story far from over. In *Advances in wave turbulence*, volume 83 of *World Sci. Ser. Nonlinear Sci. Ser. A Monogr. Treatises*, pages 1–51. World Sci. Publ., Hackensack, NJ, 2013.

[24] R. Peierls. Zur kinetischen theorie der wärmeleitung in kristallen. *Annalen der Physik*, 395(8):1055–1101, 1929.

[25] H. Spohn. The phonon Boltzmann equation, properties and link to weakly anharmonic lattice dynamics. *J. Stat. Phys.*, 124(2-4):1041–1104, 2006.

[26] V. Zakharov, V. L'vov, and G. Falkovich. *Kolmogorov spectra of turbulence I: Wave turbulence*. Springer Science & Business Media, 2012.