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Forced Convection Heat Transfer From a Particle at Small and Large Peclet Numbers

We theoretically study forced convection heat transfer from a single particle in uniform laminar flows. Asymptotic limits of small and large Peclet numbers Pe are considered. For Pe $\ll 1$ (diffusion-dominated regime) and a constant heat flux boundary condition on the surface of the particle, we derive a closed-form expression for the heat transfer coefficient that is valid for arbitrary particle shapes and Reynolds numbers, as long as the flow is incompressible. Remarkably, our formula for the average Nusselt number Nu has an identical form to the one obtained by Brenner for a uniform temperature boundary condition (Chem. Eng. Sci., vol. 18, 1963, pp. 109–122). We also present a framework for calculating the average Nu of axisymmetric and two-dimensional (2D) objects with a constant heat flux surface condition in the limits of Pe $\gg 1$ and small or moderate Reynolds numbers. Specific results are presented for the heat transfer from spheroidal particles in Stokes flow. [DOI: 10.1115/1.4046590]

1 Introduction

The transport of heat from a particle via an externally driven fluid flow is a phenomenon commonly observed in natural and man-made systems. The ubiquity and importance of forced convection heat transfer have led a large number of researchers to study various aspects of this mode of heat transport. Among the investigations conducted to date, many have focused on incompressible laminar flows. Somewhat surprisingly, however, the vast majority of theoretical studies in that area have been limited to the case of an object with a known surface temperature distribution (see, e.g., Refs. [1–12]), while little attention has been paid to the equally practical problem of convection heat transfer from a particle with a prescribed surface heat flux. Of course, in such a problem, the rate of heat transfer (i.e., the surface integral of the imposed heat flux) is already known, but what is not known, and often sought after, is the average surface temperature in response to the heat emanating from the surface of the particle. For example, envisage a scenario where heat is dissipated from an electronic element by blowing air over it. Assume that the rate at which the heat is generated by the element is known. In this system, the goal is to set the flow speed such that the average surface temperature stays well below a critical temperature, e.g., the melting temperature of the element.

It is not immediately obvious as to why cases with prescribed surface heat flux condition have been overlooked by theoreticians. However, one might surmise that the inconvenience of applying Neumann, versus Dirichlet, boundary conditions is the factor that has deterred them from considering this category of convection heat transfer problems. To partly address this deficiency in the literature, here, we examine uniform laminar flows past a single hot/cold particle whose surface is presumed to maintain a constant heat flux. Perturbation theory is used to derive approximate expressions for the Nusselt number Nu (based on the average surface temperature) in the limits of small and large Peclet numbers Pe. The accuracy of the formulas for the specific case of a spheroidal particle in axisymmetric Stokes flow is tested via comparison with finite volume numerical simulations.

Below, we first pose the mathematical problem and describe the asymptotic solutions in the limits of diffusion- and advection-dominated heat transport, respectively. Specific results are discussed next and a short summary is given in the end.

2 Problem Statement

Consider an unbounded steady laminar flow with a divergence-free velocity $\boldsymbol{u}$ past a stationary particle of arbitrary geometry and characteristic length scale ℓ (see Fig. 1). Let the undisturbed flow field be $\boldsymbol{U}_\infty = U_\infty \boldsymbol{e}$, where $U_\infty = |\boldsymbol{U}_\infty|$ is a constant and $\boldsymbol{e}$ is a unit vector. Suppose that heat is released/absorbed from the surface of the particle at a constant uniform rate q_s and that the temperature vanishes at infinity. Neglecting viscous dissipation and assuming that the fluid properties are constant, the boundary-value problem that governs the steady-state distribution of the dimensionless temperature T outside the particle is

$$\begin{aligned} &\mathrm{Pe}\,\boldsymbol{u}\cdot\nabla T = \nabla^2 T \quad \text{with} \\ &\boldsymbol{n}\cdot\nabla T = -1 \text{ for } \boldsymbol{r}\in S_p \quad \text{and} \quad \lim_{r\to\infty} T = 0 \end{aligned} \tag{1}$$

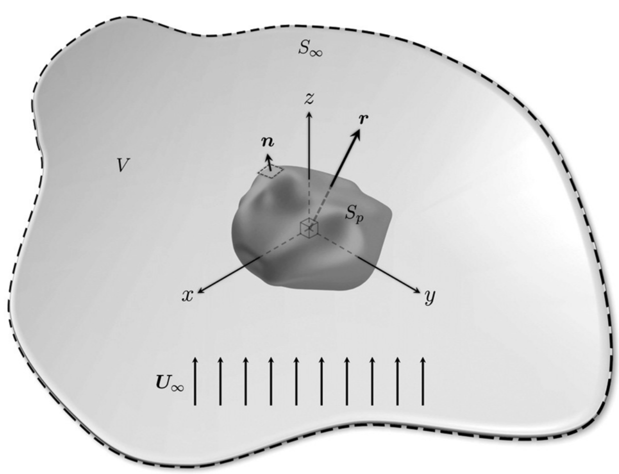

Fig. 1 Schematic of a stationary particle of arbitrary shape, with surface S_p and unit outward normal vector n, in a uniform fluid flow. The dashed line indicates an enclosing boundary in the "far field".

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where the Peclet number is defined as $\mathrm{Pe} = \rho U_\infty c_p \ell / k$, with ρ, c_p, and k being the density, specific heat, and thermal conductivity of the fluid, respectively. Also, $\boldsymbol{n}$ is the unit vector outward normal to the surface of the particle denoted by S_p and $\boldsymbol{r}$ is the position vector with magnitude $r = |\boldsymbol{r}|$ (see Fig. 1). Here, the temperature, length, and fluid velocity are nondimensionalized, respectively, by $q_s \ell / k$, ℓ, and U_∞. We reiterate that the primary novelty of our study is the consideration of a Neumann boundary condition on S_p in the energy transport Eq. (1).

One can define an average Nusselt number for the above problem as

$$\mathrm{Nu} = \frac{\mathbb{S}_p}{2\pi \bar{T}_s} \tag{2}$$

where $\mathbb{S}_p$ represents the dimensionless surface area of the particle and $\bar{T}_s$ is the mean value of T on S_p. Our primary objective is to develop approximate formulas for the variation of the Nusselt number (or equivalently $\bar{T}_s$) as a function of the Peclet number. To this end, we use the ideas of the reciprocal theorem in conjunction with the method of matched asymptotic expansions and boundary layer theory to derive expressions that are valid in the limits of $\mathrm{Pe} \ll 1$ and $\mathrm{Pe} \gg 1$. Details of the calculations are described in Secs. 3 and 4.

3 Perturbation Solution in the Limit of Conduction-Dominated Heat Transport

Suppose that the Peclet number is small, but finite. In this limit, we seek to determine the $O(\mathrm{Pe})$ contribution to the Nusselt number. It is well known that a regular perturbation expansion in terms of Pe is only valid in the vicinity of the particle, i.e., regardless of the magnitude of Pe, there exists a domain ($r/\ell \gtrsim O(\mathrm{Pe}^{-1})$), where the effect of advection outweighs that of conduction. To remedy this situation, a singular perturbation expansion is used that involves separate expansions covering regions close to and far from the particle, i.e., the inner and outer regions, respectively (see, e.g., Refs. [4], [5], and [9]). The inner and outer expansions are matched asymptotically in an intermediate region where both expansions are valid and, together, constitute a perturbation solution that is valid in the entire domain.

Specifically, the inner expansion of the temperature field takes the form of

$$T = T^{(0)} + \mathrm{Pe}\, T^{(1)} + o(\mathrm{Pe}) \tag{3}$$

which after substitution into Eq. (1) leads to

$$\nabla^2 T^{(0)} = 0 \quad \text{with} \quad \boldsymbol{n} \cdot \nabla T^{(0)} = -1 \quad \text{for}\, \boldsymbol{r} \in S_p \tag{4}$$

On the other hand, the outer expansion is expressed as

$$\tilde{T} = \mathrm{Pe}\, \tilde{T}^{(1)} + o(\mathrm{Pe}) \tag{5}$$

which results in

$$\boldsymbol{e} \cdot \tilde{\nabla} \tilde{T}^{(1)} = \tilde{\nabla}^2 \tilde{T}^{(1)} \quad \text{with} \quad \lim_{\tilde{r} \to \infty} \tilde{T}^{(1)} = 0 \tag{6}$$

The tilde overbars in Eqs. (5) and (6) denote that the temperature field and the ∇ operator are written in terms of the stretched (rescaled) position vector $\tilde{\boldsymbol{r}} = \mathrm{Pe}\, \boldsymbol{r}$ with $\tilde{r} = |\tilde{\boldsymbol{r}}|$. The remaining boundary conditions of Eqs. (4) and (6) are furnished by enforcing

$$\lim_{r \to \infty} T = \lim_{\tilde{r} \to 0} \tilde{T} \tag{7}$$

at every order of Pe.

We first consider the solution for $T^{(0)}$. To satisfy the matching requirement at the zeroth-order, $T^{(0)} \to 0$ as $r \to \infty$, which indicates that $T^{(0)}$ is the conduction ($\mathrm{Pe} = 0$) solution of the original problem described in Eq. (1). Thus, far from the particle, the zeroth-order inner solution can be written as (see, e.g., Ref. [5])

$$T^{(0)} = \frac{\mathbb{S}_p}{4\pi r} + O(r^{-3}) = \mathrm{Pe}\, \frac{\mathbb{S}_p}{4\pi \tilde{r}} + O(\mathrm{Pe}^{-3}) \tag{8}$$

This means that the far-field temperature distribution, to the leading order, may be approximated as the solution of the point source of strength $q_s \mathbb{S}_p \ell^2$ applied at the center of the particle. Next, given Eq. (8) and applying the matching condition again, we find

$$\tilde{T}^{(1)} \to \frac{\mathbb{S}_p}{4\pi \tilde{r}} \quad \text{as} \quad \tilde{r} \to 0 \tag{9}$$

which, together with Eq. (6), yields (see, e.g., Ref. [5])

$$\tilde{T}^{(1)} = \frac{\mathbb{S}_p}{4\pi \tilde{r}} \exp\left[-\frac{1}{2}(\tilde{r} - \boldsymbol{e} \cdot \tilde{\boldsymbol{r}})\right] \tag{10}$$

Note that $\tilde{T}^{(1)}$ is also the solution of the point source $q_s \mathbb{S}_p \ell^2$.

We now implement the ideas of the reciprocal theorem (see, e.g., Refs. [13–15]). We multiply the Laplace Eq. (4) by T and Eq. (1) by $T^{(0)}$ and then subtract the resulting equations to obtain, after rearranging

$$\nabla \cdot \left(T \nabla T^{(0)}\right) = \nabla \cdot \left(T^{(0)} \nabla T\right) - \mathrm{Pe}\, T^{(0)} \boldsymbol{u} \cdot \nabla T \tag{11}$$

Integrating this equation over the fluid domain V and using the divergence theorem, we arrive at

$$\int_{S_p} T \boldsymbol{n} \cdot \nabla T^{(0)}\, \mathrm{d}S = \int_{S_p} T^{(0)} \boldsymbol{n} \cdot \nabla T\, \mathrm{d}S + \mathrm{Pe} \int_V T^{(0)} \boldsymbol{u} \cdot \nabla T\, \mathrm{d}V \tag{12}$$

where the integrands decay sufficiently fast for contributions from surfaces at infinity to vanish. Application of the boundary conditions on S_p reduces Eq. (12) to

$$\begin{aligned} \mathbb{S}_p \bar{T}_s &= \int_{S_p} T^{(0)}\, \mathrm{d}S - \mathrm{Pe} \int_V T^{(0)} \boldsymbol{u} \cdot \nabla T\, \mathrm{d}V \\ &= \mathbb{S}_p \left(\bar{T}_s^{(0)} + \mathrm{Pe}\, \bar{T}_s^{(1)}\right) + o(\mathrm{Pe}) \end{aligned} \tag{13}$$

where $\bar{T}_s^{(0)}$ is the zeroth-order contribution to the average surface temperatures and $\bar{T}_s^{(1)}$ is the $O(\mathrm{Pe})$ correction to it. Given Eqs. (13), (3), and (5), we deduce

$$-\mathbb{S}_p \bar{T}_s^{(1)} = \int_V T^{(0)} \boldsymbol{u} \cdot \nabla T^{(0)}\, \mathrm{d}V + \boldsymbol{e} \cdot \int_{\mathbb{R}^3} \tilde{T}^{(0)} \tilde{\nabla} \tilde{T}^{(1)}\, \mathrm{d}\tilde{\boldsymbol{r}} \tag{14}$$

where the second integral on the right-hand side is over the entire three-dimensional (3D) real space $\mathbb{R}^3$ and $\tilde{T}^{(0)} = \mathbb{S}_p / 4\pi \tilde{r}$. The first integral on the right-hand side can be written as

$$\begin{aligned} 2\int_V T^{(0)} \boldsymbol{u} \cdot \nabla T^{(0)}\, \mathrm{d}V &= \int_V \nabla \cdot \left[\left(T^{(0)}\right)^2 \boldsymbol{u}\right] \mathrm{d}V \\ &= -\int_{S_p} \left(T^{(0)}\right)^2 \boldsymbol{n} \cdot \boldsymbol{u}\, \mathrm{d}S - \int_{S_\infty} \left(T^{(0)}\right)^2 \boldsymbol{n} \cdot \boldsymbol{u}\, \mathrm{d}S \end{aligned} \tag{15}$$

where both surface integrals are zero because of the no penetration condition on S_p and the incompressibility of the flow (i.e., $\nabla \cdot \boldsymbol{u} = 0$). Hence, utilizing integration by parts, we find

$$\bar{T}_s^{(1)} = -\frac{\mathbb{S}_p}{16\pi^2} \int_{\mathbb{R}^3} \frac{\boldsymbol{e} \cdot \boldsymbol{r}}{\tilde{r}^4} \exp\left[-\frac{1}{2}(\tilde{r} - \boldsymbol{e} \cdot \tilde{\boldsymbol{r}})\right] \mathrm{d}\tilde{\boldsymbol{r}} \tag{16}$$

Let $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, and $\boldsymbol{e}_z$ be the unit vectors of the Cartesian coordinate system (x, y, z) located at the center of the particle, see Fig. 1.

Then, expressing the position vector in spherical coordinates $(\tilde{r}, \theta, \varphi)$ as

$$\tilde{\boldsymbol{r}} = \tilde{r}\sin\theta\cos\varphi\,\boldsymbol{e}_x + \tilde{r}\sin\theta\sin\varphi\,\boldsymbol{e}_y + \tilde{r}\cos\theta\,\boldsymbol{e}_z$$

and setting the arbitrary unit vector $\boldsymbol{e}$ to $\boldsymbol{e}_z$ (for convenience), the above relation simplifies to

$$\begin{aligned}\bar{T}_s^{(1)} &= -\frac{\mathbb{S}_p}{16\pi^2}\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{\cos\theta\,\sin\theta}{\tilde{r}} \\ &\times\exp\left[-\frac{\tilde{r}}{2}(1-\cos\theta)\right]d\varphi\,d\theta\,d\tilde{r} = -\frac{\mathbb{S}_p}{8\pi}\end{aligned} \tag{17}$$

which, in turn, yields

$$\begin{aligned}\bar{T}_s &= \bar{T}_s^{(0)}\left(1 - \mathrm{Pe}\,\frac{\mathbb{S}_p}{8\pi\bar{T}_s^{(0)}}\right) + o(\mathrm{Pe}) \\ &= \bar{T}_s^{(0)}\left(1 - \mathrm{Pe}\,\frac{\mathrm{Nu}^{(0)}}{4}\right) + o(\mathrm{Pe})\end{aligned} \tag{18}$$

Substitution of Eq. (18) into Eq. (2) gives the final expression for the Nusselt number

$$\mathrm{Nu} = \frac{\mathbb{S}_p}{2\pi\bar{T}_s} = \mathrm{Nu}^{(0)} + \mathrm{Pe}\,\frac{\left(\mathrm{Nu}^{(0)}\right)^2}{4} + o(\mathrm{Pe}) \tag{19}$$

Of course, this result is consistent with the one obtained by Leal [9] for the special case of a spherical particle in a uniform Stokes flow.

There are a couple of important points to make here. First, thanks to the reciprocal theorem-inspired approach that was adopted, the derivation of Eqs. (18) and (19) did not require a detailed knowledge of the velocity field. All we utilized were the facts that the flow is divergence-free and that it does not penetrate into the particle. Even no-slip condition was not essential and we assumed no restriction on the flow Reynolds number defined as $\mathrm{Re} = \rho U_\infty \ell/\mu$, where μ is the fluid viscosity. Second, and perhaps equally notable, Eq. (19) for the dependence of the Nusselt number on the Peclet number is identical in form to the formula obtained by Brenner [5] for an isothermal particle and the expression derived by Gupalo et al. [16] for the Sherwood number Sh (analog of Nu for mass transfer problems) of a particle with a first-order chemical reaction occurring on its surface. It is noteworthy, however, that the values of $\mathrm{Nu}^{(0)}$ (or $\mathrm{Sh}^{(0)}$) corresponding to Dirichlet, Neumann, and Robin boundary conditions are not identical. A natural question to ask at this point is that over what range of Pe does Eq. (19) produce accurate results? We will answer this question for the special case of axisymmetric Stokes flow past a spheroid in Sec. 5.

4 Perturbation Solution in the Limit of Advection-Dominated Heat Transport

Suppose that the Peclet number is large (i.e., $\mathrm{Pe} \gg 1$) and the Reynolds number is small or moderate. In this limit, the temperature distribution outside the particle is mainly restricted to a thin layer around the particle (see, e.g., Fig. 2), whose thickness is proportional to $\mathrm{Pe}^{-1/3}$ (see, e.g., Ref. [6]). This scaling can be deduced by equating the order of magnitude of the advective and conductive terms in Eq. (1), while assuming a linear velocity profile next to S_p. The restriction on Re is to ensure that, unlike the temperature field, the velocity field surrounding the particle is not confined to a narrow region. Similar to Sec. 3, here also we wish to develop a two-term asymptotic approximation for the Nusselt number.

Following the standard boundary layer theory (see, e.g., Ref. [6]), we assume that the already known velocity field and the to-be-solved-for temperature distribution are axisymmetric or two-dimensional (2D) and that S_p is smooth. We also adopt the so-called boundary layer coordinate system $(\mathrm{x}, \mathrm{y}, \varphi)$, where the first two components measure, respectively, the distance along and perpendicular to the surface of the particle in the plane of flow (see Fig. 2) and the third component is the azimuthal angle for axisymmetric and the distance from the x–y plane for two-dimensional problems. The metric coefficients associated with $(\mathrm{x}, \mathrm{y}, \varphi)$ are

$$h_\mathrm{x} = 1 + \kappa(\mathrm{x})\,\mathrm{y}, \quad h_\mathrm{y} = 1, \quad h_\varphi = \varrho(\mathrm{x}) + \alpha(\mathrm{x})\,\mathrm{y} \tag{20}$$

where κ is the curvature of S_p. For 2D cases $\varrho = 1$ and $\alpha = 0$, whereas for 3D axisymmetric problems, ϱ is the rotation radius of S_p and $\alpha = \pm\sqrt{1 - (d\varrho/d\mathrm{x})^2}$ is the cosine of the angle between the axis of rotation and the tangent to S_p. Thus, $\kappa = -\left(d^2\varrho/d\mathrm{x}^2\right)/\alpha$ for axisymmetric cases.

In the vicinity of S_p, Eq. (1) can be expressed in terms of the boundary layer coordinates as

$$\begin{aligned}&\mathrm{Pe}\left(\frac{u_\mathrm{x}}{h_\mathrm{x}}\frac{\partial T}{\partial \mathrm{x}} + \frac{u_\mathrm{y}}{h_\mathrm{y}}\frac{\partial T}{\partial \mathrm{y}}\right) \\ &\quad = \frac{1}{h_\mathrm{x} h_\mathrm{y} h_\varphi}\left[\frac{\partial}{\partial \mathrm{x}}\left(\frac{h_\mathrm{y} h_\varphi}{h_\mathrm{x}}\frac{\partial T}{\partial \mathrm{x}}\right) + \frac{\partial}{\partial \mathrm{y}}\left(\frac{h_\mathrm{x} h_\varphi}{h_\mathrm{y}}\frac{\partial T}{\partial \mathrm{y}}\right)\right] \quad \text{with} \\ &\left.\frac{\partial T}{\partial \mathrm{y}}\right|_{\mathrm{y}=0} = -1 \quad \text{and} \quad \lim_{\mathrm{y}\to\infty} T = 0,\end{aligned} \tag{21}$$

where

$$u_\mathrm{x} = \frac{1}{h_\varphi h_\mathrm{y}}\frac{\partial \psi}{\partial \mathrm{y}} \quad \text{and} \quad u_\mathrm{y} = -\frac{1}{h_\varphi h_\mathrm{x}}\frac{\partial \psi}{\partial \mathrm{x}} \tag{22}$$

are the velocity components in the x and y directions, respectively, with ψ being the stream function. Remember that $\partial T/\partial\varphi$ and u_φ are zero. Since we are interested in the solution near S_p, it is useful to expand h_φ/h_x and ψ about $\mathrm{y} = 0$ as

$$h_\varphi/h_\mathrm{x} = \varrho + (\alpha - \varrho\kappa)\mathrm{y} + \left(\varrho\kappa^2 - \alpha\kappa\right)\mathrm{y}^2 + \cdots, \tag{23a}$$

$$\psi = \psi_2(\mathrm{x})\,\mathrm{y}^2 + \psi_3(\mathrm{x})\,\mathrm{y}^3 + \cdots, \tag{23b}$$

where

$$2\psi_2 = \varrho\left.\frac{\partial u_\mathrm{x}}{\partial \mathrm{y}}\right|_{\mathrm{y}=0} = \varrho\,\tau_0 \tag{24}$$

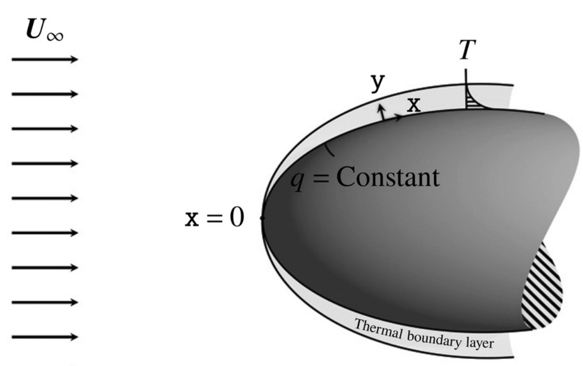

Fig. 2 Schematic of uniform flow past an axisymmetric heated object at high Peclet number. The boundary layer coordinates are represented by x and y, and the surface heat flux is denoted by *q*.

with τ_0 being the dimensionless shear stress at the surface of the particle. Note that both ψ and $\partial\psi/\partial y$ are zero at $y = 0$ due to the no-slip condition.

Given how the thickness of the boundary layer scales with the Peclet number, we consider

$$\tilde{y} = \mathrm{Pe}^{1/3}\, y$$

as the stretched coordinate and proceed with rewriting Eq. (21) as

$$\left[\mathcal{L}^{(0)} + \mathrm{Pe}^{-1/3}\,\mathcal{L}^{(1)} + O\left(\mathrm{Pe}^{-2/3}\right)\right] T = 0 \quad \text{with} \qquad \left.\frac{\partial T}{\partial \tilde{y}}\right|_{\tilde{y}=0} = -\mathrm{Pe}^{-1/3} \quad \text{and} \quad \lim_{\tilde{y}\to\infty} T = 0 \tag{25}$$

where

$$\mathcal{L}^{(0)} = 2\tilde{y}\,\psi_2\,\frac{\partial}{\partial x} - \tilde{y}^2\,\frac{d\psi_2}{dx}\,\frac{\partial}{\partial \tilde{y}} - \varrho\,\frac{\partial^2}{\partial \tilde{y}^2} \tag{26a}$$

$$\mathcal{L}^{(1)} = 3\tilde{y}^2\,\psi_3\,\frac{\partial}{\partial x} - \tilde{y}^3\,\frac{d\psi_3}{dx}\,\frac{\partial}{\partial \tilde{y}} - (\alpha + \varrho\kappa)\,\frac{\partial}{\partial \tilde{y}}\left(\tilde{y}\,\frac{\partial}{\partial \tilde{y}}\right) \tag{26b}$$

Taking the form of Eq. (25) into account, it is natural to expand the temperature field, its mean value on S_p, and the Nusselt number as

$$T = \mathrm{Pe}^{-1/3}\left(T^{(0)} + \mathrm{Pe}^{-1/3}\,T^{(1)}\right) + O\left(\mathrm{Pe}^{-1}\right) \tag{27a}$$

$$\bar{T}_s = \mathrm{Pe}^{-1/3}\left(\bar{T}_s^{(0)} + \mathrm{Pe}^{-1/3}\,\bar{T}_s^{(1)}\right) + O\left(\mathrm{Pe}^{-1}\right) \tag{27b}$$

$$\mathrm{Nu} = \mathrm{Pe}^{1/3}\,\mathrm{Nu}^{(0)} + \mathrm{Nu}^{(1)} + O\left(\mathrm{Pe}^{-1/3}\right) \tag{27c}$$

where

$$\bar{T}_s = \left(\int_0^{x_m} \varrho\, dx\right)^{-1} \int_0^{x_m} T|_{\tilde{y}=0}\,\varrho\, dx \tag{28a}$$

$$\mathrm{Nu}^{(0)} = \frac{\mathbb{S}_p}{2\pi \bar{T}_s^{(0)}} \tag{28b}$$

$$\mathrm{Nu}^{(1)} = -\left(\mathrm{Nu}^{(0)}\right)^2 \frac{2\pi\,\bar{T}_s^{(1)}}{\mathbb{S}_p} \tag{28c}$$

with $x = 0$ and x_m corresponding to the forward stagnation point of S_p and the maximum value of x, respectively. We note that, for two-dimensional problems, the quantities described by Eqs. (27) and (28) belong to the temperature boundary layer that forms over one side of the stagnation point (see Fig. 2), as the energy equation in the two layers can be treated separately, but in the same manner. Furthermore, as clearly articulated by Acrivos and Goddard [6], the above expansions are not expected to be fully applicable to all scenarios, such as where there exists a rear stagnation point on S_p. Thus, caution needs to be exercised when employing these expansions to ensure the accuracy of the results.

Below, we derive $\bar{T}_s^{(0)}$ and $\bar{T}_s^{(1)}$, and, by extension, $\mathrm{Nu}^{(0)}$ and $\mathrm{Nu}^{(1)}$. Substituting for T in Eq. (26) and requiring the energy equation and its boundary conditions to hold for all orders of Pe, we arrive at the following parabolic equations for $T^{(0)}$ and $T^{(1)}$:

$$\mathcal{L}^{(0)}\,T^{(0)} = 0 \quad \text{with} \qquad \left.\frac{\partial T^{(0)}}{\partial \tilde{y}}\right|_{\tilde{y}=0} = -1, \quad \lim_{\tilde{y}\to\infty} T^{(0)} = 0, \quad \text{and} \quad \lim_{x\to 0} T^{(0)} = 0 \tag{29a}$$

$$\mathcal{L}^{(0)}\,T^{(1)} = -\mathcal{L}^{(1)}\,T^{(0)} \quad \text{with} \qquad \left.\frac{\partial T^{(1)}}{\partial \tilde{y}}\right|_{\tilde{y}=0} = 0, \quad \lim_{\tilde{y}\to\infty} T^{(1)} = 0, \quad \text{and} \quad \lim_{x\to 0} T^{(1)} = 0 \tag{29b}$$

We first consider Eq. (29a), which is indeed the standard boundary-layer approximation of Eq. (26). One might be tempted to develop a similarity solution for this equation, as done traditionally when S_p is considered isothermal (see, e.g., Ref. [17]). However, such an approach would fail here because of the Neumann boundary condition at $\tilde{y} = 0$. Instead, we apply a coordinate transformation of the form (see, e.g., Ref. [6])

$$t = \int_0^x \sqrt{2\psi_2(s)}\,\varrho(s)\, ds \quad \text{and} \quad z = \sqrt{2\psi_2(x)}\,\tilde{y} \tag{30}$$

which turns Eq. (29a) into

$$2\psi_2\,\varrho\left(z\,\frac{\partial T^{(0)}}{\partial t} - \frac{\partial^2 T^{(0)}}{\partial z^2}\right) = 0 \quad \text{with} \qquad \left.\frac{\partial T^{(0)}}{\partial z}\right|_{z=0} = -\frac{1}{\sqrt{2\psi_2}}, \quad \lim_{z\to\infty} T^{(0)} = 0, \quad \text{and} \quad \lim_{t\to 0} T^{(0)} = 0 \tag{31}$$

According to Sutton [18], the solution of the above problem is

$$T^{(0)} = -\int_0^t \left.\frac{\partial T^{(0)}}{\partial z}\right|_{z=0} G\left(t, z; \hat{t}, 0\right)\, d\hat{t} \tag{32}$$

where

$$G\left(t, z; \hat{t}, \hat{z}\right) = \begin{cases} \dfrac{\sqrt{z\,\hat{z}}}{3(t-\hat{t})}\,\exp\left[-\dfrac{z^3+\hat{z}^3}{9(t-\hat{t})}\right] I_{-1/3}\left[\dfrac{2(z\,\hat{z})^{3/2}}{9(t-\hat{t})}\right] & \text{for} \quad t > \hat{t} \\ 0 & \text{for} \quad t \le \hat{t} \end{cases} \tag{33}$$

is the Green's function that satisfies

$$z\,\frac{\partial G}{\partial t} - \frac{\partial^2 G}{\partial z^2} = \delta\left(\hat{t}, \hat{z}\right) \quad \text{with} \qquad \left.\frac{\partial G}{\partial z}\right|_{z=0} = \lim_{z\to\infty} G = \lim_{t\to\hat{t}} G = 0 \tag{34}$$

Here, $I_{-1/3}$ is the modified Bessel function of the first kind and of order $-1/3$ and δ is the Dirac delta function. Replacing for G and the heat flux distribution in Eq. (32), we find

$$T^{(0)} = \frac{1}{3^{1/3}\,\Gamma(2/3)} \int_0^t \left(t-\hat{t}\right)^{-2/3} \left(2\hat{\psi}_2\right)^{-1/2} e^{-z^3/9(t-\hat{t})}\, d\hat{t} \tag{35}$$

where Γ is the gamma function and $\hat{\psi}_2 = \psi_2(\hat{x})$. Hence, the leading order contribution to the average surface temperature can be obtained via (see Eqs. (27b) and (28a))

$$\bar{T}_s^{(0)} = \frac{1}{3^{1/3}\,\Gamma(2/3) \displaystyle\int_0^{x_m} \varrho\, dx} \int_0^{x_m} \int_0^{x} \frac{\hat{\varrho}}{\left(t-\hat{t}\right)^{2/3}}\, d\hat{x}\,\varrho\, dx \tag{36}$$

where $\hat{\varrho} = \varrho(\hat{x})$ and $\hat{t} = t(\hat{x})$. Remember that all needed to calculate $t(x)$ is the knowledge of the shear stress distribution on S_p (see Eq. (24)).

With the Green's function known, the solution of Eq. (29b) can be written formally as

$$T^{(1)} = \int_0^{\mathrm{t}} \int_0^{\infty} f(\hat{\mathrm{t}}, \hat{\mathrm{z}})\, G(\mathrm{t}, \mathrm{z}; \hat{\mathrm{t}}, \hat{\mathrm{z}})\, \mathrm{d}\hat{\mathrm{z}}\, \mathrm{d}\hat{\mathrm{t}} \tag{37}$$

where

$$f(\mathrm{t}, \mathrm{z}) = -\frac{\mathcal{L}^{(1)} T^{(0)}}{2\psi_2\, \varrho} = -\frac{1}{\sqrt{2\psi_2}\, \varrho}\left[\mathcal{A}\, \mathrm{z} \frac{\partial^2 T^{(0)}}{\partial \mathrm{z}^2} - (\mathcal{B}\, \mathrm{z}^3 + \mathcal{C}) \frac{\partial T^{(0)}}{\partial \mathrm{z}}\right] \tag{38a}$$

$$\mathcal{A}(\mathrm{x}) = \frac{3\psi_3}{2\psi_2}\varrho - (\alpha + \varrho\kappa) \tag{38b}$$

$$\mathcal{B}(\mathrm{x}) = \frac{d}{d\mathrm{x}}\left[\frac{\psi_3}{(2\psi_2)^{3/2}}\right] \tag{38c}$$

$$\mathcal{C}(\mathrm{x}) = \alpha + \varrho\kappa \tag{38d}$$

Substituting for $T^{(0)}$ in (38*a*), multiplying the result by Eq. (33), and carrying out the $\hat{\mathrm{z}}$ integration (see, e.g., Refs. [6] and [19]), we reach, after some simplifications

$$\begin{aligned} T^{(1)} =& -\left[\frac{1}{3^{1/3}\,\Gamma(2/3)}\right]^2 \int_0^{\mathrm{x}} (\mathrm{t}-\hat{\mathrm{t}})^{1/3} \\ &\times \exp\left[-\frac{\mathrm{z}^3}{9(\mathrm{t}-\hat{\mathrm{t}})}\right] \int_0^{\hat{\mathrm{x}}} \frac{1}{(\mathrm{t}-\hat{\hat{\mathrm{t}}})^2 (\hat{\mathrm{t}}-\hat{\hat{\mathrm{t}}})^{2/3}} \\ &\times \Bigg\{ (\mathrm{t}-\hat{\mathrm{t}})\left[3\hat{\mathcal{A}} + 9(\hat{\mathrm{t}}-\hat{\hat{\mathrm{t}}})\hat{\mathcal{B}}\right] \times {}_1F_1\left(2; \frac{2}{3}; \frac{\hat{\mathrm{t}}-\hat{\hat{\mathrm{t}}}}{\mathrm{t}-\hat{\hat{\mathrm{t}}}} \frac{\mathrm{z}^3}{9(\mathrm{t}-\hat{\mathrm{t}})}\right) \\ &+ (\mathrm{t}-\hat{\hat{\mathrm{t}}})(\hat{\mathcal{C}} - 2\hat{\mathcal{A}}) \times {}_1F_1\left(1; \frac{2}{3}; \frac{\hat{\mathrm{t}}-\hat{\hat{\mathrm{t}}}}{\mathrm{t}-\hat{\hat{\mathrm{t}}}} \frac{\mathrm{z}^3}{9(\mathrm{t}-\hat{\mathrm{t}})}\right) \Bigg\} \hat{\hat{\varrho}}\, \mathrm{d}\hat{\hat{\mathrm{x}}}\, \mathrm{d}\hat{\mathrm{x}} \end{aligned} \tag{39}$$

where $\hat{\hat{\mathrm{t}}} = \mathrm{t}(\hat{\hat{\mathrm{x}}})$, $\hat{\mathcal{A}} = \mathcal{A}(\hat{\mathrm{x}})$, $\hat{\mathcal{B}} = \mathcal{B}(\hat{\mathrm{x}})$, $\hat{\mathcal{C}} = \mathcal{C}(\hat{\mathrm{x}})$, $\hat{\hat{\varrho}} = \varrho(\hat{\hat{\mathrm{x}}})$, and ${}_1F_1$ is the confluent hypergeometric function of the first kind [20]. The first correction to the mean surface temperature then becomes (see Eqs. (27*b*) and (28*a*))

$$\begin{aligned} \bar{T}_s^{(1)} =& -\frac{1}{\int_0^{\mathrm{x}_m} \varrho\, \mathrm{dx}} \left[\frac{1}{3^{1/3}\,\Gamma(2/3)}\right]^2 \int_0^{\mathrm{x}_m} \int_0^{\mathrm{x}} (\mathrm{t}-\hat{\mathrm{t}})^{1/3} \\ &\times \int_0^{\hat{\mathrm{x}}} \frac{1}{(\mathrm{t}-\hat{\hat{\mathrm{t}}})^2 (\hat{\mathrm{t}}-\hat{\hat{\mathrm{t}}})^{2/3}} \Bigg\{ \Bigg\{ a(\mathrm{t}-\hat{\mathrm{t}})\left[3\hat{\mathcal{A}} + 9(\hat{\mathrm{t}}-\hat{\hat{\mathrm{t}}})\hat{\mathcal{B}}\right] \\ &+ (\mathrm{t}-\hat{\hat{\mathrm{t}}})(\hat{\mathcal{C}} - 2\hat{\mathcal{A}}) \Bigg\} \hat{\hat{\varrho}}\, \mathrm{d}\hat{\hat{\mathrm{x}}}\, \mathrm{d}\hat{\mathrm{x}}\, \varrho\, \mathrm{dx} \end{aligned} \tag{40}$$

5 Specific Results for Spheroids in Axisymmetric Stokes Flow

We choose Stokes (zero Reynolds number) flow to exemplify the general results of the previous two sections because it is representative of many small-scale flows arising in biology, engineering, and material science. And, we choose spheroids because of their practical significance. In what follows, we first present the results for the limits of small and large Peclet numbers and then show the comparison with the full numerical solution of Eq. (1). We acknowledge that the choice of Stokes flow does not affect the low Pe analytical result. For the sake of completeness, we also provide the results for spheroids with isothermal surface condition following the works of Brenner [5] and Acrivos and Goddard [6].

5.1 Limit of Pe $\ll$ 1. Consider a spheroid of equatorial radius ℓ and aspect ratio (ratio of polar to equatorial radius) ε. As discussed in Sec. 3, when the Peclet number is small, the details of the flow field are irrelevant for calculating the Nusselt number to the leading order in Pe. In this limit, all needed is Nu corresponding to Pe = 0. As reported by Romero [21] and Jafari et al. [22], the conduction Nusselt number of a spheroid follows:

$$\mathrm{Nu}^{(0)} = \frac{\left[-\left(1+\xi_0^2\right)\right]^{3/2} \mathbb{S}_p^2}{8\pi^2 \xi_0^2 \sum_{m=0}^{\infty} 16^m (4m+1) \frac{Q_{2m}(i\xi_0)}{Q'_{2m}(i\xi_0)} \mathcal{D}_m^2} \tag{41}$$

where

$$\mathcal{D}_m = \sum_{n=0}^{m} \frac{\Gamma(m+n+1/2)}{\Gamma(2n+1)\,\Gamma(2m-2n+1)\,\Gamma(n-m+1/2)} \times \frac{{}_2\mathrm{F}_1\left(-1/2, n+1/2; n+3/2; -\xi_0^{-2}\right)}{2n+1} \tag{42}$$

$$\mathbb{S}_p = 2\pi\left(1 + \frac{\xi_0^2}{\sqrt{1+\xi_0^2}} \coth^{-1}\sqrt{1+\xi_0^2}\right) \tag{43}$$

$$\xi_0 = \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} \tag{44}$$

Here, m and n are integers, Q_{2m} are Legendre functions of the second kind [20] with $Q'_{2m}(\mathfrak{X}) = dQ_{2m}(\mathfrak{X})/d\mathfrak{X}$, and ${}_2\mathrm{F}_1$ is the hypergeometric function [20]. The summation in the denominator of Eq. (41) converges very quickly, to the extent that taking only two terms of the series produces results accurate to within 0.25% of the exact values. Furthermore, the parameter ξ_0 is real for $\varepsilon \le 1$ (oblate spheroids) and is imaginary for $\varepsilon > 1$ (prolate spheroids).

It is interesting to contrast Eq. (41) with its counterpart for the problem of conduction heat transfer from a spheroid with a uniform surface temperature T_s, which is (see, e.g., Refs. [23] and [24])

$$\mathbb{N}\mathrm{u}^{(0)} = -\frac{\int_{S_p} \boldsymbol{n} \cdot \boldsymbol{\nabla} T\, \mathrm{d}S}{2\pi\ell T_s} = \frac{2\sqrt{1-\varepsilon^2}}{\cos^{-1}\varepsilon} \tag{45}$$

Allowing for imaginary values of the square root and inverse cosine functions, this expression and also the forthcoming Eq. (48) are valid for the entire range of ε. Here, the Nusselt number is denoted by a different symbol so it is easily distinguished from its analog for the constant heat flux surface condition. Clearly, Eq. (45) is far less cumbersome than Eq. (41). That aside, the substitution of either Eqs. (41) or (45) in Eq. (19) gives the Nusselt number correct to the order of Pe for incompressible uniform laminar flows past a spheroid.

5.2 Limit of Pe $\gg$ 1. Consider the spheroid of Sec. 5.1 with its axis of revolution coinciding with the z axis of the Cartesian coordinate system, and suppose that $\boldsymbol{e} = \boldsymbol{e}_z$. To determine the $O(\mathrm{Pe}^{1/3})$ Nusselt number and its first correction, we need to calculate the integrals in Eqs. (36) and (40), which involve the functions $\varrho(\mathrm{x})$, $\mathrm{t}(\mathrm{x})$, $\mathcal{A}(\mathrm{x})$, $\mathcal{B}(\mathrm{x})$, and $\mathcal{C}(\mathrm{x})$. The first and last functions depend only on the geometry of the spheroid whereas the remaining three are additionally dependent on the flow field. It is more convenient to express these functions in the terms of a new variable η, where

$$\frac{d\mathrm{x}}{d\eta} = -h_\eta = -\sqrt{\frac{\varepsilon^2 + (1-\varepsilon^2)\eta^2}{1-\eta^2}} \quad \text{and} \quad -1 \le \eta \le 1 \tag{46}$$

with $\eta = 1$ and $\eta = -1$ corresponding to $\mathrm{x} = 0$ and $\mathrm{x} = \mathrm{x}_m$, respectively. The stream function for axisymmetric Stokes flow past a spheroid is known (see, e.g., Ref. [25]). Granted this, we find, following the definitions given in Sec. 4, that

$$\varrho(\eta) = \sqrt{1-\eta^2} \tag{47a}$$

$$\mathrm{t}(\eta) = \sqrt{\frac{F\varepsilon}{16\pi}}\left(\cos^{-1}\eta - \eta\sqrt{1-\eta^2}\right) \tag{47b}$$

$$\mathcal{A}(\eta) = \frac{1}{2h_\eta\,\varepsilon}\left[(1-\varepsilon^2) - \frac{5\varepsilon^2}{\eta^2+\varepsilon^2(1-\eta^2)}\right] \tag{47c}$$

$$\mathcal{B}(\eta) = \frac{\eta}{3h_\eta}\sqrt{\frac{\pi}{F}}\left[\frac{1}{(1-\eta^2)\varepsilon}\right]^{3/2}\left\{\frac{3\varepsilon^2}{[\eta^2+\varepsilon^2(1-\eta^2)]^2}\times\left[1+3(\varepsilon^2-1)(1-\eta^2)\right]-(1+\varepsilon^2)\right\} \tag{47d}$$

$$\mathcal{C}(\eta) = \frac{\varepsilon}{h_\eta}\left[1+\frac{1}{\eta^2+\varepsilon^2(1-\eta^2)}\right] \tag{47e}$$

where

$$F = \frac{8\pi(1-\varepsilon^2)^{3/2}}{(1-2\varepsilon^2)\cos^{-1}\varepsilon + \varepsilon\sqrt{1-\varepsilon^2}} \tag{48}$$

is the magnitude of the Stokes drag experienced by the spheroid. This quantity is made dimensionless by $\mu U_\infty \ell$. The average surface temperature and its first correction can now be calculated by replacing the foregoing relations in Eqs. (36) and (40). Substitution of $\bar{T}_s^{(0)}$ and $\bar{T}_s^{(1)}$ in Eqs. (28*b*) and (28*c*) then yields $\mathrm{Nu}^{(0)}$ and $\mathrm{Nu}^{(1)}$.

Equation (47) can also be used to determine the $O(\mathrm{Pe}^{1/3})$ and $O(1)$ contributions to the Nusselt number for high-Peclet number heat transfer from an isothermal spheroid in axisymmetric Stokes flow. Availing ourselves of the general results of Acrivos and Goddard [6], it can be shown, after much reduction, that

$$\mathbb{N}\mathrm{u}^{(0)} = \frac{1}{8\,\Gamma(4/3)}(12\pi F\varepsilon)^{1/3} \tag{49a}$$

$$\mathbb{N}\mathrm{u}^{(1)} = \mathbb{N}\mathrm{u}^{(1)}_{\mathrm{sphere}}\frac{4\varepsilon^2+1}{5\varepsilon} \tag{49b}$$

where

$$\mathbb{N}\mathrm{u}^{(1)}_{\mathrm{sphere}} = 0.92301 = \frac{5}{3}\left\{1-\frac{4\,\Gamma(2/3)}{[\Gamma(1/3)]^2}\times\int_0^\pi\left(\frac{\gamma-\frac{1}{2}\sin 2\gamma}{\pi}\right)^{-2/3}\left(1-\frac{\gamma-\frac{1}{2}\sin 2\gamma}{\pi}\right)^{1/3}\times\frac{1-\cos\gamma}{\pi}\sin^2\gamma\,\mathrm{d}\gamma\right\} \tag{50}$$

Again, simplicitywise, the contrast between the above formulas and those for $\mathrm{Nu}^{(0)}$ and $\mathrm{Nu}^{(1)}$ is quite remarkable. Equation (49*a*) was also reported by Sehlin [26], though in a different form. However, to the best of our knowledge, Eq. (49*b*) has not been reported elsewhere, and is, therefore, another original contribution of this article.

Finally, we note that the existence of a rear stagnation point on the spheroid in Stokes flow renders the perturbation expansion described by Eq. (27*a*) invalid in the vicinity of $\eta = -1$. Fortunately, however, the contribution of this singular region to the Nusselt number is beyond $O(1)$, and, hence, has no effect on $\mathrm{Nu}^{(1)}$ and $\mathbb{N}\mathrm{u}^{(1)}$ [6].

5.3 Comparison With Full Numerical Solution. To find out the true limits for which the perturbation calculations of Secs. 5.1 and 5.2 for the Nusselt number are valid, we compare our theoretical results with those obtained from the full numerical solution of Eq. (1). A second-order finite volume method as implemented in OPENFOAM (see, e.g., Ref. [27]) is used to perform the numerical calculations. The Stokes equations for the velocity field $\boldsymbol{u}$ are solved first using the SIMPLE algorithm, and the advection-diffusion equation for the temperature distribution T is treated next. The outer boundary at infinity is modeled as a large cylinder, whose center coincides with the center of the spheroid. The diameter of the cylinder is equal to its length, which is 200 times the semimajor axis of the spheroid. 2D axisymmetric meshes concentrated around S_p are employed to discretize the physical domain and grid-independence tests are performed by refining the mesh in the entire domain and repeating the simulations. In all cases considered, the computational grid is chosen such that the change in the results due to the refinement is marginal. Figure 3 shows the results of the numerical calculations for the spheroids of various aspect ratios. Interestingly, the plots of Nu versus Pe for constant heat flux and isothermal boundary conditions are very much alike, not only qualitatively but also quantitatively (compare solid and dashed lines).

Perhaps, the simplest way to construct a global approximation for the Nusselt number based on the two-term asymptotic expressions for $\mathrm{Pe} \ll 1$ and $\mathrm{Pe} \gg 1$ (see Eqs. (51) and (52)) is

$$\mathrm{Nu} = \begin{cases} \mathrm{Nu}_l^{(0)} + \mathrm{Pe}\,\dfrac{\left(\mathrm{Nu}_l^{(0)}\right)^2}{4} & \text{if} \quad \mathrm{Pe} < \mathrm{Pe}_c \\ \mathrm{Pe}^{1/3}\,\mathrm{Nu}_h^{(0)} + \mathrm{Nu}_h^{(1)} & \text{if} \quad \mathrm{Pe} \geq \mathrm{Pe}_c \end{cases} \tag{51}$$

where Pe_c is the cut-off Peclet number and the subscripts l and h indicate that the coefficients belong to the low and high Pe limits, respectively. Alternatively, the Nusselt number may be approximated over the entire range of the Peclet number by (see Refs. [23] and [28])

$$\mathrm{Nu} = \frac{\mathrm{Nu}_l^{(0)}}{2} + \left[\left(\frac{\mathrm{Nu}_l^{(0)}}{2}\right)^3 + \left(\mathrm{Nu}_h^{(0)}\right)^3\mathrm{Pe}\right]^{1/3} \tag{52}$$

which only incorporates the leading order terms in the perturbation expansions of Nu in the asymptotic limits of Pe. Figure 4 presents the percent difference between the results of Fig. 3 and

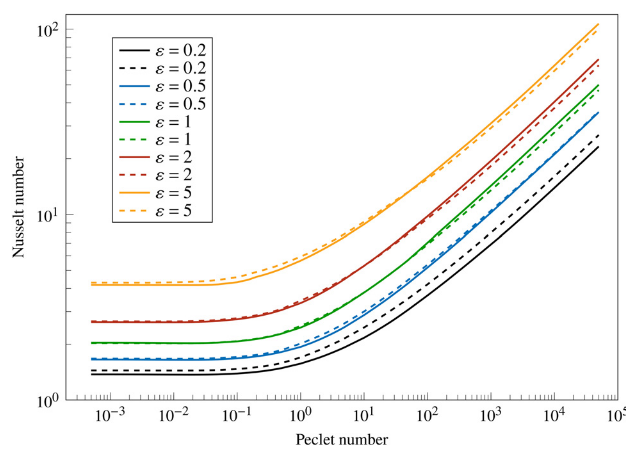

Fig. 3 Numerically calculated plots of the Nusselt number versus Peclet number for forced convection heat transfer from spheroids of various aspect ratios in an axisymmetric uniform Stokes flow subject to constant heat flux (solid lines) and isothermal (dashed lines) boundary conditions on the surface of the spheroid

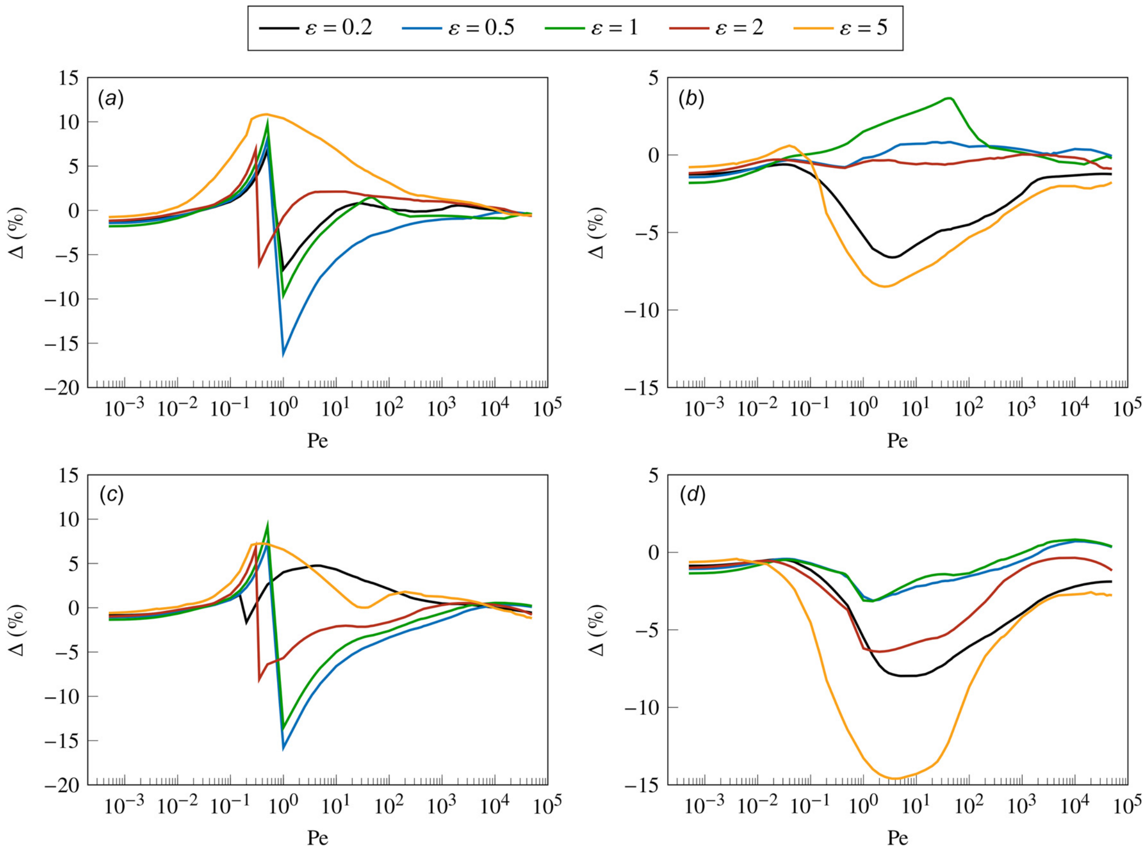

Fig. 4 Percent difference Δ between the numerical and approximate asymptotic results for the Nusselt number corresponding to (*a*) and (*b*) constant heat flux and (*c*) and (*d*) isothermal boundary conditions. (*a*)–(*c*) and (*b*)–(*d*) present Δ versus Pe curves for approximations based on Eqs. (51) and (52), respectively. The cut-off Peclet numbers for the aspect ratios $\varepsilon = 0.2, 0.5, 1, 2, 5$ in (*a*) and (*c*) are, respectively, $\mathrm{Pe}_c = 0.2, 1, 1, 0.35, 0.25$ and $\mathrm{Pe}_c = 1, 1, 1, 0.35, 0.25$.

those of Eqs. (51) and (52). Overall, we see that both approximations are quite accurate. Specifically, Eq. (51) provides more precise predictions when $\mathrm{Pe} \ll 1$ and $\mathrm{Pe} \gg 1$, whereas better estimates are given by Eq. (52) at intermediate values of Pe, where the approximations deviate the most from the numerical results. Finally, we note that the kinks in Figs. 4(*a*) and 4(*c*) are associated with transition from low- to high-Peclet-number solutions at Pe_c, which vary from 0.2 to 1 for the range of aspect ratios considered. In comparison with the plots for $\varepsilon = 0.2, 0.5, 1, 2$, the transition in the curves corresponding to $\varepsilon = 5$ is smoother as, in these cases, both asymptotic formulas overestimate the Nusselt number (i.e., Δ is positive) in the neighborhood of the cut-off point and no discontinuity exist at Pe_c.

6 Summary

We examined the problem of heat transfer from a stationary hot (or cold) particle immersed in an unbounded fluid in the presence of a uniform background flow. We used the perturbation theory to derive two-term approximations for the average Nusselt number in the asymptotic limits of the Peclet number. At small Pe, Nu was approximated as the summation of the conduction Nu and the $O(\mathrm{Pe})$ correction. We showed that for arbitrary particle shapes and flow Reynolds numbers, the correction term is equal to the square of the zeroth-order term divided by four (see Eq. (19)). At high Pe, the boundary layer theory was employed to analytically solve for the temperature distribution within the thermal boundary layer up to $O(\mathrm{Pe}^{-2/3})$. The solutions were used to calculate the $O(\mathrm{Pe}^{1/3})$ and $O(1)$ contributions to the Nusselt number. These calculations were restricted to axisymmetric and two-dimensional problems with low to moderate Reynolds numbers. It is important to note that the primary novelty of both low and high Peclet number results is due to the assumption of constant heat flux condition on the surface of the particle.

We exemplified the general perturbation calculations through the problem of axisymmetric Stokes flow past a spheroid. The specific results were then compared against those obtained from the full numerical solution of the underlying conservation of thermal energy equation. The comparisons confirmed the accuracy of the approximations for Nu over a wide range of Pe. Overall, our theoretical calculations are meant to provide a simple, yet asymptotically correct, approach for estimating the Nusselt number in a fundamental problem in heat transfer. Needless to say, the calculations are equally valid for approximating the Sherwood number in the equivalent mass transfer problem.

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Nomenclature

$\mathcal{A}$ = function of x (or η)
$\hat{\mathcal{A}}$ = $\mathcal{A}$ written in terms of $\hat{x}$
$\mathcal{B}$ = function of x (or η)

$\hat{\mathcal{B}}$ = $\mathcal{B}$ written in terms of $\hat{\mathrm{x}}$
$\mathcal{C}$ = function of x (or η)
$\hat{\mathcal{C}}$ = $\mathcal{C}$ written in terms of $\hat{\mathrm{x}}$
cp = fluid specific heat
$\mathcal{D}_m$ = function series
e = base of natural logarithm
$\boldsymbol{e}$ = unit vector
$\boldsymbol{e}_x$ = unit vector in x direction
$\boldsymbol{e}_y$ = unit vector in y direction
$\boldsymbol{e}_z$ = unit vector in z direction
f = function of t and z (or $\hat{\mathrm{t}}$ and $\hat{\mathrm{z}}$)
F = magnitude of Stokes drag
${}_1\mathrm{F}_1$ = confluent hypergeometric function of the first kind
${}_2\mathrm{F}_1$ = hypergeometric function
G = green's function
h_x = scale factor of x coordinate
h_y = scale factor of y coordinate
h_η = scale factor of η coordinate
h_φ = scale factor of φ coordinate
i = imaginary unit
$I_{-1/3}$ = modified Bessel function of first kind and order $-1/3$
k = fluid thermal conductivity
ℓ = characteristic length scale
$\mathcal{L}^{(0)}$ = differential operator
$\mathcal{L}^{(1)}$ = differential operator
m = integer variable
n = integer variable
$\boldsymbol{n}$ = outward normal unit vector
Nu = Nusselt number (constant flux boundary condition)
$\mathbb{N}\mathrm{u}$ = Nusselt number (constant temperature boundary condition)
Nu_h = Nu obtained in limit of high Pe
Nu_l = Nu obtained in limit of low Pe
$\mathbb{N}\mathrm{u}_{\mathrm{sphere}}$ = $\mathbb{N}$u for sphere
$\mathrm{Nu}^{(0)}$ = zeroth-order term in expansion of $\mathbb{N}\mathrm{u}$
$\mathbb{N}\mathrm{u}^{(0)}$ = zeroth-order term in expansion of $\mathbb{N}\mathrm{u}$
$\mathrm{Nu}^{(1)}$ = first-order term in expansion of $\mathbb{N}\mathrm{u}$
$\mathbb{N}\mathrm{u}^{(1)}$ = first-order term in expansion of $\mathbb{N}\mathrm{u}$
Pe = Peclet number
Pe_c = cut-off Peclet number
qs = heat flux at S_p
$\boldsymbol{r}$ = position vector
r = magnitude of position vector
Q = Legendre function of second kind
$\tilde{\boldsymbol{r}}$ = rescaled position vector
$\tilde{r}$ = magnitude of rescaled position vector
$\mathbb{R}^3$ = three-dimensional real space
Re = Reynolds number
s = dummy variable
S = surface
S_p = particle surface
$\mathbb{S}_p$ = surface area of particle
Sh = Sherwood number
$\mathrm{Sh}^{(0)}$ = zeroth-order term in expansion of Sh
t = first component of transformed boundary layer coordinates
$\hat{\mathrm{t}}$ = dummy coordinate variable corresponding to $\hat{\mathrm{x}}$
$\hat{\hat{\mathrm{t}}}$ = dummy coordinate variable corresponding to $\hat{\hat{\mathrm{x}}}$
T = temperature field
$\tilde{T}$ = temperature field in terms of rescaled variables
T_s = temperature on S_p
$\bar{T}_s$ = average T_s
$T^{(0)}$ = zeroth-order term in expansion of T
$T^{(1)}$ = first-order term in expansion of T
$\tilde{T}^{(1)}$ = first-order term in expansion of $\tilde{T}$
$\bar{T}_s^{(0)}$ = zeroth-order term in expansion of $\bar{T}_s$
$\bar{T}_s^{(1)}$ = first-order term in expansion of $\bar{T}_s$
$\boldsymbol{u}$ = fluid velocity
u_x = velocity component in x direction
u_y = velocity component in y direction
u_φ = velocity component in φ direction
U_∞ = magnitude of undisturbed velocity
$\boldsymbol{U}_\infty$ = undisturbed fluid velocity
V = volume of fluid domain
x = first component of Cartesian coordinates
x = tangential component of boundary layer coordinates
$\hat{\mathrm{x}}$ = dummy coordinate variable
$\hat{\hat{\mathrm{x}}}$ = dummy coordinate variable
x_m = maximum value of x on S_p
y = second component of Cartesian coordinates
y = normal component of boundary layer coordinates
$\tilde{\mathrm{y}}$ = rescaled y coordinate
z = third component of Cartesian coordinates
z = second component of transformed rescaled boundary layer coordinates
$\hat{\mathrm{z}}$ = dummy coordinate variable corresponding to $\hat{\mathrm{x}}$
α = 0 or cosine of the angle between the axis of rotation and the tangent to S_p for 3D axisymmetric particles
γ = dummy variable
Γ = gamma function
δ = Dirac delta function
Δ = percent difference between numerical and asymptotic calculations
ε = aspect ratio of spheroid
η = elliptic coordinate parameterizing surface of spheroidal particle
θ = polar angle in spherical coordinates
κ = curvature of S_p
μ = fluid viscosity
ξ_0 = function of ε
π = Pi number
ρ = fluid density
τ_0 = shear stress at S_p
φ = third component of boundary layer coordinates; azimuthal angle in spherical coordinates
ψ = stream function
ψ_2 = second-order term in expansion of ψ
$\hat{\psi}_2$ = ψ_2 written in terms of $\hat{\mathrm{x}}$
ψ_3 = third-order term in expansion of ψ
ϱ = 1 or rotation radius of S_p for 3D axisymmetric particles
$\hat{\varrho}$ = ϱ written in terms of $\hat{\mathrm{x}}$
$\hat{\hat{\varrho}}$ = ϱ written in terms of $\hat{\hat{\mathrm{x}}}$
$\mathfrak{X}$ = dummy variable
2D = two dimensional
3D = three dimensional

References

[1] Lighthill, M. J., 1950, "Contributions to the Theory of Heat Transfer Through a Laminar Boundary Layer," Proc. R. Soc. London, Ser. A, **202**(1070), pp. 359–377.
[2] Morgan, G. W., and Warner, W. H., 1956, "On Heat Transfer in Laminar Boundary Layers at High Prandtl Number," J. Aerosp. Sci., **23**(10), pp. 937–948.
[3] Morgan, G. W., Pipkin, A. C., and Warner, H. W., 1958, "On Heat Transfer in Laminar Boundary-Layer Flows of Liquids Having a Very Small Prandtl Number," J. Aerosp. Sci., **25**(3), pp. 173–180.
[4] Acrivos, A., and Taylor, T. D., 1962, "Heat and Mass Transfer From Single Spheres in Stokes Flow," Phys. Fluids, **5**(4), pp. 387–394.
[5] Brenner, H., 1963, "Forced Convection Heat and Mass Transfer at Small Peclet Numbers From a Particle of Arbitrary Shape," Chem. Eng. Sci., **18**(2), pp. 109–122.
[6] Acrivos, A., and Goddard, J. D., 1965, "Asymptotic Expansions for Laminar Forced-Convection Heat and Mass Transfer," J. Fluid Mech., **23**(02), pp. 273–291.
[7] Goddard, J. D., and Acrivos, A., 1966, "Asymptotic Expansions for Laminar Forced-Convection Heat and Mass Transfer: Part 2—Boundary-Layer Flows," J. Fluid Mech., **24**(2), pp. 339–366.
[8] Robertson, C. R., and Acrivos, A., 1970, "Low Reynolds Number Shear Flow Past a Rotating Circular Cylinder: Part 2—Heat Transfer," J. Fluid Mech., **40**(04), pp. 705–718.
[9] Leal, L. G., 2007, *Advanced Transport Phenomena: Fluid Mechanics and Convective Transport Processes*, Cambridge University Press, Cambridge, UK.

[10] Feng, Z.-G., 2013, "Forced Heat and Mass Transfer From a Slightly Deformed Sphere at Small but Finite Peclet Numbers in Stokes Flow," ASME J. Heat Transfer, **135**(8), p. 081702.
[11] Relyea, L. M., and Khair, A. S., 2017, "Forced Convection Heat and Mass Transfer From a Slender Particle," Chem. Eng. Sci., **174**, pp. 285–289.
[12] Shah, R., and Li, T., 2018, "The Thermal and Laminar Boundary Layer Flow Over Prolate and Oblate Spheroids," Int. J. Heat Mass Transfer, **121**, pp. 607–619.
[13] Vandadi, V., Jafari Kang, S., and Masoud, H., 2016, "Reciprocal Theorem for Convective Heat and Mass Transfer From a Particle in Stokes and Potential Flows," Phys. Rev. Fluids, **1**(2), p. 022001.
[14] Masoud, H., and Stone, H. A., 2019, "The Reciprocal Theorem in Fluid Dynamics and Transport Phenomena," J. Fluid Mech., **879**, p. P1.
[15] Timm, M. L., Dehdashti, E., Darban, A. J., and Masoud, H., 2019, "Evaporation of a Sessile Droplet on a Slope," Sci. Rep., **9**(1), p. 19803.
[16] Gupalo, Y. P., Ryazantsev, Y. S., and Syskov, Y. N., 1976, "Diffusion to a Reacting Particle of Arbitrary Shape Having a Liquid Flowing Around It," Fluid Dyn., **10**(2), pp. 274–281.
[17] Acrivos, A., 1960, "Solution of the Laminar Boundary Layer Energy Equation at High Prandtl Numbers," Phys. Fluids, **3**(4), pp. 657–658.
[18] Sutton, W. G. L., 1943, "On the Equation of Diffusion in a Turbulent Medium," Proc. R. Soc. London A, **182**(988), pp. 48–75.
[19] Gradshteyn, I. S., and Ryzhik, I. M., 2007, *Table of Integrals, Series, and Products*, Academic Press, Boston, MA.
[20] Abramowitz, M., and Stegun, I. A., 1972, *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables*, Dover, New York.
[21] Romero, L. A., 1995, "Low or High Péclet Number Flow Past a Prolate Spheroid in a Saturated Porous Medium," SIAM J. Appl. Math., **55**(4), pp. 952–974.
[22] Jafari Kang, S., Dehdashti, E., and Masoud, H., 2019, "Conduction Heat Transfer From Oblate Spheroids and Bispheres," Int. J. Heat Mass Transfer, **139**, pp. 115–120.
[23] Clift, R., Grace, J. R., and Weber, M. E., 1978, *Bubbles, Drops, and Particles*, Academic Press, New York.
[24] Alassar, R. S., 1999, "Heat Conduction From Spheroids," ASME J. Heat Transfer, **121**(2), pp. 497–499.
[25] Happel, J., and Brenner, H., 1983, *Low Reynolds Number Hydrodynamics, With Special Applications to Particulate Media*, Martinus Nijhoff, The Hague, The Netherlands.
[26] Sehlin, R. C., 1969, "Forced-Convection Heat and Mass Transfer at Large Péclet Numbers From an Axisymmetric Body in Laminar Flow: Prolate and Oblate Spheroids," Master's thesis, Carnegie Institute of Technology, Pittsburgh, PA.
[27] Moukalled, F., Mangani, L., and Darwish, M., 2015, *The Finite Volume Method in Computational Fluid Dynamics: An Advanced Introduction With OpenFOAM® and Matlab®*, Vol. 113, Springer, Berlin.
[28] Polyanin, A. D., and Dil'man, V. V., 1990, "The Method of Asymptotic Analogies in the Mass and Heat Transfer Theory and Chemical Engineering Science," Int. J. Heat Mass Transfer, **33**(6), pp. 1057–1072.