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Achieving Rental Harmony with a Secretive Roommate

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Abstract. Given the subjective preferences of n roommates in an n-bedroom apartment, one can use Sperner's lemma to find a division of the rent such that each roommate prefers a distinct room. At the resulting rent division, no roommate has a strictly stronger preference for a different room. We give a new elementary proof that the subjective preferences of only $n-1$ of the roommates actually suffice to achieve this envy-free rent division. Our proof, in particular, yields an algorithm to find such a division of rent. The techniques also give generalizations of Sperner's lemma including a new proof of a conjecture of Meunier.

1. INTRODUCTION. The rent in Larry's new two-bedroom apartment is \$1000, and he would like to split the cost with Moe, his new roommate. Because the two rooms are not the same size and each has its own advantages, Larry is concerned with dividing the rent between the rooms so that neither roommate will be envious of the other. He feels that splitting the rent \$600–\$400 between the two rooms is fair—the disadvantages of the second room are offset by its reduced cost. Now, when Larry offers the two rooms to Moe at these prices, it will not matter to him which room Moe chooses; Larry is content with the other room. The two new roommates will not be envious of one another and live in a state of rental harmony. Larry accomplished this envy-free rent division without taking Moe's preferences into account.

This is not a lucky accident of the two person–two bedroom situation: for a three-bedroom apartment, Larry and Moe can fairly divide the rent among the rooms without taking the preferences of a third roommate, Curly, into account. Curly can then choose an arbitrary room, and still leave Larry and Moe with sufficient options to accomplish *rental harmony*: each roommate is assigned a room that he or she prefers.

For a fixed rent division, we say that a roommate *prefers* a room if there is no other room that he or she thinks is strictly better at its designated rent. Notice that a roommate can prefer several rooms. An *envy-free division* is a division of the rent such that it is possible to assign each roommate a distinct room that he or she prefers at a rent division within one cent of the actual division. In other words, we accept approximate envy-free divisions: roommates do not care about one-cent margins. This is without loss of generality; by a compactness argument, we can recover the existence of exact envy-free rent divisions.

In general, it suffices if $n-1$ roommates know each other's preferences to fairly divide the rent of an n-bedroom apartment. In other words, an envy-free division exists even if there is a *secretive roommate*: one whose preferences are not considered while the rent is being divided among the n rooms, but whose preferences are considered in the assignment of rooms. We give an algorithm for producing such an envy-free division of rent; see Asada et al. [**1**] for the recent nonconstructive topological proof of this result.

The fact that rental harmony can always be achieved in an n-bedroom apartment (under mild conditions) if the subjective preferences of all n roommates are known

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was shown by Su [12], partially reporting on work of Simmons. The proof uses a combinatorial-geometric lemma about labelings of simplices due to Sperner [11]. This makes the proof, especially for low n, accessible to a nonexpert audience. Here our goal is to adapt Su's arguments for $n = 3$ and then give a separate elementary proof of the existence of an envy-free division of rent for n roommates, where the preferences of one roommate are unknown.

We first recall the mild conditions stipulated by Su to guarantee the existence of an envy-free rent division:

1. In any division of the rent, each roommate prefers at least one room.
2. Each roommate prefers a room that costs no rent (i.e., a free room) to a nonfree room.
3. If a roommate prefers a room for a convergent sequence of prices, then that roommate also prefers the room for the limiting price.

We remark that Su's second condition together with the third condition imply that the roommates are indifferent among free rooms, that is, if multiple rooms are free then each roommate prefers all of them. This is because for any division of the rent where multiple rooms are free, there is always a sequence of prices converging to this rent division where only one specific room is free.

Under these conditions, we give a new, elementary, and constructive proof of our main theorem. This theorem was previously proved by Asada et al. [1] using nonconstructive techniques.

Theorem 1. *With the above conditions, for an n-bedroom apartment it is sufficient to know the subjective preferences of $n - 1$ roommates to find an envy-free division of rent.*

In Section 2 we recall and prove Sperner's lemma. The proof introduces a piecewise linear map that is used in the following sections. Section 3 gives two proofs of Theorem 1 in the case where $n = 3$, the first of which was originally presented in the *PBS Infinite Series* episode "Splitting Rent with Triangles" [7]. Section 4 generalizes the second proof given in Section 3 to show the main theorem. Section 5 explains how this proof yields an algorithm to find the envy-free division of rent. Section 6 utilizes the piecewise linear map introduced in previous sections to prove two generalizations of Sperner's lemma in an elementary way—one of these generalizations had been conjectured by Meunier in his dissertation and was recently proved by Babson [3] with different methods.

2. SPERNER'S LEMMA. Begin with a triangle that is subdivided into several smaller triangles. Label the three vertices of the original triangle 1, 2, and 3, so that each vertex receives a distinct label. Then, label each vertex on an edge of the original triangle by either of the labels at the endpoints of that edge. Finally, label the interior vertices 1, 2, or 3 arbitrarily. This is a *Sperner labeling* of a triangle, as in Figure 1.

The result known as *Sperner's lemma* for a triangle states that there exists some small triangle with each of the three labels on its vertices. Even stronger, there will be an odd number of such *fully-labeled* triangles, i.e., triangles whose vertices exhibit all labels. A classic proof, which uses a "trap-door" argument, goes back to Cohen [4] and Kuhn [8], and yields an algorithm to find a fully-labeled triangle. Here, we present a different proof, based on a piecewise linear map between the vertex labels. This technique was originally presented by Le Van [9]. We use the piecewise linear map in Section 3 to prove there exists an envy-free division of rent for three roommates, one of whose preferences are secret.

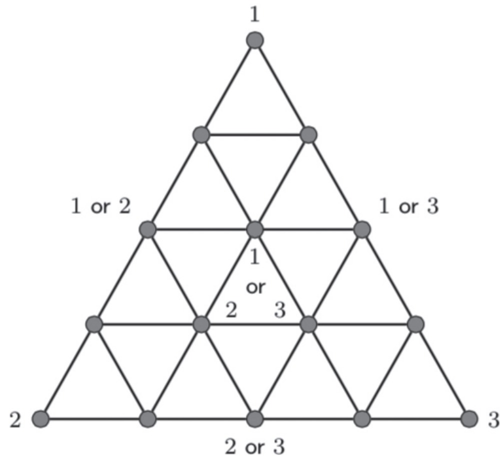

Figure 1. Options for a Sperner labeling.

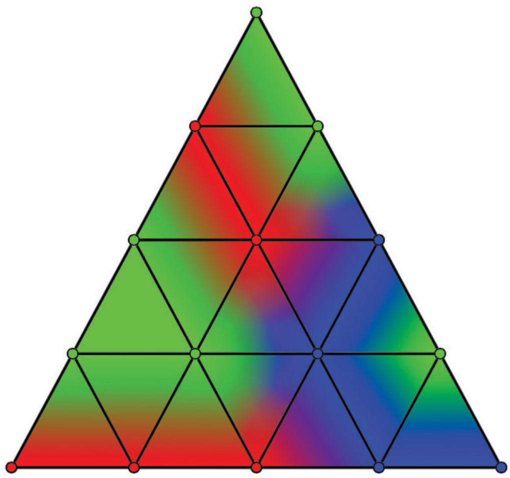

Figure 2. Piecewise linear map. The range of this map is a simplex of colors interpolating among green, red, and blue.

The proof of Sperner's lemma will use the fact that a piecewise linear self map λ of the triangle that fixes the vertices and the edges is surjective. While this is intuitive in two dimensions, we will use the same result in higher dimensions in Section 4.

Fact 1. *A piecewise linear self map λ of the $(n-1)$-simplex Δ_{n-1} that preserves faces setwise (i.e., $\lambda(\sigma) \subseteq \sigma$ for any face σ of Δ_{n-1}) is surjective. Let π be the linear extension of a permutation of the vertices of Δ_{n-1}. If $\lambda(\sigma) \subseteq \pi(\sigma)$ for all faces σ, then λ is also surjective.*

While this is easily seen using the notion of degree, we give a self-contained elementary path-following proof of Fact 1 in Section 5. (For a definition of the degree of a map and its relationship to surjectivity, we refer to Hatcher [6], and in particular to the paragraph "Degree" on page 134.)

Proof of Sperner's lemma. A Sperner labeling of a subdivided triangle can naturally be thought of as a piecewise linear map $\lambda\colon \Delta \longrightarrow \Delta$ from the triangle to itself, defined as follows. For each vertex v of the subdivision, $\lambda(v)$ is the vertex of the original triangle with the same label. Then extend this map linearly within the small triangle. See Figure 2 for an example where the labels are colors: green, red, and blue.

Typically, the labels will be numbers, not colors. For example, the main vertices are labeled 1, 2, 3. The barycenter of a small triangle labeled 1, 2, 3 maps to the barycenter of the original triangle, while the barycenter of a triangle labeled 1, 1, 2 maps to the point on the edge with endpoints labeled 1 and 2 that separates the edge in a two-to-one ratio. Note that, for any edge e of the original triangle $\lambda(e) \subseteq e$ (thus λ also fixes the vertices), and that λ is piecewise linear.

This implies that λ is surjective by Fact 1 and, in particular, that there exists a point $x \in \Delta$ such that $\lambda(x)$ is the barycenter of the original triangle, i.e., $\lambda(x) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Then τ, the smaller triangle containing x, must be fully-labeled. If τ had only two of the labels, then $\lambda(\tau)$ would be contained in the edge of the original triangle spanned by the vertices with the two labels of τ. This contradicts that $\lambda(x)$ is the barycenter. ■

Sperner's lemma easily generalizes to higher dimensions. A *triangulation* of Δ_{n-1} is a subdivision of Δ_{n-1} into smaller $(n-1)$-simplices, or, more precisely, a covering of Δ_{n-1} by a collection of $(n-1)$-simplices whose interiors are disjoint and such that any two of them intersect in a (possibly empty) common face. Given a triangulation of an $(n-1)$-dimensional simplex Δ_{n-1}, a *Sperner labeling* is a labeling of the vertices with $\{1, 2, \ldots, n\}$ such that (1) the n vertices of the original $(n-1)$-simplex receive distinct labels, and (2) a vertex on a k-face of Δ_{n-1} is labeled by one of the $k+1$ labels of that k-face. *Sperner's lemma* for higher-dimensional simplices states that any triangulated $(n-1)$-simplex with a Sperner labeling contains an odd number of smaller $(n-1)$-simplices that are *fully-labeled*, i.e., that exhibit all n labels on their vertices.

The proof of Sperner's lemma using a piecewise linear map generalizes to higher-dimensional simplices fairly directly. The map $\lambda \colon \Delta_{n-1} \longrightarrow \Delta_{n-1}$ from the $(n-1)$-simplex to itself is defined nearly identically: map each vertex of a smaller $(n-1)$-simplex to the vertex of the original simplex with the same label and then extend the map linearly inside each smaller simplex.

For any face σ of the simplex Δ_{n-1}, subdivision vertices contained in σ are only labeled by labels found at the vertices of σ by the definition of Sperner labeling, and therefore, the image $\lambda(\sigma)$ is contained in σ. Thus λ is surjective by Fact 1, and there must exist a point $x \in \Delta_{n-1}$ such that $\lambda(x) = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, that is, it is the barycenter of the simplex. (As mentioned earlier, we will give an elementary proof of this fact in Section 5.) The smaller $(n-1)$-simplex containing x must be fully-labeled.

3. RENTAL HARMONY IN THE ABSENCE OF FULL INFORMATION FOR THREE ROOMMATES. Here we give two combinatorial proofs of the $n = 3$ case of Theorem 1. The first proof is constructive and yields an algorithm to find the envy-free division of rent, since it reduces Theorem 1 to Sperner's lemma for $n = 3$. The second proof mirrors the proof we provide in Section 2. As presented in this section, the second proof only gives the existence of an envy-free division with a secret preference, but Section 5 explains how this method also yields an algorithm.

Proof 1. Form a triangle in $\mathbb{R}^3$ with vertices given by $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Because this lies in the plane $x + y + z = 1$, we can interpret each point in the triangle as a division of rent. For example, $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ indicates that rooms 1 and 2 cost one-quarter of the total rent and room 3 costs one-half of the total rent.

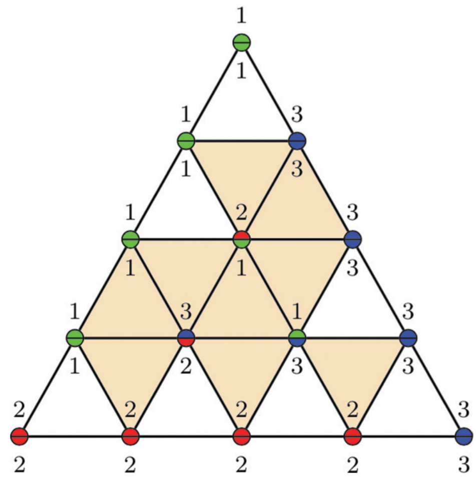

Figure 3. Example of Theorem 1. The rooms 1, 2, and 3 correspond to colors green, red, and blue, respectively. Each vertex records the preferences of Larry (top) and Moe (bottom). Beige triangles correspond to approximately envy-free divisions of rent, where Curly can decide on an arbitrary room.

Subdivide this triangle into many smaller triangles, i.e., triangulate the original triangle. If this triangulation is fine enough, then the vertices of a single small triangle represent divisions of the rent that only differ from one another by a penny or so. At each vertex we survey both Larry and Moe, asking which room they would prefer if the rent were split in this specific way. We record their preferences as an ordered pair of two integers (L, M) with $1 \leq L, M \leq 3$, where L is the number of Larry's preferred room and M is the number of Moe's preferred room. If Larry or Moe prefers multiple rooms at a given rent division, we make an arbitrary choice, except on the three main vertices of the original triangle, where there are two free rooms. On the three original vertices, we ask Larry to choose distinct rooms, i.e., to select each room once, and Moe to copy Larry's preferences.

See Figure 3 for an example, where the top half of each vertex indicates Larry's preference and the bottom half indicates Moe's preference. Note that the colors and numbers are redundant labelings.

At every vertex of the triangulation transform the label (L, M) into a single-digit label according to the following rules: label the vertex 3 if the previous label was $(1, 1)$, $(1, 2)$, or $(2, 1)$; label the vertex 1 for $(2, 2)$, $(2, 3)$, or $(3, 2)$; and label it 2 for $(3, 3)$, $(3, 1)$, or $(1, 3)$. The three original vertices of the triangle are labeled $(1, 1)$, $(2, 2)$, and $(3, 3)$, so their new labels are 3, 1, and 2. On every edge of the triangle the vertices of the triangulation have the same label—only one room is free. And this label is one of the two labels we find at the endpoints of that edge. Therefore we have a Sperner labeling.

Thus we can find a small triangle that has vertices with all three labels. The corresponding rent divisions are all within a small margin of error. We arbitrarily select one rent division from this triangle as the envy-free rent division. It is simple to check that seeing all three labels for this rent division (with small error margin) implies that Larry and Moe each prefer at least two distinct rooms, and each room is preferred by at least one of them. For example, it is impossible that Larry only likes room 1 since one vertex is labeled 1; and it is impossible that both Larry and Moe dislike room 3 since one vertex is labeled 2. See, for example, Figure 4. This means that regardless of which room Curly chooses, both Larry and Moe will be left with one of their favorite rooms. ■

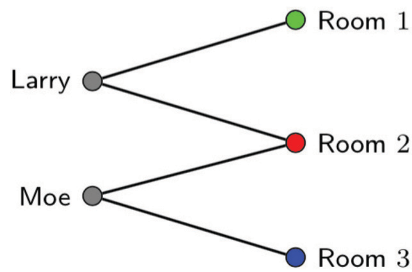

Figure 4. Bipartite graph showing an example of preferences of Larry and Moe at an envy-free rent division. Notice that, regardless of which room Curly selects, both Larry and Moe can still be assigned a preferred room.

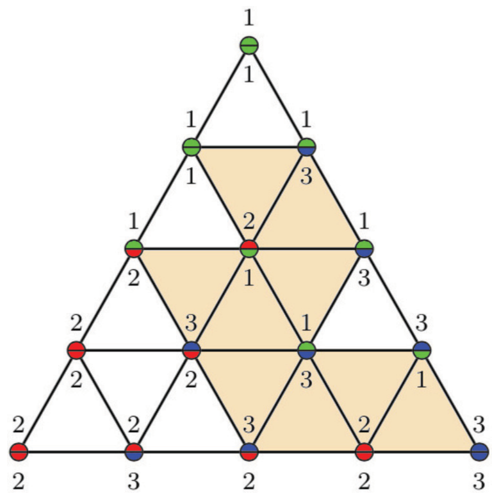

Figure 5. The reduction of Theorem 1 shows a generalization of Sperner's lemma to multiple labelings. Colored triangles exhibit all three labels across the two Sperner labelings and each labeling exhibits at least two distinct labels.

The reduction in the proof above actually proves a generalization of Sperner's lemma: given two Sperner labelings of a subdivided triangle that match up on the original vertices of the triangle, use the rules above to transform them into one Sperner labeling. A fully-labeled triangle, which exists by Sperner's lemma, now exhibits all three labels across the two Sperner labelings and both Sperner labelings exhibit at least two labels. See Figure 5 for an example, where we do not impose special labels for vertices on the boundary of the triangle as in Figure 3 (other than that labels match up at the three original vertices of the triangle). Higher-dimensional generalizations of Sperner's lemma to multiple labelings were conjectured by Meunier and proved by Babson [**3**]. We will treat these and other extensions with new and simple proofs in Section 6.

Proof 2. This second proof of Theorem 1 for $n = 3$ will generalize to the case of n roommates and yield an algorithm to find the envy-free division of rent. It is analogous to the proof of Sperner's lemma—where we used Fact 1 to guarantee the surjectivity of a piecewise linear map—except that the argument is applied to an average of $n - 1$ piecewise linear maps. The general technique of applying a topological argument to the average of functions coming from simplex labelings goes back to Gale [**5**].

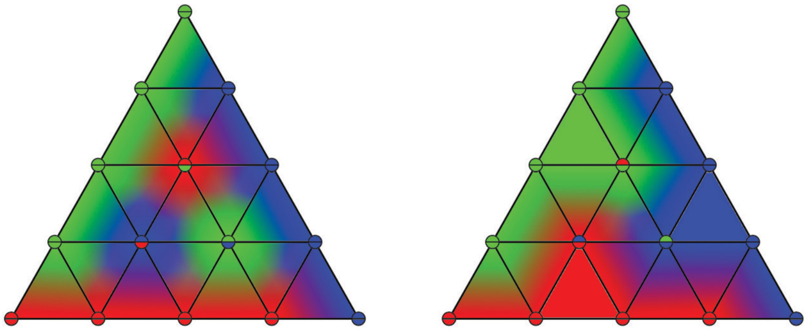

Figure 6. Piecewise linear maps λ_L and λ_M.

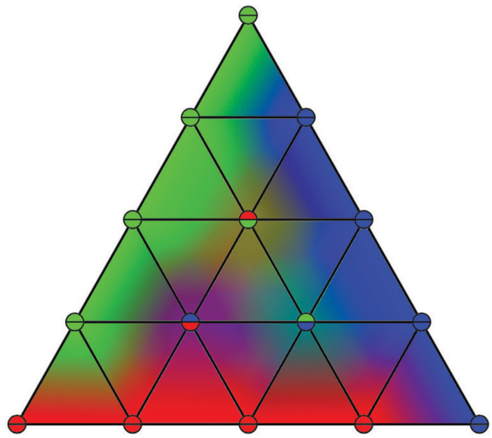

Figure 7. Average of λ_L and λ_M.

As in the first proof, consider the standard simplex in $\mathbb{R}^3$ with vertices on the standard basis e_1, e_2, and e_3. Triangulate the simplex finely enough, and at each vertex of the triangulation, survey Larry and Moe about their room preferences. Instead of recording this as an ordered pair, construct two piecewise linear maps $\lambda_L, \lambda_M \colon \Delta \longrightarrow \Delta$ that reflect the preferences of Larry and Moe, respectively. That is, for each vertex v of the triangulation, λ_L maps v to the vertex of the original triangle with the same label as one of Larry's preferred rooms at the rent division given at vertex v. The map λ_M is defined in the same way using Moe's preferences. Then λ_L and λ_M are defined within each smaller triangle as the linear extension of the values at its vertices.

Let $\lambda = \frac{1}{2}(\lambda_L + \lambda_M)$ denote their average, which again is a piecewise linear map $\lambda \colon \Delta \longrightarrow \Delta$. The map λ maps vertices of the triangulation of Δ either to one of the three original vertices of Δ (if the vertex receives the same label by both Larry and Moe) or to one of the three midpoints of edges (if the labelings do not agree on the vertex).

Continuing the example from Figure 3 and using the color labels, Figure 6 illustrates the two piecewise linear maps λ_L and λ_M associated with Larry and Moe's preferences, respectively. Figure 7 shows their average λ.

Observe that for every subdivision vertex v the vector $2\lambda(v)$ counts how often each label is exhibited in v, e.g., if $2\lambda(v) = (1, 0, 1)$ then $\lambda_L(v) = e_1$ and $\lambda_M(v) = e_3$ or vice versa. Because the roommates are indifferent among free rooms, we can suppose that on each of the original vertices of the triangle Larry and Moe choose the same room, and that they choose each room precisely once on one of the three original vertices. Then as before we check that λ maps each face σ of Δ to itself up to a permutation of the vertices, and thus by Fact 1 there is a point $x \in \Delta$ with $\lambda(x) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The point x lies in some small triangle τ.

We claim that (1) τ exhibits all three labels, and (2) both λ_L and λ_M exhibit at least two labels on τ. To see claim (1), note that if τ exhibited only two of the labels, then one of the coordinates of $\lambda(x)$ would be zero. To see claim (2), assume for contradiction that either λ_L or λ_M only exhibits one of the labels. Assume that label is 1. Then the first coordinate of $\lambda(v)$ will be at least $\frac{1}{2}$ for each vertex v of τ. And so the first coordinate of $\lambda(y)$ for any $y \in \tau$ must be at least $\frac{1}{2}$, which contradicts $\lambda(x) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

The point x corresponds to an envy-free rent division. After the secretive roommate selects their room, two remain. Since τ exhibits all three labels, each of the remaining labels will be exhibited by λ_L or λ_M. Moreover, since λ_L and λ_M each exhibit at least two distinct labels, it is impossible that one roommate prefers neither room. Therefore, the remaining two rooms can be assigned in an envy-free way. ■

4. THE GENERAL CASE. Generalizing the last section, here we give a proof of the main result, Theorem 1: it is always possible to find an envy-free division of the rent for an n-bedroom apartment with a secretive roommate, one whose preferences are considered while assigning rooms, but not while dividing rent between the rooms. That is, the rent division is based on the subjective preferences of only $n - 1$ roommates. The proof is a generalization of the second proof given in Section 3.

Proof. For n roommates, we consider the standard $(n - 1)$-simplex in $\mathbb{R}^n$. Its vertices lie on $e_1, \ldots, e_n$, the standard basis of $\mathbb{R}^n$, and $x_1 + x_2 + \cdots + x_n = 1$ for any point $(x_1, \ldots, x_n)$ in the simplex. Similar to the $n = 3$ case, each point in the simplex is a distribution of the rent and the fraction of the rent corresponding to the ith room is given by x_i. Triangulate the simplex finely enough so that the rent division in the same subdivision simplex is within a one-cent margin of error.

For each of the $n - 1$ given subjective preferences, we define a piecewise linear map $\lambda_j : \Delta_{n-1} \longrightarrow \Delta_{n-1}$ from the triangulated $(n - 1)$-simplex to itself, defined as before: for each vertex v of the triangulated simplex, $\lambda_j(v)$ maps to a vertex of the original simplex, recording the jth roommate's preference. If a roommate prefers several rooms we make an arbitrary choice at vertices in the interior of the simplex, whereas for vertices on the boundary we make a specific choice explained below. For example, if $\lambda_j(v) = e_3$, then roommate j prefers room 3 at the rent division given at v. As in the $n = 3$ case, on vertices in the boundary, we impose certain preferences, described below.

The vertices $e_1, e_2, \ldots, e_n$ of Δ_{n-1} correspond to the rooms $1, 2, \ldots, n$, where room i is the only nonfree room at e_i. We ask each roommate to decide for room $\pi(i) = i + 1$ at e_i for $i < n$ and room $\pi(n) = 1$ at e_n. For the other vertices on the boundary of Δ_{n-1}, we proceed as follows: at such a vertex v, we require that each roommate chooses a room k such that e_{k-1} is a vertex of the supporting face σ of v, but not e_k. Here, we use the convention $e_0 = e_n$. The rooms that are free for rent divisions in σ are precisely those rooms that correspond to vertices of Δ_{n-1} not contained in σ; thus v is labeled by a free room. Labeling the other vertices in the interior of Δ_{n-1} by the preferences of the roommates leads to a Sperner labeling since each vertex v on the boundary of Δ_{n-1} gets a label that appears on one of the vertices of its supporting face. After λ_j is specified on each vertex of the triangulation, define λ_j within each smaller simplex as the linear extension of its values on the vertices of the smaller simplex.

Let $\lambda\colon \Delta_{n-1} \longrightarrow \Delta_{n-1}$ denote the average, $\lambda = \frac{1}{n-1}(\lambda_1 + \lambda_2 + \cdots + \lambda_{n-1})$. Since each λ_j fixes the faces of Δ_{n-1} setwise up to the permutation π, so does their average λ and we can apply Fact 1. Thus, as before, λ is surjective. Let $x \in \Delta_{n-1}$ be such that $\lambda(x) = (\frac{1}{n}, \dots, \frac{1}{n})$, that is, x is mapped to the barycenter. Let τ be the simplex containing x.

Fix $k \in \{1, \dots, n-1\}$. We claim that any k-subset of the labelings λ_i will exhibit at least $k+1$ labels within τ. Assume for contradiction that Λ is some k-subset of the λ_i whose labels are in $\{e_{j_1}, \dots, e_{j_k}\}$. Then $\lambda(\tau)$ will be shifted toward the vertices e_{j_i} and thus will not contain the barycenter. More precisely, for any vertex v of τ and any $\lambda_i \in \Lambda$,

$$\langle \lambda_i(v), e_{j_1} + \cdots + e_{j_k} \rangle = 1$$

and hence

$$\langle \lambda(v), e_{j_1} + \cdots + e_{j_k} \rangle \geq \frac{k}{n-1}.$$

Since this holds for all vertices v of τ, it also holds for any point inside τ. But,

$$\langle \lambda(x), e_{j_1} + \cdots + e_{j_k} \rangle = \left\langle \left(\frac{1}{n}, \dots, \frac{1}{n}\right), e_{j_1} + \cdots + e_{j_k} \right\rangle = \frac{k}{n},$$

which is a contradiction.

This means any subset of k roommates prefers at least $k+1$ rooms. This implies that, for the rent division x, there is an envy-free assignment regardless of which room is picked by the secretive roommate given by the labels of τ. This is because no matter which room the secretive roommate picks, any subset of k (nonsecretive) roommates has k rooms to pick from; construct a bipartite graph with vertices corresponding to the $n-1$ roommates on the one hand and vertices corresponding to the $n-1$ untaken rooms on the other. We add an edge for a pair of roommate and room if the roommate prefers this particular room. An envy-free rent division now corresponds to a perfect matching, which exists by Hall's marriage theorem: in any bipartite graph with bipartite sets A and B there exists a matching that entirely covers A if and only if for every subset $W \subseteq A$ its neighborhood $N(W)$ satisfies $|N(W)| \geq |W|$. For a proof of Hall's marriage theorem and description of the algorithm used to find a perfect matching, see Theorem 22.1 in Schrijver [**10**]. ■

Hall's marriage theorem was also used by Azrieli and Shmaya [**2**] to prove a generalization of Su's rental harmony result to a setting where roommates can share rooms. It would be interesting to know whether Theorem 1 can be extended to this setting.

We remark that this shows the following generalization of Sperner's lemma: Given $n-1$ Sperner labelings $\lambda_1, \dots, \lambda_{n-1}$ of a triangulation of the $(n-1)$-simplex Δ_{n-1} that match up on the original n vertices of Δ_{n-1}, there is a smaller simplex τ that for all $k = 1, \dots, n-1$ exhibits at least $k+1$ labels for any subset of k of the Sperner labelings.

5. ALGORITHMIC ASPECTS. Our proof in the previous section relies on Fact 1 to find a simplex τ whose image under λ contains the barycenter of Δ_{n-1}. Here we describe a simple algorithm for how to find τ. For this we can assume, without loss of generality, that $\lambda(\sigma) \subseteq \sigma$ by first applying π. Our algorithm does not use the surjectivity of λ. In fact, our algorithm also gives an elementary proof of the surjectivity of λ.

It is instructive to first consider low-dimensional cases. The algorithm for $n = 2$ roommates just traverses the interval Δ_1 until for some edge e of (a triangulation of) Δ_1 the image $\lambda(e)$ contains the barycenter, which must exist by the intermediate value theorem.

We now describe the algorithm for a triangle $\Delta_2 = \mathrm{conv}\{e_1, e_2, e_3\}$, the convex hull of e_1, e_2, and e_3. We are given a triangulation T of Δ_2 and a map $\lambda\colon \Delta_2 \longrightarrow \Delta_2$ that interpolates linearly on every face of T. The map λ fixes the vertices and edges of Δ_2 setwise. We will construct a path that starts in the vertex e_1 of Δ_2 and ends in a triangle σ of T such that $\lambda(\sigma)$ contains the barycenter $\frac{1}{3}(e_1 + e_2 + e_3)$ of Δ_2. We will now describe the vertices and edges of a graph G such that following paths in this graph will lead to a triangle mapped to the barycenter by λ. To build this graph we assume that λ is generic in the sense that no vertex of T gets mapped to the segment connecting the barycenter of $[e_1, e_2]$ to the barycenter of Δ_2. We define the vertices of G to be:

1. the vertex e_1 of Δ_2;
2. any edge e of T that subdivides the edge $[e_1, e_2]$ of Δ_2 and such that $\lambda(e) \cap [e_1, \frac{1}{2}(e_1 + e_2)] \neq \emptyset$, i.e., the image of e under λ intersects the segment from vertex e_1 to the barycenter of $[e_1, e_2]$;
3. any triangle σ of T such that $\lambda(\sigma)$ intersects $[\frac{1}{2}(e_1 + e_2), \frac{1}{3}(e_1 + e_2 + e_3)]$, the segment connecting the barycenter of the edge $[e_1, e_2]$ to the barycenter of Δ_2.

We define the edges of G to be:

1. between e_1 and the vertex corresponding to the edge of T that contains e_1 and subdivides $[e_1, e_2]$;
2. between any two vertices corresponding to boundary edges of T that share a common vertex v such that $\lambda(v) \in [e_1, \frac{1}{2}(e_1 + e_2)]$;
3. between any two vertices corresponding to triangles of T that share a common edge e such that $\lambda(e) \cap [\frac{1}{2}(e_1 + e_2), \frac{1}{3}(e_1 + e_2 + e_3)] \neq \emptyset$;
4. between a boundary edge e of T with $\frac{1}{2}(e_1 + e_2) \in \lambda(e)$ and the incident triangle σ of T, i.e., the unique triangle σ that contains e as an edge.

We claim that G has a connected component that is a path from e_1 to some triangle σ with $\frac{1}{3}(e_1 + e_2 + e_3) \in \lambda(\sigma)$. Notice that e_1 is incident to one other vertex in G (corresponding to the unique edge of T that has e_1 as a vertex and subdivides $[e_1, e_2]$). If e is any boundary edge of T (that does not have e_1 as a vertex) with $\lambda(e) \subseteq [e_1, \frac{1}{2}(e_1 + e_2)]$, then in G it is connected to two other boundary faces of T (the ones that lie to the left and right of e on the edge $[e_1, e_2]$). If e is a boundary edge of T with $\frac{1}{2}(e_1 + e_2) \in \lambda(e)$, then precisely one of its vertices gets mapped to $[e_1, \frac{1}{2}(e_1 + e_2)]$, so e is connected to one other edge of T in G. The other neighbor of e in G is the unique triangle σ of T that contains e as an edge. Now since, generically, line segments intersect (boundaries of) triangles in either one or two edges, the vertices of G corresponding to triangles have precisely two neighbors unless the segment of points in the triangle σ that λ maps to $[\frac{1}{2}(e_1 + e_2), \frac{1}{3}(e_1 + e_2 + e_3)]$ intersects σ in only one edge. In that case $\frac{1}{3}(e_1 + e_2 + e_3) \in \lambda(\sigma)$. Thus G is a graph where all vertices have degree one or two, and have degree one if and only if they correspond to a triangle that gets mapped to the barycenter of Δ_2 or correspond to our starting point e_1. Starting in the vertex e_1 and following the edges of G we must end up in such a triangle, as desired.

To summarize the algorithm, we start walking at the vertex e_1 and traverse along the edge $[e_1, e_2]$ until we hit an edge of T that is mapped to the barycenter of $[e_1, e_2]$. From there we walk inwards into the triangle Δ_2 following a path of triangles whose image under λ intersects $[\frac{1}{2}(e_1 + e_2), \frac{1}{3}(e_1 + e_2 + e_3)]$. This either ends in a triangle of T that gets mapped to the barycenter $\frac{1}{3}(e_1 + e_2 + e_3)$, or we return to edges subdividing $[e_1, e_2]$. However, in the latter case we must leave the edge $[e_1, e_2]$ and follow a path of triangles again. After finitely many trips back to the edge $[e_1, e_2]$, we must end up in a triangle mapped to the barycenter.

This construction and algorithm easily generalize to higher dimensions. We work with the faces

$$e_1, \operatorname{conv}\{e_1, e_2\}, \operatorname{conv}\{e_1, e_2, e_3\}, \dots, \operatorname{conv}\{e_1, \dots, e_n\} = \Delta_{n-1}$$

and their barycenters $b_k = \frac{1}{k}\sum_{i=1}^k e_i$. All faces σ of the triangulation T of Δ_{n-1} that subdivide one of these faces, say $\operatorname{conv}\{e_1, \dots, e_k\}$, and such that $\lambda(\sigma)$ intersects the segment $[b_{k-1}, b_k]$ that joins the barycenter of $\operatorname{conv}\{e_1, \dots, e_{k-1}\}$ to that of $\operatorname{conv}\{e_1, \dots, e_k\}$ make up the vertices of G. The vertex e_1 of Δ_n is a vertex of G as well. Two such faces σ_1 and σ_2 of dimension k are connected by an edge in G if they share a common $(k-1)$-face τ such that $\lambda(\tau)$ intersects $[b_k, b_{k+1}]$. We assume that if $\lambda(\tau)$ intersects $[b_k, b_{k+1}]$, then there is a point x in the relative interior of τ such that $\lambda(x) \in [b_k, b_{k+1}]$. This can be achieved by slightly perturbing the barycenters b_k. Moreover, there is an edge between k-face σ and $(k-1)$-face τ if τ is a face of σ in T and $b_k \in \lambda(\tau)$.

A line segment generically cannot intersect a k-face in more than two of its $(k-1)$-faces and it intersects in precisely one $(k-1)$-face if it ends inside the k-face. Thus our reasoning for Δ_2 also applies to this higher-dimensional construction and starting in the vertex e_1 of G, we can follow edges of G to end up in an $(n-1)$-face σ of T with $\frac{1}{n}\sum_{i=1}^n e_i \in \lambda(\sigma)$.

6. GENERALIZATIONS OF SPERNER'S LEMMA. As mentioned before, the methods of Section 4 actually yield generalizations of Sperner's lemma to multiple labelings. Fix a triangulation of the $(n-1)$-simplex and several Sperner labelings $\lambda_1, \dots, \lambda_m$ of it. We will always assume that these labelings match up on the original n vertices of Δ_{n-1}. We will again think of a Sperner labeling λ as a piecewise linear map $\lambda\colon \Delta_{n-1} \longrightarrow \Delta_{n-1}$, that is, the labels of vertices are the standard basis vectors $e_1, \dots, e_n$. If no confusion can arise we will also denote the label e_i by i. By Sperner's lemma, each of the labelings λ_i has a fully-labeled simplex. It is simple to come up with examples where no pair of these respective fully-labeled simplices coincide. In attempting to understand how many labels a single simplex must exhibit across the m Sperner labelings, there are two natural questions:

1. For each vertex v in the triangulation of Δ_{n-1}, define $\lambda(v) = (j_1, \dots, j_n)$ where j_i indicates the number of times the ith label appears at v across all m Sperner labelings $\lambda_1, \dots, \lambda_m$, i.e., $\lambda = \lambda_1 + \dots + \lambda_m$. How can we constrain $\{\ell^i\}_{i=1}^n$ with $\ell^i \in \mathbb{Z}^n_{\geq 0}$ to guarantee that there exists a simplex with vertices $v_1, \dots, v_n$ such that $\lambda(v_i) = \ell^i$ for each $i = 1, \dots, n$?
2. Dually, how can we constrain m-tuples $(k_1, \dots, k_m)$ of nonnegative integers such that there is a simplex τ on which λ_i exhibits k_i distinct labels?

We will relate the first question to convex hulls of *lattice points*, i.e., points with integer coordinates, in $m \cdot \Delta_{n-1} = \{x \in \mathbb{R}^n \mid \sum x_i = m, x_i \geq 0\}$, the $(n-1)$-simplex

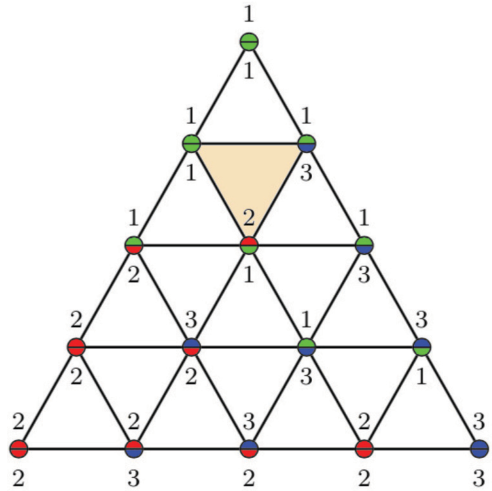

Figure 8. Example of Theorem 2. We can guarantee the existence of a triangle with one vertex labeled by (1, 1), a second vertex labeled (2, 2), (1, 2), or (2, 1), and the last vertex exhibits label 3 at least once.

scaled by m. Specifically, we establish a correspondence between the collection of m Sperner labelings on Δ_{n-1} and the lattice points in $m \cdot \Delta_{n-1}$.

We study the maximal collections of n lattice points in $m \cdot \Delta_{n-1}$ whose convex hulls all intersect in a common point y. There are only a finite number of simplex labelings that could be the preimage of these collections under λ; now Δ_{n-1} must actually have a simplex with at least one of those labelings.

For example, fix two Sperner labelings λ_1 and λ_2 of a triangulation of the triangle Δ_2. We can learn about what types of labelings must occur by studying the lattice points in $2 \cdot \Delta_2$ whose intersections all contain one point.

Consider the point $y = (2 - 3\varepsilon, 2\varepsilon, \varepsilon)$ for some small $\varepsilon > 0$. The point y is close to the vertex $(2, 0, 0)$ and even closer to the edge between $(2, 0, 0)$ and $(0, 2, 0)$ without being on it. Besides the three vertices, the other relevant integer lattices points are midpoints of edges $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$.

Any three lattice points $\ell^1, \ell^2, \ell^3 \in 2 \cdot \Delta_2$ whose convex hull contains y must be (up to reordering):

1. $\ell^1 = (2, 0, 0)$;
2. $\ell^2 = (0, 2, 0)$ or $\ell^2 = (1, 1, 0)$;
3. $\ell^3 = (0, 0, 2)$, $\ell^3 = (1, 0, 1)$, or $\ell^3 = (0, 1, 1)$.

In Theorem 2, we claim that, up to renaming, there exists a smaller triangle of Δ_2 with $\lambda(v_i) = \ell^i$ for $i = 1, 2, 3$. For example, see Figure 8. Expanding on the possible preimages of ℓ^1, ℓ^2, and ℓ^3, we have that there must exist a smaller triangle with vertices v_1, v_2, v_3 such that

1. $(\lambda_1(v_1), \lambda_2(v_1)) = (e_1, e_1)$;
2. $(\lambda_1(v_2), \lambda_2(v_2)) \in \{(e_2, e_2), (e_1, e_2), (e_2, e_1)\}$;
3. $(\lambda_1(v_3), \lambda_2(v_3)) \in \{(e_3, e_3), (e_1, e_3), (e_2, e_3), (e_3, e_1), (e_3, e_2)\}$.

Theorem 2. *Let $\lambda_1, \dots, \lambda_m$ be Sperner labelings of a triangulation T of Δ_{n-1}. Let $y \in m \cdot \Delta_{n-1}$ be some point that is not in the convex hull of any $n - 1$ lattice points in $m \cdot \Delta_{n-1}$. Then there is a simplex τ of T and an ordering of its vertices $v_1, \dots, v_n$*

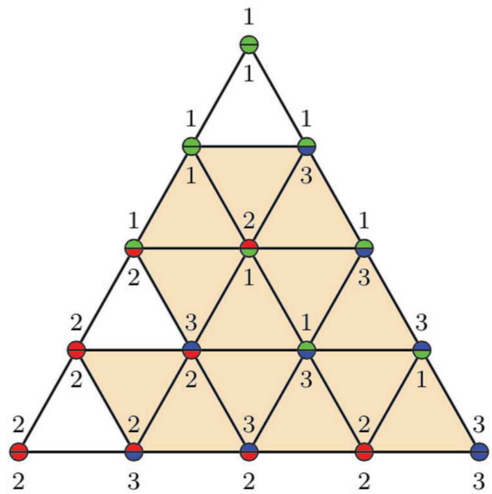

Figure 9. Example of Theorem 3. Both Sperner labelings exhibit at least two labels on the shaded triangles.

such that the point y is contained in $\mathrm{conv}\{\ell^1, \dots, \ell^n\}$*, where* $\ell^i \in m \cdot \Delta_{n-1}$ *denotes the lattice point whose* j*th coordinate is the number of times the* j*th label appears at* v_i*.*

Proof. Let $\lambda(v) = (\lambda_1 + \cdots + \lambda_m)(v)$ be a map from Δ_{n-1} to $m \cdot \Delta_{n-1}$. The map λ satisfies $\lambda(\sigma) \subseteq m \cdot \sigma$ for any face σ of Δ_{n-1} since the λ_i are Sperner labelings. Thus the average $\frac{1}{m}\lambda$ fixes faces setwise, and by Fact 1, there is an $x \in \Delta_{n-1}$ with $\lambda(x) = y$. Let τ be a face of the triangulation of Δ_{n-1} that contains x. The map λ maps vertices of the triangulation of Δ_{n-1} to lattice points of $m \cdot \Delta_{n-1}$. Since y is not in the convex hull of fewer than n lattice points in $m \cdot \Delta_{n-1}$, the vertices of τ must be mapped precisely to the elements of a set of n lattice points in $m \cdot \Delta_{n-1}$ whose convex hull captures y. ■

Question 2 can be approached in much the same way. Instead of defining the map λ as the sum or average of the piecewise linear extensions of the Sperner labelings as before, we now take a biased average with weights according to how many labels each Sperner labeling is supposed to exhibit. Meunier conjectured in his dissertation that $\sum k_i = n + m - 1$ is a valid constraint for Question 2. This was recently proved by Babson [3]. We give a different proof below in the spirit of the other proofs of this manuscript. See Figure 9 for an example.

Theorem 3. *Let $\lambda_1, \dots, \lambda_m$ be m Sperner labelings of a triangulation of Δ_{n-1} and let $k_1, \dots, k_m$ be m positive integers summing up to $n + m - 1$. Then there exists a simplex τ such that, for all j, the labeling λ_j exhibits at least k_j pairwise distinct labels on τ.*

Proof. Let $\alpha_j = \frac{1}{n}(k_j + \frac{1}{m} - 1)$ for $1 \le j \le m$. Then since $\sum_j k_j = n + m - 1$, we have that $\sum_j \alpha_j = 1$. Thus $\lambda = \sum_j \alpha_j \lambda_j$ is a map $\Delta_{n-1} \longrightarrow \Delta_{n-1}$, and λ satisfies $\lambda(\sigma) \subseteq \sigma$ for each face σ of Δ_{n-1} as usual. Let $x \in \Delta_{n-1}$ with $\lambda(x) = (\frac{1}{n}, \dots, \frac{1}{n})$ and let τ be a smaller simplex of the triangulation of Δ_{n-1} containing x. Denote the vertices of τ by $v_1, \dots, v_n$ and let $x = \sum_i \mu_i v_i$ for nonnegative μ_i with $\sum_i \mu_i = 1$.

Define for $i = 1, \ldots, n$ and $j = 1, \ldots, m$

$$\beta_{ij} = \alpha_j \cdot \sum_{\{k | \lambda_j(v_k) = e_i\}} \mu_k.$$

Since $\sum_i \mu_i = 1$, we have that $\sum_i \beta_{ij} = \alpha_j$ for every j. The choice of x, definition of λ, and piecewise linearity of the λ_j imply that

$$\left(\frac{1}{n}, \ldots, \frac{1}{n}\right) = \lambda(x) = \sum_j \alpha_j \lambda_j(x) = \sum_j \alpha_j \sum_{k=1}^{n} \mu_k \lambda_j(v_k),$$

and thus $\sum_j \beta_{ij} = \frac{1}{n}$. Since in particular $0 \leq \beta_{ij} \leq \frac{1}{n}$ and we already know that $\sum_i \beta_{ij} = \alpha_j$, for each j the number of indices i such that $\beta_{ij} > 0$ is at least $\alpha_j \cdot n > k_j - 1$. Now $\beta_{ij} > 0$ implies that there is a vertex v of τ with $\lambda_j(v) = e_i$, and thus τ receives at least k_j distinct labels by λ_j. ■

Using the same technique as in Section 5, the proofs of both Theorem 2 and Theorem 3 can be made constructive. This provides a path-following algorithm to find the simplices whose existence is shown by the theorems.

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100 Years Ago This Month in *The American Mathematical Monthly*
Edited by Vadim Ponomarenko

The Geometrical Lectures of Isaac Barrow. Translated by J. M. Child, Chicago and London, The Open Court Publishing Co., 1916. xiv+218 pages.

An English translation of so important a work as Isaac Barrow's *Lectiones geometricæ* will be greatly welcomed. Few American mathematicians have had access to a translation into English by E. Stone, published in 1735; according to a statement made by W. Whewell in the preface to his Latin edition of *The Mathematical Works of Isaac Barrow*, Cambridge, 1860, Stone's translation "was so badly executed that it cannot be of use to any one." [. . .] Child has aimed to do much more than simply to supply a translation. He has made a searching study of Barrow and has arrived at startling conclusions on the historical question relating to the first invention of the calculus. He places his conclusions in italics in the first sentence of his preface, as follows:

"Isaac Barrow *was the first inventor of the Infinitesimal Calculus; Newton got the main idea of it from Barrow by personal communication; and Leibniz was also in some measure indebted to Barrow's work, obtaining confirmation of his own original ideas, and suggestions for their further development, from the copy of Barrow's book that he purchased in 1673.*"

—Excerpted from Cajori, F. (1919)."Recent Publications." 26(1): 15–20.