A Dynamic Theory-Based Method for Computing Unstable Equilibrium Points of Power Systems

Robert Owusu-Mireku, *Member, IEEE*, Hsiao-Dong Chiang, *Fellow, IEEE*, and Matt Hin, *Student Member, IEEE*

***Abstract*—In this paper, a new theory-based dynamic method for computing unstable equilibrium points is proposed. This dynamic method is a combination of a dynamic transformation method and a trajectory-unified method for the computation of unstable equilibrium points (UEPs) of power system models. The transformation method converts a UEP into a stable equilibrium point (SEP) to expand its convergence region by creating a quotient gradient system (QGS). The resulting SEP is then calculated using a quasi–Newton form of the pseudo-transient continuation method (ψtc) that exploits the structure of the quotient gradient system for fast and reliable computation. It is shown that the proposed QGS-based ψtc can have local q-superlinear or even local quadratic convergence under certain conditions. These conditions for convergence are presented and analyzed. The proposed method is tested on the WSCC 9-bus system and the IEEE 145-bus 50-machine system. The results show that the proposed method gives accurate results, is sufficiently fast, numerically stable, and enlarges the convergence region of computed UEPs.**

***Index Terms*—Direct method, pseudo-transient continuation, quotient gradient system, transient stability, unstable equilibrium point.**

I. Introduction

THE fast and accurate computation of unstable equilibrium points (UEPs) is important in several applications in power systems [1], [2], [5]–[8], [34]. Direct methods, like the closest and controlling UEP (CUEP) methods, for transient stability assessment requires the computation of type-1 UEPs to determine the relevant critical energy level needed for stability assessment [1], [2], [8], [34]. The computation of a UEP is also required in assessing the proximity of an operating point to a system's voltage collapse limit [5]–[7] for direct voltage stability assessment [3]. These direct methods for transient stability and voltage stability analysis are important because they are faster and provide additional information about degree of stability, which power system operators can use for control decisions. The development of direct methods for power system stability assessment are also important to the modern power system engineer because as the level of uncertainty in the power system increases with the proliferation of intermittent and stochastic renewable energy sources there will be a need for faster stability assessment methods to perform scenario based transient stability or voltage stability analysis. UEPs are also computed in power flow analysis [4] when all the possible solutions of a power system are required [9]–[13]. However, the computation of UEPs, with algebraic solvers like the Newton-Raphson's method, are inherently difficult due to the generally small size of their convergence regions [1], [30] and the difficulty in finding an initial point that is sufficiently close to the UEP.

Various methods have been proposed for computing UEPs. Some methods like the corrected corner, the ray point approximation method, the MOD, and the BCU [1], [14]–[17], [26] are based on finding efficient and robust ways to determine correct initial points for computing UEP, and then solving for the exact UEP using algebraic solvers like the Newton-Raphson method. The dependence of these methods on the Newton-Raphson method makes the computation of the UEP fast when the initial guess they find are sufficiently close to the UEP. However, the initial guess they find might be relatively far from UEP being computed and out of the Newton-Raphson convergence region of the UEP. Other methods are based on continuation and homotopy methods [9]–[13], [18], [20], [22], [23] or optimization techniques [27]. These methods have been found to be intractable for larger systems.

There are significant efforts directed toward developing methods for computing UEPs or solutions of nonlinear systems by transforming the problem into the computation of stable equilibrium points (SEPs) (for example, see [19], [21], [24]). The methods proposed in [21], [24] used a spectral decomposition of the Jacobian of the original system to construct a new gradient system where the UEPs in the original system are SEPs, while [19] proposes the construction of a generic quasi-gradient system, combined with a reflected gradient system, for the computation of multiple solutions of a nonlinear system. The proposed methods in [21], [24] are only applicable to the power system network-reduction models, and the method in [19] focuses more on using the transformation and a reflected gradient system to determine the initial guesses sufficiently close to multiple solutions of a nonlinear system.

In this paper, we propose a new theory-based solver for robust computation of UEPs of a system of ordinary differential equations (ODEs) or differential algebraic equations (DAE) representing say a power system. The solver consists of two steps.

Manuscript received January 28, 2019; revised June 19, 2019 and August 12, 2019; accepted September 21, 2019. Date of publication October 8, 2019; date of current version April 22, 2020. Paper no. TPWRS-00148-2019. *(Corresponding author: Robert Owusu-Mireku.)*

R. Owusu-Mireku is with the Eaton, Cooper Powers, Franksville, WI 53126 USA (e-mail: ro82@cornell.edu).

H.-D. Chiang is with the Department of Electrical and Computer Engineering, Cornell University, Ithaca, NY 14853 USA (e-mail: hc63@cornell.edu).

M. Hin is with the Center for Applied Mathematics, Cornell University, Ithaca, NY 14853 USA (e-mail: mfh72@cornell.edu).

Color versions of one or more of the figures in this article are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TPWRS.2019.2946143

The first step transforms the original problem of computing UEPs into the problem of computing stable equilibrium points (SEP) of a surrogate system called the quotient gradient system (QGS) [28]. The second step computes the desired SEP of the QGS via the Pseudo-transient continuation method [33], a trajectory-unified method (TJU). This enlarges the convergence region and avoids geometric fractals that can manifest in convergence regions of methods like Newton-Raphson [29], [35]. Numerical results suggest that the proposed method generally has better convergence regions as compared to the Newton-Raphson (NR) method when applied to UEP computation but sacrifices some speed for robustness. The proposed method is to be used in tandem with existing methods, like the BCU method, or any of the other methods that require an algebraic solver for UEP computations. This work focuses on enlarging the connected convergence region to reduce the likelihood that the initial point determined by methods like the boundary following procedure proposed in [1] will be outside the convergence region of the UEP being computed.

The major difference between our proposed method and the method in [19], [21], [24] is that the proposed method does not require eigenvalue computations. The transformation used in our work is a subset of the class of transformations proposed in [19], but this work focuses on finding a specific UEP, given an initial condition. The usage of an inexact TJU method on a QGS surrogate is also new in this kind of power system application. In effect, this proposed method will provide a theory-based method to improve the robustness of UEP computations. The method is also independent of the network model and can be applied to problems that are unrelated to power systems. Our proposed method could be used in tandem with [19] for solving for multiple UEPs in a power system. Under certain conditions, the proposed method can have local q-superlinear or local quadratic convergence. Compared to other newton-based methods as in [30], [36]–[38] the proposed method uses an implicit integration technique after the transformation/construction of the surrogate system. This gives it an advantage of better stability compared to the other methods that use explicit integration techniques. Thus, using the proposed approach will ensure that the equilibrium point found is in fact the controlling UEP associated with the forward time fault trajectory of any initial state.

The contribution of this paper is 1) The ensemble of the transformation of the UEP to a SEP using the QGS transformation to enlarge the convergence region of the UEP with the application of a customized pseudo-transient continuation method. 2) The approximation of the pseudo-transient continuation method in a way that exploits the structure of the transformation technique and speeds up the computation of the UEP. 3) The priming of the NR method with the proposed method which exploit the larger convergence region of the latter with the speed of the former.

This paper is organized as follows. Section II presents the transformation of the original system. Section III reviews the pseudo-transient continuation ($\psi\ tc$) method for ODE systems. Section IV proposes the QGS-based pseudo-transient continuation ($\psi\ tc$) method and Section V presents the conditions for its convergence. Section VI presents two numerical examples, discusses the findings from the simulations, and proposes a method that combines the proposed method with the NR method for computing UEPs. The conclusions are then stated in Section VII.

II. System Transformation

A. Original Problem Formulation

Without loss of generality, the equilibrium equation of an ODE or DAE describing the dynamics of a power system can be represented by (1), [1].

$$F(x) = 0 \tag{1}$$

where $F : R^n \to R^n$, F is assumed to be C^2, $x \in R^n$ is a vector of equilibrium states, and $n \geq 1$. The function $F(x)$ is either the vector field of the power system ODEs or the vector field and the algebraic manifold of the power system DAEs. Regardless of what (1) represents, UEPs are inherently difficult to compute because appropriate initial guesses are difficult to determine, and numerical methods like NR [1], [29] have small convergence regions. However, like any equilibrium point or zero of a function, UEPs can be computed using algebraic solvers when an initial point sufficiently close to the UEP is provided. The size, continuity, and compactness of the convergence region of the underlying algebraic solver determines how close an initial guess must be for the solver to successfully converge to the UEP. The purpose of this work is to provide a fast solver that has a large connected convergence region, implying that the initial guesses can be further away from a UEP. One common solver used for UEP computations is the NR method.

$$\dot{x} = f(x) = -DF(x)^{-1}F(x) \tag{2}$$

Each step in the NR algorithm can also be viewed as a forward Euler step of the dynamic system (2) with a time step of 1, where $DF(x)$ is the Jacobian matrix of $F(x)$ and $-DF(x)^{-1}F(x)$ is the vector field $f(x)$ of the ODE (2) [30]. Thus, the NR method and some of its variants—for example, Iwamoto's method—are basically the forward Euler integration of the new dynamic system (2) [30]. This new system (2) is stable at all equilibrium points where $DF(x)$ is nonsingular. Consequently, in terms of the computation of UEPs, the NR method and some of its variants can be considered as a numerical technique that involves the transformation of a UEP of the original system (1) into an SEP of (2), and an application of an explicit integration method like the Euler method, to solve for the new system's SEP, which is the UEP of the system of differential equations corresponding to (1) for a given initial point.

Unlike the transformation used in this work, the transformation used in the NR method employs the inverse of a Jacobian $DF(x)$ which can be singular in the neighborhood of the resulting SEP. Implying that (2) does not always satisfy the requirements for existence and uniqueness of solutions. Some variants of the NR method, like the Continuous Newton-Raphson method (Continuous NR) presented in [30], propose the use of a much more stable explicit integration technique, such as the fourth order Runge-Kutta (RK4) method over the forward Euler method, for the integration of the dynamic system in (2).

Fig. 1. An illustration of the quotient gradient system transformation.

B. Quotient Gradient System Transformation

The Quotient Gradient System (QGS) [28] transformation used in the proposed method is illustrated in Fig. 1 and has the following form:

$$\dot{x} = Q(x) = -DF(x)^T F(x). \tag{3}$$

Observe that the QGS system has the Lyapunov function:

$$V(x) = \tfrac{1}{2} F(x)^T F(x) \tag{4}$$

Thus, the QGS surrogate is a generalized gradient system [19] with only stable equilibria.

Proposition 1: Let all the equilibrium points of (3) be hyperbolic and finite in number. If $\hat{x}$ is the solution of (1), then $\hat{x}$ is a SEP of (3).

Proof: Note that if $\hat{x}$ is a solution of (1), then it is an equilibrium point of (3). Since (3) is completely stable and thus only has stable equilibria, $\hat{x}$ must also be a SEP of (3).

Hyperbolicity of equilibrium points of a power system implies that all trajectories that converge to equilibrium points of the power system, like the steady state operating conditions, must be unique. This characteristic is dependent on how the power system is modeled. The hyperbolicity requirements for the QGS transformation limits the applicability of the proposed method to power system problems that involve non-hyperbolic equilibrium points like in [39], [40]. For applications of these sort a fast continuation method approach will be more appropriate.

The advantage of the transformation used in the proposed method over the NR method is that the stability region of the SEP resulting from the transformation exists and is smooth (connected), while there is no stability region for the SEP from the NR transformation (since the resulting system does not have unique solutions). The UEP and consequently, its corresponding stability region in the new system (3) can also be efficiently computed by solving for the SEP using implicit integration techniques, which are more stable compared to explicit ones like the Euler method.

III. The Pseudo-Transient Continuation Method

A. Introduction

After transforming the algebraic problem/power system equilibrium problem into computation of a SEP of an associated dynamic system (3), the most efficient way to solve for the equilibrium point is to use the TJU method. Since we are only interested in the steady state solution, we need a TJU method that can converge quickly to the corresponding stable steady state solution. The TJU method can be numerically implemented using explicit (Euler or Runge-Kutta) or implicit (the trapezoidal method) method. One such implicit method is the pseudo-transient continuation ($\psi\ tc$) method. The $\psi\ tc$ is an implicit TJU that employs adaptive time-stepping for the computation of steady state solutions for partial differential equations, ODEs, and semi-explicit index-one DAEs [32], [33]. It is analogous to an implicit integration of a dynamic system with increasing time steps as the system trajectory approaches the steady state solution.

The trajectory-based nature of $\psi\ tc$ makes its convergence region a better approximation of a SEP's stability region compared to the convergence region of other algebraic solvers. This characteristic of $\psi\ tc$ implies that the initial points don't have to be as close as the NR method requires. It also means trajectories are more likely to converge to a power-system relevant SEP and not to other non-physical local minima of $|Q(x)|$ [33]. The adaptive time steps used in the $\psi\ tc$ method make it faster than conventional fixed-step integration methods, since larger time steps are taken as the trajectory gets closer to the EP. The $\psi\ tc$ method is an implicit method, making it numerically more stable than explicit methods like the Euler and Runge-Kutta methods.

Given an initial value problem of the form (5):

$$\dot{x} = -G(x), x(0) = x_0 \tag{5}$$

The steady state solution can be found by integrating (5) with $\psi\ tc$. Each step in the $\psi\ tc$ method is given by (6):

$$x_{i+1} = x_i - \left(h_i^{-1} I + DG(x_i)\right)^{-1} G(x_i) \tag{6}$$

where I is an identity matrix of appropriate size, and h_i is a variable time step systematically adjusted to improve the rate of convergence to a steady state. The time step can be adjusted using the "switch evolution relaxation" (SER) (7) or the norm of the steps [33]:

$$h_i = \min\left(h_{i-1} \frac{\|G(x_{i-1})\|}{\|G(x_i)\|}, h_{\max}\right) \tag{7}$$

where $h_{\max}$ is a large upper bound of h_i. For the results in this work, (7) was used for the time step adjustments and $h_{\max}$ was set to ∞. h can be a vector of different time steps if the system of equations in (5) is stiff or has an ill-conditioned Jacobian.

$\psi\ tc$ Algorithm [33]:

1) Set $x = x_0$ and $h = h_0$. Evaluate $G(x)$.
2) While $\|G(x)\|$ is larger than a threshold:
   a) Solve $(h^{-1} I + DG(x))s = -G(x)$.
   b) Set $x = x + s$.
   c) Evaluate $G(x)$.
   d) Update h.

It has been shown that if a steady state solution exists, then ψtc for ODEs of the form (5) has a local q-superlinear or quadratic convergence under certain assumptions [33]. To improve the computational performance of the application of the $\psi\ tc$ to (3) and guarantee convergence to solutions of the power system equilibrium equations (1), we propose an inexact approach, termed the Quotient Gradient System-based Pseudo-Transient Continuation method (QGS-based $\psi\ tc$) for solving UEPs that is presented in the next section.

IV. The Quotient Gradient System-Based Pseudo-Transient Continuation Method

If we apply the $\psi\ tc$ method to (3), each step of $\psi\ tc$ method (6) can be rewritten, as shown in (8). where f_k is the kth function in $F(x)$ and $H^T(f_k)$ is the Hessian of f_k. We can then simply solve (8) at step (2a) for each iteration of the $\psi\ tc$ algorithm without any further modifications, since the UEP is now a SEP due to the QGS transformation. However, this approach requires the construction of two Jacobians at each iteration, one for the QGS transformation and another for step (2a) of the $\psi\ tc$ algorithm. This can be computationally expensive if the Jacobians are constructed numerically. Also, the analytical Jacobian for step (2a) can be complex and error-prone. The use of automatic differentiation for a Jacobian construction in step (2a) can also be quite challenging for complex functions or systems of equations like (3).

$$x_{i+1} = x_i - \left(\begin{array}{c} h_i^{-1} I + \sum\limits_{k=1}^{n} f_k H^T(f_k) \\ + \\ DF(x_i)^T DF(x_i) \end{array} \right)^{-1} DF(x_i)^T F(x_i) \tag{8}$$

If we assume that $\| - \sum_{k=1}^{n} f_k H^T(f_k) \|$ is sufficiently small, which is true as we approach the equilibrium point of (3), then we can use a quasi-Newton method approach and approximate the Jacobian of (3) with (9):

$$D\tilde{Q}(x) \approx -DF(x)^T DF(x) \tag{9}$$

$$\left(h^{-1} I + DF(x)^T DF(x)\right) s \approx -DF(x)^T F(x) \tag{10}$$

$$\min_s \left\| \begin{bmatrix} DF(x) \\ h^{-\frac{1}{2}} I \end{bmatrix} s + \begin{bmatrix} F(x) \\ 0 \end{bmatrix} \right\| \tag{11}$$

$$\left\| \left(h^{-1} I + DF(x)^T DF(x)\right) s + DF(x)^T F(x) \right\| \leq \left\| \xi DF(x)^T F(x) \right\| \tag{12}$$

$$x_{i+1} = x_i + s. \tag{13}$$

Step (2a) in the $\psi\ tc$ algorithm can then be replaced by either (10) or (11). We recommend for small dense systems to solve (10) using QR factorization, while for larger sparse systems, solve (11) using Cholesky decomposition or iterative methods like the conjugate gradient methods (precondition conjugate gradient methods). Generically, (10) and (11) can be represented by (12) where ξ could be related to the difference between the exact Jacobian of (3) and the approximate Jacobian (9) if (10) or (11) are solved using QR factorization or Cholesky decomposition, or ξ could be related to the Jacobian approximation and the inexact steps involved when iterative methods are used to solve the Newton step.

The proposed QGS-based $\psi\ tc$ method can be summarized by the following steps.

1) Transform the original algebraic problem (1) into a dynamic system (3) using the QGS transformation.
2) Starting at the given initial guess, apply the $\psi\ tc$ method to the original system (1)'s surrogate QGS system (3), solving either (10) or (11) at step (2a) of the $\psi\ tc$ algorithm.

In comparison to the NR method, the QGS-based $\psi\ tc$ method will, in most cases, require more iterations since it does not always converge quadratically to the equilibrium point. Since it will also require more computations per iteration compared to the NR method, we suggest that the QGS-based $\psi\ tc$ be used to re-start the UEP computation and then switch back to the NR method when the convergence criterion for the QGS-based $\psi\ tc$ method is below a defined threshold. It should be noted that the QGS-based $\psi\ tc$ method shares some similarities with the Levenberg-Marquardt method used for nonlinear least square problems, but the QGS-based $\psi\ tc$ method was derived independently and is a time-dependent integration instead of spatial.

V. Convergence Analysis

A. Convergence

In this section, we present the type of convergence and the conditions necessary for the convergence of the proposed QGS-based $\psi\ tc$ method. We show that the QGS-based $\psi\ tc$ can have local q-superlinear or even local quadratic convergence under certain conditions. We also show that the Jacobian approximation used in the QGS-based $\psi\ tc$ method guarantees that if the QGS-based $\psi\ tc$ method convergences it will only converge to solutions of the original power system equilibrium equations (1). Let x^* be a UEP of the dynamic system with equilibrium equations represented by (1).

Assumptions:

1) $DF(x)^T F(x)$ is everywhere defined and Lipschitz continuously Fréchet differentiable, $\|DF(x)^T F(x)\| \leq M$, $M > 0$ for all x. For physical power system models this means that the state variables like bus voltages, machine angle change continuously and the velocity and acceleration of the changes in the variables can be tracked.
2) There are $\varepsilon_2, \beta > 0$ such that if $\|x - x^*\| < \varepsilon_2$, then $\|(h^{-1} I + D(DF(x)^T F(x)))^{-1}\| \leq (1 + \beta h)^{-1}$ for all $h \geq 0$.
3) The equilibrium points of (3) are hyperbolic. (Proposition 1).
4) $Inf_i h_i > 0$. This assumption must be satisfied to prevent the QGS-based $\boldsymbol{\psi tc}$ method from stalling.

Proposition 2. (convergence analysis): Consider solving for the power system equilibrium (1). Let $F(x^*) = 0$. Let Assumptions 1, 2, 3, and 4 hold. The proposed QGS-based $\psi\ tc$ method will have a local q-superlinear convergence to x^* from an initial point x_0 lying inside the stability region of x^* for the QGS system (3), $\{h_i\}$ is of the form (8), $h_{\max} = \infty$, $\xi_i \leq \hat{\xi}$, $h_i \to h_{\max}$, and $\xi_i \to 0$. The convergence to x^* is q-linear if $h_{\max} < \infty$.

Proof Sketch: From Proposition 1, if $F(x^*) = 0$ and Assumption 3 holds, then x^* is a hyperbolic stable equilibrium point of (3). Consequently, x^* has a stability region in (3). We also know that the portion of ξ_i that is due to the approximation of the Jacobian in the QGS-based $\psi\ tc$ method is directly proportional to $\| - \sum_{k=1}^{n} f_k H^T(f_k)_i \|$, and that $\| - \sum_{k=1}^{n} f_k H^T(f_k)_i \| \to 0$

since $f_{ki} \to 0$ as you get closer to x^*. This implies that $\xi_i \to 0$ as $x \to x^*$. Now, since x^* has a stability region in the quotient gradient system, (3), the convergence of the QGS-based $\boldsymbol{\psi}\ tc$ method to x^* can be analyzed using the analysis in [33].

In summary, the QGS-based $\boldsymbol{\psi}\ tc$ with $h_{\max} = \infty$ will converge at least q-superlinearly if all the assumptions are satisfied, ξ_i is sufficiently small, and $\xi_i \to 0$ as $x_i \to x^*$. The convergence of the QGS-based $\boldsymbol{\psi}\ tc$ method to x^* is locally q-quadratic if $\xi_i = O\|Q(x_i)\|$.

A solution/equilibrium point of the QGS system (3) can be belong to one of three groups:

1) A solution x^* such that $F(x^*) = 0$, and $-DF(x^*)^T F(x^*) = 0$.
2) A solution x^* such that $F(x^*) \neq 0$, and $-DF(x^*)^T F(x^*) = 0$.
3) An x^* such that $F(x^*) = 0$, $-DF(x^*)^T F(x^*) = 0$, and $DF(x^*)^T$ is singular.

Unlike the standard $\boldsymbol{\psi}\ tc$ or the Euler method, the QGS-based $\boldsymbol{\psi}\ tc$ method when applied to (3) converges only to solutions of group 1 because of the QGS structure-dependent Jacobian approximation. This implies that approximating the QGS Jacobian guarantees that if the method converges, it will only converge to solutions of the power system equilibrium equations (1).

Theorem 3: If the QGS-based $\boldsymbol{\psi}\ tc$ method converges to x^*, then x^* is a solution belonging to group 1 and hence, a solution of the original power system equilibrium (1).

Proof: If the QGS-based $\boldsymbol{\psi}\ \boldsymbol{tc}$ method converges, then it implies that there exists a sequence of iterates $x_i \to x^*$ where $DF(x^*)^T F(x^*) = 0$. If x^* belongs to group 3, then $DF(x^*)^T DF(x^*)$ is singular. Since the function $\lambda_{\min}(DF(x)^T DF(x))$, the minimum eigenvalue of $DF(x)^T DF(x)$ is continuous in x, the sequence $\{\lambda_{i,\min} = \lambda_{\min}(DF(x_i)^T DF(x_i))\}$ converges to zero as $x_i \to x^*$. Thus, there exists a value of $i < \infty$ where $DF(x_i)^T DF(x_i)$ is singular and h_i is very large such that $(h_i^{-1} I + DF(x_i)^T DF(x_i)) \approx DF(x_i)^T DF(x_i)$, which is a contradiction since $(DF(x_i)^T DF(x_i))^{-1}$ must exist for convergence. Hence, the QGS-based $\boldsymbol{\psi tc}$ method cannot converge to solutions that belong to group 3. If x^* belongs to group 2, then the $Null(DF(x^*)^T) \neq \emptyset$, and $F(x^*) \in Null(DF(x^*)^T)$. This implies that $DF(x^*)^T DF(x^*)$ is singular, which implies that the QGS-based $\boldsymbol{\psi tc}$ method cannot converge to solutions that belong to group 2, since it will be a contradiction. Thus, if the QGS-based $\boldsymbol{\psi}\ \boldsymbol{tc}$ method converges, the solution can only belong to group 1 and hence, the solution will be a solution of (1).

VI. Numerical Examples

The numerical simulations were performed on a computer with an Intel Core i7-3630Q M CPU @2.40G Hz processor and 16 GB memory. All the simulations were performed with Matlab 7.11. Matlab's Fsolve is used as a check for accuracy. The systems simulated were the structure-preserving models of the WSCC 9-bus 3-machine system and the IEEE 145-bus 50-machine system with classical generators and constant impedance load models.

Structure Preserving Model: For n generators and m buses,

$$\dot{\tilde{\delta}}_i = \tilde{\omega}_i \tag{14}$$

$$M_i \dot{\tilde{\omega}}_i = -D_i \tilde{\omega}_i + P_{m_i} - \frac{E'_{qi} V_i \sin\left(\tilde{\delta}_i - \tilde{\theta}_i\right)}{X'_{di}} - \frac{M_i}{M_T} P_{COI} \tag{15}$$

For generator buses $i = 1, \ldots . n$:

$$(I_{di} + jI_{qi})\, e^{-j(\delta_i - \pi/2)} = \sum_{k=1}^{m} Y_{ik} e^{j\alpha_{ik}} V_k e^{j\tilde{\theta}_k}$$

$$I_{di} = \frac{E'_{qi} - V_i \cos\left(\tilde{\delta}_i - \tilde{\theta}_i\right)}{X'_{di}},\ I_{qi} = \frac{V_i \sin\left(\tilde{\delta}_i - \tilde{\theta}_i\right)}{X'_{qi}} \tag{16}$$

For load buses $i = n+1, \ldots . m$:

$$0 = \sum_{k=1}^{m} Y_{ik} e^{j\alpha_{ik}} V_k e^{j\tilde{\theta}_k} \tag{17}$$

$$\delta_0 = \frac{1}{M_T} \sum_{i=1}^{n} M_i \delta_i, \omega_0 = \frac{1}{M_T} \sum_{i=1}^{n} M_i \omega_i$$

$$M_T = \sum_{i=1}^{n} M_i,\ \tilde{\delta}_i = \delta_i - \delta_0,\ \tilde{\omega}_i = \omega_i - \omega_0\ ,$$

$$\tilde{\theta}_i = \theta_i - \theta_0 \text{ for } i = 1, \ldots . n,$$

$$P_{COI} = \sum_{i=1}^{n} P_{m_i} - \sum_{i=1}^{n} \frac{E'_{qi} V_i \sin\left(\tilde{\delta}_i - \tilde{\theta}_i\right)}{X'_{di}}$$

where $\delta_i, \omega_i,\ M_i,\ D_i, P_{m_i}, E'_{qi}, X'_{di}, V_i, \theta_i$ and $Y_{ik} e^{j\alpha_{ik}}$ are: rotor angle of machine i, speed of machine i, moment of inertia of machine i, damping of machine i, mechanical power of machine i, equivalent transient quadrature internal voltage of machine i, equivalent direct transient reactance of machine i, voltage magnitude at bus i, voltage angle at bus i, and the network admittance between buses i and k, respectively.

The size and connectedness of the convergence region of the QGS-based $\boldsymbol{\psi}\ tc$ method when used for UEP computation will be one of the metrics used in the evaluation of the proposed method's performance in comparison to other solvers. The convergence region of a UEP of a dynamic system for a numerical solver N is defined as the set of initial points that converge to the UEP for the numerical solver [1]. Obviously, a large and connected convergence region implies that the initial point for the UEP computation does not have to be very close to the UEP for the algebraic solver to converge.

The convergence region of a UEP for a classical generator model can be constructed using a dimension-reduced model in which ω is a zero vector. This construction is made by creating a grid of initial points around the UEP in the machine angle space less the reference machine angle variable. The reference machine angle and the algebraic variables corresponding to the initial points on the grid are then updated by using the COI equation for machine angles and the solutions of the network

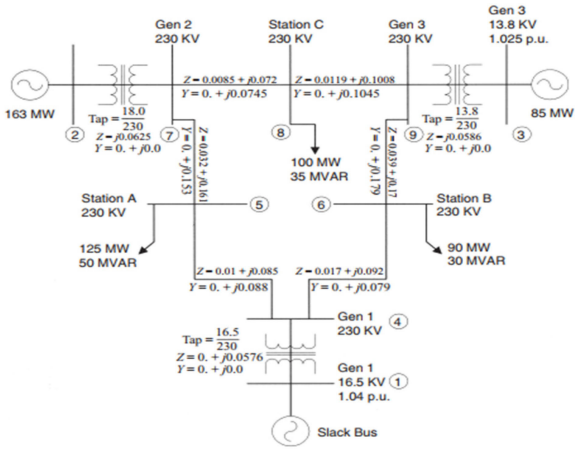


Fig. 2. The WSCC 9-bus 3-machine system. The value of Y is half the line charging.

TABLE I
CONTINGENCY LIST OF THE WSCC 9-BUS 3-MACHINE SYSTEM

Contingency Number	Fault Bus	From Bus	To Bus
1	7	7	5
2	7	8	7
3	6	4	6
4	6	6	9
5	9	9	8

equations at the grid points, respectively. The solutions of the equilibrium equations for the dynamic system, starting at these initial points, are then computed using the algebraic solver for which a convergence region of a UEP is being constructed. If the L2 norm of the difference between the computed equilibrium point and the UEP is below a defined threshold, then the initial point is in the convergence region of the UEP under the algebraic solver.

A. The WSCC 9-Bus 3-Machine System With a Classical Generator Model

The proposed method is tested on the WSCC 9-bus 3-machine system [1] (see Fig. 2), to compute a UEP on the stability boundary of a post-fault system. A uniform damping of $\lambda = 0.1$ is assumed, and the simulation is done in the Center of the Inertia (COI) reference framework. Initial time step h_0 of the $\psi\ tc$ and QGS-based $\psi\ tc$ methods are set to 0.1. Table I shows the list of contingencies used in our simulations of this system.

We first look at a simulation example where the initial point for a UEP computation is outside the convergence region of the NR method but within the convergence region of the proposed method. Fig. 3 shows a comparison of projected convergence regions of the NR method and the proposed method for the controlling UEP of contingency 1. We observe that an initial point, depicted by the black asterisk, falls within the convergence region of the proposed method but outside the convergence region of the NR method. Hence, the proposed method can converge to the CUEP of contingency 1 while the NR method diverges. Fig. 3 also shows that the convergence region of the proposed method is the largest one among the two methods and

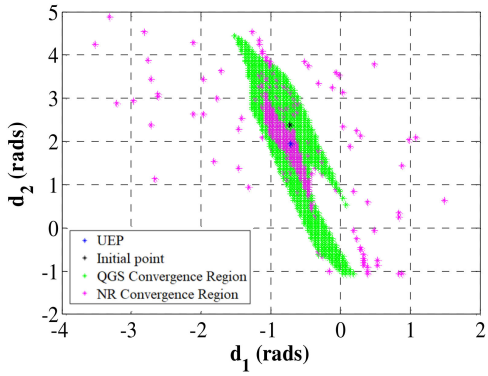


Fig. 3. Convergence regions of a UEP for the structure-preserving model (contingency 1) of a post-fault WSCC 9-bus 3-machine system. The convergence region of the NR method is superimposed on the convergence region of the QGS-based $\psi\ tc$ method, with an example of an initial point that is in the convergence region of the QGS-based ψtc for a UEP but not in the convergence region of the NR method. Note that there are several points in the convergence region that appear disconnected. However, these points are either connected via a path transverse to the delta_1 and delta_2 projection depicted or are numerical artifacts.

connected while that of the NR method is disconnected. The connected section of the convergence region of the QGS-based $\psi\ tc$ method is also larger. Consequently, the initial point for a UEP computation could be close and the NR method will still diverge or converge to the wrong UEP or SEP, a less likely case with the proposed method. Since a point in the convergence region of the proposed method is path connected to the QGS SEP, the convergence region itself is a connected set. According to the property of uniqueness of each trajectory, it follows that the QGS-based $\psi\ tc$ method allows us to compute convergence regions which are numerical approximations of the stability region/basins of attraction of the surrogate SEPs (the SEPs for the QGS constructed) with an implicit ODE solver. The multiple UEPs and SEPs in the state space of the power system model each have their own connected QGS-based $\psi\ tc$ method convergence region which are disjointed from the convergence region of other equilibrium points. Since each stability region is disjointed from any other stability region. Furthermore, these stability regions must have boundaries that are formed by the union of the stable manifolds of the UEPs of the surrogate system by virtue of its gradient vector field [1]. Note that the UEPs of the QGS system are not equilibria of the original system by Proposition 1. Therefore, in the context of the original system, we find that the convergence region of the original UEPs are bounded by singular surfaces consisting of trajectories that converge to some singularity of the original system.

Table II shows a comparison of the convergence regions of the controlling UEPs for the structure-preserving model of the post-fault WSCC 9-bus 3-machine system for the five contingencies in Table I using the NR method, the QGS-based tc, and the continuous Newton-Raphson method (CNR). The table groups the number of initial points that converges to the controlling UEPs into two groups for each of the three methods, the group of initial points that belong to the connected convergence region of the UEP and the group of initial points that fall outside the

TABLE II
COMPARISON OF CONVERGENCE REGIONS OF METHODS FOR SOLVING THE CUEPS OF THE STRUCTURE-PRESERVING MODEL OF THE WSCC 9-BUS 3-MACHINE SYSTEM $\varepsilon = 10^{-6}$

Contingency	Initial Points in Connected Convergence Region			Initial Points Outside Connected Convergence Region		
	NR	QGS ψtc	CNR	NR	QGS ψtc	CNR
1	241	1016	426	62	0	160
2	231	337	293	26	1	24
3	230	490	346	120	3	193
4	215	510	345	127	1	221
5	240	361	324	68	0	26

TABLE III
COMPARISON OF METHODS FOR SOLVING THE CUEPS OF THE STRUCTURE-PRESERVING MODEL OF THE WSCC 9-BUS 3-MACHINE SYSTEM $\varepsilon = 10^{-6}$

Method	Average Iterations	Average Computation Time (secs)
NR	4	0.023
MATLAB'S Fsolve	5	0.035
Original ψtc	9	0.200
QGS ψtc	9	0.027
QGS ψtc with NR	8	0.027
Continuous Newton	15	0.051

connected convergence region of the UEP. We have the following observations:

- Almost all of the initial points that converged to a controlling UEP using the QGS-based $\psi\ tc$ method belong to the connected convergence region. Meanwhile 10% to 37% of the initial points that converge to a controlling UEP when computed with the NR method belong to the group of initial points that fall outside the connected convergence region. Similarly, 7% to 39% of the initial points that converge to a controlling UEP computed with the CNR method also belong to the group of initial points that fall outside the connected convergence region. It should be noted that most of the initial points that fall outside the connected convergence region are closer to other equilibrium points, and as a result, should be converging to those equilibrium points. This is particularly important in controlling UEP computations where you want your numerical solver to converge to the actual UEP whose stable manifold is intersecting with the sustained fault trajectory for the contingency. These observations show that a large-section of the convergence region of each of the controlling UEP of the QGS-based $\psi\ tc$ method is connected; as compared with the NR and CNR methods, implying that initial points that are close to the controlling UEP of interest is more likely to converge to the UEP of interest when computed with the QGS-based $\psi\ tc$ method.
- Table II shows that the QGS-based $\psi\ tc$ method has a larger UEP convergence region, as compared to the NR method, for the 5 contingencies. We also observe the UEP convergence region for the QGS-based $\psi\ tc$ method is larger than that of the CNR method for 3 contingencies. However, for the other two contingencies, we find that a third of the convergence region of the CNR method is not part of the connection portion. Consequently, the connected portion of the convergence region of the QGS-based $\psi\ tc$ method is larger than the convergence region of the CNR method. Most of these disconnection section of the CNR convergence region closer to other equilibrium points in the state space of the power system. Thus, they converged to the wrong equilibrium point.
- Based on these observations we can conclude that the proposed method is more robust than the NR method and other related methods.

Table III shows a comparison of the average number of iterations and computation time in seconds for CUEP evaluations for the 5 contingencies given an initial point, using Matlab's Fsolve, the NR method, the CNR [30] method, the original $\psi\ tc$ method (thus, the $\psi\ tc$ method applied to (3) or (9)), the QGS-based $\psi\ tc$, and the QGS-based $\psi\ tc$ combined with the NR method. The table shows that, by using the QGS-based $\psi\ tc$ method, we maintain the average number of iterations while improving the UEP computational speed by 7.4 times on average compared to the original $\psi\ tc$ method. We also get a lesser number of average iterations when the QGS-based $\psi\ tc$ method is combined with the NR method. The table also shows that the computational speed of the QGS-based $\psi\ tc$ method is comparable to the computational speed of the NR method and Matlab's Fsolve and about 2 times faster compared to the CNR method for this numerical example. As expected, the QGS-based $\psi\ tc$ requires more iterations than both Matlab's Fsolve and the NR method. The CNR method requires the largest number of iterations for all the contingencies in this study.

To assess the advantage presented by the proposed QGS-based $\psi\ tc$ over the NR method the two methods were each applied in the BCU method to compute the CUEPs in the evaluation of the critical clearing time (CCT) for the 5 contingencies.

The algorithm for the BCU Method for transient stability analysis can be summarized as follows:

1) Compute the post-fault SEP of the dynamic system (power system).
2) Find the exit-point of the projected fault-on trajectory in a reduced model [1].
3) Starting from the exit-point find the minimum gradient point (MGP) using the stability boundary following procedure [1]. The MGP is the point with a lowest norm as your move along the stability boundary of post-fault SEP.
4) Compute the CUEP with the MGP as the initial guess and consequently the critical energy at the CUEP [1].
5) Compute the energy at the initial post-fault state (The state of the system right after the fault is cleared). If this energy is less than the critical energy, then the system is stable else it is unstable.

It is in step 4 that the proposed method is needed to solve for a UEP. However, the type of solver used in step 4 can also determine how the MGP is calculated. For instance, in the case of the QGS-based $\psi\ tc$ where the convergence region of the UEP/CUEP has been expanded and is more continuous, we can set the tolerance value, norm of the gradient of our MGP, to be higher since that will reduce the amount of computation

TABLE IV
COMPARISON OF THE CRITICAL CLEARING TIME (SECONDS) FOR BCU AND TIME DOMAIN SIMULATION WSCC 9-BUS 3-MACHINE SYSTEM—STRUCTURE-PRESERVING MODEL

	NR (F(x)=0)		QGS Based ψtc		
Contingency Number	CUEP Computation	BCU CCT	CUEP Computation	BCU CCT	Time Domain CCT
1	Failed	-	Success	0.164	0.166
2	Success	0.156	Success	0.156	0.185
3	Success	0.437	Success	0.437	0.460
4	Success	0.381	Success	0.381	0.400
5	Success	0.207	Success	0.207	0.240

TABLE V
CONTINGENCY LIST OF IEEE 145-BUS 50-MACHINE SYSTEM

Contingency Number	Fault Bus	From Bus	To Bus
1	6	7	6
2	72	59	72
3	116	115	116
4	100	100	72
5	91	91	75
6	112	112	69
7	101	101	73
8	6	6	1
9	59	59	103

required in the stability boundary following procedure [1]. In this numerical example we increased the MGP tolerance value from 1 to 1.2. Table IV shows the CCTs of the 5 contingencies when the QGS-based $\psi\ tc$ and NR method were each applied in the BCU method for CCT computation for the WSCC 9-bus system. The table shows that all the two methods resulted in the same CCT's for 4 of the contingencies except for contingency 1 where the NR method failed to successful find the CUEP. The computed BCU CCT's were also conservative compared to the CCT from time domain. A test of the equilibrium type of the CUEP also showed that they were all type-1 equilibrium points and are on the stability boundary of the post-fault SEP [1]. The failure of the NR method for contingency 2 can be explained by the graph in Fig. 3. It can be seen from the graph that the MGP for contingency 1 represented by the black star lies inside the convergence region of the QGS-based $\psi\ tc$ method but outside the convergence region of the NR method. Hence, the proposed QGS-based $\psi\ tc$ method can converge to the CUEP of contingency 1 while the NR method diverges. The divergence of the NR method despite the proximity of the MGP to the CUEP goes to emphasis the advantage of using the QGS-based $\psi\ tc$ method over the NR method when computing UEPs. A failure in the CUEP computation such as convergence to the wrong UEP can lead to inaccurate power system transient stability assessments of contingencies leading to potential stability issues.

B. The IEEE 145-Bus 50-Generator System With the Classical Generator Model

The proposed method is also tested on the IEEE 145-bus 50-machine system for CUEP computations. A uniform damping of $\lambda = 0.5$ is assumed and the simulation is done in the Center of Inertia (COI) reference framework. An initial time step, the same as in the previous study, is used. Table V shows the list of 9 contingencies used in the test.

TABLE VI
COMPARISON OF METHODS FOR SOLVING THE CUEPS OF THE STRUCTURE-PRESERVING MODEL OF THE IEEE 145-BUS 50-MACHINE SYSTEM $\varepsilon = 10^{-6}$

Method	Average Iterations	Average Computation Time (secs)
NR	3	0.038
MATLAB'S Fsolve	4	0.068
Original ψtc	9	10.411
QGS ψtc	9	0.078
QGS ψtc with NR	7	0.066
Continuous Newton	14	0.293

TABLE VII
COMPARISON OF THE CRITICAL CLEARING TIME (SECONDS) FOR BCU AND TIME DOMAIN SIMULATION IEEE 145-BUS 50-MACHINE SYSTEM—STRUCTURE-PRESERVING MODEL

	NR (F(x)=0)		QGS Based ψtc		
Contingency Number	CUEP Computation	BCU CCT	CUEP Computation	BCU CCT	Time Domain CCT
1	*Failed*	-	Success	0.10	0.16
2	Success	0.22	Success	0.22	0.23
3	Success	0.27	Success	0.27	0.29
4	Success	0.26	Success	0.26	0.27
5	Success	0.20	Success	0.20	0.20
6	Success	0.24	Success	0.24	0.26
7	Success	0.24	Success	0.24	0.26
8	Success	0.19	Success	0.19	0.20
9	*Failed*	-	*Success*	*0.15*	*0.24*

Table VI shows a comparison of the average number of iterations and computation time in seconds for CUEP evaluations for the 9 contingencies given an initial point, using Matlab's Fsolve, the NR method, the CNR [30] method, the proposed $\psi\ tc$ method, the QGS-based $\psi\ tc$, and the QGS-based $\psi\ tc$ combined with the NR method. The following observations are obtained from this table:

- The proposed QGS-based $\psi\ tc$ method improves the UEP computation speed by 133 times on average, as compared with the original $\psi\ tc$ method,
- The proposed QGS-based $\psi\ tc$ method, combined with the NR method, achieves a better number of average iterations and a faster UEP computation speed of 157 times the speed of the original $\psi\ tc$ method,
- The proposed QGS-based $\psi\ tc$ method is about 4 times faster compared to the CNR method, but 2 times slower than the NR method.
- The proposed QGS-based $\psi\ tc$ method is comparable in computational speed with Matlab's Fsolve for this numerical example. As expected, the QGS-based $\psi\ tc$ requires more iterations than both Matlab's Fsolve and the NR method. The CNR method requires the greatest number of iterations for all the contingencies in this study.

Table VII shows the CCTs of the 9 contingencies when the QGS-based $\psi\ tc$ and the NR method were each applied in the BCU method for CCT computation for the IEEE 145-bus 50-Machine system with MGP tolerances larger than what we usually use in our BCU implementations (15 and over). The table shows that all the three methods resulted in the same CCT's for

7 of the contingencies except for contingency 1 and 9 where the NR method failed to successful find the CUEP. The results once again show that the QGS-based $\psi\ tc$ offer an opportunity to reduce the computational requirements of the stability boundary following procedure [1] used in computing the MPG, the most time-consuming step in the BCU method. By expanding the convergence region of the CUEP/UEP the QGS-based $\psi\ tc$ provides the opportunity for users to use initial guesses (MGPs) that are further way from the UEP. All the UEPs were confirmed to be type-1 UEPs on the stability boundary of the post-fault SEP using the boundary property verification scheme proposed in [1]. The fact that BCU based CCT is always smaller than the time domain CCT is attributed to the conservative nature of the CUEP method, and an approximation in the energy function value evaluation (due to the existence of path dependent terms). We tried to reduce the CCT error by increasing the number of points used in the multistep trapezoidal scheme [1] used for the energy function computations but the CCT errors are still larger for contingencies 1 and 9.

Our numerical studies show that the proposed method achieves robustness in the computation of UEPs by having a larger connected convergence region at the expense of a decrease in speed. Proposition 1 supports this observation. It is shown that by exploiting the structure of the QGS (3), and by using an appropriate approximation of the Jacobian (11), we can significantly improve the speed of the proposed method and ensures convergence only to solutions of the original problem (1). It is due to the larger convergence region of the proposed method that all of the initial guesses lying further away from a UEP, as compared with the NR method and the CNR method, can converge to the UEP.

VII. Conclusion

In this paper, we have proposed a theory-based method that combines a QGS transformation with a TJU method for the computation of UEPs. The method converts the UEP to a SEP by changing the problem into a quotient gradient system. It then applies a quasi-Newton form of the pseudo-transient continuation method by exploiting the structure of the proposed QGS's Jacobian. The main advantages of the proposed method are: 1) the method has a larger connected convergence region than the NR method and thus, the initial guess does not have to be as close to the equilibrium point as the NR method requires; 2) it is faster than simply applying the exact pseudo-transient continuation method to the QGS; and 3) the proposed inexact pseudo-transient continuation method can always converge to solutions of the original system, unlike the exact pseudo-transient continuation method. The applicability of the proposed method to improving the robustness of CUEP computations in the direct method for transient stability analysis of power systems will be further investigated.

References

[1] H.-D. Chiang, *Direct Methods for Stability Analysis of Electric Power Systems: Theoretical Foundation, BCU Methodologies, and Applications*. Hoboken, NJ, USA: John Wiley & Sons, Mar. 2011.

[2] P. Varaiya, F. F. Wu, and R. L. Chen, "Direct methods for transient stability analysis of power systems: Recent results," *Proc. IEEE*, vol. 73, no. 12, 1985, pp. 1703–1715.

[3] T. Van Cutsem and C. Vournas, *Voltage Stability of Electric Power Systems*, vol. 441. New York, NY, USA: Springer, 1998.

[4] P. Kundur, *Power System Stability and Control*. New York, NY, USA: McGraw-HIll, 1994.

[5] C. L. DeMarco and T. J. Overbye, "An energy based security measure for assessing vulnerability to voltage collapse," *IEEE Trans. Power Syst.*, vol. 5, no. 2, pp. 419–427, May 1990.

[6] T. J. Overbye and C. L. DeMarco, "Voltage security enhancement using energy based sensitivities," *IEEE Trans. Power Syst.*, vol. 6, no. 3, pp. 1196–1202, Aug. 1991.

[7] T. J. Overbye, "Use of energy methods for on-line assessment of power system voltage security," *IEEE Trans. Power Syst.*, vol. 8, no. 2, pp. 452–458, May 1993.

[8] H.-D. Chiang, J. Tong, and Y. Tada, "On-line transient stability screening of 14,000-bus models using TEPCO-BCU: Evaluations and methods," in *Proc. IEEE Power Energy Soc. General Meeting*, 2010, pp. 1–8.

[9] W. Ma and J. Thorp, "An efficient algorithm to locate all the load flow solutions," *IEEE Trans. Power Syst.*, vol. 8, no. 3, pp. 1077–1083, Aug. 1993.

[10] H. D. Nguyen and K. S. Turitsyn, "Appearance of multiple stable load flow solutions under power flow reversal conditions," in *Proc. IEEE Power Energy Soc. General Meeting*, Harbor, MD, USA, 2014, pp. 1–5.

[11] F. M. A. Salam, L. Ni, S. Guo, and X. Sun, "Parallel processing for the load flow of power systems: The approach and applications," in *Proc. 28th IEEE Conf. Decis. Control*, Dec. 1989, vol. 3, pp. 2173–2178.

[12] C.-W. Liu, C.-S. Chang, J.-A. Jiang, and G.-H. Yeh, "Toward a CP-FLOW-based algorithm to compute all the type-1 load-flow solutions in electric power systems," *IEEE Trans. Circuits Syst. I: Regular Papers*, vol. 52, no. 3, pp. 625–630, 2005.

[13] D. Mehta, H. D. Nguyen, and K. Turitsyn, "Numerical polynomial homotopy continuation method to locate all the power flow solutions," *IET Gener. Transmiss. Distribution*, vol. 10, no. 12, pp. 2972–2980, Aug. 2016.

[14] R. T. Treinen, V. Vittal, and W. Kliemann, "An improved technique to determine the controlling unstable equilibrium point in a power system," *IEEE Trans. Circuits Syst. I: Fundam. Theory Appl.*, vol. 43, no. 3, pp. 313–323, Apr. 1996.

[15] A. A. Fouad, V. Vittal, and T. Oh, "Critical energy for direct transient stability assessment of a multimachine power system," *IEEE Trans. Power Syst.*, vol. PAS-103, no. 8, pp. 2199–2206, Aug. 1984.

[16] K. Uemura, J. Matsuki, I. Yamada, and T. Tsuji, "Approximation of an energy function in transient stability analysis of power systems," *Elect. Eng. Japan*, vol. 92, no. 6, pp. 96–100, 1972.

[17] A. A. Fouad and V. Vittal, *Power System Transient Stability Analysis Using the Transient Energy Function Method*. Upper Saddle River, NJ, USA: Prentice-Hall, 1992.

[18] L. Chen, Y. Min, F. Xu, and K.-P. Wang, "A continuation-based method to compute the relevant unstable equilibrium points for power system transient stability analysis," *IEEE Trans. Power Syst.*, vol. 24, no. 1, pp. 165–172, Feb. 2009.

[19] H.-D. Chiang and C.-C. Chu, "A systematic search method for obtaining multiple local optimal solutions of nonlinear programming problems," *IEEE Trans. Circuits Syst. I: Fundam. Theory Appl.*, vol. 43, no. 2, pp. 99–109, Feb. 1996.

[20] J. Mitra, M. Benidris, and N. Cai, "Use of homotopy-based approaches in finding controlling unstable equilibrium points in transient stability analysis," in *Proc. Power System Comput. Conf.*, 2016.

[21] C.-W. Liu and J. S. Thorp, "A novel method to compute the closest unstable equilibrium point for transient stability region estimate in power systems," *IEEE Trans. Circuits Syst.*, vol. 44, no. 7, pp. 630–635, Jul. 1997.

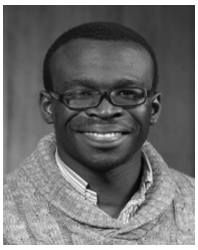
[22] J. Lee, "A novel homotopy-based algorithm for the closest unstable equilibrium point method in nonlinear stability analysis," in *Proc. Int. Symp. Circuits Syst.*, 2003, vol. 3, pp. 8–11.

[23] J. Lee and H.-D. Chiang, "A singular fixed-point homotopy method to locate the closest unstable equilibrium point for transient stability region estimate," *Circuits Syst. II: Exp. Briefs*, vol. 51, no. 4, pp. 185–189, 2004.

[24] I. Luna-Lopez, J. M. Canedo, and A. Loukianov, "Dynamical method for CUEP detection in power system transient stability assessment," in *Proc. IEEE Power Eng. Soc. Winter Meeting Conf.*, 2002, vol. 1, pp. 189–194.

[25] D. K. Molzahn, B. C. Lesieutre, and H. Chen, "Counterexample to a continuation-based algorithm for finding all power flow solutions," *IEEE Trans. Power Syst.*, vol. 28, no. 1, pp. 564–565, Feb. 2013.

[26] B. Berggren and G. Anderson, "On the nature of unstable equilibrium points in power systems," *IEEE Trans. Power Syst.*, vol. 8, no. 2, pp. 738–745, May 1993.
[27] J. Lee, "An optimization-driven framework for the computation of the controlling UEP in transient stability analysis," *IEEE Trans. Autom. Control*, vol. 49, no. 1, pp. 115–119, Jan. 2004.
[28] J. Lee and H.-D. Chiang, "A dynamical trajectory-based methodology for systematically computing multiple optimal solutions of general nonlinear programming problems," *IEEE Trans. Autom. Control*, vol. 49, no. 6, pp. 888–899, Jun. 2004.
[29] J. S. Thorp and S. A. Naqavi, "Load flow fractals," in *Proc. 28th IEEE Conf. Dec. Control*, Dec. 1989.
[30] F. Milano, "Continuous Newton's method for power flow analysis," *IEEE Trans. Power Syst.*, vol. 24, no. 1, pp. 50–56, Feb. 2009.
[31] J.-J. Deng, H.-D. Chiang, and T.-Q. Zhao, "Newton method and trajectory based-method for solving power flow problems," *Int. J. Bifurcation Chaos*, vol. 25, no. 6, 2015.
[32] X. Wang and H.-D. Chiang, "Application of pseudo-transient continuation method in dynamic stability analysis," in *Proc. IEEE PES General Meeting Conf. Expo.*, National Harbor, MD, USA, Oct. 2014, pp. 1–5.
[33] C. T. Kelley and D. E. Keyes, "Convergence analysis of pseudo-transient continuation," *SIAM J. Numer. Anal.*, vol. 35, no. 2, pp. 508–523, Apr. 1998.
[34] R. Owusu-Mireku and H.-D. Chiang, "A direct method for the transient stability analysis of transmission switching events," in *Proc. IEEE Power Energy Soc. General Meeting*, Portland, OR, USA, 2018, pp. 1–5.
[35] R. Owusu-Mireku and H.-D. Chiang, "On the dynamics and transient stability of power systems post-transmission switching," in *Proc. IEEE Power Energy Syst. General Meeting*, Chicago, IL, USA, 2017, pp. 1–5.
[36] D. Saupe, "Discrete versus continuous Newton's method: A case study," *Acta Applicandae Mathematicae*, vol. 13, pp. 59–80, 1988.
[37] S. Suresh Reddy, S. Sarat Kumar, and S. V. J. Kumar, "Load flow solution for ill-conditioned power systems using Runge–Kutta and Iwamoto methods with FACTS devices," *J. Theor. Appl. Inf. Technol.*, pp. 693–703, 2005–2009.
[38] N. Xie, E. Bompard, R. Napoli, and F. Torelli, "Widely convergent method for finding solutions of simultaneous nonlinear equations," *Electric Power Syst. Res.*, vol. 83, pp. 9–18, 2012.
[39] H. Jia, X. Yu, and P. Zhang, Topological characteristic studies on power system small signal stability region," in *Proc. IEEE Power Eng. Soc. General Meeting*, Montreal, Canada, Jun. 2006, pp. 1–7.
[40] X. Yu, S. Cao, H. Jia, and P. Zhang, Impact of the exciter voltage limit to power system small signal stability region," in *Proc. IEEE Power Eng. Soc. General Meeting*, Tampa, FL, USA, Jun. 2007, pp. 1–7.



Robert Owusu-Mireku (M'18) received the B.Sc. degree in electrical and electronic engineering from the Kwame Nkrumah University of Science and Technology, Kumasi, Ghana, and the M.Eng., M.S., and Ph.D. in electrical engineering from Cornell University, Ithaca, NY, USA.

He is currently working with the Eaton, Cooper Powers, Franksville, WI, USA, as a Power System Engineer. His current research interests include distribution power system protection, nonlinear system theory, nonlinear computation, renewable energy, and their practical applications to electric power systems.

Hsiao-Dong Chiang (M'87–SM'91–F'97) received the Ph.D. degree in electrical engineering and computer science from the University of California Berkeley, Berkeley, CA, USA.

He is a Professor with the School of Electrical and Computer Engineering, Cornell University, Ithaca, NY, USA. He and his research team have published more than 450 refereed papers with H factor of 57. His current research interests include nonlinear system theory, nonlinear computation, nonlinear optimization, and their practical applications. He holds 18 U.S. and overseas patents and several consultant positions. He is the author of two books: *Direct Methods for Stability Analysis of Electric Power Systems: Theoretical Foundation, BCU Methodologies, and Applications* (with Luis F. C. Alberto) and *Stability Regions of Nonlinear Dynamical Systems: Theory, Estimation, and Applications*.

Dr. Chiang has served as an Associate Editor for several IEEE transactions and journals and is the Founder of Bigwood Systems, Inc., Ithaca, NY, USA.

Matt Hin received the B.Sc. degree in mathematics from Harvey Mudd College, Claremont, CA, USA. He is currently working toward the Ph.D. degree at the Center for Applied Mathematics, Cornell University, Ithaca, NY, USA.

His current research interests include applications of multitime scale theory to electric power systems.