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Quantitative non-vanishing of Dirichlet L-values modulo p

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Abstract
Let p be an odd prime and k a non-negative integer. Let N be a positive integer such that $p \nmid N$ and λ a Dirichlet character modulo N. We obtain quantitative lower bounds for the number of Dirichlet character χ modulo F with the critical Dirichlet L-value $L(-k, \lambda\chi)$ being p-indivisible. Here $F \to \infty$ with $(N, F) = 1$ and $p \nmid F\phi(F)$. We explore the indivisibility via an algebraic and a homological approach. The latter leads to a bound of the form $F^{1/2}$. The p-indivisibility yields results on the distribution of the associated p-Selmer ranks. We also consider an Iwasawa variant. It leads to an explicit upper bound on the lowest conductor of the characters factoring through the Iwasawa $\mathbb{Z}_\ell$-extension of $\mathbb{Q}$ for an odd prime $\ell \neq p$ with the corresponding critical L-value twists being p-indivisible.

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1 Introduction

For a motive $\mathcal{M}$, a critical L-value is a fundamental arithmetic invariant. The vanishing or non-vanishing of the L-value is conjecturally closely related to the associated Bloch–Kato Selmer group. As the motive varies in a family, conjecturally the root numbers govern the generic non-vanishing.

Suppose $\mathcal{M}$ is defined over $\mathbb{Q}$. An instructive setup arises from the twists of $\mathcal{M}$ by Dirichlet characters. Once the L-values are expected to be generically non-vanishing, quantification of the non-vanishing twists is a natural question. The quantitative non-vanishing has been explored in various situations, typically via an analytic or an ergodic approach or a blend (for example, [2,10,18,22,23,31]). For a prime p, the L-values can often be normalised to be p-integral. One can then ask for quantitative non-vanishing modulo p. This is closely related to the distribution of the associated p-Selmer ranks.

Surprisingly, the mod p non-vanishing has only been explored in Iwasawa situations: for ℓ-power order twists of $\mathcal{M}$ with ℓ a fixed prime (for example, [4,5,11,13,15,24,26,32–34]). We are not aware of any horizontal instance: for q-power conductor twists of $\mathcal{M}$ as q varies over the primes.

In this article, we hope to initiate a broader study of quantitative non-vanishing modulo p. We consider the first case: when $\mathcal{M}$ itself arises from a Dirichlet character.

1.1 Setup

Fix an algebraic closure $\overline{\mathbb{Q}}$ and a complex embedding $\overline{\mathbb{Q}} \overset{\iota}{\hookrightarrow} \mathbb{C}$. Fix an odd prime p and a p-adic embedding $\overline{\mathbb{Q}} \overset{\iota_p}{\hookrightarrow} \mathbb{C}_p$. Let $\overline{\mathbb{Z}}$ denote the ring of algebraic integers, and $\mathfrak{p}$ the prime of $\overline{\mathbb{Z}}$ determined via ι_p.

For an integer $M > 1$, let

$$\zeta_M = \iota^{-1} \exp\left(\frac{2\pi i}{M}\right).$$

For $m \geq 2$, let μ_m denote the group of m^{th}-roots of unity in $\overline{\mathbb{Z}}$.

For a prime ℓ, let μ_{ℓ^∞} be the set of ℓ-power roots of unity in $\overline{\mathbb{Z}}$. We use the same notation to denote[1] the image of ζ_M, μ_m or μ_{ℓ^∞} under ι_p or in $\overline{\mathbb{F}}_p$.

[1] according to the context

For an abelian group G, let $\widehat{G}$ denote the group of $\overline{\mathbb{Q}}^\times$-valued characters of G. For $\chi \in \widehat{G}$, let $\mathbb{Z}[\chi]$ be the ring generated over $\mathbb{Z}$ by the image of χ and $\mathbb{Q}(\chi)$ its fraction field. Let $\overline{\chi}$ denote the conjugate character under the embedding ι. For χ with values in $\overline{\mathbb{F}}_p$, we analogously define the Hecke field $\mathbb{F}_p(\chi)$.

For χ with modulus F, let $G(\chi)$ be the Gauss sum:

$$\sum_{r=1}^{F-1} \chi(r) \exp\left(\frac{2\pi i r}{F}\right) = G(\chi). \tag{1.1}$$

Let λ and χ be (not necessarily primitive) Dirichlet characters of moduli N and F, respectively. Let k be a non-negative integer. We consider critical Dirichlet L-values $L(-k, \lambda\chi)$. If $p \nmid FN$, then $L(-k, \lambda\chi)$ is p-integral. In the introduction, we often only consider $k = 0$.

Let

$$\mathfrak{X}_{\lambda,k}(F) = \left\{\chi \in \widehat{(\mathbb{Z}/F\mathbb{Z})^\times} \,|\, L(-k, \lambda\chi) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\right\}.$$

A quantitative non-vanishing modulo p:

$$\text{to estimate } \#\mathfrak{X}_{\lambda,k}(F) \text{ as a function of } F. \tag{QN}$$

We often let $\mathfrak{X}_\lambda(\cdot)$ denote $\mathfrak{X}_{\lambda,0}(\cdot)$.

The quantification (QN) is closely related to the distribution of the associated p-Selmer ranks. For $k = 0$, the Selmer groups are a certain cyclotomic class groups. Let $\psi = \lambda\chi$, N_ψ the corresponding conductor and $O_\psi = \mathbb{Z}[\psi]$. For the Euler totient function ϕ and odd Dirichlet character ψ with $p \nmid \phi(N_\psi)$, the p-indivisibility of $L(0, \psi)$ is equivalent to the p-indivisibility of

$$\#(Cl(\mathbb{Q}(\zeta_{N_\psi})) \otimes O_\psi)^{\psi^{-1}}$$

for $Cl(\cdot)$ being the class group and $(\cdot)^{\psi^{-1}}$ the ψ^{-1}-component ([21,35]). For a general k, the Selmer groups are the second p-adic ètale cohomology groups of a certain cyclotomic integers with the coefficients being a Tate twist (of $\mathbb{Z}_p$) dependent on k ([19, Thm. 3.3]).

1.2 Horizontal non-vanishing

We present main results towards (QN). Here we only consider Dirichlet twists with prime moduli for simplicity.

Based on a homological approach, we prove

Theorem A *Let λ be a non-trivial Dirichlet character of modulus N and p an odd prime. Let F be a prime. Suppose $p \nmid \phi(F)FN$ and $(F, N) = 1$.*

If $F > N$, then

$$\#\mathfrak{X}_\lambda(F) \geq \left\lfloor \left(\frac{F}{9N}\right)^{\frac{1}{2}} \right\rfloor$$

for $\lfloor\cdot\rfloor$ the greatest integer function.

For a general k and composite moduli, we refer to Theorem 3.4.

Based on an algebraic approach, we prove

Theorem B *Let λ be a non-trivial Dirichlet character of modulus N and p an odd prime. Let F be a prime. Suppose $p \nmid \phi(F)FN$ and $(F, N) = 1$.*

If $F > N$, then

$$\#\mathfrak{X}_\lambda(F) \geq \left\lfloor \frac{2[\mathbb{F}_p(\lambda, \zeta_F) : \mathbb{F}_p(\lambda)]}{N} \right\rfloor^{\frac{1}{4}}.$$

For a general k and composite moduli, we refer to Theorem 5.2.

For a version of Theorem A and Theorem B with λ being trivial, we refer to Corollary 3.6 and 5.4.

Theorem B is evidently weaker than Theorem A. However, the approach is flexible and amenable to generalisations (for example, Theorem C).

It is instructive to compare these mod p quantitative results with characteristic zero quantitative non-vanishing (for example, [10,23,31]). It would be interesting to parlay the results in the context of arithmetic statistics: the distribution of the associated p-Selmer ranks. The Cohen–Lenstra heuristics is complementary to the setup. We are not aware of any conjecture regarding the distribution of the associated p-Selmer ranks. The recent Diophantine stability heuristics due to Mazur–Rubin seems relevant ([20]).

As for the asymptotic size of $\mathfrak{X}_\lambda(\cdot)$, the exponents in Theorem A and Theorem B are non-optimal. In a special case, we prove mod p non-vanishing of Dirichlet L-values for all the primitive characters with suitable parity (see Remark 5.2 and Corollary 5.5). The optimal exponent seems to be at least $1 - \varepsilon$ for any given $\varepsilon > 0$.

1.3 Iwasawa non-vanishing

In Iwasawa situations, mod p quantitative non-vanishing goes back to Ferrero–Washington [11] and Washington [34]. We present an explicit quantitative variant.

Let ℓ be an odd prime such that $\ell \neq p$. For a positive integer n, let

$$G_n = \frac{1 + \ell\mathbb{Z}_\ell}{1 + \ell^n\mathbb{Z}_\ell}.$$

Based on an algebraic approach, we prove

Theorem C *Let λ be a non-trivial Dirichlet character of modulus N and p an odd prime. Let ℓ be an odd prime such that $\ell \neq p$ and ω a Dirichlet character of modulus ℓ. Suppose $\lambda\omega(-1) = -1$ and $(p\ell, N) = 1$. Let M_0 be a non-negative integer such that*

$$\mu_{\ell^{M_0}} = \mathbb{Q}(\lambda) \cap \mu_{\ell^\infty}.$$

Then,

$$\#\{\chi \in \widehat{G}_n \mid L(0, \lambda\omega\chi) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} \geq (N(\ell-1))^{-\frac{1}{3}} \ell^{E_{\ell,n}}$$

for

$$E_{\ell,n} = \frac{n - M_0}{3\phi(\ell-1)} - \frac{2}{3}.$$

Theorem C is a companion to Washington's foundational result ([34]).

Washington's approach yields stronger[2] quantitative non-vanishing for characters with sufficiently large conductor. For $\chi \in \widehat{G}_n$ with sufficiently large n, the prime $\mathfrak{p} \cap \mathbb{Q}(\chi)$ becomes inert in $\mathbb{Q}(\mu_{\ell^\infty})$. Accordingly, Galois conjugation spreads the p-indivisibility. To approach the generic p-indivisibility, Washington builds on this observation and a key equidistribution result. For an effective version of Washington's result in a certain cases, we refer to [12,26,27] and references therein. These studies crucially build on Washington's approach.

We have the following evident

Corollary D *Let the notation and assumptions be as in Theorem C.*

Then, there exists a Dirichlet character χ of ℓ-power order and ℓ-power conductor less than

$$((\ell-1)\ell^2 N)^{\phi(\ell-1)\ell^{M_0+1}}$$

such that

$$L(0, \lambda\omega\chi) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p}).$$

As far as we know, the corollary is a first general result of its kind.

For a version of Theorem C with λ being trivial, we refer to Theorem 6.5.

1.4 Sketch of horizontal non-vanishing

We now describe the strategy for Theorem A and B.

[2] in fact, optimal

We begin with an expression which involves the L-values:

$$R_\lambda(r; F) = \sum_{\chi \in \widehat{(\mathbb{Z}/F\mathbb{Z})^\times}} \chi(r) c(\chi) L(0, \chi\lambda) \tag{1.2}$$

for $r \in (\mathbb{Z}/F\mathbb{Z})^\times$ and $c(\chi) \in \overline{\mathbb{Z}}_p$. Let

$$\mathfrak{X}_\lambda(F) = \{\chi_1, \cdots, \chi_t\}.$$

For an integer Q and $\mathfrak{r} = (r_i), \mathfrak{s} = (s_j) \in (\mathbb{Z}/F\mathbb{Z})^{\times Q}$,

$$R_\lambda(r_i s_j; F) \equiv \sum_{k=1}^{t} \chi_k(-r_i s_j) c(\chi_k) L(0, \chi_k\lambda) \mod \mathfrak{p}. \tag{1.3}$$

Suppose $t < Q$. Then, the vectors $(\chi_1(r_i), \cdots, \chi_t(r_i)) \in \overline{\mathbb{F}}_p^t$, $i = 1, \cdots, Q$ are linearly dependent. From (1.3), we then note

$$\det(R_\lambda(r_i s_j; F)) \equiv 0 \,(\mathrm{mod}\ \mathfrak{p})$$

for any $\mathfrak{r}, \mathfrak{s} \in (\mathbb{Z}/F\mathbb{Z})^{\times Q}$.

We deduce: if $\det(R_\lambda(r_i s_j; F))$ is non-zero modulo $\mathfrak{p}$ for some $\mathfrak{r}, \mathfrak{s} \in (\mathbb{Z}/F\mathbb{Z})^{\times Q}$, then

$$\#\mathfrak{X}_\lambda(F) \geq Q.$$

To find a Q, we consider dialectic expressions of the form (1.2): an algebraic and a homological one. The study of $\det(R_\lambda(r_i s_j; F))$ is closely tied with the nature of the expression.

Let $\Phi_\lambda(T)$ be the rational function:

$$\Phi_\lambda(T) = \frac{\sum_{r=1}^{N-1} \lambda(r) T^r}{1 - T^N}.$$

1.4.1 Algebraic approach

We begin with an expression of the form (1.2):

$$R_\lambda(r; F) = -\phi(F)\lambda(-1)\Phi_\lambda(\zeta_F^r) \tag{1.4}$$

with $c(\chi)$ being the Gauss sum $G(\overline{\chi})$. Here we utilise

$$\sum_{r=1}^{F-1} \overline{\chi}(r) \exp\left(\frac{2\pi i r s}{F}\right) = \chi(s) G(\overline{\chi})$$

and a Fourier inversion formula (see Corollary 2.3 for details).

We try to find Q, $\mathfrak{r}$ and $\mathfrak{s}$ such that

$$\det \Phi_\lambda(\zeta_F^{r_i s_j}) \not\equiv 0 \,(\bmod\ \mathfrak{p}).$$

Let $\Psi_\lambda(T)$ be the polynomial:

$$\Psi_\lambda(T) = (1 - T^N)\Phi_\lambda(T) = \sum_{r=1}^{N-1} \lambda(r) T^r.$$

For simplicity, let $\Psi_\lambda(T)$ also denote its mod p reduction. For $\mathfrak{r}, \mathfrak{s} \in (\mathbb{Z}/F\mathbb{Z})^{\times Q}$, let

$$\mathcal{D}(\lambda, \mathfrak{r}, \mathfrak{s}; T) = \sum_{\sigma \in S_Q} sgn(\sigma) \prod_{i=1}^{Q} \Psi_\lambda(T^{r_i s_{\sigma(i)}}) \prod_{j \neq \sigma(i)} \left(1 - T^{N r_i s_j}\right) \in \mathbb{F}_p(\lambda)[T]$$

and $\mathcal{Q}(\lambda, F; T)$ the irreducible polynomial of ζ_F over $\mathbb{F}_p(\lambda)$.

Equivalently, we now seek Q, $\mathfrak{r}$ and $\mathfrak{s}$ such that

$$\mathcal{D}(\lambda, \mathfrak{r}, \mathfrak{s}; T) \not\equiv 0 \,(\bmod\ \mathcal{Q}(\lambda, F; T)). \tag{1.5}$$

If the polynomial is non-zero modulo $\mathfrak{p}$ and it satisfies

$$\deg \mathcal{D}(\lambda, \mathfrak{r}, \mathfrak{s}; T) < [\mathbb{F}_p(\zeta_F, \lambda) : \mathbb{F}_p(\lambda)],$$

then (1.5) evidently holds.

Via an elementary analysis, we show

$$\deg \mathcal{D}(\lambda, \mathfrak{r}, \mathfrak{s}; T) \leq \frac{N Q^4}{2},$$

(Proposition 3.3).

This concludes the proof of Theorem B.

1.4.2 Homological approach

We begin with an expression of the form (1.2):

$$R_\lambda(r; F) = (F - 1)\lambda(F) G(\lambda) \int_{\frac{r}{F} - i\infty}^{\frac{r}{F} + i\infty} \Phi_{\overline{\lambda}}(e^{2\pi i z}) dz$$

with $c(\chi) = \overline{\chi}(N)$ (Corollary 4.3).

For $x \in \mathbb{R} \setminus \frac{1}{N}\mathbb{Z}$ modulo 1, let

$$I(\lambda; x) = \int_{x - i\infty}^{x + i\infty} \Phi_{\overline{\lambda}}(e^{2\pi i z}) dz.$$

We try to find Q, $\mathfrak{r}$ and $\mathfrak{s}$ such that

$$\det I\left(\lambda; \frac{r_i s_j}{F}\right) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p}).$$

We first note p-integrality of $I(\lambda; x)$ (Corollary 4.3). A crucial property of the integral: for an integer $1 \leq m < N$ and $\frac{m-1}{N} < x < \frac{m}{N} < y < \frac{m+1}{N}$, we have

$$I(\lambda; x) - I(\lambda; y) = \mathrm{Res}\Big(\frac{r}{N}; \Phi_{\bar{\lambda}}(e^{2\pi i z})\Big) = \frac{G(\bar{\lambda})\lambda(m)}{N\zeta_N^{-m}}$$

(see (4.2)).

Suppose there exists an integer $Q > 1$ and integral vectors $\mathfrak{r} = (r_i)$, $\mathfrak{s} = (s_i)$, $\mathfrak{s}' = (s'_j)$ satisfying: there exist integers $1 \leq m_{ij} < N$ with m_{jj} relatively prime to N for $1 \leq i, j \leq Q$ such that

(C1) $$\frac{m_{jj} - 1}{N} < \frac{r_j s_j \,(\mathrm{mod}\ F)}{F} < \frac{m_{jj}}{N} < \frac{r_j s'_j \,(\mathrm{mod}\ F)}{F} < \frac{m_{jj} + 1}{N}, 1 \leq j \leq Q,$$

(C2) $$\frac{m_{ij}}{N} < \frac{r_i s_j \,(\mathrm{mod}\ F)}{F}, \frac{r_i s'_j \,(\mathrm{mod}\ F)}{F} < \frac{m_{ij} + 1}{N}, 1 \leq j < i \leq Q$$

(see (**P**) in Section 5).

Let

$$\mathfrak{a}_j = \left(I\left(\lambda; \frac{r_1 s_j}{F}\right), \cdots, I\left(\lambda; \frac{r_Q s_j}{F}\right)\right), \ \mathfrak{a}'_j = \left(I\left(\lambda; \frac{r_1 s'_j}{F}\right), \cdots, I\left(\lambda; \frac{r_Q s'_j}{F}\right)\right).$$

Let S be the set of $Q \times Q$ matrices whose j-th row is either $\mathfrak{a}_j$ or $\mathfrak{a}'_j$.

We show at least one of the elements in S has non-zero determinant modulo $\mathfrak{p}$ and consequently

$$\#\mathfrak{X}_\lambda(F) \geq Q.$$

Indeed, from (C1) and (C2),

$$\mathfrak{a}_j - \mathfrak{a}'_j = \left(0, 0, \cdots, 0, G(\bar{\lambda})N^{-1}\lambda(m_{jj})\zeta_N^{m_{jj}}, *, \cdots, *\right) \tag{1.6}$$

where the first non-zero component is at the jth-position. Based on (1.6), a linear combination of a certain elements in S can be upper triangularised with the diagonal entries being

$$G(\bar{\lambda})\lambda(m_{jj})N^{-1}\zeta_N^{m_{jj}}$$

for $1 \leq j \leq Q$. Thus, at least one of the elements in S has non-zero determinant modulo $\mathfrak{p}$ (see Sect. 5 for details).

We now seek Q, $\mathfrak{r}$, $\mathfrak{s}$ and $\mathfrak{s}'$ which satisfy (C1) and (C2).

An elementary[3] solution: for $m_{ii} = 1$ with $1 \le i \le Q$ and $m_{ij} = 0$ with $1 \le j < i \le Q$, the conditions (C1) and (C2) hold from the inequalities:

$$r_i s_j < F \quad \text{for } 1 \le i, j \le Q$$

and

$$\begin{aligned} 1 = s_Q < \frac{F}{Nr_Q} < s_{Q-1} = s'_Q < \frac{F}{Nr_{Q-1}} < \cdots < s_i = s'_{i+1} < \frac{F}{NQ_i} \\ < s_{i-1} = s'_i < \frac{F}{Nr_{i-1}} < \cdots < \frac{F}{Nr_2} < s_1 = s'_2 < \frac{F}{Nr_1} < s'_1 < \frac{2F}{Nr_Q}. \end{aligned}$$

To find such s_i, s'_i, we let $r_i = 2Q + i - 1$ and determine an estimate for range of Q for which the inequalities

$$\frac{F}{Nr_{i-1}} - \frac{F}{Nr_i} > 1 \quad \text{and} \quad \frac{2F}{Nr_Q} - \frac{F}{Nr_1} > 1 \tag{1.7}$$

hold. As

$$\frac{F}{Nr_{i-1}} - \frac{F}{Nr_i} \ge \frac{F}{9Q^2N}$$

and $2r_1 - r_Q \ge Q$, the inequalities (1.7) evidently hold once

$$Q^2 < \frac{F}{9N}.$$

In this range, $F/Nr_Q > 1$ as $F > N$ and we let $s_Q = 1$.

This concludes the proof of Theorem A.

1.5 Sketch of Iwasawa non-vanishing

We now describe the strategy for Theorem C.

The ℓ-adic units $\mathbb{Z}_\ell^\times$ are decomposed as $\mu_{\ell-1} \times (1 + \ell\mathbb{Z}_\ell)$. Let W be a set of representatives of $\mu_{\ell-1}/\{\pm 1\}$. For $\zeta \in \mu_{\ell^\infty}$, let

$$\widetilde{\Phi}_\lambda(\zeta) = \sum_{\eta \in \mu_{\ell-1}} \overline{\omega}(\eta)\Phi_\lambda(\zeta^\eta) = 2\sum_{\eta \in W} \overline{\omega}(\eta)\Phi_\lambda(\zeta^\eta).$$

For $r \equiv 1 \pmod{\ell}$,

$$\widetilde{\Phi}_\lambda(\zeta_{\ell^n}^r) = \sum_{\chi \in \widehat{G}_n} c(\chi)\chi(r)L(0, \chi\omega\lambda) \tag{1.8}$$

[3] yet non-optimal

for a certain $c(\chi) \in \overline{\mathbb{Z}}_p$.

Suppose for an integer $Q \geq 1$, we have

$$\#\{\chi \in \widehat{G}_n \mid L(0, \chi\omega\lambda) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} < Q.$$

Then as before,

$$\det \widetilde{\Phi}_\lambda(\zeta_{\ell^n}^{r_i s_j}) \equiv 0 \,(\mathrm{mod}\ \mathfrak{p}) \tag{1.9}$$

for any $\mathfrak{r}, \mathfrak{s} \in (1+\ell\mathbb{Z})^Q$.

For $\alpha \in \mathbb{Z}_\ell$, the pseudo-monomial T^α can be regarded[4] as a function from μ_{ℓ^∞} to $\overline{\mathbb{Q}}$ or $\overline{\mathbb{F}}_p$ via $\zeta \mapsto \zeta^\alpha$. Let $\Xi_\lambda(T)$ be the pseudo-polynomial:

$$\Xi_\lambda(T) = \det \widetilde{\Phi}_\lambda(T^{r_i s_j}) \prod_{\eta \in W} \prod_{i,j=1}^{Q} (1 - T^{N s_i r_j \eta}).$$

Then, from (1.9)

$$\Xi_\lambda(\zeta_{\ell^n}) \equiv 0 \,(\mathrm{mod}\ \mathfrak{p}). \tag{1.10}$$

We fix an embedding $\mu_{\ell-1} \hookrightarrow \mathbb{C}$ following [25] and let

$$r_j = s_j = \ell(j-1) + 1,\ 1 \leq j \leq Q.$$

With these choices, we show Ξ_λ is non-trivial modulo $\mathfrak{p}$ (Proposition 6.3).

Let M_1 be the maximum of the p-adic valuation of the differences of the exponents of pseudo-monomials which appear in $\Xi_\lambda(T)$. Via a variant of the arguments in [25], we show

$$\ell^{M_1} \leq ((\ell-1)N\ell^2 Q^3)^{\phi(\ell-1)}.$$

Based on Sinnott's result, the zeros of mod p reduction of Ξ_λ lie in $\mu_{\ell^{M_0+M_1}}$ (Proposition 6.1).

In light of (1.10) and the preceeding paragraphs, we deduce

$$\#\{\chi \in \widehat{G}_n \mid L(-k, \chi\omega\lambda) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} \geq Q$$

once Q satisfies

$$\ell^n > \ell^{M_0}((\ell-1)N\ell^2 Q^3)^{\phi(\ell-1)}.$$

This concludes the proof of Theorem C.

[4] according to the context

1.6 Horizontal non-vanishing: general moduli

We include a few remarks on the horizontal non-vanishing of Dirichlet twists with general moduli (Theorems 3.4 and 5.2).

Broadly, we still follow the approach for the prime moduli. A main technical issue for the composite moduli: the expression (1.1) is no longer valid (see Remark 2.2) and consequently the expression (1.2) for the L-values in terms of the Gauss sums is not available. To get around, we formulate the approach in terms of a certain Fourier transforms and periodic zeta functions instead of the Gauss sums and Dirichlet L-functions. In particular, to represent the special L-value $L(0, \chi)$: we utilise the special zeta-value $L(0, \chi \mathbf{1}_q)$ of periodic zeta functions for $q > 1$ an auxiliary integer and $\mathbf{1}_q$ a periodic function (see (3.8)), instead of $L(0, \chi\lambda_0)$ for λ_0 the trivial character of modulus q.

Another technical issue (only) for the algebraic approach: the expression (1.2) holds only for the special zeta-values $L(0, \lambda\widehat{\chi})$ of periodic zeta functions with $\widehat{\chi}$ the Fourier transform:

$$\widehat{\chi}(r) = \sum_{s=1}^{F} \chi(s)\zeta_F^{rs}.$$

We refer to (2.7) and Remark 2.3. While employing the algebraic approach, we thus study

$$\mathfrak{Y}_\lambda(F) = \left\{\chi \in \widehat{(\mathbb{Z}/F\mathbb{Z})^\times} \mid L(0, \lambda\widehat{\chi}) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\right\}$$

instead of $\mathfrak{X}_\lambda(F)$. For further discussion, we refer to Remark 2.2. Here we only mention that the study turns out to be closely related to the non-triviality of critical L-values.

1.7 Miscellaneous remarks

The idea to use the Fourier inversion and the determinant goes back to Michel–Venkatesh [23]. More recently, it was employed in Burungale–Hida [6] (also see [8]). In fact, these articles initiated us to the current setup. The latter article considers the case of self-dual Rankin–Selberg convolution of a fixed Hecke eigenform over a totally real field with theta series arising from a CM quadratic extension of the totally real field with root number one. The authors show the characteristic zero non-vanishing of the corresponding central L-values as the CM quadratic extension varies. The key simple idea lies at the backdrop. As far as we know, the determinant approach has not been used earlier in regards to mod p non-vanishing. For a followup, we refer to [9].

In Iwasawa situations, the algebraic approach is pioneered by Sinnott [26]; and the homological one by Ferrero–Washington [11], Washington [34]. The homological approach was interpreted in the way presented in the article by the second named author [28].

The result obtained via algebraic approach is significantly weaker than the one via homological approach. Regardless of this, we believe that the algebraic approach has

its own advantage and much room for improvement. As an example, for the cyclotomic twists it currently gives a better bound than the homological approach: an upper bound for the λ-invariant of Kubota–Leopoldt p-adic L-function in [11] is weaker than the one in [1,25]. In our study, we seem to be able to show Theorem C only via the algebraic approach.

It is perhaps instructive to mention that the previous results on the mod p non-vanishing are based on various distribution results such as normality of p-adic integers, Zariski density of CM points on a Shimura variety, or equidistribution of special points ([33], [16, Ch. 1], [3,7]). The Iwasawa extension leads to a group action on the underlying p-adic integers or special points. The action plays a key role in establishing the distribution results. On the other hand, group action is perhaps missing in the horizontal situations. Nevertheless, the mod p non-vanishing in Iwasawa and horizo ntal situations seems to be mysteriously governed by closely related principles.

1.8 Layout of the article

The article is organised as follows. In Sects. 2 and 3, we present the algebraic approach. We express the critical L-values as the Fourier transform of a rational function on the one-dimensional torus $\mathbb{G}_m$ and analyse a determinant obtained from the inverse Fourier transform to approach the quantitative non-vanishing. In particular, this leads to the proof of Theorem B. In Sects. 4 and 5, we present the homological approach. We discuss an expression for the critical L-values as the Fourier transform of period integrals and analyse a determinant arising from the inverse Fourier transform. In particular, this leads to the proof of Theorem A. In Sect. 6, we introduce a variant of the algebraic approach so as to apply it to the Iwasawa setting. In particular, this leads to the proof of Theorem C. In Sect. 7, we provide numerical examples regarding the equidistribution which appears in the homological approach.

A notation. Let U and V be real-valued functions on an unbounded subset of $\mathbb{R}_{>0}$. Suppose V is non-vanishing on the domain. We say '$U \ll V$' if there exists $C \in \mathbb{R}_{>0}$ such that

$$\limsup_{x\to\infty} \left|\frac{U(x)}{V(x)}\right| \leq C.$$

If such a C depends on a parameter ε, we say '$U \ll_\varepsilon V$'.

2 Algebraic expression for Dirichlet L-values

In this section, we express the critical values of L-functions of periodic functions as Fourier transforms of rational functions.

For a periodic function $\varphi : \mathbb{Z}/M\mathbb{Z} \to \mathbb{C}$ with $M > 1$ and $s \in \mathbb{C}$ with $\Re(s) > 1$, define a Dirichlet series

$$L(s, \varphi) = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s}.$$

Let us set rational functions

$$\Phi_{\varphi}^{\circ}(T) := \frac{\sum_{r=1}^{M} \varphi(r) T^r}{1 - T^M} = \sum_{m=1}^{\infty} \varphi(m) T^m$$

and

$$\Phi_{\varphi}(T) := \frac{\sum_{r=0}^{M-1} \varphi(r) T^r}{1 - T^M} = \Phi_{\varphi}^{\circ}(T) + \varphi(0). \tag{2.1}$$

Note that $L(s, \varphi)$ can be meromorphically extended to $\mathbb{C}$ and satisfies the following contour integral representation (see [14, Section 2.3])

$$(e^{2\pi i s} - 1)\Gamma(s) L(s, \varphi) = \int_{P(\varepsilon)} \Phi_{\varphi}^{\circ}(e^{-z}) z^{s-1} dz. \tag{2.2}$$

Here $P(\varepsilon)$ is a contour that consists of a path from $+\infty$ to ε, a counter-clockwise circle around 0 with the radius ε, and a path from ε to $+\infty$.

Let us set the Fourier transform $\widehat{\varphi}$ as

$$\widehat{\varphi}(r) = \sum_{m=1}^{M} \varphi(r) \zeta_M^{rm}.$$

For any r, we have

$$\widehat{\widehat{\varphi}}(r) = M\varphi(-r).$$

Assume that $\widehat{\varphi}(0) = 0$. Then, $L(s, \varphi)$ is an analytic function on $\mathbb{C}$ and $\Phi_{\varphi}^{\circ}(e^{-z})$ is holomorphic at $z = 0$ and one can obtain that

$$L(-k, \varphi) = \left(T \frac{d}{dT} \right)^k \Phi_{\varphi}^{\circ}(T) \Big|_{T=1} \tag{2.3}$$

using (2.2) and the formula

$$\lim_{s \to -k} (e^{2\pi i s} - 1)\Gamma(s) = \frac{(-1)^k 2\pi i}{k!}$$

for integers $k \geq 0$.

If φ is a Dirichlet character, then it is well-known ([14, Section 4.2]) that for an integer a relatively prime to the conductor of φ and for integers $k \geq 0$, one has

$$(1 - a^{k+1}\varphi(a))L(-k, \varphi) \in \mathbb{Z}[\varphi].$$

Hence if p is relatively prime to the conductor of φ, then $L(-k, \varphi)$ is p-integral. We extend this to a general φ as follows.

Proposition 2.1 *Let $p \nmid M$. Suppose that*

$$\sum_{r=1}^{M} \varphi(r) = 0$$

and the values of φ are algebraic and p-integral.

Then, $L(-k, \varphi)$ is also p-integral for any integer $k \geq 0$.

Proof Observe that we have the following partial fraction decomposition of $\Phi^{\circ}_{\varphi}(T)$:

$$\Phi^{\circ}_{\varphi}(T) = \frac{1}{M} \sum_{s=1}^{M-1} \frac{\widehat{\varphi}(s)}{1 - \zeta_M^{-s} T}.$$

From (2.3), we conclude the proof of the proposition. □

In the remaining part of present paper, let us assume that the values of periodic functions are algebraic integers. For a periodic function φ, let us set

$$\Phi_{k,\varphi}(T) := \left(T\frac{d}{dT}\right)^k \Phi_{\varphi}(T) \quad \text{and} \quad \Phi^{\circ}_{k,\varphi}(T) := \left(T\frac{d}{dT}\right)^k \Phi^{\circ}_{\varphi}(T).$$

Note that if $k \geq 1$, then we have $\Phi_{k,\varphi} = \Phi^{\circ}_{k,\varphi}$. Furthermore, for $k \geq 0$ we also have

$$\Phi^{\circ}_{k,\varphi}(T^{-1}) = (-1)^{k+1}\Phi_{k,\varphi\circ -1}(T) \tag{2.4}$$

where

$$\varphi \circ m : r \mapsto \varphi(mr)$$

for an integer m.

Now we have the following.

Proposition 2.2 *Let χ be a Dirichlet character of modulus F and ψ a periodic function of a period N. Suppose $(N, F) = 1$.*

Then, we have

$$L(-k, \widehat{\chi}\psi) = \sum_{r=1}^{F-1} \chi(r)\Phi^{\circ}_{k,\psi}(\zeta_F^r) \tag{2.5}$$

$$= (-1)^{k+1} \sum_{r=1}^{F-1} \chi(-r)\Phi_{k,\psi\circ -1}(\zeta_F^r). \tag{2.6}$$

Proof For any character χ of modulus F, one obtains

$$\Phi^{\circ}_{\widehat{\chi}\psi}(T) = \sum_{m=1}^{\infty} \widehat{\chi}\psi(m)T^m = \sum_{r=1}^{F-1} \chi(r) \sum_{m=1}^{\infty} \psi(m)\zeta_F^{mr}T^m.$$

Note also that since $\sum_{r=1}^{F} \widehat{\chi}(r) = F\chi(-F) = 0$, we have

$$\sum_{r=1}^{FN} \widehat{\chi}\psi(r) = 0.$$

Therefore from (2.3), we obtain (2.5). From (2.4), we then finish the proof of the proposition. □

Remark 2.1 In Proposition 2.2, we neither suppose $\widehat{\psi}(0) = 0$ nor $\psi(0) = 0$.

Remark 2.2 In general we have

$$\widehat{\chi}(r) = G(\chi) \cdot \sum_{d|(r,\frac{F}{F^*})} d\overline{\chi}^*\left(\frac{r}{d}\right)\mu\left(\frac{F}{dF^*}\right)$$

for χ^* being the primitive character of conductor F^* which induces χ (see [30, Lem. 3.2]) and μ the Möbius function. Let ψ be a periodic function of period N with $(N, F) = 1$. If

$$\psi \circ r = \psi(r)\psi$$

for $(r, N) = 1$, we have

$$L(-k, \widehat{\chi}\psi) = G(\chi) \cdot \Big(\sum_{d|F/F^*} d^{k+1}\mu(F/dF^*)\psi(d)\Big) \cdot L(-k, \overline{\chi}^*\psi).$$

In particular, the L-values $L(-k, \widehat{\chi}\psi)$ and $G(\chi)L(-k, \overline{\chi}\psi)$ are identical for a primitive χ.

Applying the Fourier inversion to (2.6), we obtain the following

Corollary 2.3 *Suppose the same conditions for ψ and F as in Proposition 2.2. For $r \in (\mathbb{Z}/F\mathbb{Z})^\times$ and an integer $k \geq 0$, we have*

$$\phi(F)\Phi_{k,\psi\circ -1}(\zeta_F^r) = (-1)^{k+1} \sum_{\chi \in \widehat{(\mathbb{Z}/F\mathbb{Z})^\times}} \overline{\chi}(-r) L(-k, \widehat{\chi}\psi) \tag{2.7}$$

Remark 2.3 One may seek an inversion based on the following formula:

$$L(-k, \chi\psi) = \chi(-1)F \sum_{r=1}^{F} \widehat{\chi}(r)\Phi^{\circ}_{k,\psi}(\zeta_F^r).$$

An issue: it seems there is no orthogonality relation for the functions $\widehat{\chi}$. For integers r and s, we have

$$\sum_{\chi \in \widehat{(\mathbb{Z}/F\mathbb{Z})^\times}} \widehat{\chi}(r)\overline{\widehat{\chi}(s)} = \sum_{(m,F)=1} \zeta_F^{m(r-s)}.$$

and this gives an orthogonality only when F is prime.

3 Non-vanishing mod p : algebraic setting

In this section, we prove the quantitative non-vanishing of Dirichlet L-values as in Theorem B. Based on the inverse Fourier transform for the algebraic representation of L-values (Sect. 2), we approach quantitative non-vanishing modulo p for an odd prime p.

For a periodic function ψ of a period $N > 1$, let us set

$$\Psi_{k,\psi}(T) = (1 - T^N)^{k+1}\Phi_{k,\psi}(T) \in \overline{\mathbb{Z}}[T].$$

It is easy to see that if $\psi(N-1) \neq 0$, then the degree of $\Psi_{k,\psi}(T)$ is $(k+1)N - 1$ and that the leading coefficient is $(-1)^k\psi(N-1)$.

In this section, we often consider $\Psi_{k,\psi}(T)$ as a polynomial with coefficients in characteristic p. For simplicity, we denote it by the same notation.

Let F be a positive integer. For an integer $Q \geq 1$ (dependent on F), let r_i, s_i $(i = 1, \cdots, Q)$ be integers relatively prime to F such that

$$1 \leq r_i < r_{i+1} \leq F \text{ and } 1 \leq s_i < s_{i+1} \leq F \text{ for each } 1 \leq i \leq Q-1.$$

For $\mathfrak{r} = \{r_i\}$, $\mathfrak{s} = \{s_i\}$, let us set

$$\mathcal{D}_p(\psi, k, \mathfrak{r}, \mathfrak{s}; T) = \sum_{\sigma \in S_Q} sgn(\sigma) \prod_{i=1}^{Q} \Psi_{k,\psi}(T^{r_i s_{\sigma(i)}}) \prod_{j \neq \sigma(i)} \left(1 - T^{N r_i s_j}\right)^{k+1} \in \mathbb{F}_p(\psi)[T]$$

and $\mathcal{Q}_p(\psi, F; T)$ the irreducible polynomial of ζ_F over $\mathbb{F}_p(\psi)$.

We begin with a key preparatory.

Proposition 3.1 *Let ψ be a periodic function of a period $N > 1$. Let $p \nmid \phi(F)FN$ be an odd prime for F a positive integer with $(N, F) = 1$. Suppose that there exists a positive integer Q as above such that*

$$\mathcal{D}_p(\psi \circ -1, k, \mathfrak{r}, \mathfrak{s}; T) \not\equiv 0 \,(\mathrm{mod}\ \mathcal{Q}_p(\psi, F; T)). \tag{3.1}$$

Then, we have

$$\#\{\chi \in \widehat{(\mathbb{Z}/F\mathbb{Z})}^{\times} \mid L(-k, \widehat{\chi}\psi) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} \geq Q.$$

Proof Suppose that

$$\{\chi \in \widehat{(\mathbb{Z}/F\mathbb{Z})}^{\times} \mid L(-k, \widehat{\chi}\psi) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} = \{\chi_1, \chi_2, \cdots, \chi_t\} \tag{3.2}$$

for $t < Q$. From (2.7), for all $r \in (\mathbb{Z}/F\mathbb{Z})^{\times}$

$$\phi(F)\Phi_{k,\psi\circ-1}(\zeta_F^r) \equiv (-1)^{k+1} \sum_{i=1}^{t} \overline{\chi}_i(-r) L(-k, \widehat{\chi}_i\psi) \,(\mathrm{mod}\ \mathfrak{p}). \tag{3.3}$$

Let s_i, r_j be the integers given in the hypothesis. Since the vectors

$$(\chi_1(s_i r), \cdots, \chi_t(s_i r)),\ (1 \leq i \leq Q)$$

for a fixed r are linearly dependent over $\overline{\mathbb{F}}_p$, we have

$$\det(\Phi_{k,\psi\circ-1}(\zeta_F^{r_i s_j})) \equiv 0 \,(\mathrm{mod}\ \mathfrak{p}). \tag{3.4}$$

In other words

$$\sum_{\sigma \in S_Q} sgn(\sigma) \prod_{i=1}^{Q} \Phi_{k,\psi\circ-1}(\zeta_F^{r_i s_{\sigma(i)}}) \equiv 0 \,(\mathrm{mod}\ \mathfrak{p}). \tag{3.5}$$

Multiplying (3.5) by the product

$$\prod_{i,j=1}^{Q} (1 - \zeta_F^{N s_i r_j})^{k+1},$$

we obtain the following expression.

$$\sum_{\sigma \in S_Q} sgn(\sigma) \prod_{i=1}^{Q} \Psi_{k,\psi\circ-1}(\zeta_F^{r_i s_{\sigma(i)}}) \prod_{j \neq \sigma(i)} \left(1 - \zeta_F^{N r_i s_j}\right)^{k+1} \equiv 0 \,(\mathrm{mod}\ \mathfrak{p}) \tag{3.6}$$

This is a contradiction to (3.1) and we conclude the proof of the proposition. □

In a special case, we are able to compute the degree of $\mathcal{D}_p(\psi, k; \mathfrak{r}, \mathfrak{s}; T)$ explicitly. To do so, we need the rearrangement inequality. For convenience of the reader, we recall the following.

Lemma 3.2 *Let Q be a positive integer. For $1 \leq i, j \leq Q$, let r_i, s_j be real numbers such that $0 < r_1 < r_2 < \cdots < r_Q$ and $0 < s_1 < s_2 < \cdots < s_Q$.*

Then, for $\sigma \in S_Q$, the minimum of $\sum_{i=1}^{Q} s_i r_{\sigma(i)}$ is attained precisely when $\sigma(i) = Q - i + 1$.

Now we have the expression for the degree as follows.

Proposition 3.3 *Suppose $\psi(N-1) \neq 0$ in $\mathbb{F}_p(\psi)$.*

Then, we have

$$\deg \mathcal{D}_p(\psi, k; \mathfrak{r}, \mathfrak{s}; T) = (k+1)N \left(\sum_{i=1}^{Q} r_i\right)\left(\sum_{j=1}^{Q} s_j\right) - \sum_{i=1}^{Q} r_i s_{Q-i+1}. \tag{3.7}$$

Proof Note that the candidates of the degree of the polynomial in (3.6) are

$$\sum_{i=1}^{Q} \left\{ ((k+1)N-1) r_i s_{\sigma(i)} + \sum_{j \neq \sigma(i)} (k+1) N r_i s_j \right\}$$
$$= (k+1)N \left(\sum_{i=1}^{Q} r_i\right)\left(\sum_{j=1}^{Q} s_j\right) - \sum_{i=1}^{Q} r_i s_{\sigma(i)}.$$

Hence by Lemma 3.2 the degree is

$$(k+1)N \left(\sum_{i=1}^{Q} r_i\right)\left(\sum_{j=1}^{Q} s_j\right) - \sum_{i=1}^{Q} r_i s_{Q-i+1}$$

and the coefficient of the corresponding unique monomial is

$$sgn(\sigma_0)\psi(N-1)^Q(-1)^{Q(Q-1)} \neq 0$$

where $\sigma_0(i) = Q - i + 1$ for $i = 1, 2, \ldots, Q$. This finishes the proof. □

We are now ready to prove Theorem B and its generalisation.

Theorem 3.4 *Let ψ be a periodic function of a period $N > 1$ with*

$$\sum_{r=1}^{N} \psi(r) = 0$$

and $\psi(1) \neq 0$ in $\mathbb{F}_p(\psi)$. Suppose $p \nmid \phi(F) F N$ and $(N, F) = 1$.

(1) *Let F be an odd prime.*
Then,

$$\#\{\chi \in (\widehat{\mathbb{Z}/F\mathbb{Z}})^{\times} \mid L(-k, \widehat{\chi}\psi) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} \geq \left\lfloor \frac{2[\mathbb{F}_p(\psi, \zeta_F) : \mathbb{F}_p(\psi)]}{N(k+1)} \right\rfloor^{\frac{1}{4}}.$$

(2) *For any $\varepsilon > 0$,*

$$\#\{\chi \in (\widehat{\mathbb{Z}/F\mathbb{Z}})^{\times} \mid L(-k, \widehat{\chi}\psi) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} \gg_{\varepsilon,N,k} \phi(F)^{\frac{1}{4}-\varepsilon}.$$

Proof We choose specific $\mathfrak{r}$ and $\mathfrak{s}$ such that the degree (3.7) is smaller than

$$[\mathbb{F}_p(\psi, \zeta_F) : \mathbb{F}_p(\psi)].$$

As the polynomial satisfies

$$\deg \mathcal{D}_p(\psi, k; \mathfrak{r}, \mathfrak{s}; T) = \deg \mathcal{D}_p(\psi \circ -1, k; \mathfrak{r}, \mathfrak{s}; T),$$

the expression (3.1) would then evidently hold.

Let us consider the first part. Let $Q > 1$ be any integer satisfying

$$Q < \left(\frac{2[\mathbb{F}_p(\psi, \zeta_F) : \mathbb{F}_p]}{N(k+1)} \right)^{1/4}.$$

Setting $s_i = r_i = i$, note that the degree in (3.7):

$$(k+1)N\frac{Q^2(Q+1)^2}{4} - \frac{Q(Q+1)^2}{2} + \frac{Q(Q+1)(Q+2)}{6} < \frac{(k+1)NQ^4}{2}$$

is less than $[\mathbb{F}_p(\psi, \zeta_F) : \mathbb{F}_p]$.

In view of Proposition 3.1, this finishes the proof of the first part.

We now consider the second part. Let $\omega(F)$ be the number of prime divisors of F. Let $Q \geq 1$ be any integer. Among the first $Q + \omega(F)$ prime numbers $p_1, p_2, \ldots, p_{Q+\omega(F)}$, there are at least Q prime numbers that are relatively prime to F. Let us set such prime numbers as $s_i = r_i$ in the ascending order. Obviously, we obtain

$$p_i \leq s_i = r_i \leq p_{i+\omega(F)}.$$

Since

$$n < p_n \ll n \log n,$$

we have

$$\sum_{i=1}^{Q} s_i = \sum_{j=1}^{Q} r_j \ll \sum_{j=1+\omega(F)}^{Q+\omega(F)} j \log j \sim \frac{1}{2}(Q+\omega(F))^2 \log(Q+\omega(F)).$$

Furthermore,

$$\sum_{i=1}^{Q} s_i r_{Q-i+1} > \sum_{i=1}^{Q} i(Q-i+1) = \frac{1}{6}Q(Q+1)(Q+2).$$

Note that from (3.7), the degree of $\mathcal{D}_p(\psi, k; \mathfrak{r}, \mathfrak{s}; T)$ equals

$$(k+1)N\left(\sum_{i=1}^{Q} r_i\right)\left(\sum_{j=1}^{Q} s_j\right) - \sum_{i=1}^{Q} r_i s_{Q-i+1} < C\frac{(k+1)N}{4}(Q+\omega(F))^{4+\varepsilon}$$

for any $\varepsilon > 0$ and a constant $C > 0$. Choose the largest integer Q that is less than or equal to

$$\left(\frac{4}{(k+1)NC}[\mathbb{F}_p(\psi, \zeta_F) : \mathbb{F}_p(\psi)]\right)^{\frac{1}{4+\varepsilon}} - \omega(F).$$

Then, Q and r_i, s_j satisfy the condition (3.1).

Note that $\omega(F) \ll F^{o(1)}$ and that $[\mathbb{F}_p(\psi, \zeta_F) : \mathbb{F}_p(\psi)] \gg \phi(F)$. Hence, we obtain

$$Q \gg_{\varepsilon,N,k} \phi(F)^{\frac{1}{4}-\varepsilon}.$$

In view of Proposition 3.1, this finishes the proof of the second part. □

We obtain a consequence of Theorem 3.4:

Corollary 3.5 *Let λ, λ' be non-trivial Dirichlet characters of moduli $N, N' > 1$, respectively. Suppose $u \not\equiv 1 \pmod{\mathfrak{p}}$ for $u \in \overline{\mathbb{Z}}$, $p \nmid \phi(F)FNN'$ and $(NN', F) = 1$.*

(1) *Let F be an odd prime.*
Then,

$$\#\{\chi \in (\widehat{\mathbb{Z}/F\mathbb{Z}})^{\times} \mid L(-k, \widehat{\chi}\lambda) \not\equiv uL(-k, \widehat{\chi}\lambda') \pmod{\mathfrak{p}}\} \geq \left\lfloor \frac{2[\mathbb{F}_p(\lambda, \lambda', \zeta_F) : \mathbb{F}_p(\lambda, \lambda')]}{NN'(k+1)} \right\rfloor^{\frac{1}{4}}.$$

(2) *For any $\varepsilon > 0$,*

$$\#\{\chi \in (\widehat{\mathbb{Z}/F\mathbb{Z}})^{\times} \mid L(-k, \widehat{\chi}\lambda) \not\equiv uL(-k, \widehat{\chi}\lambda') \pmod{\mathfrak{p}}\} \gg_{\varepsilon,N,k} \phi(F)^{\frac{1}{4}-\varepsilon}$$

Proof For the periodic function

$$\psi = \lambda - u\lambda'$$

with period NN', we have $\psi(1) \neq 0$ in $\mathbb{F}_p$.

From the relation $L(-k, \widehat{\chi}\psi) = L(-k, \widehat{\chi}\lambda) - uL(-k, \widehat{\chi}\lambda')$, we finish the proof. □

Let $q > 1$ be a positive integer and $\mathbf{1}_q : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$ a periodic function with period q given by

$$\mathbf{1}_q(r) := \begin{cases} 1 & \text{if} \quad \nmid r \\ 1 - q & \text{otherwise} \end{cases}. \tag{3.8}$$

Observe that if q is relatively prime to a modulus of χ, we have

$$L(s, \widehat{\chi}\mathbf{1}_q) = (1 - q^{1-s}\overline{\chi}(q))L(s, \widehat{\chi}). \tag{3.9}$$

We now obtain a consequence towards the non-vanishing mod p of Dirichlet L-values from Theorem 3.4:

Corollary 3.6 *Let λ be a non-trivial Dirichlet character of modulus $N > 1$. Suppose $(N, F) = 1$ and $p \nmid \phi(F)FN$.*

(1) *Let F be an odd prime.*
Then,

$$\#\{\chi \in (\widehat{\mathbb{Z}/F\mathbb{Z}})^\times \mid L(-k, \lambda\chi) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} \geq \left\lfloor \frac{2[\mathbb{F}_p(\lambda, \zeta_F) : \mathbb{F}_p(\lambda)]}{N(k+1)} \right\rfloor^{\frac{1}{4}}.$$

Furthermore,

$$\#\{\chi \in (\widehat{\mathbb{Z}/F\mathbb{Z}})^\times \mid L(-k, \chi) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} \geq \left\lfloor \frac{F-1}{k+1} \right\rfloor^{\frac{1}{4}}.$$

(2) *For any $\varepsilon > 0$,*

$$\#\{\chi \in (\widehat{\mathbb{Z}/F\mathbb{Z}})^\times \mid L(-k, \lambda\widehat{\chi}) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} \gg_{\varepsilon,N,k} \phi(F)^{\frac{1}{4}-\varepsilon}.$$

Furthermore, for any integer $q > 1$ with $(pF, q) = 1$ we also have

$$\#\{\chi \in (\widehat{\mathbb{Z}/F\mathbb{Z}})^\times \mid L(-k, \widehat{\chi}) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} \gg_{\varepsilon,k} \phi(F)^{\frac{1}{4}-\varepsilon}.$$

Proof The first assertion for $L(-k, \lambda\chi)$ follows from Theorem 3.4, Remark 2.2, and an observation that $\widehat{\chi}_0 = \mathbf{1}_F$.

For the trivial χ_0 we have

$$L(-k, \mathbf{1}_F\lambda) = \frac{1 - F^{k+1}\lambda(F)}{1 - F^k\lambda(F)} L(-k, \chi_0\lambda).$$

For $L(-k, \chi)$, we put $\psi = \mathbf{1}_2$ in Theorem 3.4 and apply the same argument as before. Note that from (3.9) we have

$$\#\{\chi \mid L(-k, \widehat{\chi}) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} \geq \#\{\chi \mid L(-k, \widehat{\chi}\mathbf{1}_2) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\},$$

For the second part, we let $\psi = \lambda$ and $\mathbf{1}_q$ in Theorem 3.4. □

Remark 3.1 One can expect to obtain a better bound than the current one if appropriate $\mathfrak{r}$ and $\mathfrak{s}$ can be found to determine $\mathcal{D}_p(\lambda, F, k; \mathfrak{r}, \mathfrak{s})$ relatively explicitly without the assumption that the degree (3.7) is less than $[\mathbb{F}_p(\lambda, \zeta_F) : \mathbb{F}_p(\lambda)]$. The latter seems to be the main reason behind the exponent being $\frac{1}{4}$.

Remark 3.2 In Sect. 6, we consider the complementary case to part (1) (i.e. when F has a high prime power conductor) based on a variant of the algebraic approach.

4 Integral expression of Dirichlet *L*-values

In this section, we express the critical Dirichlet L-values as a Fourier transform of periods. This section is based on [29], which mainly concerns primitive Dirichlet characters. Since we consider results for imprimitive characters as well, we include the exposition with a general setting.

Let φ be a periodic function of period M. For a polynomial $P(z)$, we have a meromorphic function $\Phi^\circ_\varphi(e^{2\pi iz})P(z)$ on $\mathbb{C}$ (Sect. 2) with poles $z = \frac{r}{M}$, $r \in \mathbb{Z}$ and the residue is given by

$$\mathrm{Res}\left(\frac{r}{M}; \Phi^\circ_\varphi(e^{2\pi iz})P(z)\right) = \frac{\widehat{\varphi}(r)}{M} P\left(\frac{r}{M}\right).$$

Here $\mathrm{Res}\big(z_0; f(z)\big)$ is the residue of $f(z)$ at $z = z_0$.

Observe that if $\sum_{s=1}^{M} \varphi(s) = 0$, i.e., $\widehat{\varphi}(M) = 0$, then the function $\Phi_{\widehat{\varphi}}(e^{2\pi iz})P(z) = \Phi^\circ_{\widehat{\varphi}}(e^{2\pi iz})P(z)$ is exponentially decreasing as $\Im(z) \to \pm\infty$. Therefore the contour integral

$$\int_{x-i\infty}^{x+i\infty} \Phi_{\widehat{\varphi}}(e^{2\pi iz})P(z)dz \tag{4.1}$$

is well-defined for $x \in \mathbb{R} - \frac{1}{M}\mathbb{Z}$. Furthermore if x and y satisfy

$$\frac{r-1}{M} < x < \frac{r}{M} \quad \text{and} \quad \frac{r}{M} < y < \frac{r+1}{M},$$

then we have

$$\int_{x-i\infty}^{x+i\infty} \Phi_{\widehat{\varphi}}(e^{2\pi iz})P(z)dz - \int_{y-i\infty}^{y+i\infty} \Phi_{\widehat{\varphi}}(e^{2\pi iz})P(z)dz \tag{4.2}$$
$$= \mathrm{Res}\Big(\frac{r}{M}; \Phi_{\widehat{\varphi}}(e^{2\pi iz})P(z)\Big) = \varphi(-r)P\left(\frac{r}{M}\right).$$

Note also that the integral (4.1) is well-defined even for $x = 0$ if $\varphi(0) = 0$.

When $P(z) = z^k$, we obtain the integral representation of L-values. In the remaining section, by p-integral we mean algebraic and p-integral.

Proposition 4.1 *Let $k \geq 0$ and φ be a periodic function with a period M such that $\varphi(M) = 0$ and $\widehat{\varphi}(M) = 0$.*

(1) *We have*

$$L(-k, \varphi) = -M^k \int_{-i\infty}^{i\infty} \Phi_{\widehat{\varphi}}(e^{2\pi iz})z^k dz. \tag{4.3}$$

(2) *Let $p \nmid M$, $x \in (0, 1) \setminus \frac{1}{M}\mathbb{Z}$, and $P(z)$ a polynomial with p-integral coefficients. Then, the difference*

$$\int_{-i\infty}^{i\infty} \Phi_{\widehat{\varphi}}(e^{2\pi iz})P(z)dz - \int_{x-i\infty}^{x+i\infty} \Phi_{\widehat{\varphi}}(e^{2\pi iz})P(z)dz$$

is p-integral.
In particular, if $L(-k, \varphi)$ is p-integral, then so are the integrals $\int_{x-i\infty}^{x+i\infty} \Phi_{\widehat{\varphi}}(e^{2\pi iz})z^k dz$.

Remark 4.1 Even though the definition of $\widehat{\varphi}$ depends on a choice of period, it can be easily seen that the R.H.S. of (4.3) does not depend on the choice (see [29, §3 3]).

Proof For the proof of the first assertion, see [29, Prop. 1].

For the second assertion, observe that from (4.2), we obtain

$$\int_{x-i\infty}^{x+i\infty} \Phi_{\widehat{\varphi}}(e^{2\pi iz})P(z)dz = \int_{-i\infty}^{i\infty} \Phi_{\widehat{\varphi}}(e^{2\pi iz})P(z)dz + \sum_{r=1}^{\lfloor Nx \rfloor} \varphi(-r)P\left(\frac{r}{M}\right). \tag{4.4}$$

Hence we obtain the proposition. □

In the homological setting, we have the following analogue of Proposition 2.2.

Proposition 4.2 *Let ψ be a periodic function with a period N. Let χ be a Dirichlet character of modulus F. Suppose $\widehat{\psi}(N) = 0$ and $(F, N) = 1$.*

Then, we have

$$L(-k, \chi\psi) = (FN)^k \sum_{r=1}^{F-1} \chi(Nr) \int_{\frac{r}{F}-i\infty}^{\frac{r}{F}+i\infty} \Phi_{\widehat{\psi \circ F}}(e^{2\pi iz}) \left(z - \frac{r}{F}\right)^k dz. \tag{4.5}$$

Proof Observe that

$$\widehat{\chi\psi} = \widehat{\chi \circ N} \cdot \widehat{\psi \circ F}.$$

Hence we have $\widehat{\chi\psi}(NF) = \chi\psi(NF) = 0$. From the formula (4.3):

$$L(-k, \chi\psi) = -(FN)^k \int_{-i\infty}^{i\infty} \Phi_{\widehat{\chi\psi}}(e^{2\pi i z}) z^k dz.$$

Note also that

$$\Phi_{\widehat{\chi\psi}}(T) = \sum_{m=1}^{\infty} \widehat{\chi \circ N}(m)\widehat{\psi \circ F}(m) T^m.$$

This is equal to

$$\sum_{r=1}^{F} \chi(Nr) \sum_{m=1}^{\infty} \widehat{\psi \circ F}(m)\zeta_F^{rm} T^m = \sum_{r=1}^{F} \chi(Nr)\Phi_{\widehat{\psi \circ F}}(T\zeta_F^r)$$

and we deduce (4.5). □

Remark 4.2 In the proposition, it is inessential to suppose $\psi(N) = 0$.

For a periodic function ψ of a period N with $\widehat{\psi}(N) = 0$, a polynomial $P(z) \in \mathbb{Q}[z]$ with p-integral coefficients, and a real number $x \in (0, 1) \setminus \frac{1}{N}\mathbb{Z}$, let us set

$$I(\psi, P; x) = \int_{x-i\infty}^{x+i\infty} \Phi_{\widehat{\psi}}(e^{2\pi i z}) P(z) dz.$$

Applying the inversion to (4.5) we have:

Corollary 4.3 *Let $F > 1$ be an integer and ψ a periodic function of a period $N > 1$ such that $\widehat{\psi}(N) = 0$ and $(N, F) = 1$.*

Then, we have

$$\phi(F)(FN)^k I\left(\psi \circ F, \left(z - \frac{r}{F}\right)^k; \frac{r}{F}\right) = \sum_{\chi \in (\mathbb{Z}/F\mathbb{Z})^\times} \chi(\overline{N}r) L(-k, \chi\psi) \tag{4.6}$$

for $1 \le r < F$ with $(r, F) = 1$ and $\overline{N}$ the inverse of N modulo F.

Hence if $p \nmid \phi(F)FN$, then $I\left(\psi \circ F, (z - \frac{r}{F})^k; \frac{r}{F}\right)$ is p-integral.

Remark 4.3 Note that the function $\psi \circ -1$ does not appear in (4.6). An explanation is that as $\widehat{\psi}(N) = 0$, we have $\Phi_{\widehat{\psi}} = \Phi^\circ_{\widehat{\psi}}$. Thus, we need not resort to a conversion like (2.6).

5 Non-vanishing mod p: homological setting

In this section, we prove the quantitative non-vanishing of Dirichlet L-values as in Theorem B. Based on the inverse Fourier transform for the integral expression of the L-values, we approach quantitative non-vanishing modulo p.

As in Sect. 3, we first present conditions for a lower bound in regards to the quantitative non-vanishing.

Let N and F be positive integers. We consider an integer $Q \geq 1$ (dependent on N and F). For an integer r, let $[r]$ be the residue of r modulo F. Let us specify a certain conditions on Q as follows.

(**P**) There exist integers $1 \leq r_i, s_j, s'_j < F$ and $1 \leq m_{ij} \leq N-1$ for $1 \leq i, j \leq Q$ such that

- The integers r_j, s_j, s'_j are relatively prime to F.
- The integers m_{ii} are relatively prime to N.
- $\dfrac{m_{jj}-1}{N} < \dfrac{[r_j s_j]}{F} < \dfrac{m_{jj}}{N} < \dfrac{[r_j s'_j]}{F} < \dfrac{m_{jj}+1}{N}$,
- $\dfrac{m_{ij}}{N} < \dfrac{[r_i s_j]}{F}, \dfrac{[r_i s'_j]}{F} < \dfrac{m_{ij}+1}{N}$ for all $i > j$.

(**D**) If $k > 0$, then $[r_i s_j], [r_i s'_j] \equiv 0 \,(\mathrm{mod}\ p)$ for each $i, j = 1, \cdots, Q$.

An intention of conditions (**P**) and (**D**) is to control the contours and values of the integrations $I(\psi, z^k, x)$ in order to obtain suitable residues of $\Phi_{\widehat{\psi}}(e^{2\pi i z}) z^k dz$ based on the expression (4.2).

We begin with a key preparatory.

Proposition 5.1 *Let N, F be positive integers and p an odd prime. Let ψ be a periodic function of period N. Let $Q \geq 1$ be an integer dependent on N, F such that there exist integers $1 \leq r_i, s_j, s'_j < F$ and $1 \leq m_{ij} \leq N-1$ for $1 \leq i, j \leq Q$ satisfying (**P**) and (**D**) as above. Suppose that*

$$\widehat{\psi}(N) = 0 \text{ and } \psi(-F m_{ii}) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p}) \text{ for each } i.$$

Further, suppose $p \nmid \phi(F) F N$ and $(F, N) = 1$.

Then, we have

$$\#\{\chi \in (\widehat{\mathbb{Z}/F\mathbb{Z}})^\times \mid L(-k, \chi\psi) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} \geq Q.$$

Proof For convenience, let us omit the bracket for $[r]$ in the proof unless any confusion arises and set $\psi_F = \psi \circ F$.

Assume the contrary. In other words,

$$\{\chi \in (\widehat{\mathbb{Z}/F\mathbb{Z}})^\times \mid L(-k, \psi\chi) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} = \{\chi_1, \cdots, \chi_t\}$$

for $t < Q$.

From (4.6),

$$\phi(F)(FN)^k I\left(\psi_F, \left(z-\frac{r}{F}\right)^k; \frac{r}{F}\right) \equiv \sum_{i=1}^{t} \chi_i(\overline{N}r) L(-k, \chi_i\psi) \,(\mathrm{mod}\ \mathfrak{p}).$$

Hence for any $u_i, v_j \in (\mathbb{Z}/F\mathbb{Z})^\times$ $(1 \le i, j \le Q)$, we obtain the congruence

$$\det\left(I\left(\psi_F, \left(z-\frac{u_i v_j}{F}\right)^k; \frac{u_i v_j}{F}\right)\right) \equiv 0 \,(\mathrm{mod}\ \mathfrak{p}) \tag{5.1}$$

in the same way as in the proof of Proposition 3.1. Recall that the congruence holds due to the linear dependence over $\overline{\mathbb{F}}_p$ of the vectors $(\chi_i(r))_i$, $r \in (\mathbb{Z}/F\mathbb{Z})^\times$.

Now let us consider r_i, s_i, s_i' satisfying the conditions (**P**) and (**D**). For each i and j, let

$$a_{ij} = \begin{cases} I\left(\psi_F, \left(z-\frac{r_i s_1}{F}\right)^k; \frac{r_i s_1}{F}\right) & \text{if } j = 1 \\ I\left(\psi_F, \left(z-\frac{r_i v_j}{F}\right)^k; \frac{r_i v_j}{F}\right) & \text{if } j \ge 2 \end{cases}$$

and

$$a'_{ij} = \begin{cases} I\left(\psi_F, \left(z-\frac{r_i s_1'}{F}\right)^k; \frac{r_i s_1'}{F}\right) & \text{if } j = 1 \\ a_{ij} & \text{if } j \ge 2 \end{cases}.$$

Note here that $v_2, v_3, \ldots, v_L$ are arbitrary.

Then, from (5.1) we note the congruence

$$\det(a_{ij}) \equiv \det(a'_{ij}) \equiv 0 \,(\mathrm{mod}\ \mathfrak{p}).$$

Thus,

$$\det(a_{ij}) - \det(a'_{ij}) = \det\begin{pmatrix} a_{11}-a'_{11} & a_{12} & \cdots & a_{1Q} \\ a_{21}-a'_{21} & a_{22} & \cdots & a_{2Q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{Q1}-a'_{Q1} & a_{Q2} & \cdots & a_{QQ} \end{pmatrix} \equiv 0 \,(\mathrm{mod}\ \mathfrak{p}). \tag{5.2}$$

From the condition (**P**), we have

$$\frac{m_{11}-1}{N} < \frac{r_1 s_1}{F} < \frac{m_{11}}{N} < \frac{r_1 s_1'}{F} < \frac{m_{11}+1}{N}.$$

Note that from the condition (**D**) and Proposition 4.1(2),

$$a_{i1} = I\left(\psi_F, \left(z - \frac{r_i s_1}{F}\right)^k; \frac{r_i s_1}{F}\right) \equiv I\left(\psi_F, z^k; \frac{r_i s_1}{F}\right) \pmod{\mathfrak{p}}$$

and the same congruence for a'_{i1} with s'_1.

Hence from (4.2) we obtain

$$\begin{aligned} a_{11} - a'_{11} &\equiv I\left(\psi_F, z^k; \frac{r_1 s_1}{F}\right) - I\left(\psi_F, z^k; \frac{r_1 s'_1}{F}\right) \\ &= \mathrm{Res}\left(\frac{m_{11}}{N}, \Phi_{\widehat{\psi}_F}(e^{2\pi i z}) z^k\right) = \frac{\psi(-F m_{11})}{N^{k-1}} \pmod{\mathfrak{p}}. \end{aligned} \tag{5.3}$$

Similarly for each $i \geq 2$,

$$a_{i1} - a'_{i1} \equiv I\left(\lambda, z^k; \frac{r_i s_1}{F}\right) - I\left(\lambda, z^k; \frac{r_i s'_1}{F}\right) = 0 \pmod{\mathfrak{p}}. \tag{5.4}$$

In summary, the congruence (5.2) can be rephrased as

$$\det \begin{pmatrix} \frac{\psi(-F m_{11})}{N^{k-1}} & a_{12} & \cdots & a_{1Q} \\ 0 & a_{22} & \cdots & a_{2Q} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{Q2} & \cdots & a_{Q\ell} \end{pmatrix} \equiv 0 \pmod{\mathfrak{p}} \tag{5.5}$$

for any $v_2, v_3, \ldots, v_Q$.

We now apply the previous process with $v_2 = s_2$, $v_2 = s'_2$, and arbitrary $v_3, \ldots, v_Q$ to the remaining part

$$\begin{pmatrix} a_{22} & \cdots & a_{2Q} \\ \vdots & \ddots & \vdots \\ a_{Q2} & \cdots & a_{QQ} \end{pmatrix}. \tag{5.6}$$

Since $r_2 s_2, \cdots, r_Q s_2$ and $r_2 s'_2, \cdots, r_Q s'_2$ satisfies the condition (**P**), we obtain the same result as (5.5) for the first column of (5.6).

Repeating this process, we obtain a congruence for a determinant of an upper triangular matrix

$$\begin{aligned} &\det \begin{pmatrix} \frac{\psi(-F m_{11})}{N^{k-1}} & * & \cdots & * \\ 0 & \frac{\psi(-F m_{22})}{N^{k-1}} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\psi(-F m_{QQ})}{N^{k-1}} \end{pmatrix} \\ &= N^{-Q(k-1)} \psi(-F m_{11}) \cdots \psi(-F m_{QQ}) \equiv 0 \pmod{\mathfrak{p}}. \end{aligned}$$

This contradiction finishes the proof. □

Our approach to find a suitable Q involves the following.

Jacobthal function. For an integer $F > 1$, let $H(F) > 0$ be the Jacobthal function, i.e., the smallest integer h such that for any $a \in \mathbb{Z}$ with $[a, a+h) \subset [1, F)$, there is an integer $r \in [a, a+h)$ and $(r, F) = 1$.

Then, it is well-known (see [17]) that

$$H(F) \ll (\omega(F) \log \omega(F))^2 \ll (\log F)^{O(1)}. \tag{5.7}$$

We are now ready to prove Theorem A and its generalisation.

Theorem 5.2 *Let N, F be positive integers and p an odd prime. Let ψ be a periodic function of period N. Suppose*

$$\widehat{\psi}(N) = 0 \text{ and } \psi(-F) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p}).$$

Further, suppose $p \nmid \phi(F) F N$ and $(F, N) = 1$.

(1) *For any $\varepsilon > 0$,*

$$\#\{\chi \in \widehat{(\mathbb{Z}/F\mathbb{Z})^\times} \mid L(-k, \chi\psi) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} \gg_{\varepsilon, N, k} F^{\frac{1}{2}-\varepsilon}.$$

(2) *Let F be an odd prime. Set $\epsilon = 1$ if $k > 0$ and $\epsilon = 0$ if $k = 0$.*
If $F > N/p^\epsilon$, then

$$\#\{\chi \in \widehat{(\mathbb{Z}/F\mathbb{Z})^\times} \mid L(-k, \chi\psi) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} \geq \left\lfloor \left(\frac{F}{9Np^\epsilon}\right)^{\frac{1}{2}} \right\rfloor.$$

Proof Choose any positive integer Q satisfying the inequality

$$Q < \left(\frac{F}{9Np^\epsilon H^3}\right)^{\frac{1}{2}} \tag{5.8}$$

where $H = H(F)$.

By the definition of H, there exists a positive integer $r_i^\circ < F$ such that

$$2QH + (i-1)H \leq r_i^\circ < 2QH + iH \tag{5.9}$$

and $(r_i^\circ, F) = 1$ for all $i = 1, \ldots, Q$.

Let

$$r_i = p^\epsilon r_i^\circ.$$

From (5.9), we note that r_i is an increasing sequence satisfying

$$r_i \geq r_{i-1} + p^{\epsilon} \text{ and } 2r_1 - r_Q > QHp^{\epsilon} > 0. \tag{5.10}$$

From (5.8), we obtain

$$\frac{F}{Nr_{i-1}} - \frac{F}{Nr_i} > \frac{F}{9Np^{\epsilon}Q^2H^2} \geq H$$

for $i > 2$.

Furthermore, we have

$$\frac{2F}{Nr_Q} - \frac{F}{Nr_1} > \frac{F}{3Np^{\epsilon}(2Q+1)H} \geq \frac{F}{9Np^{\epsilon}Q^2H^2} \geq H.$$

The first inequality arises from (5.9), (5.10) and the second one is evident.

In light of the discussion: for a sufficiently large F there exist integers s_i and s'_j such that

$$0 < s_Q = 1 < \frac{F}{Nr_Q} < s_{Q-1} < \frac{F}{Nr_{Q-1}} < \cdots < \frac{F}{Nr_2} < s_1 < \frac{F}{Nr_1} < \frac{2F}{Nr_Q}$$

and

$$\frac{F}{Nr_Q} < s'_Q < \frac{F}{Nr_{Q-1}} < \cdots < \frac{F}{Nr_2} < s'_2 < \frac{F}{Nr_1} < s'_1 < \frac{2F}{Nr_Q}.$$

Then, the sequences r_i, s_i, s'_i satisfy the desired conditions (**P**) and (**D**) with $m_{ij} = 1$ for all $i \geq j$.

From Proposition 5.1, we thus deduce

$$\#\{\chi \in \widehat{(\mathbb{Z}/F\mathbb{Z})^{\times}} \mid L(-k, \chi\psi) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} \geq \left\lfloor \left(\frac{F}{9Np^{\epsilon}H^3}\right)^{\frac{1}{2}} \right\rfloor. \tag{5.11}$$

The bound (5.7) finally yields the first part.

If $F > N/p^{\epsilon}$, then $F > Nr_Q$ and we may accordingly choose $s_Q = 1$. As $H(F) = 1$, this conclude the proof of the second part. □

Remark 5.1 In the proof, we choose a certain r_i and s_j such that

$$s_i r_j \ll F \text{ and } [r_i s_j] = r_i s_j.$$

In particular, $Q^2 \ll F$.

Remark 5.2 Let D be the decomposition group of prime $\mathfrak{p} \cap \mathbb{Q}(\psi\chi)$ over $\mathbb{Q}(\psi)$. Applying the element σ of D, we have

$$L(0, \chi\psi)^{\sigma} \equiv L(0, \psi\chi^{\sigma}) \,(\mathrm{mod}\ \mathfrak{p}).$$

Therefore, if the lower bound in Theorem 5.2 is smaller than $|D|$, the theorem is nothing but the existence of a character giving us the non-vanishing of L-values.

Remark 5.3 Unlike Theorem B, the lower bounds in Theorem 5.2 do not depend on k. An explaination is that the quantifier Q in the conditions (**P**) and (**D**) turns out to be independent of k.

As ψ is a general periodic function, we may set ψ as a difference of two Dirichlet characters and obtain the following:

Corollary 5.3 *Let λ, λ' be non-trivial Dirichlet characters of moduli $N, N' > 1$, respectively. Suppose $p \nmid \phi(F)FNN'$, $(F, NN') = 1$ and*

$$\lambda(-F) \not\equiv u\lambda'(-F) \,(\mathrm{mod}\ \mathfrak{p})$$

for $u \in \overline{\mathbb{Z}}$.

(1) *For any $\varepsilon > 0$,*

$$\#\{\chi \in \widehat{(\mathbb{Z}/F\mathbb{Z})}^{\times} \mid L(-k, \lambda\chi) \not\equiv uL(-k, \lambda'\chi) \,(\mathrm{mod}\ \mathfrak{p})\} \gg_{\varepsilon, N, N', k} F^{\frac{1}{2}-\varepsilon}.$$

(2) *Let F be an odd prime. Set $\epsilon = 1$ if $k > 0$ and $\epsilon = 0$ if $k = 0$.*
If $F > NN'/p^{\epsilon}$, then

$$\#\{\chi \in \widehat{(\mathbb{Z}/F\mathbb{Z})}^{\times} \mid L(-k, \lambda\chi) \not\equiv uL(-k, \lambda'\chi) \,(\mathrm{mod}\ \mathfrak{p})\} \geq \left\lfloor \left(\frac{F}{9NN'p^{\epsilon}}\right)^{\frac{1}{2}} \right\rfloor.$$

Recall that if q is relatively prime to a modulus of χ,

$$L(s, \chi\mathbf{1}_q) = (1 - q^{1-s}\chi(q))L(s, \chi) \tag{5.12}$$

Here is another consequence of Theorem 5.2.

Corollary 5.4 *Let λ be a non-trivial Dirichlet characters of modulus $N > 1$. Suppose $p \nmid \phi(F)FN$ and $(F, N) = 1$.*

(1) *For any $\varepsilon > 0$,*

$$\#\{\chi \in \widehat{(\mathbb{Z}/F\mathbb{Z})}^{\times} \mid L(-k, \lambda\chi) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} \gg_{\varepsilon, N, k} F^{\frac{1}{2}-\varepsilon}.$$

Moreover,

$$\#\{\chi \in \widehat{(\mathbb{Z}/F\mathbb{Z})}^{\times} \mid L(-k, \chi) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} \gg_{\varepsilon, k} F^{\frac{1}{2}-\varepsilon}.$$

(2) *Let F be an odd prime.*
If $F > N/p^{\epsilon}$, then

$$\#\{\chi \in \widehat{(\mathbb{Z}/F\mathbb{Z})}^{\times} \mid L(-k, \lambda\chi) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} \geq \left\lfloor \left(\frac{F}{9Np^{\epsilon}}\right)^{\frac{1}{2}} \right\rfloor.$$

If $F > 2/p^{\epsilon}$, then

$$\#\{\chi \in \widehat{(\mathbb{Z}/F\mathbb{Z})}^{\times} \mid L(-k, \chi) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} \gg_{\varepsilon,k} \left\lfloor \left(\frac{F}{18p^{\epsilon}}\right)^{\frac{1}{2}} \right\rfloor.$$

Proof Let $q > 1$ be an integer with $(pF, q) = 1$.

In regards to $L(-k, \chi)$, it suffices to observe that for an integer m with $q \nmid m$, we have

$$\mathrm{Res}\left(\frac{m}{q}, \Phi_{\widehat{\mathbf{1}}_q}(e^{2\pi i z})z^k\right) = -\frac{1}{q^{k-1}}.$$

Note that

$$\widehat{\mathbf{1}}_q(r) = \begin{cases} 0 & \text{if } q \mid r, \\ -q & \text{otherwise.} \end{cases}$$

and $\widehat{\mathbf{1}_q \circ F} = \widehat{\mathbf{1}}_q$.

Using (5.12) and (4.6), we deduce the result for $L(-k, \chi)$ with an odd prime F and $q = 2$. □

In a special case, we note the following stronger non-vanishing.

Corollary 5.5 *Suppose p is inert in $\mathbb{Q}(\zeta_{\phi(F)})$ for F a sufficiently large prime.*
Then,

$$L(0, \chi) \not\equiv 0 \,(\mathrm{mod}\ p)$$

for all odd primitive character χ of conductor F.

Remark 5.4 The prime F is necessarily of the form $2p^m + 1$ for $m \geq 1$.

The corollary seems to suggest that the exponents in our non-vanishing results are non-optimal.

It would be an interesting problem to find the largest value of Q such that there exist integers r_i, s_j, s'_j satisfying (**P**) and (**D**) for the case $Q^2 > F$. In Sect. 7, a table of numerical examples for $k = 0$ and prime F is presented. In the table, the integers r_i, s_j, and s'_j are not the ones in the proof of Theorem 5.3 as they satisfy $Q^2 > F$.

In light the numerical data, we raise the following:

Question 5.6 *Is it possible to show that Q can be chosen to be greater than a positive proportion of F?*

It seems tempting to say 'Yes'.

6 Cyclotomic twists

In this section, we refine our algebraic approach so as to apply it to the Iwasawa setting leading to the proof of Theorem C.

Based on the homological approach, we can also show a similar result to Theorem 5.3 (2) for prime power conductors. However it does not seem to imply any consequence along the lines of Washington's theorem ([34]) in which the base character (of the prime modulus) is fixed. Instead, we present a complementary argument to discuss a result in the context of Washington's theorem.

Let p and ℓ be distinct odd primes. We consider variation over the cyclotomic $\mathbb{Z}_\ell$-extension of the rationals. Let v_ℓ be the ℓ-adic valuation on $\mathbb{Q}_\ell$ such that $v_\ell(\ell) = 1$.

We commence with the following proposition on the zero set of a pseudo-polynomial that is due to Sinnott:

Proposition 6.1 (Sinnott [26]) *For $1 \leq i \leq m$, let $\alpha_i \in \mathbb{Z}_\ell$ be pairwise distinct and $c_i \in \overline{\mathbb{F}}_p$ non-zero. Let M_0 and M_1 be the integers such that*

$$\mathbb{F}_p(c) \cap \mu_{\ell^\infty} = \mu_{\ell^{M_0}} \text{ and } M_1 = \max_{i \neq j}(v_\ell(\alpha_i - \alpha_j))$$

for $\mathbb{F}_p(c)$ the extension of $\mathbb{F}_p$ generated by $c_i's$.

Then, the solutions of the equation

$$c_1 T^{\alpha_1} + c_2 T^{\alpha_2} + \cdots + c_m T^{\alpha_m} = 0$$

in μ_{ℓ^∞} are all contained in $\mu_{\ell^{M_0+M_1}}$.

In particular, a non-trivial pseudo-polynomial is a non-trivial function on μ_{ℓ^∞}.

In this section, let χ be a Dirichlet character of ℓ-power conductor and ℓ-power order. Let ω be a Dirichlet character of modulus ℓ. Let $\mu_{\ell-1}$ be the torsion subgroup of $\mathbb{Z}_\ell^\times$ and W a set of representatives of $\mu_{\ell-1}/\{\pm 1\}$. The character ω can be regarded as a character on $\mu_{\ell-1}$ in a canonical way.

For a periodic function ψ (Sect. 2), we define

$$\widetilde{\Phi}^\circ_{k,\psi}(\omega; T) = \sum_{\eta \in \mu_{\ell-1}} \overline{\omega}(\eta) \Phi^\circ_{k,\psi}(T^\eta)$$

with the complex conjugation $\overline{\omega}$ as before. Let us also set

$$\widetilde{\Phi}_{k,\psi}(\omega; T) = 2 \sum_{\eta \in W} \overline{\omega}(\eta) \Phi_{k,\psi}(T^\eta).$$

For $n \geq 1$, recall that

$$G_n = \frac{1+\ell\mathbb{Z}}{1+\ell^n\mathbb{Z}}.$$

We have the following miscellaneous result on the expression for Dirichlet L-values in terms of rational functions:

Lemma 6.2 *Let $k \geq 0$ be an integer.*

(1) *Let $\chi \in \widehat{G}_n$ be non-trivial and ψ a periodic function with $\widehat{\psi}(0) = 0$. Then,*

$$L(-k, \widehat{\chi\omega}\psi) = \sum_{r \in G_n} \chi(r)\widetilde{\Phi}^{\circ}_{k,\psi}(\omega; \zeta^r_{\ell^n}). \tag{6.1}$$

(2) *Let λ be a non-trivial Dirichlet character of modulus N. Then,*

$$\widetilde{\Phi}^{\circ}_{k,\lambda}(\omega; T) = \begin{cases} 0 & \text{if } \omega\lambda(-1) = (-1)^k \\ \widetilde{\Phi}_{k,\lambda}(\omega; T) & \text{otherwise} \end{cases}.$$

(3) *Let $q \geq 2$ an integer with $(q, \ell) = 1$. Then,*

$$\widetilde{\Phi}^{\circ}_{k,\mathbf{1}_q}(\omega; T) = \begin{cases} L_{k,W}(q) & \text{if } \omega(-1) = (-1)^k \\ \widetilde{\Phi}_{k,\mathbf{1}_q}(\omega; T) + L_{k,W}(q) & \text{otherwise} \end{cases}$$

where $L_{0,W}(q) = (q-1)\sum_{\eta \in W} \overline{\omega}(\eta)$ and $L_{k,W}(q) = 0$ if $k \geq 1$.

Proof The first assertion is a consequence of (2.5).

For the other assertions note that for $k \geq 1$, we obtain from (2.4) and equality $\Phi^{\circ}_{k,\psi} = \Phi_{k,\psi}$ that

$$\widetilde{\Phi}_{k,\psi}(\omega; T) = \sum_{\eta \in W} \overline{\omega}(\eta)\Big(\Phi_{k,\psi}(T^{\eta}) + \omega(-1)(-1)^{k+1}\Phi_{k,\psi\circ -1}(T^{\eta})\Big).$$

As $\Phi^{\circ}_{\psi} = \Phi_{\psi} - \psi(0)$, we have

$$\widetilde{\Phi}_{0,\psi}(\omega; T) = \sum_{\eta \in W} \overline{\omega}(\eta)\Big(\Phi_{\psi}(T^{\eta}) - \omega(-1)\Phi_{\psi\circ -1}(T^{\eta}) - \psi(0)\Big).$$

From these equalities, all the assertions follow. This finishes the proof. □

Let n be a positive integer. For a positive integer Q (dependent on ℓ^n), let r_i, s_i $(i = 1, \ldots, Q)$ be integers such that

$$1 \le r_i < r_{i+1} \le \ell^n, 1 \le s_i < s_{i+1} \le \ell^n, \ \ell \nmid r_i s_i \text{ for each } 1 \le i \le Q-1, \text{ and} \tag{6.2}$$

$$s_Q \le s_{Q-1} + s_1. \tag{6.3}$$

In this setting, for $\mathfrak{r} = (r_i)$ and $\mathfrak{s} = (s_j)$, let

$$\Xi_{k,\psi}(T) = \Xi_{k,\psi}(\mathfrak{r}, \mathfrak{s}; T) = \det \widetilde{\Phi}_{k,\psi}(\omega; T^{r_i s_j}) \prod_{\eta \in W} \prod_{i,j=1}^{Q} (1 - T^{N s_i r_j \eta})^{k+1}.$$

Note that $\Xi_{k,\psi}(T)$ is a pseudo-polynomial, which can be regarded as a $\overline{\mathbb{Z}}_p$-valued function on μ_{ℓ^∞}.

Let λ be a non-trivial Dirichlet character of modulus N such that $(p\ell, N) = 1$ and $q > 0$ an integer such that $(p\ell, q) = 1$. To show $\Xi_{k,\psi}$ is non-trivial modulo $\mathfrak{p}$ for $\psi = \lambda$ or $\psi = \mathbf{1}_q$, our approach is based on Rosenberg [25] and Anglés [1]. In other words, we find a specific pseudo-monomial in the expansion of $\Xi_{k,\psi}(T)$, for which the exponent is unique and the corresponding coefficient is non-zero modulo $\mathfrak{p}$.

In the remaining part of this section, let us fix the embedding $\iota_\ell : \mathbb{C}_\ell \to \mathbb{C}$ and choose W such that

$$\Im(\iota_\ell(\eta)) \ge 0$$

for all $\eta \in W$, i.e., we have

$$\iota_\ell(W) = \left\{ \exp\left(\frac{2\pi i k}{\ell - 1}\right) \,\middle|\, 0 \le k < \frac{\ell - 1}{2} \right\}.$$

Let us use the same symbol η for $\iota_\ell(\eta)$. Then we obtain:

Proposition 6.3 *Suppose the conditions* (6.2) *and* (6.3).
Then, the pseudo-polynomials $\Xi_{k,\lambda}(T)$ *and* $\Xi_{k,\mathbf{1}_q}(T)$ *are non-trivial modulo* $\mathfrak{p}$.

Proof Let us consider $\widetilde{\Phi}_{k,\lambda}$ first. Expanding the determinant via Lemma 6.2, we have

$$\det \widetilde{\Phi}_{k,\lambda}(\omega; T^{r_i s_j}) = \sum_{\sigma \in S_Q} sgn(\sigma) \prod_{i=1}^{Q} \widetilde{\Phi}_{k,\lambda}(\omega; T^{r_i s_{\sigma(i)}}) \tag{6.4}$$

and an expression of $\Xi_{k,\lambda}(t)$ as follows:

$$\Xi_{k,\lambda}(T) = 2^Q \sum_{\sigma \in S_Q} sgn(\sigma) \sum_{\underline{\eta} \in W^Q} \overline{\omega}(\underline{\eta}) \prod_{i=1}^{Q} \Psi_{k,\lambda}(T^{r_i s_{\sigma(i)} \eta_i}) \prod_{\eta \in W \setminus \{\eta_i\}} \prod_{j \ne \sigma(i)} (1 - T^{N r_i s_j \eta})^{k+1}$$

where $\underline{\eta} = (\eta_1, \cdots, \eta_Q)$ and $\overline{\omega}(\underline{\eta}) = \overline{\omega}(\eta_1)\cdots\overline{\omega}(\eta_Q)$.

Recall that

$$\Psi_{k,\lambda}(T) = \sum_{v=0}^{D-1} c(v)T^v$$

where $D = (k+1)N$ and $c((k+1)N-1) = \lambda(N-1)$ (Sect. 3). Thus, the exponents of $\Xi_{k,\lambda}(T)$ are of the form

$$\sum_{i=1}^{Q}\left\{v_i r_i s_{\sigma(i)}\eta_i + N r_i \sum_{\eta\in W\setminus\{\eta_i\}}\sum_{j\neq\sigma(i)} s_j \eta w_{\eta,j}\right\} \tag{6.5}$$

where

$$(\eta_1, \cdots, \eta_Q) \in W^Q,\ 1 \le v_i < D \text{ and } 0 \le w_{\eta,j} \le k+1.$$

The corresponding coefficient is

$$sgn(\sigma)\prod_{i=1}^{Q} c(v_i)\overline{\omega}(\eta_i) \prod_{\eta\neq\eta_i,\, j\neq\sigma(i)} (-1)^{w_{\eta,j}}\binom{k+1}{w_{\eta,j}}.$$

Then the imaginary part of (6.5) is at most

$$\sum_{i=1}^{Q}\left\{(D-1)r_i s_{\sigma(i)}\Im(\eta_i) + D\sum_{\eta\in W\setminus\{\eta_i\}}\sum_{j\neq\sigma(i)} r_i s_j \Im(\eta)\right\} \tag{6.6}$$

and the equality holds only when $v_i = D-1$ and $w_{\eta,j} = k+1$ for all i, η, j. Let us set

$$R = \textstyle\sum_i r_i,\ S = \sum_j s_j, \text{ and } Y = \sum_{\eta\in W}\Im(\eta).$$

Then (6.6) equals

$$\begin{aligned} &DRSY + (2D-1)\Big(\sum_i r_i s_{\sigma(i)}\Im(\eta_i)\Big) \\ &\quad - DY\Big(\sum_i r_i s_{\sigma(i)}\Big) - DS\Big(\sum_i r_i \Im(\eta_i)\Big). \end{aligned} \tag{6.7}$$

From the condition (6.3), we have $S \ge 2s_i$ for each i. Observe that if $\sigma \in S_L$ is fixed, then the following part of (6.7):

$$\sum_i r_i\Big\{(2D-1)s_{\sigma(i)} - DS\Big\}\Im(\eta_i)$$

attains the maximum only when $\eta_i = 1$ for all i. Hence the expression (6.7) with a fixed σ and $\eta_i = 1$ attains the maximum

$$DRSY + (2D - 1 - YD)\Big(\sum_i r_i s_{\sigma(i)}\Big) - DSR. \tag{6.8}$$

If $2D - 1 - YD \geq 0$, then (6.8) attains the maximum only when $\sigma = id$. If $2D - 1 - YD < 0$, the maximum is attained only when $\sigma(i) = Q - 1 - i$ by Lemma 3.2. In this case the coefficient of the unique pseudo-monomial with the exponent (6.8) is

$$sgn(\sigma)\omega\lambda(-1)^Q(-1)^{(k+1)\frac{p-3}{2}(Q-1)} = \pm 1.$$

The non-triviality of $\Xi_{k,\lambda}(T)$ thus follows.

The case for $\widetilde{\Phi}_{k,\mathbf{1}_q}$ can be proven in the same way since $\Xi_{k,\mathbf{1}_q}$ has the unique pseudo-monomial with the same exponent (6.8) as $\widetilde{\Phi}_{k,\lambda}$ with $D = (k+1)q$, of which coefficient is

$$sgn(\sigma)\omega\mathbf{1}_q(-1)^Q(-1)^{(k+1)\frac{p-3}{2}(Q-1)}$$

for σ as in the previous case.

Hence we conclude the proof. □

In the Iwasawa theoretic setting, the non-standard L-values $L(-k, \widehat{\chi\omega}\psi)$ turn out to be the Dirichlet ones. With the notation as in Remark 2.2, we have:

Lemma 6.4 *For $n \geq 2$, let ℓ^n and ℓ^f be modulus and conductor of $\chi\omega$, respectively. Then, if $\chi\omega$ is non-trivial*

$$L(-k, \widehat{\chi\omega}\psi) = G((\chi\omega)^*)\ell^{(n-f)(k+1)}L(-k, \overline{\chi\omega}\cdot\psi\circ\ell^{n-f}).$$

Furthermore, if $\chi\omega$ is the trivial χ_0 of modulus ℓ^n,

$$L(-k, \widehat{\chi}_0\psi) = \ell^{n(k+1)}L(-k, \psi\circ\ell^n) - \ell^{(n-1)(k+1)}L(-k, \psi\circ\ell^{n-1}).$$

In particular, if $\psi = \lambda$ or $\mathbf{1}_q$ (as in §3), then

$$L(-k, \widehat{\chi\omega}\psi) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p}) \implies L(-k, \overline{\chi\omega}\psi) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p}).$$

Proof For any integer m and non-trivial $\chi\omega$, we have

$$\begin{aligned}\widehat{\chi\omega}(m) &= \sum_{r=1}^{\ell^n-1}(\chi\omega)^*(r)\zeta_{\ell^n}^{rm} \\ &= \begin{cases}\ell^{n-f}G((\chi\omega)^*)\overline{\chi\omega}(m) & \text{if } m = \ell^{n-f}m' \text{ and } \ell\nmid m' \\ 0 & \text{otherwise}\end{cases}.\end{aligned}$$

If $\chi\omega = \chi_0$ is trivial,

$$\widehat{\chi_0}(m) = \begin{cases} \ell^n - \ell^{n-1} & \text{if } \ell^n \mid m \\ -\ell^{n-1} & \text{if } \ell^n \nmid m,\ \ell^{n-1} \mid m \\ 0 & \text{otherwise} \end{cases}$$

From these calculations, we conclude the proof. □

We are now ready to prove Theorem C.

Theorem 6.5 *Let p and ℓ be distinct odd primes. Let ω be a Dirichlet character of modulus ℓ.*

(1) *Let k be a non-negative integer such that $\omega\lambda(-1) = (-1)^{k+1}$. Let M_0 be the non-negative integer such that*

$$\mu_{\ell^{M_0}} = \mathbb{Q}(\lambda) \cap \mu_{\ell^\infty}.$$

Then,

$$\#\{\chi \in \widehat{G}_n \mid L(-k, \chi\omega\lambda) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} \geq (N(k+1)(\ell-1))^{-\frac{1}{3}} \ell^{E_{\ell,n}} - 1$$

where

$$E_{\ell,n} = \frac{n - M_0}{3\phi(\ell-1)} - \frac{2}{3}.$$

(2) *Suppose $k \geq 1$ with $\omega(-1) = (-1)^{k+1}$.*
Then,

$$\#\{\chi \in \widehat{G}_n \mid L(-k, \chi\omega) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} \geq (2(k+1)(\ell-1))^{-\frac{1}{3}} \ell^{E'_{\ell,n}} - 1$$

where

$$E'_{\ell,n} = \frac{n}{3\phi(\ell-1)} - \frac{2}{3}.$$

(3) *Suppose $\omega(-1) = -1$.*
Then,

$$\#\{\chi \in \widehat{G}_n \mid L(0, \chi\omega) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} \geq (2(\ell-1))^{-\frac{1}{3}} \ell^{E'_{\ell,n}} - 2.$$

Proof Let us first consider the case of $L(-k, \chi\omega\lambda)$.

By Lemma 6.4, it suffices to consider the set of $\chi \in \widehat{G}$ such that $L(-k, \widehat{\chi\omega}\lambda) \not\equiv 0$ modulo $\mathfrak{p}$.

Let $Q \geq 1$ be an integer. Let us assume

$$\#\{\chi \in \widehat{G}_n \mid L(-k, \widehat{\chi\omega}\lambda) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} = \{\chi_1, \chi_2, \ldots, \chi_t\} \tag{6.9}$$

and $t < Q$. Similarly as (2.7) using the inversion of (6.1), for all $r \equiv 1 \,(\mathrm{mod}\ \ell)$ we have

$$\ell^{n-1}\widetilde{\Phi}^{\circ}_{k,\lambda}(\zeta^r_{\ell^n}) \equiv \sum_{i=1}^{t} \chi_i(r) L(-k, \widehat{\chi_i\omega}\lambda) \,(\mathrm{mod}\ \mathfrak{p}). \tag{6.10}$$

Note that $\widetilde{\Phi}^{\circ}_{k,\lambda}(T) = \widetilde{\Phi}_{k,\lambda}(T)$. An earlier argument (proof of Proposition 3.1) enables us to conclude from (6.10) that

$$\det(\widetilde{\Phi}_{k,\lambda}(\zeta^{r_i s_j}_{\ell^n})) \equiv 0 \,(\mathrm{mod}\ \mathfrak{p}). \tag{6.11}$$

for any $r_i, s_j \equiv 1 \,(\mathrm{mod}\ \ell)$. In other words,

$$\Xi_{k,\lambda}(\zeta_{\ell^n}) \equiv 0 \,(\mathrm{mod}\ \mathfrak{p}) \tag{6.12}$$

Let us set $r_j = s_j = \ell(j-1)+1$, $j = 1, \ldots, Q$. Then the integers r_i, s_j satisfy the conditions (6.2) and (6.3). Hence, $\Xi_{k,\lambda}$ is non-trivial modulo $\mathfrak{p}$. In order to apply Proposition 6.1, we need to find the maximum M_1 of ℓ-adic valuations of the differences of two of the following exponents of $\Xi_{k,\lambda}(T)$

$$\sum_{i=1}^{Q} \left\{ v_i r_i s_{\sigma(i)} \eta_i + N r_i \sum_{\eta \in W \setminus \{\eta_i\}} \sum_{j \neq \sigma(i)} s_j \eta w_{\eta,j} \right\} = \sum_{\eta \in W} c(\eta)\eta, \ \text{say}. \tag{6.13}$$

Here we have $(\eta_1, \ldots, \eta_Q) \in W^Q$, $1 \leq v_i < D$ and $0 \leq w_{\eta,j} \leq k+1$ with $D = N(k+1)$. Furthermore,

$$|c(\eta)| \leq D\ell^2 Q^3.$$

Let γ be a difference of two exponents (6.13). Under the previous embeddings ι and ι_p, the ℓ-adic number γ can be regarded as an element in $\mathbb{Q}(\mu_{\ell-1})$ and we have

$$|\gamma^{\sigma}| \leq (\ell-1)D\ell^2 Q^3$$

for all $\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_{\ell-1})/\mathbb{Q})$. Observe that if $\gamma \equiv 0 \,(\mathrm{mod}\ \ell^M)$ for $M \geq 1$, then

$$\ell^M \mid N_{\mathbb{Q}(\mu_{\ell-1})/\mathbb{Q}}(\gamma).$$

Hence if M_1 is the maximum of $v_\ell(\gamma)$, then

$$\ell^{M_1} \leq ((\ell-1)D\ell^2 Q^3)^{\phi(\ell-1)}. \tag{6.14}$$

Observe that

$$((\ell-1)D\ell^2Q^3)^{\phi(\ell-1)}\ell^{M_0} \geq \ell^n.$$

Otherwise from Proposition 6.1, (6.12), and (6.14), we are able to deduce that $\Xi_{k,\lambda}(T)$ is trivial modulo $\mathfrak{p}$, which is a contradiction. This finishes the proof of the first part.

For the second part, the proof goes in the same way as above, but with the values $L(-k, \widehat{\chi\omega}\mathbf{1}_2)$ via the expression (5.12).

For the last part, i.e., when $k = 0$ and $Q \geq 2$, let us set

$$\{\chi \in \widehat{G}_n \mid L(-k, \widehat{\chi\omega}\mathbf{1}_2) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p})\} = \{\chi_1, \chi_2, \ldots, \chi_t\}$$

Similarly as before, by Lemma 6.2,

$$\ell^{n-1}\widetilde{\Phi}_{0,\mathbf{1}_2}(\zeta_{\ell^n}^r) \equiv \sum_{i=1}^{t} \chi_i(r)L(0, \widehat{\chi_i\omega}\mathbf{1}_2) - \ell^{n-1}L_0(W) \,(\mathrm{mod}\ \mathfrak{p}).$$

This implies that for the $r_i, s_j \in 1+\ell\mathbb{Z}$ with $i, j = 1, \cdots, Q$ and $t+1 < Q$, we obtain

$$\det(\widetilde{\Phi}_{0,\mathbf{1}_2}(\zeta_{\ell^n}^{r_i s_j})) \equiv 0 \,(\mathrm{mod}\ \mathfrak{p}).$$

In the same way as the first part, we conclude the proof. □

We obtain a non-vanishing result that is independent of the inertness as follows.

Corollary 6.6 *Let the notation and conditions be in Theorem 6.5.*

(1) *There exists a character χ of ℓ-power order and ℓ-power conductor less than*

$$(8(\ell-1)\ell^2N(k+1))^{\phi(\ell-1)}\ell^{M_0+1}$$

such that

$$L(-k, \chi\omega\lambda) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{p}).$$

(2) *In this case, the lower bound is* $(16(\ell-1)\ell^2(k+1))^{\phi(\ell-1)}\ell$.

(3) *In this case, the lower bound is* $(54(\ell-1)\ell^2))^{\phi(\ell-1)}\ell$.

Proof Let us first consider the case of $L(-k, \chi\omega\lambda)$.

Note that

$$(N(k+1)(\ell-1))^{-\frac{1}{3}}\ell^{E_{\ell,n}} - 1 \geq 1$$

if and only if

$$\ell^n \geq (8N(k+1)\ell^2(\ell-1))^{\phi(\ell-1)}\ell^{M_0}.$$

The computations for the other cases are similar. □

7 Numerical examples for $\mathfrak{r}$, $\mathfrak{s}$, and $\mathfrak{s}'$

In this section, we present a table of examples of $\mathfrak{r}$, $\mathfrak{s}$, and $\mathfrak{s}'$ that satisfy (**P**) for the case of $k = 0$ and prime F.

$F = 113$, $N = 4$, $Q = 16$
r_i 4, 2, 7, 39, 37, 44, 12, 20, 10, 22, 17, 15, 49, 5, 32, 54
s_j 1, 54, 36, 15, 7, 19, 24, 3, 6, 8, 4, 2, 9, 12, 10, 11
s'_j 5, 18, 80, 62, 82, 50, 103, 74, 40, 53, 83, 51, 28, 31, 35, 20
$F = 113$, $N = 5$, $Q = 15$
r_i 2, 1, 18, 7, 56, 32, 31, 13, 12, 6, 37, 50, 38, 25, 44
s_j 1, 20, 39, 55, 98, 66, 105, 97, 6, 106, 68, 99, 88, 85, 34
s'_j 19, 38, 57, 37, 80, 48, 10, 2, 24, 11, 104, 4, 93, 44, 22
$F = 113$, $N = 6$, $Q = 14$
r_i 5, 19, 34, 14, 6, 47, 46, 33, 27, 26, 13, 7, 53, 40
s_j 1, 53, 51, 30, 52, 13, 16, 4, 10, 15, 3, 8, 2, 6
s'_j 7, 98, 40, 12, 24, 21, 80, 5, 82, 57, 37, 17, 39, 35
$F = 173$, $N = 5$, $Q = 21$
r_i 2, 1, 7, 24, 13, 55, 18, 19, 50, 56, 80, 25, 86, 43, 12, 6, 49, 74, 62, 31, 37
s_j 1, 75, 44, 85, 34, 64, 134, 37, 9, 10, 7, 13, 15, 16, 14, 11, 5, 8, 17, 18, 19
s'_j 6, 3, 2, 57, 68, 12, 4, 70, 113, 137, 169, 28, 78, 154, 96, 99, 135, 163, 69, 62, 29
$F = 173$, $N = 6$, $Q = 19$
r_i 4, 2, 62, 21, 9, 69, 74, 44, 76, 67, 16, 37, 46, 23, 14, 53, 7, 83, 60
s_j 1, 83, 60, 45, 36, 15, 66, 24, 5, 10, 3, 9, 4, 8, 2, 6, 12, 11, 14
s'_j 7, 64, 20, 18, 95, 30, 166, 120, 29, 94, 160, 140, 76, 121, 73, 61, 147, 79, 77
$F = 197$, $N = 5$, $Q = 23$
r_i 2, 1, 7, 56, 19, 18, 13, 50, 92, 25, 98, 61, 62, 24, 31, 86, 12, 6, 68, 49, 43, 74, 37
s_j 1, 168, 50, 19, 90, 78, 27, 58, 11, 14, 15, 21, 25, 3, 5, 7, 8, 10, 13, 17, 4, 9, 20
s'_j 6, 194, 2, 12, 193, 37, 173, 105, 67, 167, 23, 81, 108, 151, 109, 174, 138, 16, 73, 56, 159, 39, 46
$F = 197$, $N = 6$, $Q = 21$
r_i 5, 33, 6, 74, 21, 13, 81, 47, 20, 48, 75, 82, 88, 41, 14, 7, 54, 95, 27, 61, 68
s_j 1, 102, 68, 51, 38, 17, 106, 73, 42, 20, 11, 8, 9, 4, 10, 6, 12, 3, 5, 13, 2
s'_j 7, 124, 154, 178, 71, 34, 136, 98, 81, 195, 179, 44, 28, 128, 151, 63, 26, 75, 67, 119, 139
$F = 229$, $N = 6$, $Q = 24$
r_i 4, 2, 46, 9, 90, 53, 16, 58, 14, 97, 23, 95, 88, 44, 51, 21, 7, 67, 60, 81, 30, 104, 111, 74
s_j 1, 196, 195, 64, 162, 130, 50, 82, 215, 16, 214, 10, 23, 105, 125, 46, 191, 86, 210, 172, 51, 141, 168, 43
s'_j 131, 97, 65, 163, 32, 31, 180, 181, 85, 146, 115, 109, 153, 6, 26, 176, 92, 117, 142, 73, 20, 5, 112, 185

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