FOURIER RESTRICTION TO A HYPERBOLIC CONE

Abstract. Using a bilinear restriction theorem of Lee and a bilinear-to-linear argument of Stovall, we obtain the conjectured range of Fourier restriction estimates for a conical hypersurface in $\mathbb{R}^4$ with hyperbolic cross sections.

1. Introduction

In this article, we resolve the Fourier restriction problem for the conical hypersurface

$$\Gamma := \left\{ \left(\zeta, \sigma, \frac{\zeta_1\zeta_2}{\sigma}\right) : \zeta \in [-1,1]^2,\ \sigma \in [1,2] \right\}$$

in $\mathbb{R}^4$. In this case, the problem asks, for which exponents p, q is the extension (adjoint restriction) operator

$$\mathcal{E}f(x,x',t) := \iint_{[-1,1]^2\times[1,2]} e^{i(x,x',t)\cdot(\zeta,\sigma,\frac{\zeta_1\zeta_2}{\sigma})} f(\zeta,\sigma) d\zeta d\sigma$$

of strong type $(p, 2q)$? The restriction problem for the light cone in $\mathbb{R}^4$ was solved by Wolff [6], while for other conical hypersurfaces, such as those with negatively curved cross sections, it has remained open. In the case of Γ, nearly optimal results are known: Greenleaf [1] proved that $\mathcal{E}$ is of strong type $(p, 2q)$ for $p \geq q'$ and $q \geq 2$, and Lee [2] extended that range to $q > 3/2$ and $p > q'$. The main result of this article is the boundedness of $\mathcal{E}$ on the scaling line $p = q'$ for $3/2 < q < 2$, solving the remaining part of the restriction problem for Γ.

Theorem 1.1. *The operator $\mathcal{E}$ is of strong type $(q', 2q)$ for $3/2 < q < 2$.*

The surface Γ looks like (a compact piece of) a cone whose cross sections are hyperbolic paraboloids. Strong type $(q', 2q)$ restriction estimates for the hyperbolic paraboloid in $\mathbb{R}^3$ are known for $q > 13/8$; see [3] and the references therein. A simple argument using Minkowski's and Hölder's inequalities shows that any such estimate implies the corresponding one for Γ. Therefore, the estimate in Theorem 1.1 is known for $q > 13/8$ and holds conditionally for smaller q, pending further estimates for the hyperbolic paraboloid. The superior bilinear restriction theory for Γ, in relation to that of the hyperbolic paraboloid, allows us to prove Theorem 1.1 unconditionally.

Terminology and notation. A positive constant is *admissible* if it depends only on q. We write $A \lesssim B$ to mean $A \leq CB$ for some admissible constant C, which is allowed to change from line to line. We denote the one-dimensional Hausdorff measure by $\mathcal{H}^1$. We write log for the base 2 logarithm. An interval of the form $[n2^{-j}, (n+1)2^{-j})$ for some $j, n \in \mathbb{Z}$ is *dyadic*, and $\mathcal{I}_j$ denotes the set of dyadic intervals of length 2^{-j}. The product of two dyadic intervals is a *tile*, and $\mathcal{T}_{j,k}$ denotes the set of $2^{-j} \times 2^{-k}$ tiles. Given $\tau \in \mathcal{T}_{j,k}$, we set $\tilde{\tau} := \tau \times [1,2]$. We denote

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by $\pi_{i,3}$ and π_i, respectively, the projections $(\zeta,\sigma)\mapsto(\zeta_i,\sigma)$ and $(\zeta_i,\sigma)\mapsto\zeta_i$, for $i=1,2$ and $(\zeta,\sigma)\in\mathbb{R}^2\times\mathbb{R}$. If π is one of these projections and S a subset of the domain of π, the *π-projection of S* refers to the set $\pi(S)$, and a *π-fiber of S* is any set of the form $\pi^{-1}(\pi(s))\cap S$ with $s\in S$. *Horizontal* and *vertical* refer to the directions in $\mathbb{R}^2$ parallel to the standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively. Finally, the *extension of a set* refers to the Fourier extension of the set's characteristic function.

Outline of the proof. We adapt an argument of Stovall [3] which showed that, for $3/2<q<2$, the extension operator associated to the hyperbolic paraboloid in $\mathbb{R}^3$ is of strong type $(q',2q)$, provided an appropriate $L^{p_0}\times L^{p_0}\to L^{q_0}$ bilinear restriction inequality holds for some $q_0<q$ and $p_0/2<q_0<p_0'$. A bilinear estimate suitable for running Stovall's argument on the hypersurface Γ is already known:

Theorem 1.2 (Lee [2])**.** *Let $\tau,\kappa\subseteq[-1,1]^2$ be squares with unit separation in both the horizontal and vertical directions. If $q>3/2$, then*

$$\|\mathcal{E}f\mathcal{E}g\|_q\lesssim\|f\|_2\|g\|_2$$

for all bounded measurable functions f and g supported in $\tau\times[1,2]$ and $\kappa\times[1,2]$, respectively.

To prove Theorem 1.1, it suffices to show that $\mathcal{E}$ is of restricted strong type $(q',2q)$ for every $3/2<q<2$. Thus, we aim to prove that

$$\|\mathcal{E}\chi_\Omega\|_{2q}\lesssim|\Omega|^{\frac{1}{q'}} \tag{1.1}$$

for an arbitrary measurable set $\Omega\subseteq[-1,1]^2\times[1,2]$. In Section 2, we use Theorem 1.2 and a bilinear-to-linear argument of Vargas [4] to show that sets having roughly constant $\pi_{1,3}$- (or $\pi_{2,3}$-) fiber length obey (1.1). In Section 3, we solve a related inverse problem: For which sets Ω of constant fiber length can the inequality in (1.1) be reversed? Oversimplified, our answer is that Ω must be a box of the form $\tilde{\tau}$; proving (1.1) then becomes a matter of bounding the extension of a union of boxes, which we do in Section 4. Our real answer, however, is quantitative: We show that Ω is approximately a union of boxes, where the number of boxes in the union and the tightness of the approximation relate to the quantity $\mathrm{C}(\Omega)$, defined thus:

Definition 1.3. For measurable sets $\Omega_1\subseteq\Omega_2\subseteq[-1,1]^2\times[1,2]$, let $\mathrm{C}(\Omega_1,\Omega_2)$ denote the smallest number ε, either dyadic, zero, or infinite, such that $\|\mathcal{E}\chi_{\Omega_1'}\|_{2q}\leq\varepsilon|\Omega_2|^{1/q'}$ for every measurable set $\Omega_1'\subseteq\Omega_1$, and let $\mathrm{C}(\Omega_1):=\mathrm{C}(\Omega_1,\Omega_1)$.

Finally, in Section 5, we start with a generic set Ω, decompose it into sets $\Omega(K)$ of fiber length roughly 2^{-K}, sorted thence according to the value of $\mathrm{C}(\Omega(K))$, and apply the restriction estimates of Sections 3 and 4 to obtain (1.1).

While much of our argument resembles Stovall's in [3], we include full details for the convenience of the reader.

2. Extensions of sets of constant fiber length

In this section, we prove a scaling line restriction estimate for characteristic functions of sets of constant $\pi_{1,3}$-fiber length, arguing à la Vargas [4]. By symmetry, the same estimate then holds for sets of constant $\pi_{2,3}$-fiber length.

Definition 2.1. Given a measurable set $\Omega \subseteq [-1,1]^2 \times [1,2]$ and an integer $K \geq 0$, let

$$\Omega(K) := \{(\zeta, \sigma) \in \Omega : \mathcal{H}^1(\pi_{1,3}^{-1}(\zeta_1, \sigma) \cap \Omega) \sim 2^{-K}\}.$$

Proposition 2.2. *Suppose that $\Omega = \Omega(K)$ for some K. Then $\mathrm{C}(\Omega) \lesssim 1$.*

Proof. Let $\Omega' \subseteq \Omega$ be measurable. Given $\tau, \kappa \in \mathcal{T}_{j,k}$, we write $\tau \sim \kappa$ if τ and κ are separated by a distance of $\sim 2^{-j}$ in the horizontal direction and $\sim 2^{-k}$ in the vertical direction. Up to a set of measure zero, we have

$$([-1,1]^2 \times [1,2])^2 = \bigcup_{j,k} \bigcup_{\substack{\tau,\kappa \in \mathcal{T}_{j,k} \\ \tau \sim \kappa}} \tilde{\tau} \times \tilde{\kappa}.$$

Consequently, by the triangle inequality and Lemma 6.1 in [5] (using that $q < 2$),

$$\|\mathcal{E}\chi_{\Omega'}\|_{2q}^2 \lesssim \sum_{j,k} \Bigg(\sum_{\substack{\tau,\kappa \in \mathcal{T}_{j,k} \\ \tau \sim \kappa}} \|\mathcal{E}\chi_{\Omega' \cap \tilde{\tau}} \mathcal{E}\chi_{\Omega' \cap \tilde{\kappa}}\|_q^q \Bigg)^{\frac{1}{q}}.$$

By rescaling, Theorem 1.2 implies that

$$\|\mathcal{E}\chi_{\Omega' \cap \tilde{\tau}} \mathcal{E}\chi_{\Omega' \cap \tilde{\kappa}}\|_q \lesssim 2^{-(j+k)(1-\frac{2}{q})} |\Omega' \cap \tilde{\tau}|^{\frac{1}{2}} |\Omega' \cap \tilde{\kappa}|^{\frac{1}{2}} \leq 2^{-(j+k)(1-\frac{2}{q})} |\Omega \cap \tilde{\tau}|^{\frac{1}{2}} |\Omega \cap \tilde{\kappa}|^{\frac{1}{2}}$$

for $\tau, \kappa \in \mathcal{T}_{j,k}$ with $\tau \sim \kappa$. Given $\tau \in \mathcal{T}_{j,k}$, there are admissibly many κ such that $\tau \sim \kappa$, and for each such κ, we have $10\tau \supseteq \kappa$. Thus,

$$\begin{aligned} \|\mathcal{E}\chi_{\Omega'}\|_{2q}^2 &\lesssim \sum_{j,k} 2^{-(j+k)(1-\frac{2}{q})} \Bigg(\sum_{\tau \in \mathcal{T}_{j,k}} |\Omega \cap 10\tilde{\tau}|^q \Bigg)^{\frac{1}{q}} \\ &\lesssim \sum_{j,k} 2^{-(j+k)(1-\frac{2}{q})} \max_{\tau \in \mathcal{T}_{j,k}} |\Omega \cap 10\tilde{\tau}|^{1-\frac{1}{q}} |\Omega|^{\frac{1}{q}}. \end{aligned} \tag{2.1}$$

Let J be an integer such that $|\pi_{1,3}(\Omega)| \sim 2^{-J}$. Then, by Fubini's theorem, $|\Omega| \sim 2^{-J-K}$ and

$$\max_{\tau \in \mathcal{T}_{j,k}} |\Omega \cap 10\tilde{\tau}| \lesssim \min\{2^{-J}, 2^{-j}\} \min\{2^{-K}, 2^{-k}\}. \tag{2.2}$$

We split the right-hand side of (2.1) into four parts: summation over j, k satisfying (i) $j \leq J$, $k \leq K$; (ii) $j \leq J$, $k > K$; (iii) $j > J$, $k \leq K$; (iv) $j > J$, $k > K$. Each part is estimated simply by applying (2.2) and summing a geometric series. We obtain the desired bound in this way. □

3. An inverse problem related to Proposition 2.2

In this section, we answer quantitatively the following question: If Ω extremizes the inequality in Proposition 2.2, what structure must Ω have?

Proposition 3.1. *Suppose that $\Omega = \Omega(K)$ for some K, let J be an integer such that $|\Omega| \sim 2^{-J-K}$, and let $\varepsilon := \mathrm{C}(\Omega)$. Up to a set of measure zero, there exists a decomposition*

$$\Omega = \bigcup_{0 < \delta \lesssim \varepsilon^{1/5}} \Omega_\delta,$$

where the union is taken over dyadic numbers, such that

(i) $\mathrm{C}(\Omega_\delta, \Omega) \lesssim \delta^{1/3}$*, and*

(ii) $\Omega_\delta \subseteq \bigcup_{\tau\in\mathcal{T}_\delta} \tilde{\tau}$*, where* $\mathcal{T}_\delta \subseteq \mathcal{T}_{J,K}$ *with* $\#\mathcal{T}_\delta \lesssim \delta^{-C_0}$ *for some admissible constant* C_0*.*

Proof of Proposition 3.1. The construction of the sets Ω_δ consists of five steps. We will begin by dividing Ω into sets Ω^1_α whose $\pi_{1,3}$-projections have constant π_1-fiber length α, respectively. That simple step enables us to adapt then the decomposition scheme employed in [3]. We divide each Ω^1_α into sets $\Omega^2_{\alpha,\eta}$ whose respective projections to the ζ_1-axis are contained in η^{-1} intervals in $\mathcal{I}_J$. In our third step, we divide each $\Omega^2_{\alpha,\eta}$ into sets $\Omega^3_{\alpha,\eta,\rho}$ of constant $\pi_{2,3}$-fiber length $\rho\eta^{-1}2^{-J}$. To each $\Omega^3_{\alpha,\eta,\rho}$ we may then apply variants of the first two steps wherein the roles of the coordinates ζ_1, ζ_2 are reversed. Indeed, were $\pi_{1,3}$ replaced by $\pi_{2,3}$ in Definition 2.1, each $\Omega^3_{\alpha,\eta,\rho}$ would be of the form $\Omega^3_{\alpha,\eta,\rho}(J + \log(\rho^{-1}\eta))$. In the end, we obtain sets $\Omega^5_{\alpha,\eta,\rho,\beta,\delta}$ whose respective projections to the ζ_2-axis are contained in δ^{-1} intervals in $\mathcal{I}_K$. For fixed δ, we define Ω_δ to be the union of the sets $\Omega^5_{\alpha,\eta,\rho,\beta,\delta}$, of which there will be at most $(\log\delta^{-1})^4$ by construction. Appearing in the argument below, there are of course constants and minor technical adjustments missing from this summary.

Step 1. For each dyadic number $0 < \alpha \le 1$, define

$$\Omega^1_\alpha := \{(\zeta,\sigma) \in \Omega : \mathcal{H}^1(\pi_1^{-1}(\zeta_1) \cap \pi_{1,3}(\Omega)) \sim \alpha^A\},$$

where A is an admissible constant to be chosen momentarily.

Lemma 3.2. *For every* $0 < \alpha \le 1$*, we have* $\mathrm{C}(\Omega^1_\alpha, \Omega) \lesssim \alpha$*.*

Proof of Lemma 3.2. Let $\Omega' \subseteq \Omega^1_\alpha$ be measurable, and let J_α be an integer such that $|\pi_{1,3}(\Omega_\alpha)| \sim \alpha^A 2^{-J_\alpha}$. We record the bound

$$\alpha^A 2^{-J_\alpha} \lesssim 2^{-J}. \tag{3.1}$$

Following the proof of Proposition 2.2, we have

$$\|\mathcal{E}\chi_{\Omega'}\|^2_{2q} \lesssim \sum_{j,k} 2^{-(j+k)(1-\frac{2}{q})} \max_{\tau\in\mathcal{T}_{j,k}} |\Omega^1_\alpha \cap 10\tilde{\tau}|^{1-\frac{1}{q}} |\Omega|^{\frac{1}{q}}. \tag{3.2}$$

By Fubini's theorem,

$$\begin{aligned} |\Omega^1_\alpha \cap 10\tilde{\tau}| &\lesssim |\pi_{1,3}(\Omega^1_\alpha \cap 10\tilde{\tau})| \min\{2^{-K}, 2^{-k}\} \\ &\lesssim \alpha^A \min\{2^{-J_\alpha}, 2^{-j}\} \min\{2^{-K}, 2^{-k}\} \end{aligned} \tag{3.3}$$

for every $\tau \in \mathcal{T}_{j,k}$. As in the proof of Proposition 2.2, we split the right-hand side of (3.2) into four parts: summation over j, k satisfying (i) $j \le J_\alpha$, $k \le K$; (ii) $j \le J_\alpha$, $k > K$; (iii) $j > J_\alpha$, $k \le K$; (iv) $j > J_\alpha$, $k > K$. Using (3.3) and (3.1), we bound the sum corresponding to (i) by

$$\begin{aligned} \sum_{\substack{j\le J_\alpha\\ k\le K}} 2^{-(j+k)(1-\frac{2}{q})} (\alpha^A 2^{-J_\alpha-K})^{1-\frac{1}{q}} |\Omega|^{\frac{1}{q}} &\sim \alpha^{A(1-\frac{1}{q})} 2^{-(J_\alpha+K)(2-\frac{3}{q})} |\Omega|^{\frac{1}{q}} \\ &\lesssim \alpha^{A(\frac{2}{q}-1)} 2^{-(J+K)(2-\frac{3}{q})} |\Omega|^{\frac{1}{q}} \\ &\sim \alpha^{A(\frac{2}{q}-1)} |\Omega|^{\frac{2}{q'}}. \end{aligned}$$

Using the same steps, the sum corresponding to (ii) is at most

$$\sum_{\substack{j\le J_\alpha\\ k> K}} 2^{-j(1-\frac{2}{q})} 2^{-k(2-\frac{3}{q})} \alpha^{A(1-\frac{1}{q})} 2^{-J_\alpha(1-\frac{1}{q})} |\Omega|^{\frac{1}{q}} \sim \alpha^{A(1-\frac{1}{q})} 2^{-(J_\alpha+K)(2-\frac{3}{q})} |\Omega|^{\frac{1}{q}}$$

$$\lesssim \alpha^{A(\frac{2}{q}-1)}|\Omega|^{\frac{2}{q'}}.$$

The sums corresponding to (iii) and (iv) can be handled in essentially the same way, leading to the estimate

$$\|\mathcal{E}\chi_{\Omega'}\|_{2q} \lesssim \alpha^{A(\frac{1}{q}-\frac{1}{2})}|\Omega|^{\frac{1}{q'}}.$$

We conclude the proof by setting $A := (\frac{1}{q} - \frac{1}{2})^{-1}$. □

Step 2. For each $0 < \alpha \leq 1$, let $S_\alpha := \pi_1(\pi_{1,3}(\Omega^1_\alpha))$, and note that $|S_\alpha| \sim 2^{-J_\alpha}$ with J_α as in the proof of Lemma 3.2. Given a dyadic number $0 < \eta \leq \alpha$ and a Lebesgue point ζ_1 of S_α, let $I_{\alpha,\eta}(\zeta_1)$ be the maximal dyadic interval I such that $\zeta_1 \in I$ and

$$\frac{|I \cap S_\alpha|}{|I|} \geq \eta^B, \tag{3.4}$$

where B is an admissible constant to be chosen later; such an interval exists by the Lebesgue differentiation theorem. Without loss of generality, we assume that S_α is equal to its set of Lebesgue points. Let

$$T_{\alpha,\eta} := \{\zeta_1 \in S_\alpha : |I_{\alpha,\eta}(\zeta_1)| \geq \eta^B 2^{-J_\alpha}\}.$$

If $\alpha < \varepsilon$, define $S_{\alpha,\alpha} := T_{\alpha,\alpha}$ and $S_{\alpha,\eta} := T_{\alpha,\eta} \setminus T_{\alpha,2\eta}$ for $\eta < \alpha$, and let

$$\Omega^2_{\alpha,\eta} := \Omega^1_\alpha \cap \pi^{-1}_{1,3}(\pi^{-1}_1(S_{\alpha,\eta})).$$

For $\varepsilon \leq \alpha \leq 1$, define $S_{\alpha,\varepsilon} := T_{\alpha,\varepsilon}$ and $S_{\alpha,\eta} := T_{\alpha,\eta} \setminus T_{\alpha,2\eta}$ for $\eta < \varepsilon$. For $\eta \leq \varepsilon$, let

$$\Omega^2_{\varepsilon,\eta} := \bigcup_{\varepsilon \leq \alpha \leq 1} \tilde{\Omega}^2_{\alpha,\eta},$$

where $\tilde{\Omega}^2_{\alpha,\eta} := \Omega^1_\alpha \cap \pi^{-1}_{1,3}(\pi^{-1}_1(S_{\alpha,\eta}))$.

Remark 3.3. We note that $\Omega^2_{\alpha,\eta} \subseteq \Omega^1_\alpha$ for $\alpha < \varepsilon$ and $\tilde{\Omega}^2_{\alpha,\eta} \subseteq \Omega^1_\alpha$ for $\varepsilon \leq \alpha \leq 1$, while in general $\Omega^2_{\varepsilon,\eta}$ is not contained in Ω^1_ε. We do have

$$\Omega = \bigcup_{0<\alpha\leq 1} \Omega^1_\alpha = \bigcup_{0<\alpha\leq\varepsilon} \bigcup_{0<\eta\leq\alpha} \Omega^2_{\alpha,\eta}.$$

Lemma 3.4. *For every $0 < \eta \leq \alpha \leq \varepsilon$, the set $\Omega^2_{\alpha,\eta}$ is contained in a union of $O(\eta^{-3B-A-1})$ boxes of the form $\tilde{\tau}$, with $\tau \in \mathcal{T}_{J,0}$, and satisfies $\mathrm{C}(\Omega^2_{\alpha,\eta}, \Omega) \lesssim \eta^{1/2}$.*

Proof of Lemma 3.4. We argue first under the hypothesis that $\alpha < \varepsilon$, then indicate the changes needed when $\alpha = \varepsilon$. By its definition, $S_{\alpha,\eta}$ is covered by dyadic intervals I of length $|I| \gtrsim \eta^B |S_\alpha|$, in each of which S_α has density obeying (3.4). The density of each such I in S_α is

$$\frac{|I \cap S_\alpha|}{|S_\alpha|} = \frac{|I \cap S_\alpha|}{|I|} \cdot \frac{|I|}{|S_\alpha|} \gtrsim \eta^{2B}.$$

Therefore, if $\mathcal{C}$ is a minimal-cardinality covering of $S_{\alpha,\eta}$ by these I (consisting necessarily of pairwise disjoint intervals), then $\#\mathcal{C} \lesssim \eta^{-2B}$. Moreover, (3.4) and (3.1) imply that

$$|I| \lesssim \eta^{-B} 2^{-J_\alpha} \lesssim \eta^{-B} \alpha^{-A} 2^{-J} \leq \eta^{-B-A} 2^{-J}$$

for every $I \in \mathcal{C}$. Thus, $S_{\alpha,\eta}$ is covered by $O(\eta^{-3B-A})$ intervals in $\mathcal{I}_J$. Since $\alpha < \varepsilon$, it immediately follows that $\Omega^2_{\alpha,\eta}$ is contained in a union of $O(\eta^{-3B-A})$ boxes of the form claimed.

We turn to the restriction estimate. If $\eta = \alpha$, the result follows from Lemma 3.2 and Remark 3.3. Thus, we may assume that $\eta < \alpha$. We proceed by optimizing the proof of Proposition 2.2, as in [3]. Let $\Omega' \subseteq \Omega^2_{\alpha,\eta}$ be measurable. From the proof of (2.1), we see that

$$\|\mathcal{E}\chi_{\Omega'}\|^2_{2q} \lesssim \sum_{j,k} 2^{-(j+k)(1-\frac{2}{q})} \max_{\tau \in \mathcal{T}_{j,k}} |\Omega' \cap 10\tilde{\tau}|^{1-\frac{1}{q}} |\Omega|^{\frac{1}{q}}. \tag{3.5}$$

Fix $\tau \in \mathcal{T}_{j,k}$. By Fubini's theorem and the definition of Ω^1_α (with $\alpha < \varepsilon$), we have

$$\begin{aligned} |\Omega' \cap 10\tilde{\tau}| &\lesssim |\pi_{1,3}(\Omega' \cap 10\tilde{\tau})| \min\{2^{-K}, 2^{-k}\} \\ &\lesssim \alpha^A \min\{|\pi_1(\pi_{1,3}(\Omega'))|, |\pi_1(\pi_{1,3}(10\tilde{\tau}))|\} \min\{2^{-K}, 2^{-k}\} \\ &\lesssim \alpha^A \min\{2^{-J_\alpha}, 2^{-j}\} \min\{2^{-K}, 2^{-k}\}. \end{aligned} \tag{3.6}$$

For certain j, the definition of $\Omega^2_{\alpha,\eta}$ leads to a better estimate. We claim that if $|j - J_\alpha| < \frac{B}{4} \log \eta^{-1}$, then

$$|\Omega' \cap 10\tilde{\tau}| \lesssim \eta^{\frac{3B}{4}} \alpha^A \min\{2^{-J_\alpha}, 2^{-j}\} \min\{2^{-K}, 2^{-k}\}. \tag{3.7}$$

Fix such a j. Note that 10τ is contained in a union of four tiles κ in $\mathcal{T}_{j-4,k-4}$, so it suffices to prove (3.7) with κ in place of 10τ. Let $\kappa =: I_{j-4} \times I_{k-4}$, where $I_{j-4} \in \mathcal{I}_{j-4}$ and $I_{k-4} \in \mathcal{I}_{k-4}$. We have

$$|I_{j-4}| = 2^{-j+4} \geq 16\eta^{\frac{B}{4}} 2^{-J_\alpha} \geq (2\eta)^B 2^{-J_\alpha},$$

provided η is sufficiently small. Suppose that $I_{j-4} \cap S_{\alpha,\eta} \neq \emptyset$. Then there exists $\zeta_1 \in I_{j-4} \cap S_{\alpha,\eta}$ such that $\zeta_1 \notin T_{\alpha,2\eta}$, whence

$$|I_{\alpha,2\eta}(\zeta_1)| < (2\eta)^B 2^{-J_\alpha} \leq |I_{j-4}|.$$

Consequently, by the maximality of $I_{\alpha,2\eta}(\zeta_1)$ and the fact that $2^{-j} \leq \eta^{-\frac{B}{4}} 2^{-J_\alpha}$, we have

$$|I_{j-4} \cap S_{\alpha,\eta}| \leq |I_{j-4} \cap S_\alpha| \leq (2\eta)^B |I_{j-4}| = 16(2\eta)^B 2^{-j} \lesssim \eta^{\frac{3B}{4}} \min\{2^{-J_\alpha}, 2^{-j}\}.$$

Thus, by Fubini's theorem,

$$|\Omega' \cap \tilde{\kappa}| \lesssim \alpha^A |S_{\alpha,\eta} \cap I_{j-4}| \min\{2^{-K}, 2^{-k}\} \lesssim \eta^{\frac{3B}{4}} \alpha^A \min\{2^{-J_\alpha}, 2^{-j}\} \min\{2^{-K}, 2^{-k}\},$$

as claimed.

Now, to bound (3.5), we split the sum into eight parts determined by the conditions (a) $k \leq K$, (b) $k > K$ and (i) $j \leq J_\alpha - \frac{B}{4} \log \eta^{-1}$, (ii) $J_\alpha - \frac{B}{4} \log \eta^{-1} < j \leq J_\alpha$, (iii) $J_\alpha < j < J_\alpha + \frac{B}{4} \log \eta^{-1}$, (iv) $J_\alpha + \frac{B}{4} \log \eta^{-1} \leq j$. In each case, we use (3.7) if it applies, otherwise (3.6). Summing geometric series and using (3.1) and the fact that $|\Omega| \sim 2^{-J-K}$, it is straightforward to deduce the bound

$$\|\mathcal{E}\chi_{\Omega'}\|_{2q} \lesssim \eta^{B'} |\Omega|^{\frac{1}{q'}},$$

where B' is an admissible constant determined by B. We may choose B so that $B' = 1$; this better-than-required exponent will be utilized in the next paragraph.

Suppose now that $\alpha = \varepsilon$. For $\eta < \varepsilon$, the preceding arguments work equally well with $\Omega^2_{\alpha,\eta}$ replaced by $\tilde{\Omega}^2_{\alpha',\eta}$, where $\varepsilon \leq \alpha' \leq 1$. In particular, each such $\tilde{\Omega}^2_{\alpha',\eta}$ is contained in a union of $O(\eta^{-3B-A})$ boxes $\tilde{\tau}$, with $\tau \in \mathcal{T}_{J,0}$, and satisfies

$\mathrm{C}(\tilde\Omega^2_{\alpha',\eta},\Omega)\lesssim\eta$. The case $\eta=\varepsilon$ is similar, but with the bound $\mathrm{C}(\tilde\Omega^2_{\alpha',\varepsilon},\Omega)\lesssim\varepsilon$ following directly from the definition of ε. Since the number of sets $\tilde\Omega^2_{\alpha',\eta}$ is $O(\log\varepsilon^{-1})=\log(\eta^{-1/2})$ and their union is $\Omega^2_{\varepsilon,\eta}$, the lemma holds for $\alpha=\varepsilon$ as well. □

Step 3. For dyadic $0<\eta\le\alpha\le\varepsilon$ and $0<\rho\lesssim\eta^{1/5}$, define

$$\Omega^3_{\alpha,\eta,\rho}:=\{(\zeta,\sigma)\in\Omega^2_{\alpha,\eta}:\mathcal{H}^1(\pi_{2,3}^{-1}(\zeta_2,\sigma)\cap\Omega^2_{\alpha,\eta})\sim\rho^{5C}\eta^{-3B-A-1-C}2^{-J}\},$$

where C is an admissible constant to be chosen later. Lemma 3.4 implies that $\mathcal{H}^1(\pi_{2,3}^{-1}(\zeta_2,\sigma)\cap\Omega^2_{\alpha,\eta})\lesssim\eta^{-3B-A-1}2^{-J}$ for every $(\zeta,\sigma)\in\Omega^2_{\alpha,\eta}$. Thus,

$$\Omega^2_{\alpha,\eta}=\bigcup_{0<\rho\lesssim\eta^{1/5}}\Omega^3_{\alpha,\eta,\rho}.$$

Lemma 3.5. *For every $0<\eta\le\alpha\le\varepsilon$ and $0<\rho\lesssim\eta^{1/5}$, we have $\mathrm{C}(\Omega^3_{\alpha,\eta,\rho},\Omega)\lesssim\rho$.*

Proof of Lemma 3.5. If $\rho^{5C}\eta^{-3B-A-1-C}\ge\rho^{2C}$, then by Lemma 3.4, we have

$$\mathrm{C}(\Omega^3_{\alpha,\eta,\rho},\Omega)\lesssim\eta^{\frac12}\le\rho^{\frac{3C}{2(3B+A+1+C)}}\lesssim\rho$$

for C chosen sufficiently large. Thus, we may assume that $\rho^{5C}\eta^{-3B-A-1-C}\le\rho^{2C}$. Given a measurable set $\Omega'\subseteq\Omega^3_{\alpha,\eta,\rho}$ and $\tau\in\mathcal{T}_{j,k}$, the set $\Omega'\cap10\tilde\tau$ has $\pi_{1,3}$- and $\pi_{2,3}$-fibers of length at most $\min\{2^{-K},2^{-k}\}$ and $\min\{\rho^{2C}2^{-J},2^{-j}\}$, respectively, and it has $\pi_{1,3}$- and $\pi_{2,3}$-projections of measure at most $\min\{2^{-J},2^{-j}\}$ and 2^{-k}, respectively. Therefore, by Fubini's theorem,

$$|\Omega'\cap10\tilde\tau|\lesssim\min\{2^{-J-K},2^{-j-K},2^{-j-k},\rho^{2C}2^{-J-k}\}.\tag{3.8}$$

Following [3], we define

$$R_1:=\{(j,k):J-C\log\rho^{-1}\ge j,\ K\ge k\}\cup\{(j,k):J\ge j,\ K-C\log\rho^{-1}\ge k\}$$
$$R_2:=\{(j,k):j\ge J+C\log\rho^{-1},\ K\ge k\}\cup\{(j,k):j\ge J,\ K-C\log\rho^{-1}\ge k\}$$
$$R_3:=\{(j,k):j\ge J+C\log\rho^{-1},\ k\ge K\}\cup\{(j,k):j\ge J,\ k\ge K+C\log\rho^{-1}\}$$
$$R_4:=\{(j,k):J+C\log\rho^{-1}\ge j,\ k+C\log\rho^{-1}\ge K\}.$$

Each (j,k) belongs to some R_i, $1\le i\le4$, so by (3.5) and (3.8), we have

$$\|\mathcal{E}\chi_{\Omega'}\|^2_{2q}\lesssim\sum_{(j,k)\in R_1}2^{-(j+k)(1-\frac2q)}2^{-(J+K)(1-\frac1q)}|\Omega|^{\frac1q}+\sum_{(j,k)\in R_2}2^{-(j+k)(1-\frac2q)}2^{-(j+K)(1-\frac1q)}|\Omega|^{\frac1q}$$
$$+\sum_{(j,k)\in R_3}2^{-(j+k)(1-\frac2q)}2^{-(j+k)(1-\frac1q)}|\Omega|^{\frac1q}+\sum_{(j,k)\in R_4}2^{-(j+k)(1-\frac2q)}\rho^{2C(1-\frac1q)}2^{-(J+k)(1-\frac1q)}|\Omega|^{\frac1q}.$$

Summing these geometric series leads to the bound $\|\mathcal{E}\chi_{\Omega'}\|_{2q}\lesssim\rho^{C'}|\Omega|^{1/q'}$, where C' is an admissible constant determined by C; increasing C if necessary, we can make $C'\ge1$. □

As indicated above, the final two steps of our construction are variants of the first two, wherein the roles of the coordinates ζ_1,ζ_2 are reversed. Below, we briefly explain how the argument in Steps 1 and 2 transfers, without rewriting all the details. In short, $\Omega^3_{\alpha,\eta,\rho}$ has constant $\pi_{2,3}$-fiber length by construction and thus may replace Ω, and ρ may replace ε by Lemma 3.5.

Step 4. For each dyadic number $0<\beta\le1$, define

$$\Omega^4_{\alpha,\eta,\rho,\beta}:=\{(\zeta,\sigma)\in\Omega^3_{\alpha,\eta,\rho}:\mathcal{H}^1(\pi_1^{-1}(\zeta_2)\cap\pi_{2,3}(\Omega^3_{\alpha,\eta,\rho}))\sim\beta^A\}.$$

Lemma 3.6. *For every $0 < \beta \leq 1$, $0 < \eta \leq \alpha \leq \varepsilon$, and $0 < \rho \lesssim \eta^{1/5}$, we have* $\mathrm{C}(\Omega^4_{\alpha,\eta,\rho,\beta}, \Omega) \lesssim \beta$.

Proof of Lemma 3.6. Since $\Omega^3_{\alpha,\eta,\rho}$ has constant $\pi_{2,3}$-fiber length, we can imitate the proof of Lemma 3.2 to show that $\beta \gtrsim \mathrm{C}(\Omega^4_{\alpha,\eta,\rho,\beta}, \Omega^3_{\alpha,\eta,\rho}) \geq \mathrm{C}(\Omega^4_{\alpha,\eta,\rho,\beta}, \Omega)$. □

Step 5. For each $0 < \beta \leq 1$, let $S_{\alpha,\eta,\rho,\beta} := \pi_1(\pi_{2,3}(\Omega^4_{\alpha,\eta,\rho,\beta}))$, and let $K_{\alpha,\eta,\rho,\beta}$ be an integer such that $|S_{\alpha,\eta,\rho,\beta}| \sim 2^{-K_{\alpha,\eta,\rho,\beta}}$. Given a dyadic number $0 < \delta \leq \beta$ and a Lebesgue point ζ_2 of $S_{\alpha,\eta,\rho,\beta}$, let $I_{\alpha,\eta,\rho,\beta,\delta}(\zeta_2)$ be the maximal dyadic interval I such that $\zeta_2 \in I$ and

$$\frac{|I \cap S_{\alpha,\eta,\rho,\beta}|}{|I|} \geq \delta^B.$$

As before, we may assume that $S_{\alpha,\eta,\rho,\beta}$ is equal to its set of Lebesgue points. Let

$$T_{\alpha,\eta,\rho,\beta,\delta} := \{\zeta_2 \in S_{\alpha,\eta,\rho,\beta} : |I_{\alpha,\eta,\rho,\beta,\delta}(\zeta_2)| \geq \delta^B 2^{-K_{\alpha,\eta,\rho,\beta}}\}.$$

If $\beta < \rho$, define $S_{\alpha,\eta,\rho,\beta,\beta} := T_{\alpha,\eta,\rho,\beta,\beta}$ and $S_{\alpha,\eta,\rho,\beta,\delta} := T_{\alpha,\eta,\rho,\beta,\delta} \setminus T_{\alpha,\eta,\rho,\beta,2\delta}$ for $\delta < \beta$, and let

$$\Omega^5_{\alpha,\eta,\rho,\beta,\delta} := \Omega^4_{\alpha,\eta,\rho,\beta} \cap \pi_{2,3}^{-1}(\pi_1^{-1}(S_{\alpha,\eta,\rho,\beta,\delta})).$$

If $\rho \leq \beta \leq 1$, define $S_{\alpha,\eta,\rho,\beta,\rho} := T_{\alpha,\eta,\rho,\beta,\rho}$ and $S_{\alpha,\eta,\rho,\beta,\delta} := T_{\alpha,\eta,\rho,\beta,\delta} \setminus T_{\alpha,\eta,\rho,\beta,2\delta}$ for $\delta < \rho$. For $\delta \leq \rho$, let

$$\Omega^5_{\alpha,\eta,\rho,\rho,\delta} := \bigcup_{\rho \leq \beta \leq 1} \tilde{\Omega}^5_{\alpha,\eta,\rho,\beta,\delta},$$

where $\tilde{\Omega}^5_{\alpha,\eta,\rho,\beta,\delta} := \Omega^4_{\alpha,\eta,\rho,\beta} \cap \pi_{2,3}^{-1}(\pi_1^{-1}(S_{\alpha,\eta,\rho,\beta,\delta}))$.

Admittedly, the subscripts have become awkward. However, all we have done is repeated Step 2, replacing Ω^1_α and ε by $\Omega^4_{\alpha,\eta,\rho,\beta}$ and ρ, respectively, and projecting onto the ζ_2-axis instead of the ζ_1-axis. We note that

$$\Omega^3_{\alpha,\eta,\rho} = \bigcup_{0<\beta\leq\rho} \bigcup_{0<\delta\leq\beta} \Omega^5_{\alpha,\eta,\rho,\beta,\delta}.$$

Lemma 3.7. *For every $0 < \eta \leq \alpha \leq \varepsilon$ and $0 < \delta \leq \beta \leq \rho \lesssim \eta^{1/5}$, the set $\Omega^5_{\alpha,\eta,\rho,\beta,\delta}$ is contained in a union of $O(\delta^{-18B-6A-5C-6})$ boxes of the form $\tilde{\tau}$, with $\tau \in \mathcal{T}_{J,K}$, and satisfies $\mathrm{C}(\Omega^5_{\alpha,\eta,\rho,\beta,\delta}, \Omega) \lesssim \delta^{1/2}$.*

Proof of Lemma 3.7. Let $K_{\alpha,\eta,\rho}$ be an integer such that $|\pi_{2,3}(\Omega^3_{\alpha,\eta,\rho})| \sim 2^{-K_{\alpha,\eta,\rho}}$. Imitating the proof of Lemma 3.4, we can show that $\Omega^5_{\alpha,\eta,\rho,\beta,\delta}$ is covered by $O(\delta^{-3B-A-1})$ boxes of the form $\tilde{\tau}$, where $\tau \in \mathcal{T}_{0,K_{\alpha,\eta,\rho}}$. Since $\Omega^3_{\alpha,\eta,\rho}$ has $\pi_{2,3}$-fibers of length $\rho^{5C}\eta^{-3B-A-1-C}2^{-J}$ and volume at most 2^{-J-K}, it follows that $2^{-K_{\alpha,\eta,\rho}} \lesssim \rho^{-5C}2^{-K}$. Thus, $\Omega^5_{\alpha,\eta,\rho,\beta,\delta}$ is covered by $O(\rho^{-5C}\delta^{-3B-A-1}) = O(\delta^{-3B-A-5C-1})$ boxes $\tilde{\tau}$, with $\tau \in \mathcal{T}_{0,K}$. Since $\Omega^5_{\alpha,\eta,\rho,\beta,\delta} \subseteq \Omega^2_{\alpha,\eta}$ and $\eta \gtrsim \delta^5$, Lemma 3.4 now implies that $\Omega^5_{\alpha,\eta,\rho,\beta,\delta}$ is covered by $O(\delta^{-18B-6A-5C-6})$ boxes $\tilde{\tau}$, with $\tau \in \mathcal{T}_{J,K}$.

To obtain the restriction estimate, we can adapt the proof of Lemma 3.4. □

Finally, we are equipped to finish the proof of Proposition 3.1. We have

$$\Omega = \bigcup_{0<\alpha\leq\varepsilon} \bigcup_{0<\eta\leq\alpha} \bigcup_{0<\rho\lesssim\eta^{1/5}} \bigcup_{0<\beta\leq\rho} \bigcup_{0<\delta\leq\beta} \Omega^5_{\alpha,\eta,\rho,\beta,\delta} = \bigcup_{0<\delta\lesssim\varepsilon^{1/5}} \Omega_\delta,$$

where

$$\Omega_\delta := \bigcup_{\delta\le\beta\lesssim\varepsilon^{1/5}} \bigcup_{\beta\le\rho\lesssim\varepsilon^{1/5}} \bigcup_{\rho^5\lesssim\eta\le\varepsilon} \bigcup_{\eta\le\alpha\le\varepsilon} \Omega^5_{\alpha,\eta,\rho,\beta,\delta}.$$

Since for fixed δ there are $O((\log\delta^{-1})^4)$ sets $\Omega^5_{\alpha,\eta,\rho,\beta,\delta}$, properties (i) and (ii) in the proposition follow from Lemma 3.7.

□

4. Extensions of near unions of boxes

For each K, let $J(K)$ be an integer such that $|\Omega(K)| \sim 2^{-J(K)-K}$. For each dyadic number ε, let $\mathcal{K}(\varepsilon)$ denote the collection of all integers $K\ge 0$ for which $\varepsilon = \mathrm{C}(\Omega(K))$. For each $K\in\mathcal{K}(\varepsilon)$, Proposition 3.1 gives a decomposition $\Omega(K) = \bigcup_{0<\delta\lesssim\varepsilon^{1/5}}\Omega(K)_\delta$ such that for each δ, we have $\Omega(K)_\delta \subseteq \bigcup_{\tau\in\mathcal{T}(K)_\delta}\tilde\tau$ for some $\mathcal{T}(K)_\delta \subseteq \mathcal{T}_{J(K),K}$ with $\#\mathcal{T}(K)_\delta \lesssim \delta^{-C_0}$.

Lemma 4.1. *For every $0<\delta\lesssim\varepsilon^{1/5}$, we have*

$$\Big\|\sum_{K\in\mathcal{K}(\varepsilon)}\mathcal{E}\chi_{\Omega(K)_\delta}\Big\|_{2q}^{2q} \lesssim (\log\delta^{-1})^{2q}\sum_{K\in\mathcal{K}(\varepsilon)}\|\mathcal{E}\chi_{\Omega(K)_\delta}\|_{2q}^{2q} + \delta|\Omega|^{\frac{2q}{q'}}.$$

Proof of Lemma 4.1. Let A be an admissible constant to be chosen later, and divide $\mathcal{K}(\varepsilon)$ into $O(\log\delta^{-1})$ subsets $\mathcal{K}$ such that each is $A\log\delta^{-1}$-separated. It suffices to prove that

$$\Big\|\sum_{K\in\mathcal{K}}\mathcal{E}\chi_{\Omega(K)_\delta}\Big\|_{2q}^{2q} \lesssim \sum_{K\in\mathcal{K}}\|\mathcal{E}\chi_{\Omega(K)_\delta}\|_{2q}^{2q} + \delta^2|\Omega|^{\frac{2q}{q'}}$$

for each $\mathcal{K}$. Since $q<2$, we have

$$\begin{aligned}\Big\|\sum_{K\in\mathcal{K}}\mathcal{E}\chi_{\Omega(K)_\delta}\Big\|_{2q}^{2q} &= \int\Big|\sum_{\boldsymbol{K}\in\mathcal{K}^4}\prod_{i=1}^4\mathcal{E}\chi_{\Omega(K_i)_\delta}\Big|^{\frac{q}{2}}\\ &\lesssim \sum_{K\in\mathcal{K}}\|\mathcal{E}\chi_{\Omega(K)_\delta}\|_{2q}^{2q} + \sum_{\boldsymbol{K}\in\mathcal{K}^4\setminus D(\mathcal{K}^4)}\Big\|\prod_{i=1}^4\mathcal{E}\chi_{\Omega(K_i)_\delta}\Big\|_{\frac{q}{2}}^{\frac{q}{2}}, \end{aligned} \tag{4.1}$$

where $D(\mathcal{K}^4) := \{\boldsymbol{K}\in\mathcal{K}^4 : K_1=\cdots=K_4\}$. To control the latter sum, we have the following lemma.

Lemma 4.2. *For all $K,K'\in\mathcal{K}$, we have*

$$\|\mathcal{E}\chi_{\Omega(K)_\delta}\mathcal{E}\chi_{\Omega(K')_\delta}\|_q \lesssim 2^{-c_0|K-K'|}\max\{|\Omega(K)|,|\Omega(K')|\}^{\frac{2}{q'}} \tag{4.2}$$

for some admissible constant $c_0>0$.

Proof of Lemma 4.2. By the Cauchy–Schwarz inequality and Proposition 2.2,

$$\|\mathcal{E}\chi_{\Omega(K)_\delta}\mathcal{E}\chi_{\Omega(K')_\delta}\|_q \lesssim |\Omega(K)|^{\frac{1}{q'}}|\Omega(K')|^{\frac{1}{q'}}.$$

For $J:=J(K)$ and $J':=J(K')$, we have

$$|\Omega(K)|^{\frac{1}{q'}}|\Omega(K')|^{\frac{1}{q'}} \lesssim 2^{-\frac{|K-K'|}{q'}}\max\{|\Omega(K)|,|\Omega(K')|\}^{\frac{2}{q'}}$$

whenever either (i) $K=K'$, (ii) $J=J'$, (iii) $J<J'$ and $K<K'$, or (iv) $J>J'$ and $K>K'$; in these cases, (4.2) follows immediately.

Thus, by symmetry, it suffices to prove (4.2) for $K < K'$ and $J > J'$. By the bound $\#(\mathcal{T}(K)_\delta \times \mathcal{T}(K')_\delta) \lesssim \delta^{-2C_0}$ and the separation condition on $\mathcal{K}$ (with A sufficiently large), it suffices to prove that

$$\|\mathcal{E}\chi_{\Omega(K)_\delta\cap\tilde\tau}\mathcal{E}\chi_{\Omega(K')_\delta\cap\tilde\kappa}\|_q \lesssim 2^{-c|K-K'|}|\Omega(K)|^{\frac{1}{q'}}|\Omega(K')|^{\frac{1}{q'}} \tag{4.3}$$

for all $\tau \in \mathcal{T}(K)_\delta$, $\kappa \in \mathcal{T}(K')_\delta$, and some admissible constant c.

Fix two such tiles τ, κ, and note that τ must be taller than κ and κ wider than τ. By translation, we may assume that the ζ_2- and ζ_1-axes intersect the centers of τ and κ, respectively. Define

$$\tau_k := \begin{cases} \tau \cap \{\zeta : |\zeta_2| \sim 2^{-k}\}, & k < K', \\ \tau \cap \{\zeta : |\zeta_2| \lesssim 2^{-K'}\}, & k = K' \end{cases} \quad \text{and} \quad \kappa_j := \begin{cases} \kappa \cap \{\zeta : |\zeta_1| \sim 2^{-j}\}, & j < J, \\ \kappa \cap \{\zeta : |\zeta_1| \lesssim 2^{-J}\}, & j = J \end{cases},$$

so that

$$\tau = \bigcup_{k=0}^{K'} \tau_k \qquad \text{and} \qquad \kappa = \bigcup_{j=0}^{J} \kappa_j.$$

By the two-parameter Littlewood–Paley square function estimate and fact that $q < 2$, we have

$$\begin{aligned}\|\mathcal{E}\chi_{\Omega(K)_\delta\cap\tilde\tau}\mathcal{E}\chi_{\Omega(K')_\delta\cap\tilde\kappa}\|_q^q &\lesssim \int \Big(\sum_{k=0}^{K'}\sum_{j=0}^{J}|\mathcal{E}\chi_{\Omega(K)_\delta\cap\tilde\tau_k}\mathcal{E}\chi_{\Omega(K')_\delta\cap\tilde\kappa_j}|^2\Big)^{\frac{q}{2}} \\ &\lesssim \sum_{k=0}^{K'}\sum_{j=0}^{J}\|\mathcal{E}\chi_{\Omega(K)_\delta\cap\tilde\tau_k}\mathcal{E}\chi_{\Omega(K')_\delta\cap\tilde\kappa_j}\|_q^q, \end{aligned}\tag{4.4}$$

where $\tilde\tau_k := \tau_k \times [1,2]$ and $\tilde\kappa_j := \kappa_j \times [1,2]$. We first sum the terms with $k = K'$. By the Cauchy–Schwarz inequality and Proposition 2.2, we have

$$\sum_{j=0}^{J}\|\mathcal{E}\chi_{\Omega(K)_\delta\cap\tilde\tau_{K'}}\mathcal{E}\chi_{\Omega(K')_\delta\cap\tilde\kappa_j}\|_q^q \lesssim \sum_{j=0}^{J}|\tilde\tau_{K'}|^{\frac{q}{q'}}|\tilde\kappa_j|^{\frac{q}{q'}}.$$

Since κ has width $2^{-J'}$, there are at most two nonempty κ_j with $j \le J'$. This fact and the bound

$$|\tilde\kappa_j| \le \min\{2^{-(j-J')}, 1\}|\tilde\kappa| \tag{4.5}$$

imply that $\sum_{j=0}^{J}|\tilde\kappa_j|^{\frac{q}{q'}} \lesssim |\kappa|^{\frac{q}{q'}}$. Since $|\tilde\tau_{K'}| \lesssim 2^{-(K'-K)}|\tilde\tau|$, $|\tilde\tau| \sim |\Omega(K)|$, and $|\tilde\kappa| \sim |\Omega(K')|$, we altogether have

$$\sum_{j=0}^{J}\|\mathcal{E}\chi_{\Omega(K)_\delta\cap\tilde\tau_{K'}}\mathcal{E}\chi_{\Omega(K')_\delta\cap\tilde\kappa_j}\|_q^q \lesssim 2^{-(K'-K)\frac{q}{q'}}|\Omega(K)|^{\frac{q}{q'}}|\Omega(K')|^{\frac{q}{q'}}.$$

A similar argument shows that

$$\begin{aligned}\sum_{k=0}^{K'}\|\mathcal{E}\chi_{\Omega(K)_\delta\cap\tilde\tau_k}\chi_{\Omega(K')_\delta\cap\tilde\kappa_J}\|_q^q &\lesssim 2^{-(J-J')\frac{q}{q'}}|\Omega(K)|^{\frac{q}{q'}}|\Omega(K')|^{\frac{q}{q'}} \\ &\sim 2^{-(K'-K)\frac{q}{q'}}|\Omega(K)|^{\frac{2q}{q'}}.\end{aligned}$$

We now consider the terms with $k < K'$ and $j < J$. In this case, τ_k is a subset of four tiles in $\mathcal{T}_{J,\max\{K,k\}}$ and κ_j is a subset of four tiles in $\mathcal{T}_{\max\{J',j\},K'}$.

Moreover, these tiles are separated by a distance of 2^{-k} and 2^{-j} in the vertical and horizontal directions, respectively. Thus, by Theorem 1.2 (rescaled, as in the proof of Proposition 2.2),

$$\|\mathcal{E}\chi_{\Omega(K)_\delta\cap\tilde{\tau}_k}\mathcal{E}\chi_{\Omega(K')_\delta\cap\tilde{\kappa}_j}\|_q \lesssim 2^{-(j+k)(1-\frac{2}{q})}|\Omega(K)\cap\tilde{\tau}_k|^{\frac12}|\Omega(K')\cap\tilde{\kappa}_j|^{\frac12}.$$

Using (4.5) and the analogous bound for $|\tilde{\tau}_k|$, we now get

$$\sum_{k=0}^{K'-1}\sum_{j=0}^{J-1}\|\mathcal{E}\chi_{\Omega(K)_\delta\cap\tilde{\tau}_k}\mathcal{E}\chi_{\Omega(K')_\delta\cap\tilde{\kappa}_j}\|_q^q \lesssim 2^{-(J'-K)(q-2)}|\tilde{\tau}|^{\frac{q}{2}}|\tilde{\kappa}|^{\frac{q}{2}}$$
$$\sim 2^{(J'-J+K-K')(1-\frac{q}{2})}|\Omega(K)|^{\frac{q}{q'}}|\Omega(K')|^{\frac{q}{q'}}.$$

By the relations $K<K'$, $J>J'$ and (4.4), we have now proved (4.3). □

Returning to the proof of Lemma 4.1, we consider the second sum in (4.1). Given $\boldsymbol{K}\in\mathcal{K}^4\setminus D(\mathcal{K}^4)$, let $p(\boldsymbol{K})=(p_i(\boldsymbol{K}))_{i=1}^4$ be a permutation of $\boldsymbol{K}$ such that $|\Omega(p_1(\boldsymbol{K}))|$ is maximal among $|\Omega(K_i)|$, $1\le i\le 4$, and such that $|K_i-K_j|\le 2|p_1(\boldsymbol{K})-p_2(\boldsymbol{K})|$ for all $1\le i,j\le 4$. Then by the Cauchy–Schwarz inequality, Lemma 4.2, the separation condition on $\mathcal{K}$, the fact that $q'<2q$, and choosing A sufficiently large, we get

$$\sum_{\boldsymbol{K}\in\mathcal{K}^4\setminus D(\mathcal{K}^4)}\Big\|\prod_{i=1}^4\mathcal{E}\chi_{\Omega(K_i)_\delta}\Big\|_{\frac{q}{2}}^{\frac{q}{2}} \lesssim \sum_{\substack{\boldsymbol{K}\in\mathcal{K}^4\setminus D(\mathcal{K}^4)\\ \boldsymbol{K}=p(\boldsymbol{K})}} 2^{-c_0|p_1(\boldsymbol{K})-p_2(\boldsymbol{K})|}|\Omega(p_1(\boldsymbol{K}))|^{\frac{2q}{q'}}$$
$$\lesssim \sum_{K_1\in\mathcal{K}}\sum_{K_2\in\mathcal{K}}|K_1-K_2|^2 2^{-c_0|K_1-K_2|}|\Omega(K_1)|^{\frac{2q}{q'}}$$
$$\lesssim \delta^{\frac{c_0A}{2}}\sum_{K_1\in\mathcal{K}}|\Omega(K_1)|^{\frac{2q}{q'}}$$
$$\lesssim \delta^2|\Omega|^{\frac{2q}{q'}}.$$

□

5. Proof of Theorem 1.1

In this final section, we prove our main result. We recall our setup: For $\Omega\subseteq[-1,1]^2\times[1,2]$ a measurable set, we have divided Ω into sets $\Omega(K)$ of constant fiber length 2^{-K}, partitioned the indices K into sets $\mathcal{K}(\varepsilon)$ according to the value of $\varepsilon := \mathrm{C}(\Omega(K))$, and decomposed each $\Omega(K)$ into near unions of boxes $\Omega(K)_\delta$ for $0<\delta\lesssim\varepsilon^{1/5}$. Thus,

$$\Omega=\bigcup_{0<\varepsilon\lesssim 1}\ \bigcup_{0<\delta\lesssim\varepsilon^{1/5}}\ \bigcup_{K\in\mathcal{K}(\varepsilon)}\Omega(K)_\delta.$$

(Actually, there may be K such that $\mathrm{C}(\Omega(K))=0$; however, those terms contribute nothing to the left-hand side below.)

Proof of Theorem 1.1. By the triangle inequality, Lemma 4.1, Proposition 3.1, and the fact that $q'<2q$, we have

$$\|\mathcal{E}\chi_\Omega\|_{2q}\le\sum_{0<\varepsilon\lesssim 1}\ \sum_{0<\delta\lesssim\varepsilon^{1/5}}\Big\|\sum_{K\in\mathcal{K}(\varepsilon)}\mathcal{E}\chi_{\Omega(K)_\delta}\Big\|_{2q}$$

$$\lesssim \sum_{0<\varepsilon\lesssim 1} \sum_{0<\delta\lesssim\varepsilon^{1/5}} \left((\log\delta^{-1})^{2q} \sum_{K\in\mathcal{K}(\varepsilon)} \|\mathcal{E}\chi_{\Omega(K)_\delta}\|_{2q}^{2q} + \delta|\Omega|^{\frac{2q}{q'}} \right)^{\frac{1}{2q}}$$

$$\lesssim \left[\sum_{0<\varepsilon\lesssim 1} \sum_{0<\delta\lesssim\varepsilon^{1/5}} (\log\delta^{-1})\delta^{\frac{1}{3}} \left(\sum_{K\in\mathcal{K}(\varepsilon)} |\Omega(K)|^{\frac{2q}{q'}} \right)^{\frac{1}{2q}} \right] + |\Omega|^{\frac{1}{q'}}$$

$$\lesssim \left[\sum_{0<\varepsilon\lesssim 1} \sum_{0<\delta\lesssim\varepsilon^{1/5}} (\log\delta^{-1})\delta^{\frac{1}{3}} |\Omega|^{\frac{1}{q'}} \right] + |\Omega|^{\frac{1}{q'}}$$

$$\lesssim |\Omega|^{\frac{1}{q'}},$$

proving (1.1). □

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