

Graph-Theoretic Analyses of MIMO Channels in Diffusive Networks

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Abstract—In this work, we characterize the finite-zero and infinite-zero structure of a multi-input multi-output channel in a standard model for network synchronization. To do so, we first develop an algebraic analysis of the zeros based on a input-to-output transformation known as the special coordinate basis. This decomposition then allows us to develop topological results on the zeros, i.e. characterizations in terms of the network graph and the input/output locations relative to the graph. Specifically, our results show how the relative locations and interactions among multiple input-output pairs in a network influence the locations of the finite and invariant zeros. As a whole, the study contributes to the analysis of dynamical networks from an input-output perspective, rather than only in terms of internal or emergent behaviors.

I. INTRODUCTION

Researchers have studied the dynamics and control of complex networks, for applications ranging from distributed decision-making or consensus in autonomous-agent systems to wide-area management of large-scale infrastructures [1]–[4]. Although the analysis and control of large scale networks is based the traditional systems and control theory, it also requires methods to combat the inherent complexity and high dimensionality of the dynamics [5], [6]. One common theme in network analysis and control has been to develop simple insights in terms of the network’s topology or graph, which can be used in lieu of numerical procedures for analysis/design. Indeed, it has been shown that the network’s graph topology plays a central role in the intrinsic dynamics of complex networks, and also modulates controller design, closed-loop performance, and state/parameter estimation. These graph-theoretic results are useful for a number of engineering tasks, including new technology deployment (e.g. sensor/actuator placement), model development and identification, and threat or risk assessment [7]–[9].

In the literature, the graph-theoretic analysis of network dynamics has largely focused on internal modal behaviors (including stability and stabilization notions) [10]–[13], input-to-state properties such as controllability [10], [14], and state-to-output properties such as observability [10], [15], [16]. However, input-output properties of a control channel are also of importance in control design as well as state estimation, and hence graph-theoretic analyses of these properties are also needed. Among various input-output properties, the invariant-zero structure is critical for controller design, because it imposes fundamental limits on control and estimation in linear systems [17]–[20]. Recently, single-input

single-output channels (SISO) in canonical network models have been characterized in terms of the network’s graph and the measurement/actuation locations [21]–[32]. These analyses demonstrate that the zero structure, and particularly the presence or absence of non-minimum-phase zeros, exhibit a sophisticated dependence on the paths in the network graph between the input and output. Although these analyses are a promising starting point for an input-output analysis of networks, control channels in many network analyses are multi-input multi-output (MIMO), i.e. they involve the coordination of distributed sensing and actuation resources. Input-output analyses of MIMO network dynamics from a graph-theoretic perspective are limited: some graph-theoretic results on relative degree and generic number have been obtained [33]–[35], but control-relevant properties such as zeros have not been studied from a graph-theoretic perspective. In this work, we focus on characterizing the invariant-zero structure for a multi-input multi-output channel in a canonical dynamical network model.

In this work, we focus on a standard model for network synchronization or consensus or diffusion, wherein inputs are provided at a subset of nodes, and outputs are taken at a (possibly different) subset. Our goal is to characterize the zero structure of this multi-input multi-output channel. Specifically, we exploit an algebraic analysis of the zeros based on the special coordinate basis, as a means to develop graph-theoretic results of the zeros. Specifically, the invariant-zero structure of the linear network dynamics (including the presence or absence of nonminimum-phase dynamics) is characterized in the terms of the network graph and input/output locations. The results can be used both for analysis and design purposes. From a design perspective, the developed results are a starting point for sensor and actuator placement to shape the zero structure, and hence facilitate control and estimation. In a broad sense, our results indicate that the topology of the network graph, and certain of its induced sub-graphs, specify the zero locations and hence modulate control and estimation.

II. MODELING AND PROBLEM FORMULATION

We consider a standard linear diffusive network model defined on a digraph, where the model is augmented to represent a multi-input multi-output (MIMO) channel of interest. Our goal is to analyze the input-output properties of the MIMO channel. In particular, we are interested in characterizing the finite-zero and infinite-zero structures of the MIMO channel.

Formally, a network with n components or nodes, specified by the set $\mathcal{N} = \{1, 2, \dots, n\}$, is considered. Each node j

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is associated with a scalar state x_j that evolves in continuous time. The nodes' states are nominally governed by a linear dynamical model with diffusive state matrix A . The diffusive network model is defined by $\dot{\mathbf{x}} = A\mathbf{x}$, where $\mathbf{x} = [x_1 \dots x_n]^T$ is the **full state** of the network. The state matrix A , which need not be symmetric, is assumed to have a diffusive form (i.e., to be the negative of a M-matrix or an essentially-nonnegative matrix). That is, the off-diagonal entries of A are assumed to be nonnegative, while the diagonal entries are negative and satisfy $A_{l,l} \leq -\sum_{j=1, j \neq l}^n A_{l,j}$. Since the state matrix A encodes the network's topology, it is referred to as the *graph matrix*. The diffusive-network model encompasses standard models for synchronization/consensus, diffusion, and spread in dynamical networks (e.g., [36]–[38]), in which nodes have scalar states.

The goal of this paper is to characterize the input-output properties of a channel of interest. A multi-input multi-output channel is considered, which is defined by m inputs $u_1, \dots, u_m$ and m outputs $y_1, \dots, y_m$ where: 1) each input u_i is applied at one network component $\hat{u}_i \in \mathcal{N}$, and each network component is actuated directly by at most one input, i.e. $\hat{u}_i \neq \hat{u}_j$ for $i \neq j$; 2) each output y_i measures the state of one component $\hat{y}_i \in \mathcal{N}$, and each node's state is measured by at most one output, i.e. $\hat{y}_i \neq \hat{y}_j$ for $i \neq j$. The full model of the system, with the MIMO channel of interest included, is thus given by:

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} &= C\mathbf{x}\end{aligned}\quad (1)$$

where $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$, $C = \begin{bmatrix} \mathbf{e}_{\hat{y}_1}^T \\ \vdots \\ \mathbf{e}_{\hat{y}_m}^T \end{bmatrix}$, $B = [\mathbf{e}_{\hat{u}_1}, \dots, \mathbf{e}_{\hat{u}_m}]$, and $\mathbf{e}_i$ is 0–1 indicator vector with length n and i th entry equal to 1.

Since topological results are sought, it is convenient to associate a graph with the network dynamics. Specifically, a weighted digraph $\mathcal{G}$ with n vertices is defined, where each vertex $l = 1, 2, \dots, n$ in the graph corresponds to the network node l . For simplicity, we use the same label for a node and its corresponding vertex in graph $\mathcal{G}$. Formally, an arc (directed edge) is drawn from vertex l to vertex j in the graph (l, j distinct) if and only if $A_{j,l} \neq 0$, and is assigned a weight of $A_{j,l}$. The vertices corresponding to the input and output network nodes are referred to as the input and output vertices. The state matrix A can be viewed as a grounded Laplacian matrix associated with the directed graph. In this paper, the notation d_{ab} is used for the distance from vertex a to vertex b in the digraph $\mathcal{G}$ (i.e., the minimum number of edges among directed paths from a to b).

We focus on a restricted set of models, where the number of inputs and outputs are equal, and each input and output pair is in some sense local or regional. General models are very complicated to study, and the results developed here do not extend to the general case. Nevertheless, the restricted model class considered in this study is interesting because in many application domains, controls are initially developed

with regional goals in mind; however, as control interactions become more pronounced and network-wide management becomes critical, there is a need to consider the merged control channel. For instance, in the electric power grid, it is natural that substations or regional control authorities may have traditionally used limited measurement and actuation to enact a control. However, given more stress on the grid and also new technological advances, there is a motivation to assimilate these measurement and actuation capabilities from different substations to design and implement a wide-area MIMO controller to improve the performance of the network. To formalize this notion, we impose some constraints on the locations of the inputs and outputs relative to the network graph. Specifically, in this paper, considering the input-output model (1) with corresponding graph $\mathcal{G}$, we assume that there exist a set of m independent special input-output pairs, as specified by the following two definitions:

Definition 1: In graph $\mathcal{G}$, we consider a pair consisting of an input vertex $\hat{u}_i$ and an output vertex $\hat{y}_j$ as a *special input-output pair* $(\hat{u}_i, \hat{y}_j)$, if the distance from the input vertex $\hat{u}_i$ to the output vertex $\hat{y}_j$ is strictly less than the distance from the input vertex $\hat{u}_i$ to any other output vertex. Also, we consider a *special input-output path* $(\hat{u}_i, \hat{y}_j)$ as a directed path with shortest distance for the special input-output pair $(\hat{u}_i, \hat{y}_j)$.

Definition 2: In graph $\mathcal{G}$, we call a set of special input-output pairs as independent, if these pairs have no common input or output vertices.

Remark: It can easily be shown that special input-output paths corresponding to a set of independent special input-output pairs do not have any vertices or edges in common, i.e. the paths are disjoint.

Next, without loss of generality, we assume a specific ordering for inputs, outputs, network nodes, and graph vertices to simplify the analysis of the network model.

Special Ordering: Consider input-output model (1) with corresponding graph $\mathcal{G}$, and assume that there exist a set of m independent special input-output pairs. Without loss of generality by reordering the inputs and outputs, we consider the set of m independent special input-output pairs as $\mathcal{P} = \{(\hat{u}_1, \hat{y}_1), (\hat{u}_2, \hat{y}_2), \dots, (\hat{u}_m, \hat{y}_m)\}$. Additionally, without loss of generality, we assume a particular ordering of the original state vector and the corresponding network nodes and graph vertices as follows:

- 1) Label vertices not included in any special input-output paths as $\{1, 2, \dots, n_a\}$ where $n_a = n - n_d$ where $n_d = \sum_{i=1}^m (d_{\hat{u}_i \hat{y}_i} + 1)$.
- 2) Label vertices in the special input-output path $(\hat{u}_1, \hat{y}_1)$ as $\{n_a + 1, n_a + 2, \dots, n_a + d_{\hat{u}_1 \hat{y}_1} + 1\}$ starting from output vertex $\hat{y}_1$ as $n_a + 1$ and incrementally label others by increasing their distance from output vertex $\hat{y}_1$, up to label $n_a + q_1$ for input vertex $\hat{u}_1$ where $q_1 = d_{\hat{u}_1 \hat{y}_1} + 1$.
- 3) Similarly for $i = 2, \dots, m$, label vertices on the special input-output path $(\hat{u}_i, \hat{y}_i)$ as $\{n_a + \sum_{j=1}^{i-1} (q_j) + 1, n_a + \sum_{j=1}^{i-1} (q_j) + 2, \dots, n_a + \sum_{j=1}^{i-1} (q_j) + d_{\hat{u}_i \hat{y}_i} + 1\}$ starting from output vertex $\hat{y}_i$ as $n_a + \sum_{j=1}^{i-1} (q_j) + 1$ and

incrementally label others by increasing their distance from output vertex $\hat{y}_i$, up to label $n_a + \sum_{j=1}^{i-1} (q_j) + q_i$ for input vertex $\hat{u}_i$ where $q_i = d_{\hat{u}_i, \hat{y}_i} + 1$.

The focus of our study is to characterize the finite zeros and infinite zeros of the MIMO channel with input vector $\mathbf{u}$ and output vector $\mathbf{y}$. This channel of interest may be a control channel in the network, or may capture other input-output behaviors (e.g., a disturbance response that is a concern to network operators). We notice, while the modes of the system are internal properties of the network, the zeros are particular to the channel of interest. Crucially, the finite zeros are invariants to feedback on the same channel, and hence they place limits on control performance and constrain channel response characteristics [17]. The presence of nonminimum-phase zeros (i.e., zeros in the right half plane) is of special concern, since these zeros place hard limits on control performance (e.g., reference tracking or disturbance rejection error). Thus, we seek to characterize the finite-zero and infinite-zero structure of the MIMO channel of interest. Our primary effort here is to develop a topological understanding of the zeros, by analyzing them in terms of the network graph $\mathcal{G}$ and the input/output locations relative to the graph.

III. RESULTS

In this section, we characterize the infinite- and finite-zero structures of the introduced MIMO network model. We first develop algebraic characterizations of the zeros, using a transformation known as the special coordinate basis. These algebraic results are a starting point for characterizing the zero structure in terms of the network's graph topology; these topological results are the main contribution of the work.

A. Algebraic and Structural Characterizations

We obtain algebraic characterizations of finite-invariant-zeros and infinite-zero structures of the introduced MIMO network model, and also use these to gain basic structural insights into the network model's dynamics. To develop this analysis, we decompose the model dynamics using the *special coordinate basis* (SCB) transformation, in the case where the model has m independent special input-output pairs. The procedure of developing the special coordinate basis decomposition is explained in detail in [39]. We do not present most of the details here, but maintain the notations used in [20], [39] and refer the readers to these studies for the details. The key step in developing the structural decomposition is to find a nonsingular state transformation Γ_s , a nonsingular output transformation Γ_o , and a nonsingular input transformation Γ_i which together expose the finite- and infinite-zero structure. The iterative procedure for finding the transformations is quite complicated in the general case, see [20], [39], [40]. Here, based on the special ordering assumption (without loss of generality), it turns out that the transformation is achieved using $\Gamma_o = I$ and $\Gamma_i = I$. The state transformation Γ_s needed to place the dynamics in the special coordinate basis, and the decomposed dynamics in this new basis, is given in the following theorem. Because

the result directly uses the procedure in [39], a detailed proof is not given, however a brief sketch of the proof which highlights some subtleties is included.

Theorem 1: Consider the MIMO LTI system defined in (1), and assume it has m independent special input-output pairs as defined in Definitions 1 and 2. Also assume that the special label ordering for the network nodes, inputs, and outputs has been used. There exists a non-singular state transformation matrix Γ_s which decomposes the state space into two sub-spaces $\bar{\mathbf{x}} = \begin{bmatrix} \bar{\mathbf{x}}_0 \\ \bar{\mathbf{x}}_d \end{bmatrix}$ corresponding to the finite- and infinite-zero structure of the system. The state transformation matrix Γ_s and the system in the transformed coordinates $\bar{\mathbf{x}} = \Gamma_s \mathbf{x}$ is described by the following set of equations:

$$\begin{aligned} \dot{\bar{\mathbf{x}}} &= \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{B}}\bar{\mathbf{u}} \\ \mathbf{y} &= C\bar{\mathbf{x}} \end{aligned} \quad (2)$$

where:

$$\bar{\mathbf{x}} = \begin{bmatrix} \bar{\mathbf{x}}_0 \\ \bar{\mathbf{x}}_d \end{bmatrix}; \bar{\mathbf{x}}_0 \in \mathbb{R}^{n_a}; \bar{\mathbf{x}}_d \in \mathbb{R}^{n_d} \quad (3)$$

$$\bar{\mathbf{u}} = \begin{bmatrix} \beta_1 u_1 \\ \vdots \\ \beta_m u_m \end{bmatrix}; B\bar{\mathbf{u}} = \Gamma_s^{-1} B \mathbf{u} \quad (4)$$

$$\bar{\mathbf{A}} = \begin{bmatrix} \bar{A}_{aa} & \bar{A}_{ad} \\ \bar{A}_{da} & \bar{A}_{dd} \end{bmatrix} = \Gamma_s^{-1} A \Gamma_s; \bar{A}_{aa} \in \mathbb{R}^{n_a \times n_a}; \bar{A}_{dd} \in \mathbb{R}^{n_d \times n_d} \quad (5)$$

$$\bar{\mathbf{A}} = \begin{bmatrix} \bar{A}_{na} & \bar{A}_{nd} \\ \bar{A}_{da} & \bar{A}_{dd} \end{bmatrix}; \bar{A}_{na} \in \mathbb{R}^{n_a \times n_a}; \bar{A}_{nd} \in \mathbb{R}^{n_d \times n_d} \quad (6)$$

$$\Gamma_s = \begin{bmatrix} Z_0 \\ Z_d \end{bmatrix} \quad (7)$$

$$Z_0 = \begin{bmatrix} \mathbf{I}_{n_a} & \mathbf{0}_{n_a \times n_d} \end{bmatrix} \quad (8)$$

$$Z_d = \begin{bmatrix} Z_{q_1} \\ Z_{q_2} \\ \vdots \\ Z_{q_m} \end{bmatrix} \quad (9)$$

$$Z_{q_i} = \begin{bmatrix} \mathbf{e}_{\hat{y}_i}^T \\ [A]_{\hat{y}_i, :} \\ [A^2]_{\hat{y}_i, :} \\ \vdots \\ [A^{q_i-1}]_{\hat{y}_i, :} \end{bmatrix} \quad (10)$$

Sketch of Proof for Theorem 1: This theorem follows from the iterative procedure to develop the special coordinate basis transformation described in [39]. Crucial to the transformation, the existence of m independent special input-output pairs guarantees that the submatrices Z_{q_i} for $i \in \{1, \dots, m\}$ are full row rank, and also the matrices Z_d and Γ_s are full rank, hence a state transformation matrix Γ_s that is non-singular is found which yields the desired decomposition of the dynamics. ■

Remark: From the properties of the special coordinate basis, the finite invariant zeros of the MIMO channel of interest are the eigenvalues of zero-dynamics state matrix

$\bar{A}_{aa}$. By simplifying Equation (5), this matrix can be written as:

$$\bar{A}_{aa} = Z_0 A \Gamma_s^{-1} Z_0^T = A_{n_a} - A_{n_{ad}} Z_{n_d}^{-1} Z_{n_{da}} \quad (11)$$

where $Z_{n_{da}}$ and Z_{n_d} are submatrices of $Z_d = \begin{bmatrix} Z_{n_{da}} & Z_{n_d} \end{bmatrix}$ for $Z_{n_{da}} \in \mathbb{R}^{n_d \times n_a}$ and $Z_{n_d} \in \mathbb{R}^{n_d \times n_d}$, and A_{n_a} is a submatrix of matrix A .

In Theorem 1, we developed the special coordinated basis decomposition for the system, and characterized the finite-invariant-zero structure of the system from an algebraic standpoint. In the following two theorems, by using the properties of special coordinate basis representation, the invertibility structure as well as the infinite-zero structure of the system is clarified. Before presenting the results, let us review the definition of invertibility given in [20], in the context of the model considered here. Consider the system (1), and let $\mathbf{u}_1$ and $\mathbf{u}_2$ be any inputs to the system, and consider $\mathbf{y}_1$ and $\mathbf{y}_2$ as the corresponding outputs (for the same initial conditions). The system is said to be left invertible, if $\mathbf{y}_1(t) = \mathbf{y}_2(t)$ for all $t \geq 0$ implies that $\mathbf{u}_1(t) = \mathbf{u}_2(t)$ for all $t \geq 0$. The system is said to be right invertible if, for any $\mathbf{y}_{ref}(t)$ defined on $[0, \infty)$, a $\mathbf{u}(t)$ and a choice of $\mathbf{x}(0)$ exist such that $\mathbf{y}(t) = \mathbf{y}_{ref}(t)$ for all $t \in [0, \infty)$. The system is said to be invertible if the system is both left and right invertible. The special coordinate basis representation provides a tool for examining invertibility as discussed in following theorem; again, the proof of the result is only sketched, as it follows from the standard methodology for the special coordinate basis.

Theorem 2: Consider the MIMO LTI system defined in (1). Assume that the network model has m independent special input-output pairs, as indicated in Definitions 1 and 2. Also assume that the special label ordering for the network nodes, inputs, and outputs has been used. The described system is invertible (i.e. both left and right invertible).

Sketch of Proof for Theorem 2: The transformed dynamics given in Theorem 1 indicates that the model does not contain either a non-left-invertible subspace or a non-right-invertible subspace. In particular, in the notation of [40], the special coordinate basis representation does not contain either a set of states x_b which correspond to a non-right-invertible dynamics, nor a set of states x_c which correspond to a non-right-invertible dynamics. Thus, it is both left and right invertible, and hence invertible. ■

Theorem 2 shows that a sufficient condition for system (1) to be invertible is for the network to have m independent special input-output pairs. This theorem thus connects the invertibility properties of the network to the location of inputs and outputs in the network graph, and hence implicitly provides a graph-theoretic characterization of invertibility. The result shows how a MIMO channel in the network can be designed to be invertible, or extra inputs or outputs (actuators or sensors) can be added to an existing channel to make it invertible.

The next theorem relates the length of the special input-output paths to the infinite-zero structure of the MIMO channel of interest. Again, a sketch of the proof is given.

Theorem 3: Consider the MIMO LTI system defined in (1). Assume that the model has m independent special input-output pairs, as indicated in Definitions 1 and 2. Also, assume that the special label ordering for the network nodes, inputs, and outputs has been used. The system has m infinite zeros of order equal to the numbers of vertices on each of the m special input-output paths, i.e. $q_1, q_2, \dots, q_m$.

Proof of Theorem 3: According to [39], [40], in the special coordinate basis representation of the system, the number of integrators in the integrator chain from an input u_d to its corresponding output y_d determines the infinite zero structure. From Theorem 1, it is immediate that q_k is the number of integrators in the chain between input u_k and output y_k . ■

Theorem 3 characterizes the infinite-zero structure – i.e. number of infinite zeros and their corresponding orders – in terms of the graph topology, and specifically the distances between the input and output vertices in the graph. This result can be used to design input/output locations in the network graph to such that the MIMO channel has a specific infinite-zero structure, as may be required for the control design process.

B. Graph-Theoretic Characterizations

In the previous section, using the SCB decomposition, algebraic and structural characterizations of the network model's invariant-zero dynamics was given. These characterizations also enable a simple graph-theoretic analysis of the infinite-zero structure. The purpose of this section is to develop a graph-theoretic analysis of the finite zeros of the network model, including whether the model is minimum-phase or non-minimum phase; these results are a primary contribution of the work.

Broadly, the analysis of finite zeros is undertaken as follows. The analysis is based on Equation (11), which indicates that finite invariant zeros of the channel are equal to the eigenvalues of a certain matrix $\bar{A}_{aa}$. This matrix $\bar{A}_{aa}$ is equal to a matrix A_{n_a} which is a submatrix of the state matrix A , plus some perturbation (equal to $-A_{n_{ad}} Z_{n_d}^{-1} Z_{n_{da}}$). The matrix A_{n_a} is specified by the subgraph of $\mathcal{G}$ which includes all vertices except those on the special input-output path, and hence its zero-nonzero pattern is known from the graph. We undertake to characterize the entries in the perturbation $A_{n_{ad}} Z_{n_d}^{-1} Z_{n_{da}}$ in terms of the graph topology, which thus gives a graph-theoretic characterization of the matrix $\bar{A}_{aa}$. This analysis then allows us to characterize the finite-invariant-zero structure from a graph theoretic perspective.

The following theorem gives structural insights into the zero-dynamics state matrix $\bar{A}_{aa}$, and hence enables us to develop graph-theoretic results on the zeros. More specifically, this theorem determines the dependence of the entries in $\bar{A}_{aa}$ on the network's graph topology.

Theorem 4: Consider the MIMO LTI network model defined in (1). Assume that the network has m independent special input-output pairs, as indicated in Definitions 1 and 2. Also assume that the special label ordering for

the network nodes, inputs, and outputs is used. Consider $A_q = -A_{n_{ad}}Z_{n_d}^{-1}Z_{n_{da}}$, which is the perturbation matrix in the algebraic expression developed for zero-state matrix in Equation (11). Then $[A_q]_{i,j} = 0$ if one of the following conditions hold for the graph $\mathcal{G}$:

- 1) There is no directed edge to vertex i from any of vertices included in the special input-output paths.
- 2) For $\forall k \in \{1, \dots, m\}$: if there is a directed edge to vertex i from any vertices included in special input-output path $(\hat{u}_k, \hat{y}_k)$, then $d_{j, \hat{y}_k} > d_{r, \hat{y}_k}$ holds, where:
 - a) V_1 is a set of all vertices included in the special input-output path $(\hat{u}_k, \hat{y}_k)$ such that there is a directed edge from them to vertex i .
 - b) vertex $r \in V_1$ is the vertex with minimum distance from the input vertex $\hat{u}_k$, i.e. $d_{\hat{u}_k, r} \leq d_{\hat{u}_k, l}$ for $\forall l \in V_1$.

Proof of Theorem 4: In the case that there is no directed edge to vertex i from any of vertices included in special input-output paths, all entries in the i th row of matrix $A_{n_{ad}}$ are equal to zero. Hence, all entries in the i th row of matrix $A_{n_{ad}}Z_{n_d}^{-1}Z_{n_{da}}$ are equal to zero.

Now, consider the case that there are directed edges to vertex i from some vertices included in special input-output path $(\hat{u}_k, \hat{y}_k)$. Consider the i th row of matrix $A_{n_{ad}}$, and consider entries corresponding to special input-output path $(\hat{u}_k, \hat{y}_k)$, i.e., entries at columns $\sum_{t=1}^{k-1}(q_t) + 1$ through $\sum_{t=1}^{k-1}(q_t) + q_k$. Among these entries, the ones at columns $\sum_{t=1}^{k-1}(q_t) + 1$ to $\sum_{t=1}^{k-1}(q_t) + d_{r, \hat{y}_k}$ to $\sum_{t=1}^{k-1}(q_t) + q_k$ are all equal zero. In addition, the matrix Z_{n_d} is a block diagonal matrix, which has m lower triangular blocks associated with the special input-output paths; these blocks have dimensions $q_1 \times q_1, q_2 \times q_2, \dots, q_m \times q_m$. Consequently, $Z_{n_d}^{-1}$ is a block diagonal matrix including m lower triangular blocks. Hence, in the i th row of matrix $A_{n_{ad}}Z_{n_d}^{-1}$, among entries corresponding to special input-output path $(\hat{u}_k, \hat{y}_k)$, i.e., entries at columns $\sum_{t=1}^{k-1}(q_t) + 1$ through $\sum_{t=1}^{k-1}(q_t) + q_k$, the ones at columns $\sum_{t=1}^{k-1}(q_t) + 1$ to $\sum_{t=1}^{k-1}(q_t) + d_{r, \hat{y}_k}$ to $\sum_{t=1}^{k-1}(q_t) + q_k$ are all equal to zero.

On the other hand, in the j th column of matrix $Z_{n_{da}}$, among entries corresponding to special input-output path $(\hat{u}_k, \hat{y}_k)$, i.e., entries at rows $\sum_{t=1}^{k-1}(q_t) + 1$ through $\sum_{t=1}^{k-1}(q_t) + q_k$, the entries at rows $\sum_{t=1}^{k-1}(q_t) + 1$ through $\sum_{t=1}^{k-1}(q_t) + d_{j, \hat{y}_k}$ are all equal to zero.

In conclusion, if the second assumption in theorem statement holds, based on our discussion on two matrices $A_{n_{ad}}Z_{n_d}^{-1}$ and $Z_{n_{da}}$, it follows that $[A_q]_{i,j} = [-A_{n_{ad}}Z_{n_d}^{-1}Z_{n_{da}}]_{i,j} = 0$. ■

Theorem 4 gives a graphical interpretation for the entries of the matrix $\bar{A}_{aa}$ relative to those of A_{n_a} . Specifically, the theorem shows that entry $[\bar{A}_{aa}]_{i,j}$ is different from entry $[A_{n_a}]_{i,j}$ only if vertex i is next to one of the special input-output paths, and the distance from vertex j to the associated output vertex is sufficiently small. This interpretation provides a tool to characterize the finite zeros by relating the finite-zero dynamics with the network graph topology.

The following theorems characterize the zeros state matrix $\bar{A}_{aa}$ and the perturbation matrix A_q from a graph topological

perspective for classes of network models, and hence give insight into channel zeros for these models. First, we discuss cases where the condition in Theorem 4 holds for all entries in A_q , and hence the perturbation matrix $A_q = 0$. In this case, the finite zeros of the channel are equal to the eigenvalues of a submatrix of the state matrix A ; this submatrix is negative of a M-matrix and hence has stable eigenvalues, so the channel is minimum phase. Theorem 5 shows that, if each special input-output path is connected to the rest of the graph through only one of its vertices, then the perturbation matrix $A_q = 0$. Thus, in this case, the zeros state matrix is equal a submatrix of state matrix A , i.e. $\bar{A}_{aa} = A_{n_a}$. We note that for a collocated special input-output pair, its special input output path includes only one vertex and is connected to the rest of the graph through one vertex, and hence the condition holds automatically in this case. Here is the result:

Theorem 5: Consider the MIMO LTI network model defined in (1). Assume that the network has m independent special input-output pairs, as indicated in Definitions 1 and 2. Also assume that the special label ordering for the network nodes, inputs, and outputs is used. Suppose that each special input-output path is connected to the rest of network graph only through one of its vertices, then:

- $\bar{A}_{aa} = A_{n_a}$, i.e. $A_q = 0$.
- The system is minimum phase.

Proof of Theorem 5: Consider each special input-output pairs $\hat{u}_k, \hat{y}_k$ with $\forall k \in \{1, \dots, m\}$. Then consider any $i, j \in \{1, \dots, n_a\}$ and consider the corresponding vertex r as defined in Theorem 4. We always have $d_{j, \hat{y}_k} \geq d_{r, \hat{y}_k} + 1$ since each special input-output path is connected to the rest of network graph only through one of its vertices (which is necessarily the vertex r). Hence the condition in Theorem 4 holds for all entries in matrix A_q , and we have $A_q = 0$. ■

Now, we discuss cases where some entries of A_q are non-zero but eigenvalues of $\bar{A}_{aa}$ are the same as the eigenvalues of matrix A_{n_a} . In this case, the finite zeros of the channel can be simply obtained from a submatrix of the state matrix A which is negative of a M-matrix and has stable eigenvalues, so the channel is minimum phase. Before presenting the theorem, let us define a new notation.

Definition 3: Consider the MIMO network model, and assume that it has m independent special input-output pairs as indicated in Definition 1 and 2. Consider the network graph $\mathcal{G}$. For a non-collocated input-output pair $(\hat{u}_i, \hat{y}_i)$, we define an *independent input-output path* $(\hat{u}_i, \hat{y}_i)$ as a path from vertex $\hat{u}_i$ to vertex $\hat{y}_i$ which does not cross neither 1) any directed edge which is part of other special input-output paths, or 2) a vertex with collocated special input-output pair. It is obvious that any special input-output path is an independent input-output path.

Next, we present the theorem.

Theorem 6: Consider the MIMO LTI network model defined in (1). Assume that the network has m independent special input-output pairs, as indicated in Definitions 1 and 2. Also assume that the special label ordering for the network nodes, inputs, and outputs is used. Suppose that each non-collocated special input-output pair $(\hat{u}_i, \hat{y}_i)$ has only one

independent input-output path (as indicated in Definition 3), which is the same as its special input-output path. In this case:

- The eigenvalues of matrix $\bar{A}_{aa}$ are equal to the eigenvalues of the matrix A_{n_a} .
- The system is minimum phase.

Proof of Theorem 6: The submatrix A_{n_a} corresponds to the subgraph $\bar{\mathcal{G}}$ of graph $\mathcal{G}$. Graph $\bar{\mathcal{G}}$ includes anything except the special input-output paths' vertices and their adjacent edges in graph $\mathcal{G}$. Note that vertices in subgraph $\bar{\mathcal{G}}$ are the vertices in the graph $\mathcal{G}$ which are not included in any special input-output paths and are labeled as $\{1, \dots, n_a\}$ (based on special ordering).

Now, consider the assumption that each non-located special input-output pair $(\hat{u}_i, \hat{y}_i)$ has only one independent input-output path. This assumption follows that 1) graph $\bar{\mathcal{G}}$ consists of several separated subgraphs which we label them as $\bar{\mathcal{G}}_1, \bar{\mathcal{G}}_2, \dots$, and 2) if there is any directed edges between vertices in graph $\bar{\mathcal{G}}_j$ (for any j) and vertices in the special input-output path $(\hat{u}_i, \hat{y}_i)$ (for any i), then all of these directed edges are connected only to one of the vertices in the special input-output path $(\hat{u}_i, \hat{y}_i)$. Based on this assumption and the result presented in the Theorem 4, one can show in the perturbation matrix A_q , entries that both their row and column is corresponding to the vertices in the same separated subgraph $\bar{\mathcal{G}}_i$ are equal zero.

Consider notation $n_{\bar{\mathcal{G}}_i}$ representing the number of vertices in graph $\bar{\mathcal{G}}_i$. Next, without loss of generality, let us label vertices included in graph $\bar{\mathcal{G}}$ in following order: 1) label vertices in separated subgraph $\bar{\mathcal{G}}_1$ as $\{1, 2, \dots, n_{\bar{\mathcal{G}}_1}\}$, 2) label vertices in separated subgraph $\bar{\mathcal{G}}_2$ as $\{n_{\bar{\mathcal{G}}_1} + 1, n_{\bar{\mathcal{G}}_1} + 2, \dots, n_{\bar{\mathcal{G}}_1} + n_{\bar{\mathcal{G}}_2}\}$, 3) label other separated subgraphs in similar way. Considering this ordering assumption in node labeling as well as the assumption of special input-output paths have only one independent input-output path, it follows that matrix A_{n_a} is block diagonal and entries corresponding to each separated subgraph are located in each of these block diagonal submatrices. Previously, we discussed that in the perturbation matrix A_q , entries that both their row and column is corresponding to the vertices in the same separated subgraph $\bar{\mathcal{G}}_i$ are equal zero. This follows that in matrix A_q , entries corresponding to block diagonals in matrix A_{n_a} are equal zero. Considering ordering assumption discussed before, one can show that the matrix $A_q = -A_{n_{ad}}Z_{n_d}^{-1}Z_{n_{da}}$ is a strictly block upper-triangular (or lower-triangular) matrix, and further that its nonzero entries have no overlap with block diagonal entries of A_{n_a} . Hence, the eigenvalues of matrix $\bar{A}_{aa}$ are equal to the eigenvalues of matrix A_{n_a} . ■

Theorem 6 shows that if alternative input-output paths between each special input-output pair are blocked by other special input-output paths, the finite invariant zeros are the eigenvalues of matrix A_{n_a} , which are in the left half plane. This result also suggests means for adding new inputs and outputs to the network, so as to make the MIMO channel minimum phase. For example, this can be added by adding collocated input-output pairs at locations that block alternative paths between non-located special input-output

pairs. The next theorem shows that, if the special input-output paths are sufficiently strong (have large enough edge weights) compared to alternative paths, the finite invariant zeros are arbitrarily close to the eigenvalues of A_{n_a} . Since the eigenvalues of A_{n_a} are in the left half plane, it follows that the network model is minimum phase in this case.

Theorem 7: Consider MIMO LTI network model as defined in (1). Assume that the network has m independent special input-output pairs, as indicated in Definitions 1 and 2. Also assume that the special label ordering for the network nodes, inputs, and outputs is used. For special input-output pairs that have more than one independent input-output path, suppose that the weights of edges included in their special input-output paths are scaled by a positive factor κ . For sufficiently large factor κ :

- The eigenvalues of the zeros state matrix $\bar{A}_{aa}$ are arbitrary close to the eigenvalues of the submatrix A_{n_a} .
- The system is minimum phase.

Proof of Theorem 7: The matrix A_{n_a} has no dependence on the scaling factor κ . Similarly, the matrix $A_{n_{ad}}$ has no dependence on κ . The matrix Z_{n_d} is a block diagonal which includes m lower triangular blocks associated with the special input-output paths that have dimensions $q_1 \times q_1, q_2 \times q_2, \dots, q_m \times q_m$. The entries in row t in each of these blocks corresponding to special input-output pairs with more than one independent input-output path, are bounded by κ^{t-1} . Consequently, matrix $Z_{n_d}^{-1}$ is a block diagonal matrix including m lower triangular blocks with dimensions $q_1 \times q_1, q_2 \times q_2, \dots, q_m \times q_m$. Entries in column t in each of those blocks that corresponds to special input-output pairs with more than one independent input-output path, are bounded by $\kappa^{-(t-1)}$. On the other hand, the matrix $Z_{n_{da}}$ consists of m submatrices associated with the special input-output paths with dimensions $q_1 \times n_a, q_2 \times n_a, \dots, q_m \times n_a$ assembled in vertical order. Entries in row t in each of those submatrices associated with special input-output pairs with more than one independent input-output path are bounded by $\kappa^{(t-2)}$. Hence, it can be shown that all entries in rows associated with special input-output pairs with more than one independent input-output path are bounded by $\kappa^{(-1)}$.

Now consider subgraph $\bar{\mathcal{G}}$ of graph $\mathcal{G}$ as described in the proof for theorem 6. Graph $\bar{\mathcal{G}}$ consists of several separated subgraphs which we label them as $\bar{\mathcal{G}}_1, \bar{\mathcal{G}}_2, \dots$. Considering this ordering assumption in node labeling as well as the assumption of special input-output paths have only one independent input-output path, it follows that matrix A_{n_a} is block diagonal and entries corresponding to each separated subgraph are located in each of these block diagonal submatrices.

Since the assumption in theorem 6 does not hold here, hence for some of the graphs $\bar{\mathcal{G}}_j$ specified with set Φ (for some $j \in \Phi$) we have: if there is any directed edges between vertices in graph $\bar{\mathcal{G}}_j$ and vertices in the special input-output path $(\hat{u}_i, \hat{y}_i)$ (for any i), then all of these directed edges are connected only to one of the vertices in the special input-output path $(\hat{u}_i, \hat{y}_i)$. This discussion only holds for the graphs $\bar{\mathcal{G}}_j$ when $j \in \Phi$, and does not hold for $j \notin \Phi$. Next, without

loss of generality, let us label vertices included in graph $\bar{\mathcal{G}}_j$ in the same order as discussed in the proof for theorem 6.

Based on previous discussions and the result presented in the Theorem 4, one can show in the perturbation matrix A_q , entries that both their row and column is corresponding to the vertices in the same separated subgraph $\bar{\mathcal{G}}_j$ for $j \in \Phi$ are equal zero. Similarly, one can show that in the perturbation matrix A_q , entries that both their row and column is corresponding to the vertices in the same separated subgraph $\bar{\mathcal{G}}_j$ for $j \notin \Phi$ are bounded by $\kappa^{(-1)}$. In general we can say in the perturbation matrix A_q , entries that both their row and column is corresponding to the vertices in the same separated subgraph $\bar{\mathcal{G}}_j$ for $\forall j$ are bounded by $\kappa^{(-1)}$. On the other hand, considering ordering assumption discussed before, one can show that in the matrix $A_q = -A_{n_{ad}}Z_{n_d}^{-1}Z_{n_{da}}$ entries that are not bounded by $\kappa^{(-1)}$ are at a strictly block upper-triangular (or lower-triangular) matrix form, and further that these entries have no overlap with block diagonal entries of A_{n_a} . Hence, entries of matrix A_q which correspond to the block diagonal entries of matrix A_{n_a} are arbitrarily close to zero for sufficiently large scaling factor κ , and other entries are in the form of a strictly block upper-triangular (or lower-triangular) matrix form. Hence, the eigenvalues of matrix $\bar{A}_{aa}$ are equal to the eigenvalues of matrix A_{n_a} . Since each diagonal block of A_{n_a} is a strict M matrix, it follows that the system is minimum phase for sufficiently large κ . ■

Theorem 7 shows that if input and output locations are chosen such that there are sufficiently strong paths between them, the obtained channel is assured to be minimum phase. Alternately, if the network's edge weights can be designed, strengthening the direct (shortest) path between each input and output compared to other independent paths ensures minimum-phase dynamics.

IV. EXAMPLE

An example is used to illustrate the developed results developed in previous section. A network model of the form (1) with 9 nodes, 3 inputs, and 3 outputs and model is considered. The network graph for this example, with vertex numbers (equivalently network node numbers) and input and output locations labeled, is shown in Figure 1. In this graph, there are 3 independent special input-output pairs $(\hat{u}_1, \hat{y}_1), (\hat{u}_2, \hat{y}_2), (\hat{u}_3, \hat{y}_3)$. The inputs, outputs, and graph vertices (network nodes) are labeled according to the special ordering. Also, the special input-output paths and their corresponding vertices are shown by green and red colors respectively. The state matrix A , input matrix B , and output matrix C are shown below:

$$A = \begin{bmatrix} -5 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & -5 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & -3 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & -4 & 1 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & -4 \end{bmatrix}$$

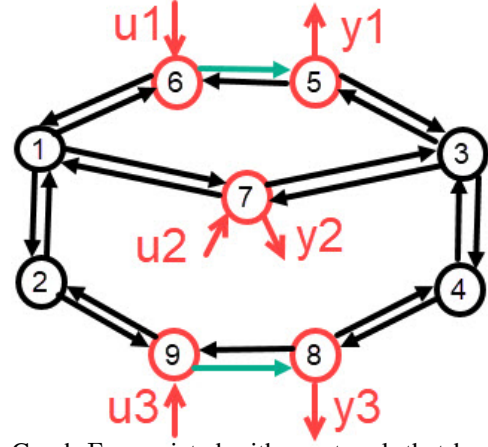


Fig. 1: Graph Γ associated with a network that has 9 nodes and 3 independent special input-output pairs.

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Using Theorem 2, we know that the system is invertible. The lengths of these special input-output paths are 1, 0, 1, so $q_1 = 2$, $q_2 = 1$, and $q_3 = 2$. Theorem 3 indicates that this system has 2 infinite zeros of order 2, and 1 infinite zero of order 1. From theorem 1, a non-singular state transformation matrix Γ_s (as shown below) can be used to transform the system to the special coordinate basis.

$$\Gamma_s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & -4 & 1 \end{bmatrix}$$

By calculating the matrices A_{n_a} and A_q we have:

$$A_{n_a} = \begin{bmatrix} -5 & 1 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ 0 & 0 & -4 & 2 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

$$A_q = \begin{bmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Based on the result presented in Theorem 4, only entries $[A_q]_{1,3}$ and $[A_q]_{2,4}$ can be non-zero. In addition, based on

the result presented in in Theorem 6, matrix A_{n_a} consists of two diagonal blocks and matrix A_q is strictly block upper-triangular relative to the non-zero diagonal blocks in A_{n_a} . Consequently, $\text{eig}(\bar{A}_{aa}) = \text{eig}(A_{n_a})$, so the finite zeros of the channel are equal to the eigenvalues of A_{n_a} matrix. These eigenvalues, and hence the finite zeros, are $-5.7321, -2.2679, -2.5858, -5.4142$.

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