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Journal of Statistical Physics

ISSN 0022-4715
Volume 179
Number 2

J Stat Phys (2020) 179:502-534
DOI 10.1007/s10955-020-02540-0

Springer

Journal of Statistical Physics (2020) 179:502–534
https://doi.org/10.1007/s10955-020-02540-0

Dissipation in Parabolic SPDEs

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Received: 19 March 2019 / Accepted: 31 March 2020 / Published online: 17 April 2020

Abstract
The study of intermittency for the parabolic Anderson problem usually focuses on the moments of the solution which can describe the high peaks in the probability space. In this paper we set up the equation on a finite spatial interval, and study the other part of intermittency, i.e., the part of the probability space on which the solution is close to zero. This set has probability very close to one, and we show that on this set, the supremum of the solution over space is close to 0. As a consequence, we find that almost surely the spatial supremum of the solution tends to zero exponentially fast as time increases. We also show that if the noise term is very large, then the probability of the set on which the supremum of the solution is very small has a very high probability.

Keywords Intermittency · Stochastic partial differential equations · White noise · Dissipation

Mathematics Subject Classification Primary: 60H15 · Secondary: 35R60

1 Introduction, Background, and Main Results

Consider the solution u to the parabolic stochastic PDE (SPDE, for short),

$$\partial_t u = \partial_x^2 u + \sigma(u)\xi \tag{1.1}$$

where $u = u(t\,,x)$, $t > 0$, x lies in the torus $\mathbb{T} := [-1\,,1]$, $\sigma : \mathbb{R} \to \mathbb{R}$ is non-random and Lipschitz continuous, and $\xi = \xi(t\,,x)$ denotes space-time white noise. The initial profile $u_0(x) := u(0\,,x)$ is assumed to be non-random, and to satisfy

Communicated by Eric A. Carlen.

Research supported in part by the NSF Grant DMS-1307470 [D.K.], the NRF Grants 2019R1A5A1028324 and 2020R1A2C4002077 [K.K.], a Simons Grant [C.M.], and MOST Grant MOST103-2628-M-008-003-MY4 [S.-Y. S.]. Parts of this research were funded by the NSF Grant DMS-1440140 while three of the authors [D.K., K.K., and C.M.] were in residence at the Mathematical Sciences Research Institute at UC Berkeley in Fall of 2015. Three of us [D.K., K.K., and C.M.] wish to also thank the Banff International Research Center for financial support under the Research in Teams Program.

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$$0 < \inf_{x\in\mathbb{T}} u_0(x) \leqslant \sup_{x\in\mathbb{T}} u_0(x) < \infty. \tag{1.2}$$

The Laplace operator ∂_x^2 in (1.1) is endowed with periodic boundary conditions on $\mathbb{T}$.

According to the standard theory of SPDEs, there exists a unique almost surely continuous random field u that satisfies

$$\sup_{t\in(0,T)} \sup_{x\in\mathbb{T}} \mathrm{E}\left(|u(t\,,x)|^k\right) < \infty \qquad \text{for all } T > 0 \text{ and } k \geqslant 2,$$

that solves (1.1); see [7,16,28]. See also Sect. 2 below for further details.

In addition, we suppose that there exist two real numbers $\mathrm{Lip}_\sigma \geqslant \mathrm{L}_\sigma > 0$ such that[1]

$$\mathrm{L}_\sigma \leqslant \left|\frac{\sigma(a)}{a}\right| \leqslant \mathrm{Lip}_\sigma \qquad \text{for every } a \in \mathbb{R}\setminus\{0\}. \tag{1.3}$$

Because the cone condition (1.3) implies that $\sigma(0) = 0$, the positivity principle for SPDEs implies that

$$\mathrm{P}\left\{u(t\,,x) > 0 \text{ for every } t \geqslant 0 \text{ and } x \in \mathbb{T}\right\} = 1;$$

see [23].

One of the interesting properties of (1.1) is that its solution is *intermittent* in the sense of [2,10]. More precisely, intermittency (or moment intermittency) can be defined as the property that

$$k \mapsto \frac{\overline{\gamma}(k)}{k} \quad \text{and/or} \quad k \mapsto \frac{\underline{\gamma}(k)}{k} \quad \text{is strictly increasing on } [2\,,\infty), \tag{1.4}$$

where $\underline{\gamma}, \overline{\gamma} : [2\,,\infty) \to [-\infty\,,\infty]$ are given by

$$\underline{\gamma}(k) := \liminf_{t\to\infty} \frac{1}{t} \inf_{x\in\mathbb{T}} \log \mathrm{E}\left(|u(t\,,x)|^k\right) \quad \text{and} \quad \overline{\gamma}(k) := \limsup_{t\to\infty} \frac{1}{t} \sup_{x\in\mathbb{T}} \log \mathrm{E}\left(|u(t\,,x)|^k\right).$$

Here $\underline{\gamma}$ and $\overline{\gamma}$ are called *lower and upper moment Lyapunov exponents* respectively.

As a result of Jensen's inequality, it is easy to see that both $k \mapsto \underline{\gamma}(k)/k$ and $k \mapsto \overline{\gamma}(k)/k$ are monotonically nondecreasing. So the defining feature of intermittency is the *strictness* of this monotonicity. Indeed, Jensen's inequality for moments is strict iff the random variable is not constant over the probability space. In the setting of this paper, intermittency is implied by the following, more easy-to-check, *weak intermittency* condition:

$$0 < \underline{\gamma}(k) \leqslant \overline{\gamma}(k) < \infty \quad \text{for all } k \geqslant 2; \tag{1.5}$$

see [10] for the relation between (1.4) and (1.5) and also see [11,17,18,25,29] for the moments and weak intermittency of the solution u to (1.1) on bounded intervals with various boundary conditions. We have set things up so that (1.5) is in fact equivalent to the strict monotonicity of both $k \mapsto \underline{\gamma}(k)/k$ and $k \mapsto \overline{\gamma}(k)/k$.

In order to see intuitively how moments yield information about the peaks of the solution, assume that $u_0(x) = \text{constant}$ for all $x \in \mathbb{T}$. It can be shown that this assumption implies that the distribution of $u(t\,,x)$ does not depend on x. We also suppose that $\underline{\gamma} = \overline{\gamma}$, and call their common value γ. This means that, for every $k \geqslant 2$,

$$\mathrm{E}\left(|u(t\,,x)|^k\right) \approx \mathrm{e}^{t\gamma(k)}, \tag{1.6}$$

[1] Clearly, (1.3) is equivalent to the condition that $\sigma(0) = 0$ and $\inf_{a\neq0} |\sigma(a)/a| > 0$. Therefore, we can always choose Lip_σ to be the Lipschitz constant of σ. This remark also justifies the notation for Lip_σ in (1.3).

where $\approx$ denotes logarithmic equivalence, i.e., $f(t) \approx g(t)$ means $\lim_{t\to\infty}(\log f(t) - \log g(t))/t = 0$.

Because $k \mapsto \gamma(k)/k$ is strictly increasing on $[2\,,\infty)$, there exist constants $2 \leqslant k_1 < k_2 < \cdots$, all strictly increasing, and events $A_1(t), A_2(t), \ldots$ (one for every $t > 0$), and constants $C_1, C_2, \ldots > 0$ such that:

(**I.1**) $\mathrm{P}(A_n(t)) \leqslant \exp(-C_n t)$ for all $n \geqslant 1$ and all large $t \geqslant 1$; and
(**I.2**) For all $n \geqslant 1$, $\mathrm{E}\left(|u(t\,,x)|^{k_n}\right) \approx \mathrm{E}\left(|u(t\,,x)|^{k_n}\,;\; A_n(t)\right)$.

Indeed, by (1.4) we can find for every $n \geqslant 1$ real numbers a_n such that

$$\frac{\gamma(k_{n-1})}{k_{n-1}} < a_n < \frac{\gamma(k_n)}{k_n}, \tag{1.7}$$

then set

$$A_n(t) := \left\{\omega \in \Omega :\; \mathrm{e}^{a_n t} \leqslant |u(t\,,x)(\omega)|\right\},$$

and finally apply Chebyshev's inequality to deduce (**I.1**):

$$\begin{aligned} \mathrm{P}\left(A_n(t)\right) &\leqslant \exp\left(-a_n k_{n-1} t\right) \mathrm{E}\left(|u(t\,,x)|^{k_{n-1}}\right) \\ &\approx \exp\left(-a_n k_{n-1} t + \gamma(k_{n-1}) t\right) \qquad [\text{see } (1.6)] \\ &\approx \exp(-C_n t) \;\; \text{for some } C_n > 0 \; [\text{see } (1.7)]. \end{aligned}$$

We deduce (**I.2**) by noticing that

$$\begin{aligned} \mathrm{E}\left(|u(t\,,x)|^{k_n}\,;\; [A_n(t)]^c\right) &\leqslant \exp\left(k_n a_n t\right) \\ &\ll \exp\left(\gamma(k_n) t\right) \qquad [\text{see } (1.7)] \\ &\approx \mathrm{E}\left(|u(t\,,x)|^{k_n}\right) \quad [\text{see } (1.6)], \end{aligned}$$

where $f(t) \ll g(t)$ denotes $\lim_{t\to\infty}(g(t)/f(t)) = \infty$. From this simple heuristic about the Lyapunov exponents, we learn a good deal about the high peaks of u, namely, that:

1. The moments of the solution grow exponentially rapidly as $t \to \infty$, and nearly all of the contribution to the k_n-th moment of $u(t\,,x)$ comes from a small part $[A_n(t)]$ of the probability space where $u(t\,,x)$ is unduly large; and
2. The k_1-th, k_2-th, ...moments of $u(t\,,x)$ are influenced by decreasing small parts of the underlying probability space.

In other words, the high peaks tend to appear at large times, and they tend to be highly localized in the probability space. This picture describes one part of "physical intermittency" in probability space where physical intermittency usually refers to the property that the solution u tends to develop "*tall peaks*," "*distributed over small islands*," and "*separated by large areas where u is small* (voids)" (see [1–3,6,13,14,19,20,22,30–33]).

The main goal of the present paper is to study the part of physical intermittency that does *not* seem to be a natural consequence of conditions such as (1.4) or (1.5). Namely, we currently propose to analyze the "*voids*" (the event where u is small). One of the key steps toward this goal is the following result, which is the counterpart to (1.5).

Theorem 1.1 *There exist $t_0 \geqslant 1$, events $B(t)$ for every $t \geqslant t_0$, and constant $c > 0$ which is independent of t such that for every $k \geqslant 2$, there exist $c_{1,k}, c_{2,k} > 0$ such that:*

1. $\mathrm{P}(B(t)) \geqslant 1 - c\,\exp(-ct)$ *for all $t \geqslant t_0$; and*

2. *For all* $t \geqslant t_0$,

$$c_{2,k}\mathrm{e}^{-c_{1,k}t} \leqslant \mathrm{E}\left(\inf_{x\in\mathbb{T}} |u(t\,,x)|^k\,;\; B(t)\right) \leqslant \mathrm{E}\left(\sup_{x\in\mathbb{T}} |u(t\,,x)|^k\,;\; B(t)\right) \leqslant c_{1,k}\mathrm{e}^{-c_{2,k}t}.$$

Loosely speaking, $B(t)$ of Theorem 1.1 denotes the event that $u(t\,,\cdot)$ is exponentially small in a sense that will be made precise in (8.4) below.

Theorem 1.1 will be proved in Sect. 8.

We learn from Theorem 1.1 the following property which contrasts with the earlier discussion about moment intermittency and its consequences: For large values of t, only a tiny part of the probability space contributes to the moments of $u(t\,,x)$. In some sense, this property and moment intermittency give us a complete mathematical description of the "physical intermittency" of the solution u in probability space.

In this connection, let us also mention a more precise result. The following is a non-trivial pathwise variation of Theorem 1.1, which gives precise bounds on the a.s. dissipation of the solution to (1.1), viewed as the solution to a semi-linear heat-flow problem in the random environment ξ.

Theorem 1.2 *With probability one,*

$$-\infty < \liminf_{t\to\infty} \frac{1}{t} \log \inf_{x\in\mathbb{T}} u(t\,,x) \leqslant \limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in\mathbb{T}} u(t\,,x) < 0.$$

In particular, the positive random variable $\sup_{x\in\mathbb{T}} u(t\,,x)$ *converges a.s. to zero [fast] as* $t \to \infty$.

Theorem 1.2 tells us that the solution to (1.1) decays exponentially rapidly as $t \to \infty$. One can understand why this might happen intuitively as follows: Let us specialize (1.1) to the case that $\sigma(x) = x$ for all $x \in \mathbb{R}$, and consider the SPDE,

$$\partial_t v(t\,,x) = \partial_x^2 v(t\,,x) + v(t\,,x)\partial_t W(t) \qquad \text{for all } t > 0,\ x \in \mathbb{T},$$

with periodic boundary condition on $\mathbb{T}$, where $W = \{W(t)\}_{t\geqslant 0}$ denotes a Brownian motion that does not depend on the spatial index x. For simplicity, let us also assume that the initial data is a constant $v_0 \neq 0$. In that case, it is possible to check that the Itô–Walsh type solution to the preceding SPDE is

$$v(t\,,x) = v(t) = v_0 \exp\left(W(t) - \tfrac{1}{2}t\right).$$

The law of large numbers for W immediately implies that $v(t\,,x) \to 0$ exponentially fast as $t \to \infty$. Here, the noise is much more regular than space-white noise. But since the state space $\mathbb{T}$ is bounded, one might hope that the large-time behavior of v might be roughly similar to that of u. Theorem 1.2 is a rigorous way to say that the preceding is indeed the case.

Remark 1.3 (SHE with a linear reaction term) Theorem 1.2 and its proof can teach us about the asymptotic behavior of the solution to other type of SPDEs as well. For instance, consider the following reaction-diffusion equation with multiplicative noise:

$$\partial_t v = \partial_x^2 v + b(v) + \lambda v \xi, \tag{1.8}$$

where $v(0\,,x) = 1$ for all $x \in \mathbb{T}$, the reaction term $b : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous functions, and $b(0) \leqslant 0$ (say). The SPDE (1.8) comes up, for example, in Zimmerman et al [33] as a toy physical model for a random field that is predicted to have spatio-temporal intermittency. Because $b(z) \leqslant cz$ for all $z \in \mathbb{R}$, where c denotes the Lipschitz constant of b, it follows from the comparison principle [27] that $v(t\,,x) \leqslant V(t\,,x)$ for all $t > 0$ and $x \in \mathbb{T}$ a.s., where V satisfies the SPDE,

$$\partial_t V = \partial_x^2 V + cV + \lambda V\xi,$$

with initial condition $V(0\,,x) = 1$ for all $x \in \mathbb{T}$, and $\lambda > 0$ is a nonrandom constant that denotes the level of the noise. One can verify that $V(t\,,x) = \mathrm{e}^{ct}u(t\,,x)$ for all $x \in \mathbb{T}$ and $t > 0$, where u solves the following specialization of (1.1):

$$\partial_t u = \partial_x^2 u + \lambda u\xi,$$

subject to $u(0\,,x) = 1$ for all $x \in \mathbb{T}$. The proof of Theorem 1.2 shows that, for all large t,

$$\mathrm{e}^{-c_1 f(\lambda)t} \leqslant u(t\,,x) \leqslant^{-c_2 f_2(\lambda)t},$$

where $c_1 > 0$, $c_2 > 0$ are nonrandom real numbers that do not depend on λ, $\lim_{\lambda\to 0} f_1(\lambda)/\lambda^4 = 1$, and $\lim_{\lambda\to\infty} f_2(\lambda)/\lambda^2 = 1$. We can assemble these remarks as follows: On one hand, if λ is large, then $0 \leqslant v(t\,,x) \leqslant \exp[-(c_2 f_2(\lambda) - c)t] \to 0$ exponentially fast as $t \to \infty$. On the other hand, if λ is small, then $v(t\,,x) \geqslant \exp[(c - c_1 f_1(\lambda))t] \to \infty$ exponentially rapidly as $t \to \infty$. This example yields partial rigorous proof of some of the physical/computational predictions of Zimmerman et al [33].

Our analysis of Theorems 1.1 and 1.2 hinges on a novel L^1/L^∞ interpolation inequality, see Proposition 5.2, which is interesting in its own right. Roughly speaking, we can control the supremum of our solution by its L^1 norm, and we can show using martingale arguments that, with high probability, the L^1 norm declines exponentially.

Our analysis has other consequences too. For example, we can describe the system (1.1) in the "high-noise" setting. That is, consider the SPDE (1.1) where we replace $\sigma(u)$ by $\lambda\sigma(u)$ for a large constant $\lambda > 0$, as follows:

$$\partial_t u(t\,,x\,;\lambda) = \tfrac{1}{2}\partial_x^2 u(t\,,x\,;\lambda) + \lambda\sigma(u(t\,,x\,;\lambda))\xi(t\,,x), \tag{1.9}$$

with periodic boundary conditions on $\mathbb{T}$ and initial value u_0, as before. In other words, we simply replace the function σ by $\lambda\sigma$, and add λ to the notation for u to help keep better track of this change. Since $\lambda\sigma$ is also Lipschitz continuous and satisfies (1.3), all of this is merely recording a change in the notation.

Now we can state a result about the large-noise behavior of the solution to (1.1), equivalently the large-λ behavior of the solution to (1.9). Roughly speaking, the following theorem states that if the level λ of the noise is high then voids take over rapidly, with very high probability. More precisely, we have

Theorem 1.4 (Large-noise regime) *For every $t > 0$,*

$$\limsup_{\lambda\uparrow\infty} \frac{1}{\lambda^2} \log \mathrm{P}\left\{\sup_{x\in\mathbb{T}} u(t\,,x\,;\lambda) > \exp\left(-\frac{\mathrm{L}_\sigma^2\lambda^2 t}{64}\right)\right\} \leqslant -\frac{\mathrm{L}_\sigma^2 t}{64}.$$

In particular, for every $t > 0$ fixed, the positive random variable $\sup_{x\in\mathbb{T}} u(t\,,x\,;\lambda)$ converges in probability to zero [fast] as $\lambda \to \infty$.

We conclude the Introduction by setting forth some notation that will be used throughout the paper.

In order to simplify some of the formulas, we distinguish between the spaces $L^k(\mathbb{T})$ and $L^k(\mathrm{P})$ by writing the former as

$$L^k := L^k(\mathbb{T}) \qquad [1 \leqslant k < \infty].$$

Thus, for example, if $f \in L^k$ for some $1 \leqslant k < \infty$, then $\|f\|_{L^k} = [\int_{-1}^{1} |f(x)|^k\,\mathrm{d}x]^{1/k}$. We will abuse notation slightly and write $\|f\|_{L^\infty} := \sup_{x\in\mathbb{T}} |f(x)|$, in place of the more customary essential supremum. The $L^k(\mathrm{P})$-norm of a random variable $Z \in L^k(\mathrm{P})$ is denoted by $\|Z\|_k := \{\mathrm{E}\left(|Z|^k\right)\}^{1/k}$ for all $1 \leqslant k < \infty$.

2 The Mild Solution

Consider the SPDEs (1.1) and (1.9). Because $u(t\,,x) = u(t\,,x\,;1)$, it suffices to consider only the SPDE (1.9) for a general $\lambda > 0$. We shall do so tacitly from here on.

Let $W = \{W(t\,,x)\}_{t\geqslant 0, x\in\mathbb{T}}$ denote a two-parameter Brownian sheet; that is, W is a two-parameter, centered, generalized Gaussian random field with

$$\mathrm{Cov}\,[W(t\,,x)\,,\; W(s\,,y)] = \min(s\,,t)\min(x\,,y) \qquad \text{for all } s, t \geqslant 0 \text{ and } x, y \in \mathbb{T}.$$

It is well known (see [28, Theorem 1.1]) that W has continuous trajectories (up to a modification). Therefore,

$$\xi(t\,,x) = \partial_t \partial_x W(t\,,x)$$

exists as a generalized random function. This ξ is space-time white noise, and was mentioned already in the Introduction.

Let $(\tau\,;x\,,y) \mapsto p_\tau(x\,,y)$ denote the fundamental solution to the heat operator $\partial_t - \partial_x^2$ on $(0\,,\infty)\times\mathbb{T}$ with periodic boundary conditions and initial data $p_0(x, y) = \delta(x-y)$, where δ is the Dirac delta function. That is,

$$p_\tau(x\,,y) := \sum_{n=-\infty}^{\infty} G_\tau(x - y + 2n) \qquad [\tau > 0,\ x, y \in \mathbb{T}], \tag{2.1}$$

where G is the heat kernel in free space; that is,

$$G_\tau(a) := (4\pi\tau)^{-1/2} \exp\left(-\frac{a^2}{4\tau}\right) \qquad \text{for all } \tau > 0 \text{ and } a \in \mathbb{R}. \tag{2.2}$$

Also, let $\{P_t\}_{t\geqslant 0}$ denote the corresponding heat semigroup. That is, $P_0 f := f$ for every measurable and bounded function $f : \mathbb{T} \to \mathbb{R}_+$, and

$$(P_t f)(x) := \int_{-1}^{1} p_t(x\,,y) f(y)\,\mathrm{d}y, \tag{2.3}$$

for all $t > 0$ and $x \in \mathbb{T}$.

With the preceding notation in place, we then follow Walsh [28, Chap. 3] and interpret (1.9) in mild/integral form as follows:

$$u(t\,,x\,;\lambda) = (P_t u_0)(x) + \mathcal{I}_t(x\,;\lambda), \tag{2.4}$$

where $\mathcal{I}$ is defined pointwise as the Walsh stochastic integral,

$$\mathcal{I}_t(x\,;\lambda) := \lambda \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y)\sigma(u(s\,,y\,;\lambda))\,W(\mathrm{d}s\,\mathrm{d}y), \tag{2.5}$$

for every $t, \lambda > 0$ and $x \in \mathbb{T}$. As was mentioned in the Introduction (for $\lambda = 1$), it is well known [28, Th. 3.2, p. 313, and Cor. 3.4, p. 318] that there exists a unique weak solution of (1.9) that is continuous and satisfies (2.4), as well as the following moment condition:

$$\sup_{\substack{x\in\mathbb{T}\\ t\in(0,T]}} \mathrm{E}\left(|u(t\,,x\,;\lambda)|^k\right) < \infty \qquad [0 < T < \infty,\ 1 \leqslant k < \infty]. \tag{2.6}$$

Moreover, for every $\lambda > 0$,

$$\mathrm{P}\{u(t\,,x\,;\lambda) > 0 \text{ for all } t \geqslant 0 \text{ and } x \in \mathbb{T}\} = 1. \tag{2.7}$$

In the case that $\sigma(z) := \text{const} \cdot z$ for all $z \in \mathbb{R}$, this follows from Theorem 1 of Mueller [23]. The general case follows by making modifications to the proof of that theorem; see the proof of Theorem 1.7 of Conus et al [5].

3 The Total Mass Process

We may integrate both sides of (2.4) [dx] in order to see that

$$\|u(t\,,\cdot\,;\lambda)\|_{L^1} = \|u_0\|_{L^1} + \lambda \int_{(0,t]\times\mathbb{T}} \sigma(u(s\,,y\,;\lambda))\, W(\mathrm{d}s\,\mathrm{d}y) \qquad [t \geqslant 0]. \tag{3.1}$$

The interchange of the integrals is justified by an appeal to a stochastic Fubini theorem [28, Th. 2.6, p. 296]. Thus, it follows from (1.3), (2.6), (2.7) and (3.1) that $t \mapsto \|u(t\,,\cdot\,;\lambda)\|_{L^1}$ defines a positive, continuous, L^2-martingale. The following result ensures that the said martingale decays exponentially rapidly at rate not less than a fixed multiple of λ^2.

Proposition 3.1 *For every $t, \lambda > 0$ and $\varepsilon \in (0\,,1)$,*

$$\mathrm{P}\left\{\|u(s\,,\cdot\,;\lambda)\|_{L^1} \geqslant \|u_0\|_{L^1} \exp\left(-\frac{(1-\varepsilon)\lambda^2\mathrm{L}_\sigma^2 s}{4}\right) \text{ for some } s \geqslant t\right\} \leqslant \exp\left(-\frac{\varepsilon^2\lambda^2\mathrm{L}_\sigma^2 t}{16}\right).$$

The proof of Proposition 3.1 requires a basic lemma about continuous martingales, which might be of independent interest.

Lemma 3.2 *Let $X = \{X_t\}_{t\geqslant 0}$ be a continuous $L^2(\mathrm{P})$ martingale, and suppose there is a nonrandom $c > 0$ such that $\langle X\rangle_t \geqslant ct$ for all $t \geqslant 0$, a.s. Then, for all nonrandom constants $\varepsilon, T > 0$,*

$$\mathrm{P}\{X_t \geqslant \varepsilon\langle X\rangle_t \text{ for some } t \geqslant T\} \leqslant \exp\left(-\frac{cT\varepsilon^2}{2}\right).$$

Proof Recall that a continuous local martingale such as X is a time-change of a Brownian motion $\{B(s)\}_{s\geqslant 0}$ (see [26, Theorem 1.6, p. 181]), so that $X_t = B(\langle X\rangle_t)$ for all $t \geqslant 0$, a.s. We first note that

$$\mathrm{P}\{X_t \geqslant \varepsilon\langle X\rangle_t \text{ for some } t \geqslant T\} \leqslant \mathrm{P}\left\{\sup_{s\geqslant cT} \frac{B(s)}{s} \geqslant \varepsilon\right\}.$$

Next we note that $\{B(s)/s\}_{s>0}$ has the same law as $\{B(1/s)\}_{s>0}$ thanks to Brownian time inversion. Thus

$$\mathrm{P}\{X_t \geqslant \varepsilon\langle X\rangle_t \text{ for some } t \geqslant T\} \leqslant \mathrm{P}\left\{\sup_{r\leqslant 1/(cT)} B(r) \geqslant \varepsilon\right\}.$$

Because $(2\pi)^{-1/2}\int_a^\infty \exp(-x^2/2)\,\mathrm{d}x \leqslant (1/2)\exp(-a^2/2)$ for all $a > 0$, the reflection principle implies the result. □

Armed with Lemma 3.2, we conclude the section with the following.

Proof of Proposition 3.1 In the case that $\sigma(z) \equiv \text{const} \cdot z$ for all $z \in \mathbb{R}$ and the SPDE (1.1) has Dirichlet—instead of periodic—boundary conditions, Mueller and Nualart [24, Theorem 2] have proved that $\mathrm{E}(|u(t\,,x\,;\lambda)|^{-k}) < \infty$ for all $1 \leqslant k < \infty$, $t > 0$, and $x \in \mathbb{T}$. Their argument, in fact, proves that, in the present setting,[2]

$$\mathrm{E}\left(\left|\inf_{x\in\mathbb{T}} u(t\,,x\,;\lambda)\right|^{-k}\right) < \infty \qquad [t \geqslant 0,\ 1 \leqslant k < \infty]. \tag{3.2}$$

Let us define

$$M_t := \|u(t\,,\cdot\,;\lambda)\|_{L^1} = \int_{-1}^{1} u(t\,,x\,;\lambda)\,\mathrm{d}x \qquad [t \geqslant 0],$$

and infer from (3.2) that

$$\mathrm{E}\left(M_t^{-k}\right) < \infty \qquad \text{for all } t \geqslant 0 \text{ and } 1 \leqslant k < \infty, \tag{3.3}$$

We will use (3.3) several times, sometimes tacitly, in the sequel.

We can apply Itô's formula in order to see that, a.s.,

$$\log M_t = \log M_0 + \int_0^t M_s^{-1}\,\mathrm{d}M_s - \frac{1}{2}\int_0^t M_s^{-2}\,\mathrm{d}\langle M\rangle_s \qquad \text{for all } t \geqslant 0.$$

Define

$$N_t := \int_0^t M_s^{-1}\,\mathrm{d}M_s \qquad \text{for all } t \geqslant 0.$$

Let $\{\mathscr{F}_t\}_{t\geqslant 0}$ denote the filtration generated by $W(s\,,\cdot)$ for $s \leqslant t$. Then, clearly, $N := \{N_t\,,\mathscr{F}_t\}_{t\geqslant 0}$ is a continuous L^2-martingale with quadratic variation $\langle N\rangle_t = \int_0^t M_s^{-2}\,\mathrm{d}\langle M\rangle_s$ at time $t > 0$. In other words, $\log M_t = \log M_0 + N_t - \frac{1}{2}\langle N\rangle_t$ a.s. for all $t > 0$; this is another way to say that

$$M_t = M_0 \exp\left(N_t - \tfrac{1}{2}\langle N\rangle_t\right) \qquad \text{a.s. for all } t > 0. \tag{3.4}$$

That is, M is the exponential martingale of the martingale N, and M is initialized at M_0.

We examine the quadratic variation of N more closely next:

$$\begin{aligned}\frac{\langle N\rangle_t}{t} &= \frac{\lambda^2}{t}\int_0^t \frac{\mathrm{d}s}{M_s^2}\int_{-1}^{1}\mathrm{d}y\,[\sigma(u(s\,,y\,;\lambda))]^2\\ &\geqslant \frac{\lambda^2}{t}\int_0^t \frac{\mathrm{d}s}{M_s^2}\int_{-1}^{1}\mathrm{d}y\,\mathrm{L}_\sigma^2|u(s\,,y\,;\lambda)|^2\\ &\geqslant \frac{\lambda^2\mathrm{L}_\sigma^2}{2},\end{aligned} \tag{3.5}$$

owing to Condition (1.3) and the Cauchy–Schwarz inequality. In light of (3.4),

$$\mathrm{P}\left\{\exists s \geqslant t:\ M_s \geqslant M_0\mathrm{e}^{-\beta s}\right\} = \mathrm{P}\left\{\exists s \geqslant t:\ N_s \geqslant \tfrac{1}{2}\langle N\rangle_s - \beta s\right\},$$

[2] There is an extension of the method of Mueller and Mueller–Nualart [23,24]—see the proof of Theorem 1.7 of Conus et al [5]—that proves (3.2) in the present more general choices of σ for SPDEs on $\mathbb{R}_+ \times \mathbb{R}$. The latter argument works in exactly the same way in the present setting.

for all $\beta, t > 0$. Therefore, we may first use (3.5) and then appeal to Lemma 3.2 in order to see that, as long as $0 < \beta < \lambda^2 \mathrm{L}_\sigma^2/4$,

$$\begin{aligned} \mathrm{P}\left\{\exists s \geqslant t :\ M_s \geqslant M_0 \mathrm{e}^{-\beta s}\right\} &\leqslant \mathrm{P}\left\{N_s \geqslant \left(\frac{1}{2} - \frac{2\beta}{\lambda^2 \mathrm{L}_\sigma^2}\right) \langle N\rangle_s \text{ for some } s \geqslant t\right\} \\ &\leqslant \exp\left(-\frac{\lambda^2 \mathrm{L}_\sigma^2 t}{4}\left(\frac{1}{2} - \frac{2\beta}{\lambda^2 \mathrm{L}_\sigma^2}\right)^2\right) \\ &= \exp\left(-t\lambda^{-2}\mathrm{L}_\sigma^{-2}\left(\frac{\lambda^2 \mathrm{L}_\sigma^2}{4} - \beta\right)^2\right). \end{aligned}$$

Substitute $\beta = \frac{1}{4}(1-\varepsilon)\lambda^2 \mathrm{L}_\sigma^2$ to deduce Proposition 3.1. □

4 Regularity

In order to prove the announced regularity properties of the solution u to (1.9) we first require a moment bound, with explicit constants, for the solution u.

Proposition 4.1 *Choose and fix a real number $c > 48$. Then, for all real numbers $k \geqslant 2$ and $\lambda > 0$ that satisfy $k\lambda^2 \geqslant (c Lip_\sigma^2)^{-1}$, the following holds: Uniformly for all $t > 0$,*

$$\sup_{x\in\mathbb{T}} \mathrm{E}\left(|u(t\,,x\,;\lambda)|^k\right) \leqslant 2^{k/2}\left(1 - \frac{48}{c}\right)^{-k/2} \|u_0\|_{L^\infty}^k \cdot \exp\left(\frac{c^2}{2} Lip_\sigma^2 k^3 \lambda^4 t\right).$$

Proposition 4.1 implies also (2.6).

Proof We modify some of the ideas of Foondun and Khoshnevisan [10], but need to make a series of modifications. Define

$$\vartheta := c^2 \mathrm{Lip}_\sigma^4 k^2 \lambda^4, \tag{4.1}$$

where, $c > 48$ is large enough to ensure that $\vartheta \geqslant 1$ whenever $k\lambda^2 \geqslant (c\mathrm{Lip}_\sigma^2)^{-1}$ holds.

For all $t \geqslant 0$ and $-1 \leqslant x \leqslant 1$, let $u_t^{(0)}(x) := u_0(x\,;\lambda)$ and define iteratively for all $n \geqslant 0$,

$$u_t^{(n+1)}(x\,;\lambda) = (P_t u_0)(x) + \mathcal{I}_t^{(n)}(x), \tag{4.2}$$

where $\{P_t\}_{t\geqslant 0}$ continues to denote the heat semigroup—see (2.3)—and

$$\mathcal{I}_t^{(n)}(x) = \mathcal{I}_t^{(n)}(x\,;\lambda) := \lambda \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y)\sigma(u_s^{(n)}(y))\, W(\mathrm{d}s\,\mathrm{d}y).$$

The random field $(t\,,x) \mapsto u_t^{(n)}(x)$ is the nth-stage Picard-iteration approximation of $u(t\,,x\,;\lambda)$.

It is well known (see [28, Chap. 3]) that

$$\lim_{n\to\infty} \sup_{t\in(0,T]} \sup_{x\in\mathbb{T}} \mathrm{E}\left(\left|u_t^{(n)}(x) - u(t\,,x)\right|^k\right) = 0, \tag{4.3}$$

and

$$\lim_{n\to\infty} \sup_{t\in(0,T]} \sup_{x\in\mathbb{T}} \mathrm{E}\left(\left|\mathcal{I}_t^{(n)}(x) - \mathcal{I}_t(x\,;\lambda)\right|^k\right) = 0,$$

for all $T \in (0\,,\infty)$ and $k \in [1\,,\infty)$.

Since the semigroup $\{P_t\}_{t\geqslant 0}$ is conservative, $(P_t u_0)(x) \leqslant \|u_0\|_{L^\infty}$ for all $t \geqslant 0$ and $x \in \mathbb{T}$. Therefore, (4.2) implies that for all integers $n \geqslant 0$ and real numbers $k \in [2\,,\infty)$, $t > 0$, and $x \in \mathbb{T}$,

$$\left\| u_t^{(n+1)}(x) \right\|_k \leqslant \|(P_t u_0)(x)\|_k + \left\| \mathcal{I}_t^{(n)}(x) \right\|_k \leqslant \|u_0\|_{L^\infty} + \left\| \mathcal{I}_t^{(n)}(x) \right\|_k. \tag{4.4}$$

A Burkholder-Davis-Gundy-type inequality for stochastic convolutions (see [16, Pr. 4.4, p. 36]) then yields the following inequality:

$$\begin{aligned} \left\| \mathcal{I}_t^{(n)}(x) \right\|_k &\leqslant \sqrt{4k\lambda^2 \int_0^t \mathrm{d}s \int_{-1}^1 \mathrm{d}y\ \left[p_{t-s}(x\,,y)\right]^2 \left\| \sigma(u_s^{(n)}(y)) \right\|_k^2} \\ &\leqslant \mathrm{Lip}_\sigma \sqrt{4k\lambda^2 \int_0^t \mathrm{d}s \int_{-1}^1 \mathrm{d}y\ \left[p_{t-s}(x\,,y)\right]^2 \left\| u_s^{(n)}(y) \right\|_k^2}. \end{aligned} \tag{4.5}$$

By the Chapman–Kolmogorov equation and symmetry,

$$\begin{aligned} \int_{-1}^1 \left[p_{t-s}(x\,,y)\right]^2 \mathrm{d}y &= \int_{-1}^1 p_{t-s}(x\,,y) p_{t-s}(y\,,x)\,\mathrm{d}y = p_{2(t-s)}(x\,,x) \\ &\leqslant 2\left(\frac{1}{\sqrt{t-s}} + 1\right); \end{aligned} \tag{4.6}$$

the final estimate is justified by Lemma B.1 below. Let us define

$$\psi^{(n)}(t) := \sup_{x\in\mathbb{T}} \left\| u_t^{(n)}(x) \right\|_k^2 \qquad \text{for all } t \geqslant 0 \text{ and integers } n \geqslant 0.$$

We can combine (4.5) and (4.6) and use the elementary inequality $(a+b)^2 \leqslant 2a^2 + 2b^2$ to see that

$$\psi^{(n+1)}(t) \leqslant 2\|u_0\|_{L^\infty}^2 + 16\mathrm{Lip}_\sigma^2 k\lambda^2 \int_0^t \psi^{(n)}(s) \left(\frac{1}{\sqrt{t-s}} + 1\right) \mathrm{d}s.$$

Multiply both sides by $\exp(-\vartheta t)$ in order to see that

$$\Psi_n := \sup_{t\geqslant 0} \left[\mathrm{e}^{-\vartheta t} \psi^{(n)}(t) \right]$$

satisfies

$$\begin{aligned} \Psi_{n+1} &\leqslant 2\|u_0\|_{L^\infty}^2 + 16\mathrm{Lip}_\sigma^2 \Psi_n k\lambda^2 \sup_{t\geqslant 0} \int_0^t \mathrm{e}^{-\vartheta(t-s)} \left(\frac{1}{\sqrt{t-s}} + 1\right) \mathrm{d}s \\ &\leqslant 2\|u_0\|_{L^\infty}^2 + 16\mathrm{Lip}_\sigma^2 \Psi_n k\lambda^2 \int_0^\infty \mathrm{e}^{-\vartheta r} \left(\frac{1}{\sqrt{r}} + 1\right) \mathrm{d}r \\ &= 2\|u_0\|_{L^\infty}^2 + 16\mathrm{Lip}_\sigma^2 \Psi_n k\lambda^2 \left(\sqrt{\frac{\pi}{\vartheta}} + \frac{1}{\vartheta}\right). \end{aligned}$$

Because $\vartheta \geqslant 1$, we have $\sqrt{\pi/\vartheta} + \vartheta^{-1} \leqslant 3/\sqrt{\vartheta}$, and hence

$$\Psi_{n+1} \leqslant 2\|u_0\|_{L^\infty}^2 + \frac{48\mathrm{Lip}_\sigma^2 k\lambda^2}{\sqrt{\vartheta}} \Psi_n = 2\|u_0\|_{L^\infty}^2 + \frac{48}{c}\Psi_n \quad \text{for all } \alpha \geqslant 1 \text{ and } n \geqslant 0.$$

The second line follows from the first, thanks to (4.1) and the fact that $c > 48$. Because $\Psi_0 = \sup_{x\in\mathbb{T}} u_0(x)$ is finite, the preceding implies that $\sup_{n\geqslant 0}\Psi_n < \infty$, and

$$\limsup_{n\to\infty}\Psi_n \leqslant 2\|u_0\|_{L^\infty}^2\left(1-\frac{48}{c}\right)^{-1}.$$

According to (4.3) and Fatou's lemma,

$$\limsup_{n\to\infty}\Psi_n \geqslant \sup_{t\geqslant 0}\sup_{x\in\mathbb{T}}\left[\mathrm{e}^{-\vartheta t}\|u(t\,,x\,;\lambda)\|_k^2\right].$$

Therefore, we may combine the preceding two displays, all the time remembering our choice of ϑ, in order to conclude that

$$\|u(t\,,x\,;\lambda)\|_k^2 \leqslant 2\|u_0\|_{L^\infty}^2\mathrm{e}^{\vartheta t}\left(1-\frac{48}{c}\right)^{-1},$$

uniformly for all $-1 \leqslant x \leqslant 1$ and $t > 0$, and all $k \geqslant 2$ and $\lambda > 0$ that ensure that $\vartheta \geqslant 1$. This is another way to state the proposition. □

We now use our moment bound [Proposition 4.1] to establish the regularity of $\lambda \mapsto u(t\,,x\,;\lambda)$.

Proposition 4.2 *Choose and fix a real number $c > 48$. Then, for all real numbers $k \geqslant 2$ and $\alpha, \beta > 0$ that satisfy $k(\alpha\vee\beta)^2 \geqslant (c\mathrm{Lip}_\sigma^2)^{-1}$, the following holds: Uniformly for all $t > 0$,*

$$\sup_{x\in\mathbb{T}}\mathrm{E}\left(|u(t\,,x\,;\alpha)-u(t\,,x\,;\beta)|^k\right) \leqslant L_c^{k/2}\|u_0\|_{L^\infty}^k\exp\left(\frac{c^2}{2}\mathrm{Lip}_\sigma^4 k^3(\alpha\vee\beta)^4 t\right)\cdot\frac{|\alpha-\beta|^k}{(\alpha\wedge\beta)^k},$$

with $L_c := (96/c)(1-(48/c))^{-2}$.

Remark 4.3 Standard methods—see [28, Chap. 3]—show that $(t\,,x) \mapsto u(t\,,x\,;\lambda)$ has a continuous modification for every $\lambda > 0$. In fact, for every $\varepsilon \in (0\,,1)$, $k \geqslant 2$, $T > t_0 > 0$ and $\Lambda > 0$,

$$\sup_{\lambda\in(0\,,\Lambda)}\left\|\sup_{\substack{-1\leqslant x\neq y\leqslant 1\\ t_0<s\neq t<T}}\frac{|u(t\,,x\,;\lambda)-u(s\,,y\,;\lambda)|}{|x-y|^{(1-\varepsilon)/2}+|s-t|^{(1-\varepsilon)/4}}\right\|_k < \infty.$$

One has to be somewhat careful here since, unlike the standard theory [28], we may not choose t_0 to be zero here. The details can be found in Proposition 5.1 below. In any case, we can see from Proposition 4.2 and an appeal to the Kolmogorov continuity theorem [i.e., a chaining argument] that: (i) $(t\,,x\,,\lambda) \mapsto u(t\,,x\,;\lambda)$ has a Hölder-continuous modification on $\mathbb{R}_+\times\mathbb{T}\times(0\,,\infty)$; and (ii) That modification satisfies the following for every $p, q, r \in (0\,,1)$, $k \geqslant 2$, $\Lambda > \lambda > 0$, and $T > t_0 > 0$:

$$\left\|\sup_{\substack{-1\leqslant x\neq y\leqslant 1\\ t_0\leqslant s\neq t\leqslant T\\ \lambda\leqslant\alpha\neq\beta\leqslant\Lambda}}\frac{|u(t\,,x\,;\alpha)-u(s\,,y\,;\beta)|}{|x-y|^{p/2}+|s-t|^{q/4}+|\alpha-\beta|^r}\right\|_k < \infty.$$

To paraphrase Walsh [28], the process $\lambda \mapsto u(t\,,x\,;\lambda)$ comes tantalizingly close to being Lipschitz continuous. One can elaborate on this further as follows: Define

$$\mathscr{D}(t\,,x\,;\lambda) := \frac{\partial}{\partial\lambda} u(t\,,x\,;\lambda),$$

where the λ-derivative is understood in the sense of distributions, and exists because u is a continuous function of λ [up to a modification]; see the preceding remark. According to Rademacher's theorem, because σ is Lipschitz continuous, it has a weak derivative $\sigma' \in L^\infty(\mathbb{T})$. Then, one can appeal to a stochastic Fubini argument in order to see that $\mathscr{D}$ is the unique solution to the λ-a.e.-defined stochastic integral equation,

$$\begin{aligned}\mathscr{D}(t\,,x\,;\lambda) = &\int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y)\sigma(u(s\,,y\,;\lambda))\, W(\mathrm{d}s\,\mathrm{d}y)\\ &+\lambda \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y)\sigma'(u(s\,,y\,;\lambda))\mathscr{D}(s\,,y\,;\lambda)\, W(\mathrm{d}s\,\mathrm{d}y).\end{aligned}$$

It is not difficult to show that if σ has additional regularity properties—for instance, if σ' is Lipschitz continuous—then $\mathscr{D}$ is almost surely Hölder-continuous in its three variables [up to a modification]. This proves the following:
Proposition. If $\sigma \in C^1(\mathbb{R})$ has a Lipschitz-continuous derivative, then $\lambda \mapsto u(t\,,x\,;\lambda)$ is a.s. continuously differentiable for every $t \geqslant 0$ and $x \in \mathbb{T}$.

We do not know whether the Lipschitz-continuity of σ is really needed for this differentiability result.

Proof of Proposition 4.2 Without loss of generality, we assume throughout that $\alpha > \beta$.

We can write

$$u(t\,,x\,;\alpha) - u(t\,,x\,;\beta) = \mathcal{I}_t(x\,;\alpha) - \mathcal{I}_t(x\,;\beta) = \mathcal{T}_1 + \mathcal{T}_2,$$

where

$$\begin{aligned}\mathcal{T}_1 &:= \alpha \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y)\,[\sigma(u(s\,,y\,;\alpha)) - \sigma(u(s\,,y\,;\beta))]\, W(\mathrm{d}s\,\mathrm{d}y),\\ \mathcal{T}_2 &:= (\alpha-\beta) \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y)\sigma(u(s\,,y\,;\beta))\, W(\mathrm{d}s\,\mathrm{d}y).\end{aligned}$$

Although $\mathcal{T}_1$ and $\mathcal{T}_2$ both depend on $(x\,,t\,,\alpha,\beta)$, we have not written those parameter dependencies explicitly in order to ease the typography.

Define

$$\mathcal{D}^2 := \sup_{s\geqslant 0}\sup_{y\in\mathbb{T}} \left[\mathrm{e}^{-\vartheta s}\, \|u(s\,,y\,;\alpha) - u(s\,,y\,;\beta)\|_k^2\right],$$

where ϑ is defined as in (4.1), but with a small difference; namely,

$$\vartheta := c^2 \mathrm{Lip}_\sigma^4 k^2 \alpha^4.$$

Our condition on c is that $c > 48$ is large enough to ensure that $\vartheta \geqslant 1$.

We apply the Burkholder-Davis-Gundy-type inequality, [16, Pr. 4.4, p. 36], in order to see that

$$\begin{aligned}\|\mathcal{T}_1\|_k^2 &\leqslant 4k\alpha^2\mathrm{Lip}_\sigma^2\int_0^t \mathrm{d}s\int_{-1}^1 \mathrm{d}y\,[p_{t-s}(x\,,y)]^2\,\|u(s\,,y\,;\alpha)-u(s\,,y\,;\beta)\|_k^2\\ &\leqslant 4k\alpha^2\mathrm{Lip}_\sigma^2\mathcal{D}^2\mathrm{e}^{\vartheta t}\int_0^t \mathrm{e}^{-\vartheta s}\,\mathrm{d}s\int_{-1}^1 \mathrm{d}y\,[p_s(x\,,y)]^2\\ &= 4k\alpha^2\mathrm{Lip}_\sigma^2\mathcal{D}^2\mathrm{e}^{\vartheta t}\int_0^t p_{2s}(x\,,x)\mathrm{e}^{-\vartheta s}\,\mathrm{d}s,\end{aligned}$$

thanks to the Chapman–Kolmogorov equation and argument in Proposition 4.1. Lemma B.1 below ensures the following:

$$\begin{aligned}\|\mathcal{T}_1\|_k^2 &\leqslant 8k\alpha^2\mathrm{Lip}_\sigma^2\mathcal{D}^2\mathrm{e}^{\vartheta t}\int_0^\infty\left(\frac{1}{\sqrt{s}}+1\right)\mathrm{e}^{-\vartheta s}\,\mathrm{d}s\\ &= 8k\alpha^2\mathrm{Lip}_\sigma^2\mathcal{D}^2\mathrm{e}^{\vartheta t}\left(\sqrt{\frac{\pi}{\vartheta}}+\frac{1}{\vartheta}\right)\\ &\leqslant \frac{24\alpha^2k\mathrm{Lip}_\sigma^2\mathcal{D}^2\mathrm{e}^{\vartheta t}}{\sqrt{\vartheta}}\\ &= \frac{24}{c}\mathcal{D}^2\mathrm{e}^{\vartheta t}.\end{aligned}$$

We proceed in like manner to estimate the moments of $\mathcal{T}_2$. First, note that, because $\alpha > \beta$, a Burkholder-Davis-Gundy bound and Proposition 4.1 together imply that

$$\begin{aligned}\|\mathcal{T}_2\|_k^2 &\leqslant 4k(\alpha-\beta)^2\mathrm{Lip}_\sigma^2\int_0^t \mathrm{d}s\int_{-1}^1 \mathrm{d}y\,[p_{t-s}(x\,,y)]^2\|u(s\,,y\,;\beta)\|_k^2\\ &\leqslant 8k\left(1-\frac{48}{c}\right)^{-1}(\alpha-\beta)^2\mathrm{Lip}_\sigma^2\|u_0\|_{L^\infty}^2\int_0^t \mathrm{d}s\; e^{\vartheta s}\int_{-1}^1 \mathrm{d}y\,[p_{t-s}(x\,,y)]^2.\end{aligned}$$

Therefore, after making a change of variables, we appeal first to the Chapman–Kolmogorov and then to Lemma B.1 below in order to deduce the following:

$$\begin{aligned}\|\mathcal{T}_2\|_k^2 &\leqslant 8\left(1-\frac{48}{c}\right)^{-1}k(\alpha-\beta)^2\mathrm{Lip}_\sigma^2\|u_0\|_{L^\infty}^2\mathrm{e}^{\vartheta t}\int_0^t \mathrm{e}^{-\vartheta s}p_{2s}(x\,,x)\,\mathrm{d}s\\ &\leqslant 16\left(1-\frac{48}{c}\right)^{-1}k(\alpha-\beta)^2\mathrm{Lip}_\sigma^2\|u_0\|_{L^\infty}^2\mathrm{e}^{\vartheta t}\int_0^\infty \mathrm{e}^{-\vartheta s}\left(\frac{1}{\sqrt{s}}+1\right)\mathrm{d}s\\ &\leqslant \frac{48}{c\alpha^2}\left(1-\frac{48}{c}\right)^{-1}(\alpha-\beta)^2\|u_0\|_{L^\infty}^2\mathrm{e}^{\vartheta t}.\end{aligned}$$

We can now collect terms to find that

$$\begin{aligned}\|u(t\,,x\,;\alpha)-u(t\,,x\,;\beta)\|_k^2 &\leqslant 2\|\mathcal{T}_1\|_k^2+2\|\mathcal{T}_2\|_k^2\\ &\leqslant \frac{48}{c}\mathcal{D}^2\mathrm{e}^{\vartheta t}+\frac{96}{c\alpha^2}\left(1-\frac{48}{c}\right)^{-1}(\alpha-\beta)^2\|u_0\|_{L^\infty}^2\mathrm{e}^{\vartheta t}.\end{aligned}$$

This bound holds pointwise. Therefore, we can divide both sides by $\exp(\vartheta t)$ and optimize both sides over $x \in \mathbb{T}$ in order to conclude that

$$\mathcal{D}^2 \leqslant \frac{48}{c}\mathcal{D}^2+\frac{96}{c\alpha^2}\left(1-\frac{48}{c}\right)^{-1}(\alpha-\beta)^2\|u_0\|_{L^\infty}^2.$$

Since $\alpha > \beta$ and $\vartheta \geqslant 1$, we may appeal to Proposition 4.1—with λ there replaced by α here—in order to see that $\mathcal{D} < \infty$. In particular, because $c > 48$, we find that

$$\mathcal{D}^2 \leqslant \frac{96\|u_0\|_{L^\infty}^2}{c}\left(1-\frac{48}{c}\right)^{-2}\frac{(\alpha-\beta)^2}{\alpha^2}.$$

This is another way to state the proposition. □

5 Improved Regularity via Interpolation

In this section we use interpolation arguments to improve the moments estimates of the preceding sections and introduce new moment estimates that, among other things, justify also Remark 4.3. One of the consequences of the matter that follows is this:

Proposition 5.1 *The process $(t\,,x\,,\lambda)\mapsto u(t\,,x\,;\lambda)$ has a continuous modification, indexed also by $(0\,,\infty)\times\mathbb{T}\times(0\,,\infty)$, that weakly solves (1.9) outside of a null set that does not depend on $(t\,,x\,,\lambda)$.*

The following will be the main result of this section.

Proposition 5.2 *There exists $\varepsilon_0=\varepsilon_0(Lip_\sigma)\in(0\,,1)$, small enough, such that for every $\varepsilon\in(0\,,\varepsilon_0)$ and $t_0\geqslant 1$ there exist finite constants $C_1=C_1(\varepsilon\,,Lip_\sigma)>0$ and $C_2=C_2(Lip_\sigma)>0$—neither depending on u_0—such that uniformly for all real numbers $\lambda\geqslant 1$, $k\geqslant 2$, and $t\geqslant t_0$,*

$$\begin{aligned}&\mathrm{E}\left(\sup_{x\in\mathbb{T}}\sup_{s\in[t_0,t]}|u(s\,,x\,;\lambda)|^k\right)\\&\quad\leqslant C_1^k k^{k/2}(1+|t-t_0|)^{(\varepsilon k+2)/2}\exp\left(\frac{C_2k^3\lambda^4t}{\varepsilon^2}\right)\|u_0\|_{L^\infty}^{k\varepsilon}\|u_0\|_{L^1}^{k(1-\varepsilon)}.\end{aligned}$$

For us, the key feature of the preceding formula is the particular way in which the expectation on the left-hand side is controlled by the L^1 and L^∞ norms of u_0 on the right. Still, we do have to be somewhat careful about the other intervening constants in order to be sure that they are not too large for our later use [they fortunately are not].

We will use Proposition 5.2 and the related Proposition 5.9 in the following way. First we shift time so that we can replace u_0 by $u(t\,,\cdot\,;\lambda)$ and replace $u(t\,,\cdot\,;\lambda)$ by $u(t+h\,,\cdot\,;\lambda)$. Then, by using our propositions, we can control $u(t+h\,,\cdot\,;\lambda)$ by the product of $\|u(t\,,\cdot\,;\lambda)\|_{L^\infty}$ to a small power, $\|u(t\,,\cdot\,;\lambda)\|_{L^1}$ to a large power, and by $\exp(-Ch)$ for some positive constant C. In fact, we would rather have a negative exponential involving $t+h$, that is, $\exp\{-C(t+h)\}$. To move from h to $t+h$, we let h be a multiple of t. But we still need a negative exponent. Proposition 3.1 shows that with high probability, $\|u(t\,,\cdot\,;\lambda)\|_{L^1}$ declines exponentially fast in t, and hence also in $t+h$. We also have to deal with $\|u(t\,,\cdot\,;\lambda)\|_{L^\infty}$ raised to a small power. But here we can use Propositions 5.2 and 5.9 once more, and the small power of $\|u(t\,,\cdot\,;\lambda)\|_{L^\infty}$ means that we have introduced a slowly-growing exponential $\exp(ct)$, which is comparable to $\exp\{c'(t+h)\}$ for a small constant c'. We will see that the negative exponential wins out, with the result that $\|u(t\,,\cdot\,;\lambda)\|_\infty$ is small with high probability.

The proof of Proposition 5.2 hinges on a series of intermediary results, some of which imply Proposition 5.1 as well. We will use the mild form (2.4) to estimate $u(t\,,x\,;\lambda)$. Our first technical result is an elementary interpolation fact about the heat semigroup $\{P_t\}_{t\geqslant 0}$, defined earlier in (2.3). This result will allow us to estimate P_tu_0, the first term on the right side of (2.4). Then we will use an argument related to Gronwall's lemma to estimate the second term $\mathcal{I}_t(x\,;\lambda)$ on the right side of (2.4). In fact, $\mathcal{I}_t(x\,;\lambda)$ is an integral containing terms which also involve the heat semigroup.

Lemma 5.3 *For every $t>0$ and $\varepsilon\in(0\,,1)$,*

$$\|P_t u_0\|_{L^\infty} \leqslant 2\left(t^{-1/2}\vee 1\right)^{1-\varepsilon}\|u_0\|_{L^\infty}^{\varepsilon}\|u_0\|_{L^1}^{1-\varepsilon}.$$

Proof We first observe that

$$\|P_t u_0\|_{L^\infty} \leqslant \min\left(\|u_0\|_{L^\infty}\,,2\left[t^{-1/2}\vee 1\right]\|u_0\|_{L^1}\right). \tag{5.1}$$

Indeed, since the semigroup $\{P_t\}_{t\geqslant 0}$ is conservative, we clearly have $(P_t u_0)(x)\leqslant \|u_0\|_{L^\infty}$ for every $x\in\mathbb{T}$. And Lemma B.1 below implies that $(P_t u_0)(x)\leqslant 2(t^{-1/2}\vee 1)\|u_0\|_{L^1}$ for every $x\in\mathbb{T}$. Now that we have verified (5.1) we deduce the lemma from (5.1) and the elementary fact that $\min(a\,,b)\leqslant a^\varepsilon b^{1-\varepsilon}$ for every $a,b>0$ and $\varepsilon\in(0\,,1)$. □

Next we establish an improvement to Proposition 4.1. The following is indeed an improvement in the sense that it shows how one can control the moments of the solution to (1.9) by using both the L^∞ and the L^1 norms of the initial data, and not just the L^∞ norm of u_0. This added improvement does cost a little at small times. This latter fact is showcased by the appearance of a negative power of t in the following.

Proposition 5.4 *Let $c:=208\sqrt{2}\approx 294.2$. Then, for all real numbers $k\geqslant 2$, $\varepsilon\in(0\,,1)$, and $\lambda>0$ that satisfy $k\lambda^2\geqslant\varepsilon(c\mathrm{Lip}_\sigma^2)^{-1}$, the following holds uniformly for all $t>0$:*

$$\sup_{x\in\mathbb{T}}\mathrm{E}\left(|u(t\,,x\,;\lambda)|^k\right)\leqslant \frac{4^k}{t^{k(1-\varepsilon)/2}}\exp\left(\frac{c^2}{\varepsilon^2}k^3\lambda^4\mathrm{Lip}_\sigma^4 t\right)\|u_0\|_{L^\infty}^{k\varepsilon}\|u_0\|_{L^1}^{k(1-\varepsilon)}.$$

Proof Let $\{u^{(n)}\}_{n=0}^\infty$ be the Picard approximants of u (see (4.2)). Thanks to Lemma 5.3, we can now write the following variation of (4.4): For all integers $n\geqslant 0$ and real numbers $k\in[2\,,\infty)$, $t>0$, and $x\in\mathbb{T}$,

$$\left\|u_t^{(n+1)}(x\,;\lambda)\right\|_k \leqslant 2\left(t^{-(1-\varepsilon)/2}\vee 1\right)\|u_0\|_{L^\infty}^{\varepsilon}\|u_0\|_{L^1}^{1-\varepsilon}+\left\|\mathcal{I}_t^{(n)}(x\,;\lambda)\right\|_k. \tag{5.2}$$

The latter quantity is estimated in (4.5). If we use that estimate in (5.2), then the elementary inequality, $(a+b)^2\leqslant 2a^2+2b^2$, valid for all $a,b\in\mathbb{R}$, yields the following:

$$\begin{aligned}
&\left\|u_t^{(n+1)}(x\,;\lambda)\right\|_k^2\\
&\leqslant 8\left(t^{-(1-\varepsilon)}\vee 1\right)\|u_0\|_{L^\infty}^{2\varepsilon}\|u_0\|_{L^1}^{2(1-\varepsilon)}\\
&\quad+8k\lambda^2\mathrm{Lip}_\sigma^2\int_0^t\mathrm{d}s\int_{-1}^1\mathrm{d}y\,\left[p_{t-s}(x\,,y)\right]^2\left\|u_s^{(n)}(y\,;\lambda)\right\|_k^2\\
&\leqslant 8\left(t^{-(1-\varepsilon)}\vee 1\right)\|u_0\|_{L^\infty}^{2\varepsilon}\|u_0\|_{L^1}^{2(1-\varepsilon)}\\
&\quad+16k\lambda^2\mathrm{Lip}_\sigma^2\int_0^t\sup_{y\in\mathbb{T}}\left\|u_s^{(n)}(y\,;\lambda)\right\|_k^2\left(\frac{1}{\sqrt{t-s}}+1\right)\mathrm{d}s.
\end{aligned} \tag{5.3}$$

We have appealed to (4.6) in the last line. The preceding motivates us to consider the temporal functions, $U^{(0)}, U^{(1)},\ldots$, defined via

$$U^{(n)}(t):=\sup_{x\in\mathbb{T}}\left\|u_t^{(n)}(x\,;\lambda)\right\|_k^2\qquad[t\geqslant 0],$$

in order to obtain a recursive inequality. We can see immediately from (5.3) that, for all $n \geqslant 0$ and $t > 0$,

$$\begin{aligned} U^{(n+1)}(t) &\leqslant 8\left(t^{-(1-\varepsilon)} \vee 1\right) \|u_0\|_{L^\infty}^{2\varepsilon} \|u_0\|_{L^1}^{2(1-\varepsilon)} + 16k\lambda^2 \mathrm{Lip}_\sigma^2 \int_0^t \frac{U^{(n)}(s)}{\sqrt{t-s}}\, \mathrm{d}s \\ &\quad + 16k\lambda^2 \mathrm{Lip}_\sigma^2 \int_0^t U^{(n)}(s)\, \mathrm{d}s. \end{aligned} \tag{5.4}$$

In order to understand this recursion more deeply, let us first note that

$$\sup_{t>0}\left[\frac{t^{1-\varepsilon}}{\mathrm{e}^{\beta t}}\left(t^{-(1-\varepsilon)} \vee 1\right)\right] \leqslant 1 \vee \frac{1}{\beta^{1-\varepsilon}} \qquad \text{for all } \beta > 0. \tag{5.5}$$

This is true simply because $t^{1-\varepsilon}\mathrm{e}^{-t} \leqslant 1$ for all $t \geqslant 0$. Therefore, we may define

$$\mathscr{U}^{(m)}(\beta) := \sup_{t \geqslant 0}\left[\frac{t^{1-\varepsilon}}{\mathrm{e}^{\beta t}} U^{(m)}(t)\right] \qquad [\beta > 0,\ m \geqslant 0],$$

in order to deduce the following recursive inequality from (5.4) and (5.5):

$$\mathscr{U}^{(n+1)}(\beta) \leqslant 8\left(1 \vee \frac{1}{\beta^{1-\varepsilon}}\right) \|u_0\|_{L^\infty}^{2\varepsilon} \|u_0\|_{L^1}^{2(1-\varepsilon)} + 16k\lambda^2 \mathrm{Lip}_\sigma^2\, [C(\beta) + D(\beta)]\, \mathscr{U}^{(n)}(\beta);$$

where we have defined, for all $0 < \varepsilon < 1$ and $\beta > 0$,

$$C(\beta) := \sup_{t \geqslant 0}\left[\int_0^t \left(\frac{t}{s}\right)^{1-\varepsilon} \frac{\mathrm{e}^{-\beta(t-s)}}{\sqrt{t-s}}\, \mathrm{d}s\right] \text{ and } D(\beta) := \sup_{t \geqslant 0}\left[\int_0^t \left(\frac{t}{s}\right)^{1-\varepsilon} \mathrm{e}^{-\beta(t-s)}\, \mathrm{d}s\right].$$

It is possible to check [see Lemma A.1 of the appendix] that

$$C(\beta) + D(\beta) \leqslant \frac{13}{\varepsilon\sqrt{\beta}} \qquad \text{for all } \varepsilon \in (0\,,1) \text{ and } \beta \geqslant 1.$$

Thus, we obtain the recursive inequalities,

$$\mathscr{U}^{(n+1)}(\beta) \leqslant 8\|u_0\|_{L^\infty}^{2\varepsilon} \|u_0\|_{L^1}^{2(1-\varepsilon)} + \frac{208k\lambda^2 \mathrm{Lip}_\sigma^2}{\varepsilon\sqrt{\beta}} \mathscr{U}^{(n)}(\beta),$$

valid for all integers $n \geqslant 0$, and reals $\beta \geqslant 1$ and $\varepsilon \in (0\,,1)$. We can replace β with

$$\beta_* := \frac{173056k^2\lambda^4 \mathrm{Lip}_\sigma^4}{\varepsilon^2},$$

in order to see that for all $n \geqslant 0$,

$$\mathscr{U}^{(n+1)}(\beta_*) \leqslant 8\|u_0\|_{L^\infty}^{2\varepsilon} \|u_0\|_{L^1}^{2(1-\varepsilon)} + \tfrac{1}{2}\mathscr{U}^{(n)}(\beta_*), \tag{5.6}$$

for all $n \geqslant 0$, provided that $\beta_* \geqslant 1$. Note that

$$\mathscr{U}^{(0)}(\beta) = \sup_{t \geqslant 0}\left[t^{1-\varepsilon}\mathrm{e}^{-\beta t}\right] \|u_0\|_{L^\infty}^2 \leqslant \|u_0\|_{L^\infty}^2,$$

for every $\varepsilon \in (0\,,1)$ and $\beta \geqslant 1$. In particular, (5.6) implies that: (i) $\sup_{n \geqslant 0} \mathscr{U}^{(n)}(\beta_*) < \infty$; and (ii) For all $n \geqslant 0$, and provided that $\beta_* \geqslant 1$,

$$\limsup_{n \to \infty} \mathscr{U}^{(n+1)}(\beta_*) \leqslant 16\|u_0\|_{L^\infty}^{2\varepsilon} \|u_0\|_{L^1}^{2(1-\varepsilon)},$$

The left-hand side is greater than or equal to $t^{1-\varepsilon}\mathrm{e}^{-\beta_* t}\|u(t\,,x\,;\lambda)\|_k^2$ uniformly for all $t>0$. This is thanks to Fatou's lemma and (4.3). Therefore, for all $x\in\mathbb{T}$ and $t\geqslant 0$,

$$\|u(t\,,x\,;\lambda)\|_k^2 \leqslant 16t^{-1+\varepsilon}\mathrm{e}^{\beta_* t}\cdot\|u_0\|_{L^\infty}^{2\varepsilon}\|u_0\|_{L^1}^{2(1-\varepsilon)},$$

provided that k is large enough to ensure that $\beta_*\geqslant 1$. This is equivalent to the assertion of the proposition. □

For our next technical result, let us recall the random field $\mathcal{I}$ from (2.5).

Lemma 5.5 *Let $c:=208\sqrt{2}\approx 294.2$. For every $\varepsilon\in(0\,,1)$ and $\delta\in(0\,,1)$ there exists a finite constant $C=C(\varepsilon\,,\delta\,,Lip_\sigma)>0$—not depending on u_0—such that uniformly for all real numbers $\lambda>0$, $x,y\in\mathbb{T}$, $k\geqslant 2$, and $t>0$ that satisfy $k\lambda^2\geqslant\varepsilon(cLip_\sigma^2)^{-1}$,*

$$\mathrm{E}\left(|\mathcal{I}_t(x\,;\lambda)-\mathcal{I}_t(y\,;\lambda)|^k\right)\leqslant C^k k^{k/2}|x-y|^{\delta k/2}\max\left\{1\,,\frac{1}{t^{k/2}}\right\}\exp\left(\frac{c^2}{\varepsilon^2}k^3\lambda^4 Lip_\sigma^4 t\right)\|u_0\|_{L^\infty}^{k\varepsilon}\|u_0\|_{L^1}^{k(1-\varepsilon)}.$$

Proof We apply the Burkholder-Davis-Gundy-type inequality, as in the proof of Proposition 4.2, in order to see that

$$\begin{aligned}
\|\mathcal{I}_t(x\,;\lambda)-\mathcal{I}_t(y\,;\lambda)\|_k^2 &\leqslant 4k\lambda^2\int_0^t\mathrm{d}s\int_{-1}^1\mathrm{d}z\,\left[p_{t-s}(x\,,z)-p_{t-s}(y\,,z)\right]^2\|\sigma(u(s\,,z\,;\lambda))\|_k^2\\
&\leqslant 4k\lambda^2\mathrm{Lip}_\sigma^2\int_0^t\mathrm{d}s\int_{-1}^1\mathrm{d}z\,\left[p_{t-s}(x\,,z)-p_{t-s}(y\,,z)\right]^2\|u(s\,,z\,;\lambda)\|_k^2\\
&\leqslant 64k\mathrm{Lip}_\sigma^2\exp\left(\frac{2c^2}{\varepsilon^2}k^2\lambda^4\mathrm{Lip}_\sigma^4 t\right)\|u_0\|_{L^\infty}^{2\varepsilon}\|u_0\|_{L^1}^{2(1-\varepsilon)}\\
&\quad\cdot\int_0^t\frac{\mathrm{d}s}{s^{1-\varepsilon}}\int_{-1}^1\mathrm{d}z\,\left[p_{t-s}(x\,,z)-p_{t-s}(y\,,z)\right]^2.
\end{aligned}$$

The final inequality is a consequence of Proposition 5.4, which is why we need the condition $k\lambda^2\geqslant\varepsilon(c\mathrm{Lip}_\sigma^2)^{-1}$. Apply Lemma B.4 to see that

$$\begin{aligned}
&\int_0^t\frac{\mathrm{d}s}{s^{1-\varepsilon}}\int_{-1}^1\mathrm{d}z\,\left[p_{t-s}(x\,,z)-p_{t-s}(y\,,z)\right]^2\\
&\leqslant C'|x-y|^\delta\int_0^t\frac{\mathrm{d}s}{s^{1-\varepsilon}\times\left((t-s)^{(\delta+1)/2}\wedge(t-s)^{\delta/2}\right)}\\
&\leqslant C''\frac{|x-y|^\delta}{t^{\delta/2-\varepsilon}}\max\left\{1\,,1/\sqrt{t}\right\}\\
&\leqslant C''|x-y|^\delta\max\{1\,,1/t\}\,,
\end{aligned}$$

where C' and C'' are finite constants that depend only on ε and δ. The second inequality can be obtained by split the integral into $\int_0^{t/2}\cdots\,\mathrm{d}s$ and $\int_{t/2}^t\cdots\,\mathrm{d}s$. The first integral is less than

$$\int_0^{t/2}\frac{\mathrm{d}s}{s^{1-\varepsilon}\min\left((t/2)^{(\delta+1)/2}\,,(t/2)^{\delta/2}\right)},$$

and the second one is less than the same bound by a similar argument. The last inequality above comes from the fact that $(\delta/2)-\varepsilon+(1/2)<1$ for all $\delta\in(0\,,1)$ and $\varepsilon\in(0\,,1)$, so that $(1/t)^{\delta/2-\varepsilon+1/2}\leqslant 1/t$ for $t<1$. We combine the preceding two displays to conclude the proof of the proposition. □

We can combine Lemmas B.5 and 5.5 together with (2.4) in order to deduce the following.

Proposition 5.6 *Let $c := 208\sqrt{2} \approx 294.2$. For every $\varepsilon \in (0\,,1)$ and $\delta \in (0\,,1)$ there exists a finite constant $C = C(\varepsilon\,,\delta\,,Lip_\sigma) > 0$—not depending on u_0—such that uniformly for all real numbers $\lambda > 0$, $x, y \in \mathbb{T}$, $k \geqslant 2$, and $t > 0$ that satisfy $k\lambda^2 \geqslant \varepsilon(cLip_\sigma^2)^{-1}$,*

$$\begin{aligned}&\mathrm{E}\left(|u(t\,,x\,;\lambda) - u(t\,,y\,;\lambda)|^k\right)\\&\leqslant C^k k^{k/2}\left(|x-y|^{\delta k/2} + |x-y|^{\varepsilon k/2}\right)\\&\quad \max\left\{1\,,\frac{1}{t^{k/2}}\right\}\exp\left(\frac{c^2}{\varepsilon^2}k^3\lambda^4 Lip_\sigma^4 t\right)\|u_0\|_{L^\infty}^{k\varepsilon}\|u_0\|_{L^1}^{k(1-\varepsilon)}.\end{aligned}$$

The preceding is a moment continuity result about $x \mapsto u(t\,,x\,;\lambda)$. The following matches that result with a moment continuity estimate for $t \mapsto u(t\,,x\,;\lambda)$.

Proposition 5.7 *Let $c := 208\sqrt{2} \approx 294.2$. For all real numbers $k \geqslant 2$, $\lambda > 0$, $\delta \in (0\,,1/4)$, $\varepsilon \in (0\,,1)$, and $t > 2\delta$ that satisfy $k\lambda^2 \geqslant \varepsilon(cLip_\sigma^2)^{-1}$,*

$$\begin{aligned}&\mathrm{E}\left(|u(t+\delta\,,x\,;\lambda) - u(t\,,x\,;\lambda)|^k\right)\\&\leqslant 74^k\left[\frac{\delta^\varepsilon}{t^{1+\varepsilon}} + \frac{\lambda^2 k\sqrt{\delta}\,Lip_\sigma^2}{\varepsilon t^{1-\varepsilon}}\right]^{k/2}\exp\left(\frac{c^2}{\varepsilon^2}k^3\lambda^4 Lip_\sigma^4 t\right)\\&\quad\cdot\|u_0\|_{L^\infty}^{k\varepsilon}\|u_0\|_{L^1}^{k(1-\varepsilon)}.\end{aligned}$$

Proof In accord with (2.4) we can write

$$\|\mathcal{I}_{t+\delta}(x\,;\lambda) - \mathcal{I}_t(x\,;\lambda)\|_k^2 \leqslant 2\lambda^2\left(\|T_1\|_k^2 + \|T_2\|_k^2\right), \tag{5.7}$$

where

$$\begin{aligned}T_1 &:= \int_{(t,t+\delta)\times\mathbb{T}} p_{t-s+\delta}(x\,,z)\sigma(u(s\,,z\,;\lambda))\,W(\mathrm{d}s\,\mathrm{d}z),\\T_2 &:= \int_{(0,t)\times\mathbb{T}}\left[p_{t-s+\delta}(x\,,z) - p_{t-s}(x\,,z)\right]\sigma(u(s\,,z\,;\lambda))\,W(\mathrm{d}s\,\mathrm{d}z).\end{aligned}$$

Now we apply the Burkholder-Davis-Gundy-type inequality, as in the proof of Proposition 4.2, in order to see that

$$\begin{aligned}\|T_1\|_k^2 &\leqslant 4k\mathrm{Lip}_\sigma^2\int_t^{t+\delta}\mathrm{d}s\int_{-1}^1\mathrm{d}z\,\left[p_{t-s+\delta}(x\,,z)\right]^2\|u(s\,,z\,;\lambda)\|_k^2\\&\leqslant \frac{64k\mathrm{Lip}_\sigma^2}{t^{1-\varepsilon}}\exp\left(\frac{2c^2}{\varepsilon^2}k^2\lambda^4\mathrm{Lip}_\sigma^4 t\right)\|u_0\|_{L^\infty}^{2\varepsilon}\|u_0\|_{L^1}^{2(1-\varepsilon)}\int_t^{t+\delta}\mathrm{d}s\int_{-1}^1\mathrm{d}z\,\left[p_{t-s+\delta}(x\,,z)\right]^2;\end{aligned}$$

consult Proposition 5.4 for the last line. We appeal first to the semigroup property of p_t and then to Lemma B.1 below in order to see from the bound $\delta \in (0\,,1/4)$ that

$$\int_t^{t+\delta}\mathrm{d}s\int_{-1}^1\mathrm{d}z\,\left[p_{t-s+\delta}(x\,,z)\right]^2 \leqslant 2\int_0^\delta\left(\frac{1}{\sqrt{s}}+1\right)\mathrm{d}s \leqslant 4\sqrt{\delta}+2\delta < 5\sqrt{\delta},$$

whence it follows that

$$\|T_1\|_k^2 \leqslant \frac{320k\mathrm{Lip}_\sigma^2}{t^{1-\varepsilon}}\exp\left(\frac{2c^2}{\varepsilon^2}k^2\lambda^4\mathrm{Lip}_\sigma^4 t\right)\sqrt{\delta}\cdot\|u_0\|_{L^\infty}^{2\varepsilon}\|u_0\|_{L^1}^{2(1-\varepsilon)}. \tag{5.8}$$

Similarly, we have

$$\begin{aligned}\|T_2\|_k^2 &\leqslant 4k\mathrm{Lip}_\sigma^2 \int_0^t \mathrm{d}s \int_{-1}^1 \mathrm{d}z\ \left[p_{t-s+\delta}(x\,,z) - p_{t-s}(x\,,z)\right]^2 \|u(s\,,z\,;\lambda)\|_k^2 \\ &\leqslant 64k\mathrm{Lip}_\sigma^2 \exp\left(\frac{2c^2}{\varepsilon^2}k^2\lambda^4\mathrm{Lip}_\sigma^4 t\right) \|u_0\|_{L^\infty}^{2\varepsilon}\|u_0\|_{L^1}^{2(1-\varepsilon)} \times Q_t(\delta),\end{aligned} \tag{5.9}$$

where

$$Q_t(\delta) := \int_0^t \frac{\mathrm{d}s}{s^{1-\varepsilon}} \int_{-1}^1 \mathrm{d}z\ \left[p_{t-s+\delta}(x\,,z) - p_{t-s}(x\,,z)\right]^2.$$

Lemma B.6 below tells us that

$$\begin{aligned}Q_t(\delta) &\leqslant \sqrt{\frac{\pi}{2}} \int_0^t \frac{\mathrm{d}s}{s^{1-\varepsilon}\sqrt{t-s}} \min\left(1\,, \frac{\delta}{t-s}\right) \\ &= \sqrt{\frac{\pi}{2}} t^{\varepsilon-(1/2)} \int_0^1 \frac{\mathrm{d}r}{r^{1-\varepsilon}\sqrt{1-r}} \min\left(1\,, \frac{\delta/t}{1-r}\right).\end{aligned}$$

If $t > 2\delta$, then we write

$$Q_t(\delta) \leqslant \sqrt{\frac{\pi}{2}}\delta t^{\varepsilon-(3/2)} \int_0^{1-(\delta/t)} \frac{\mathrm{d}r}{r^{1-\varepsilon}(1-r)^{3/2}} + \sqrt{\frac{\pi}{2}} t^{\varepsilon-(1/2)} \int_{1-(\delta/t)}^1 \frac{\mathrm{d}r}{r^{1-\varepsilon}\sqrt{1-r}}.$$

We can write the first integral as

$$\begin{aligned}\int_0^{1/2} \frac{\mathrm{d}r}{r^{1-\varepsilon}(1-r)^{3/2}} + \int_{1/2}^{1-(\delta/t)} \frac{\mathrm{d}r}{r^{1-\varepsilon}(1-r)^{3/2}} &\leqslant \frac{2^{(3/2)-\varepsilon}}{\varepsilon} + 2^{1-\varepsilon}\int_{\delta/t}^\infty \frac{\mathrm{d}r}{r^{3/2}} \\ &= \frac{2^{(3/2)-\varepsilon}}{\varepsilon} + 2^{2-\varepsilon}(\delta/t)^{-1/2} \\ &\leqslant \frac{8}{\varepsilon}(\delta/t)^{-1/2}.\end{aligned}$$

[We have used the bound $t > 2\delta$ in the last line.] And the second integral is bounded from above by

$$\left(1 - \frac{\delta}{t}\right)^{-1+\varepsilon} \int_0^{\delta/t} \frac{\mathrm{d}r}{\sqrt{r}} = 2\left(1 - \frac{\delta}{t}\right)^{-1+\varepsilon} \sqrt{\delta/t} < 4\sqrt{\delta/t}.$$

This yields

$$Q_t(\delta) \leqslant \frac{16}{\varepsilon} \cdot \frac{\sqrt{\delta}}{t^{1-\varepsilon}}.$$

We may apply this inequality in (5.9) in order to see that

$$\|T_2\|_k^2 \leqslant \frac{1024k\mathrm{Lip}_\sigma^2}{\varepsilon \cdot t^{1-\varepsilon}} \exp\left(\frac{2c^2}{\varepsilon^2}k^2\lambda^4\mathrm{Lip}_\sigma^4 t\right) \sqrt{\delta} \cdot \|u_0\|_{L^\infty}^{2\varepsilon}\|u_0\|_{L^1}^{2(1-\varepsilon)}.$$

We combine this with (5.8) and (5.7) in order to deduce the following:

$$\left\|\mathcal{I}_{t+\delta}(x\,;\lambda) - \mathcal{I}_t(x\,;\lambda)\right\|_k^2 \leqslant \lambda^2 \frac{2688k\mathrm{Lip}_\sigma^2}{t^{1-\varepsilon}} \exp\left(\frac{2c^2}{\varepsilon^2}k^2\lambda^4\mathrm{Lip}_\sigma^4 t\right) \frac{\sqrt{\delta}}{\varepsilon} \cdot \|u_0\|_{L^\infty}^{2\varepsilon}\|u_0\|_{L^1}^{2(1-\varepsilon)}.$$

This and the argument in Lemma B.7 below together yield

$$\begin{aligned}
&\|u(t+\delta\,,x\,;\lambda) - u(t\,,x\,;\lambda)\|_k^2\\
&\leqslant 2\|(P_{t+\delta}u_0)(x) - (P_t u_0)(x)\|_k^2 + 2\,\|\mathcal{I}_{t+\delta}(x\,;\lambda) - \mathcal{I}_t(x\,;\lambda)\|_k^2\\
&\leqslant \left[\frac{8\delta^\varepsilon}{t^{1+\varepsilon}} + \frac{5376k\sqrt{\delta}\,\lambda^2\mathrm{Lip}_\sigma^2}{\varepsilon t^{1-\varepsilon}}\exp\left(\frac{2c^2}{\varepsilon^2}k^2\lambda^4\mathrm{Lip}_\sigma^4 t\right)\right]\cdot\|u_0\|_{L^\infty}^{2\varepsilon}\|u_0\|_{L^1}^{2(1-\varepsilon)}.
\end{aligned}$$

This easily implies the result. □

Before we derive Proposition 5.2—the main result of this section—we pause and quickly establish Proposition 5.1.

Proof of Proposition 5.1 We can combine Propositions 4.2, 5.6, and 5.7 together with the Kolmogorov continuity theorem in order to see that $(t\,,x\,,\lambda)\mapsto u(t\,,x\,;\lambda)$ has a continuous modification on $(0\,,\infty)\times\mathbb{R}\times(0\,,\infty)$. The proofs of Propositions 4.2, 5.6, and 5.7 also imply, implicitly, the fact that two quantities on the right-hand side of (2.4)—viewed as random functions of $(t\,,x\,,\lambda)$—have continuous modifications on $(0\,,\infty)\times\mathbb{R}\times(0\,,\infty)$. It follows that (2.4) holds for all $(t\,,x\,,\lambda)\in(0\,,\infty)\times\mathbb{R}\times(0\,,\infty)$ off a single null set. This and a stochastic Fubini argument together imply the result. □

We are finally ready to prove Proposition 5.2. Before we commence, however, it might be helpful to explicitly state the following well-known chaining argument [8]. It might help to recall that an *upright box* in $\mathbb{R}^N$ has the form $\prod_{i=1}^N[a_i\,,b_i]$, where $a_i\leqslant b_i$ are real numbers $[1\leqslant i\leqslant N]$.

Proposition 5.8 *Suppose $\{X(t)\}_{t\in T}$ is a real-valued stochastic process, where T is a bounded upright box in $\mathbb{R}^N$ for some $N\geqslant 1$. Suppose also that there exists $Q\in(0\,,\infty)$ such that for every integer $K\geqslant 2$*

$$B_K := \mathrm{E}\left(|X(s)|^K\right) < \infty \quad \textit{for some } s\in T \textit{ and} \quad C_K := \sup_{\substack{s,t\in T\\ s\neq t}} \mathrm{E}\left(\frac{|X(t)-X(s)|^K}{|t-s|^{KQ}}\right) < \infty,$$

where $|\tau|$ denotes any one of the ℓ^p-norms on $\tau\in\mathbb{R}^N$ $[0<p<\infty]$. Then, there exists a finite constant D— depending also on the diameter of T, N, Q and K with $QK>N$, —such that

$$\mathrm{E}\Bigg(\sup_{\substack{s,t\in T\\ s\neq t}}|X(t)-X(s)|^K\Bigg)\leqslant D^K C_K \quad \textit{and hence} \quad \mathrm{E}\left(\sup_{t\in T}|X(t)|^K\right)\leqslant 2^K(B_K + D^K C_K).$$

Proof of Proposition 5.2 Combine Propositions 5.6, 5.7, and 5.8, all the time keeping track of the various [explicit] constants. □

Let us observe also the following fixed-time result, which is proved exactly as Proposition 5.2 was, but without the t-uniformity.

Proposition 5.9 *There exists $\varepsilon_0=\varepsilon_0(Lip_\sigma)\in(0\,,1)$ such that for every $\varepsilon\in(0\,,\varepsilon_0)$ there exist finite constants $C_1=C_1(\varepsilon\,,Lip_\sigma)>0$ and $C_2=C_2(Lip_\sigma)>0$—not depending on u_0—such that uniformly for all real numbers $\lambda\geqslant 1$, $k\geqslant 2$ that satisfy $k\lambda^2\geqslant\varepsilon(cLip_\sigma)^{-1}$, and for every $t>0$,*

$$\mathrm{E}\left(\sup_{x\in\mathbb{T}}|u(t\,,x\,;\lambda)|^k\right)\leqslant C_1^k k^{k/2}\left(1+\frac{1}{t^{k/2}}\right)\exp\left(\frac{C_2k^3\lambda^4 t}{\varepsilon^2}\right)\|u_0\|_{L^\infty}^{k\varepsilon}\|u_0\|_{L^1}^{k(1-\varepsilon)}.$$

Proof of Proposition 5.9 Combine Propositions 5.4, 5.6 and 5.8, all the time keeping track of the various [explicit] constants. □

6 Proof of Theorem 1.4

Define $\mathscr{F} := \{\mathscr{F}_t^0\}_{t>0}$ denote the filtration of sigma-algebras that is defined via

$$\mathscr{F}_t^0 := \sigma\left\{\int_{(0,t)\times\mathbb{T}} \phi(s\,,y)\,W(\mathrm{d}s\,\mathrm{d}y) :\ \phi \in L^2((0\,,t]\times\mathbb{T})\right\},$$

for every $t \geqslant 0$. Let P^{u_0} denote the law of the process $\{u(t\,,x\,;\lambda)\}_{t\geqslant 0, x\in\mathbb{T}}$, conditional on the initial state being u_0. Then we can define

$$\mathscr{F}_t := \bigcap_{s>t} \overline{\mathscr{F}_s^0} \qquad [t \geqslant 0], \tag{6.1}$$

where $\overline{\mathscr{F}_s^0}$ denotes the completion of $\mathscr{F}_s^0$ with respect to the family $\{\mathrm{P}^{u_0}\}_{u_0\in L^\infty}$ of probability measures. Intuitively speaking, the filtration $\mathscr{F} := \{\mathscr{F}_t\}_{t\geqslant 0}$ is the Brownian filtration that corresponds to the infinite-dimensional Brownian motion $t \mapsto W(t\,,\cdot)$.

It is well known, see [12, Theorem 9.15, p. 256], that the process $t \mapsto u(t\,,\cdot\,;\lambda)$ is a Markov process, with values in $C(\mathbb{T})$, with respect to the filtration $\mathscr{F} := \{\mathscr{F}_t\}_{t\geqslant 0}$ and initial measures $\{\mathrm{P}^{u_0}\}_{u_0\in L^\infty}$. That is,

$$\mathrm{E}\left[\Phi(u(t+\tau\,,\cdot\,;\lambda)) \mid \mathscr{F}_t\right] = \mathrm{E}^{u_t}\left[\Phi(u(\tau\,,\cdot\,;\lambda))\right] \qquad \text{a.s.},$$

for every bounded functional $\Phi : C(\mathbb{T}) \to \mathbb{R}_+$, and all $t, \tau \geqslant 0$, where we have suppressed the notational dependence on λ to keep the notation simple. We can restate this fact as follows: Choose and fix $t \geqslant 0$ and define $v(\tau\,,x) := u(t+\tau\,,x\,;\lambda)$ for all $\tau \geqslant 0$ and $x \in \mathbb{T}$. Then, conditioned on $\mathscr{F}_t$, the random field $\{v(\tau\,,x)\}_{\tau\geqslant 0, x\in\mathbb{T}}$ solves the SPDE (1.9) [in law], started at $v(0\,,x) := u(t\,,x\,;\lambda)$, where now the noise W is replaced by a Brownian sheet $W^{(t)}$ that is independent of $\mathscr{F}_t$. In particular, we may appeal to Proposition 5.9, conditionally, as follows: There exists $\varepsilon_0 = \varepsilon_0(\mathrm{Lip}_\sigma) \in (0\,,1/2)$ such that for every $\varepsilon \in (0\,,\varepsilon_0)$ there exist finite constants $C_1 = C_1(\varepsilon\,,\mathrm{Lip}_\sigma) > 0$ and $C_2 = C_2(\mathrm{Lip}_\sigma) > 0$—not depending on u_0—such that uniformly for all real numbers $\lambda \geqslant 2$ and $t > 0$ and $h \in (0, 1)$,

$$\mathrm{E}\left(\sup_{x\in\mathbb{T}} |u(t+h\,,x\,;\lambda)|^2 \,\middle|\, \mathscr{F}_t\right) \leqslant \frac{C_1 \mathrm{e}^{C_2\lambda^4 h/\varepsilon^2}}{h} \|u(t\,,\cdot\,;\lambda)\|_{L^\infty}^{2\varepsilon} \|u(t\,,\cdot\,;\lambda)\|_{L^1}^{2(1-\varepsilon)}, \tag{6.2}$$

almost surely.

Now consider the event,

$$\mathbf{A}(t\,;\lambda) := \left\{\omega \in \Omega :\ \|u(t\,,\cdot\,;\lambda)(\omega)\|_{L^1} \leqslant \|u_0\|_{L^1} \exp\left(-\frac{\lambda^2 \mathrm{L}_\sigma^2 t}{8}\right)\right\}. \tag{6.3}$$

According to Proposition 3.1 [with $\varepsilon := 1/2$],

$$\mathrm{P}(\mathbf{A}(t\,;\lambda)) \geqslant 1 - \mathrm{e}^{-\lambda^2 \mathrm{L}_\sigma^2 t/64}. \tag{6.4}$$

Also, (6.2) implies that

$$\begin{aligned} &\mathrm{E}\left(\sup_{x\in\mathbb{T}}|u(t+h\,,x\,;\lambda)|^2\,;\ \mathbf{A}(t\,;\lambda)\right)\\ &\quad\leqslant \frac{C_1 \mathrm{e}^{C_2\lambda^4 h/\varepsilon^2}\|u_0\|_{L^1}^{2(1-\varepsilon)}}{h}\exp\left(-\frac{(1-\varepsilon)\lambda^2\mathrm{L}_\sigma^2 t}{4}\right)\mathrm{E}\left(\|u(t\,,\cdot\,;\lambda)\|_{L^\infty}^{2\varepsilon}\right). \end{aligned} \tag{6.5}$$

Since $\varepsilon \in (0\,,\varepsilon_0) \subset (0\,,1)$, Jensen's inequality shows that

$$\mathrm{E}\left(\|u(t\,,\cdot\,;\lambda)\|_{L^\infty}^{2\varepsilon}\right) \leqslant \left|\mathrm{E}\left(\|u(t\,,\cdot\,;\lambda)\|_{L^\infty}^{2}\right)\right|^{\varepsilon}. \tag{6.6}$$

Proposition 5.9 implies the following [set $k := 2$ and $\varepsilon := 1/2$ in the statement of the proposition]: There exists a positive and finite constant C_3 such that for all $t > 0$ and $\lambda^2 \geqslant (4c\mathrm{Lip}_\sigma)^{-1}$,

$$\mathrm{E}\left(\|u(t\,,\cdot\,;\lambda)\|_{L^\infty}^{2}\right) \leqslant \frac{C_3\mathrm{e}^{C_3\lambda^4 t}}{t}\|u_0\|_{L^\infty}\|u_0\|_{L^1}.$$

We plug this estimate into (6.6), and then appeal to (6.5), and the fact that $\varepsilon < \varepsilon_0 < 1/2$, in order to see that

$$\mathrm{E}\left(\sup_{x\in\mathbb{T}}|u(t+h\,,x\,;\lambda)|^2\,;\ \mathbf{A}(t\,;\lambda)\right) \leqslant \frac{C_4\mathrm{e}^{C_5\lambda^4[(h/\varepsilon^2)+\varepsilon t]}\|u_0\|_{L^1}^{2-\varepsilon}\|u_0\|_{L^\infty}^{\varepsilon}}{ht^\varepsilon}\mathrm{e}^{-\lambda^2\mathrm{L}_\sigma^2 t/8}, \tag{6.7}$$

where $C_4 := C_1 C_3^\varepsilon$ and $C_5 := \max(C_2\,, C_3)$. Note that the implied constants do not depend on $(t\,,h\,,\lambda)$.

We now specialize the preceding to the following choice of ε and h:

$$\varepsilon := \frac{\mathrm{L}_\sigma^2}{32C_5\lambda^2} \quad\text{and}\quad h := \frac{\mathrm{L}_\sigma^6 t}{(32C_5\lambda^2)^3}.$$

This choice is permissible, provided that $\varepsilon < \varepsilon_0 < 1/2$; since if λ is large enough so that $\varepsilon < \varepsilon_0$ and $h < 1$. Because ε_0 does not depend on t, it follows that for every $t > 0$,

$$\limsup_{\lambda\uparrow\infty}\frac{1}{\lambda^2}\log\mathrm{E}\left(\sup_{x\in\mathbb{T}}\left|u\left(t+C_6\lambda^{-6}\,,x\,;\lambda\right)\right|^2;\ \mathbf{A}(t\,;\lambda)\right) \leqslant -\frac{\mathrm{L}_\sigma^2 t}{16},$$

where $C_6 = \mathrm{L}_\sigma^6(32C_5)^{-3}t$. By the Chebyshev inequality,

$$\limsup_{\lambda\uparrow\infty}\frac{1}{\lambda^2}\log\mathrm{P}\left\{\sup_{x\in\mathbb{T}}u\left(t+C_6\lambda^{-6},x\,;\lambda\right) \geqslant \exp\left(-\frac{\mathrm{L}_\sigma^2\lambda^2 t}{64}\right);\ \mathbf{A}(t\,;\lambda)\right\} \leqslant -\frac{\mathrm{L}_\sigma^2 t}{32}.$$

It is easy to see, after a change of variables in the preceding quantitative bounds [before we apply the limsup], that the preceding holds also with $u(t\,,x\,;\lambda)$ in place of $u(t+C_6\lambda^{-6}\,,x\,;\lambda)$. For otherwise, we simply replace t by $t - C_6\lambda^{-6}$ in all of the formulas before we let $\lambda\uparrow\infty$. In this way, we can combine the above estimate with (6.4) in order to deduce the theorem. □

7 Proof of Theorem 1.2

Proof of Theorem 1.2: Upper bound Throughout, we choose and hold $\lambda > 0$ fixed.

The proof of (6.2) shows also the following variation, thanks to Proposition 5.2: There exists $\varepsilon_0 = \varepsilon_0(\mathrm{Lip}_\sigma) \in (0\,,1/2)$, small enough, such that for every $\varepsilon \in (0\,,\varepsilon_0)$ there exist finite constants $C_1 = C_1(\varepsilon\,,\mathrm{Lip}_\sigma) > 0$ and $C_2 = C_2(\mathrm{Lip}_\sigma) > 0$—not depending on u_0—such that uniformly for all real numbers $\lambda \geqslant 2$ and $\eta \in (0\,,1)$ and $t \geqslant t_0 := 1$,

$$\mathrm{E}\left(\sup_{x\in\mathbb{T}}\sup_{h\in[1,\eta(t+1)+1]} |u(t+h\,,x\,;\lambda)|^2 \,\middle|\, \mathscr{F}_t\right) \leqslant C_1\,[1+\eta(t+1)]^{3/2}\,\mathrm{e}^{C_2\lambda^4(\eta(t+1)+1)/\varepsilon^2}\,\|u(t\,,\cdot\,;\lambda)\|_{L^\infty}^{2\varepsilon}\,\|u(t\,,\cdot\,;\lambda)\|_{L^1}^{2(1-\varepsilon)},$$

almost surely. We appeal to this bound with $\eta := \varepsilon^3$ in order to see that for all real numbers $\lambda \geqslant 2$, $\eta \in (0\,,1)$, and $t \geqslant t_0 := 1$,

$$\mathrm{E}\left(\sup_{x\in\mathbb{T}}\sup_{h\in[1,\varepsilon^3(t+1)+1]} |u(t+h\,,x\,;\lambda)|^2 \,\middle|\, \mathscr{F}_t\right) \leqslant C_1\left[1+\varepsilon^3(t+1)\right]^{3/2}\mathrm{e}^{C_2\lambda^4/\varepsilon^2}\mathrm{e}^{C_2\lambda^4\varepsilon(t+1)}\,\|u(t\,,\cdot\,;\lambda)\|_{L^\infty}^{2\varepsilon}\,\|u(t\,,\cdot\,;\lambda)\|_{L^1}^{2(1-\varepsilon)},$$

almost surely. It follows from this inequality that, for the same set $\mathbf{A}(t\,;\lambda)$ as was defined in (6.3), the following variation of (6.7) holds:

$$\begin{aligned}\mathrm{E}&\left(\sup_{x\in\mathbb{T}}\sup_{h\in[1,\varepsilon^3(t+1)+1]} |u(t+h\,,x\,;\lambda)|^2;\ \mathbf{A}(t\,;\lambda)\right)\\ &\leqslant C_7\left[1+\varepsilon^3(t+1)\right]^{3/2}\mathrm{e}^{2C_5\lambda^4\varepsilon t}\,\|u_0\|_{L^1}^{2-\varepsilon}\,\|u_0\|_{L^\infty}^{\varepsilon}\,\mathrm{e}^{-\lambda^2\mathrm{L}_\sigma^2 t/8}\\ &= C_7\left[1+\varepsilon^3(t+1)\right]^{3/2}\exp\left(-\lambda^2 t\left[\frac{\mathrm{L}_\sigma^2}{8}-2C_5\lambda^2\varepsilon\right]\right)\|u_0\|_{L^1}^{2-\varepsilon}\,\|u_0\|_{L^\infty}^{\varepsilon},\end{aligned}$$

provided, additionally, that $0 < \varepsilon < \varepsilon_0$; here, $C_7 = C_7(\varepsilon\,,\mathrm{Lip}_\sigma,\lambda)$ is a positive and finite constant, and C_5 is the same constant that appeared in Sect. 6. We use the preceding inequality with the following special choice:

$$\varepsilon := \min\left(\frac{\varepsilon_0}{2}\,,\,\frac{\mathrm{L}_\sigma^2}{32C_5\lambda^2}\right).$$

For this particular choice of ε, we have

$$\mathrm{E}\left(\sup_{x\in\mathbb{T}}\sup_{s\in[t+1,(1+\varepsilon^3)(t+1)]} |u(s\,,x\,;\lambda)|^2;\ \mathbf{A}(t\,;\lambda)\right) \leqslant C_8\left[1+\varepsilon^3(t+1)\right]^{3/2}\mathrm{e}^{-\mathrm{L}_\sigma^2\lambda^2(t+1)/16},$$

uniformly for all $t \geqslant 1$, where $C_8 := C_7\exp(\lambda^2\mathrm{L}_\sigma^2/16)\|u_0\|_{L^1}^{2-\varepsilon}\|u_0\|_{L^\infty}^{\varepsilon}$ is a finite constant that does not depend on t. Define

$$\mu := \log(1+\varepsilon^3).$$

For large integers N, we replace t by $\exp(N\mu) - 1$ to get that

$$\mathrm{E}\left(\sup_{x\in\mathbb{T}}\sup_{s\in[\exp(N\mu),\exp([N+1]\mu)]} |u(s\,,x\,;\lambda)|^2;\ \mathbf{A}(\mathrm{e}^{N\mu}-1\,;\lambda)\right) \leqslant C_8\left[1+\varepsilon^3\mathrm{e}^{N\mu}\right]^{3/2}\mathrm{e}^{-\mathrm{L}_\sigma^2\lambda^2\mathrm{e}^{N\mu}/16}.$$

In particular, Chebyshev's inequality shows that for all $\rho > 0$,

$$\begin{aligned} &\mathrm{P}\left\{\sup_{x\in\mathbb{T}}\sup_{s\in[\exp(N\mu),\exp([N+1]\mu)]}\left|\frac{u(s\,,x\,;\lambda)}{\mathrm{e}^{-\mathrm{L}_\sigma^2\lambda^2 s/64}}\right| \geqslant \rho\,;\mathbf{A}(\mathrm{e}^{N\mu}-1\,;\lambda)\right\} \\ &\quad\leqslant \mathrm{P}\left\{\sup_{x\in\mathbb{T}}\sup_{s\in[\exp(N\mu),\exp([N+1]\mu)]}|u(s\,,x\,;\lambda)| \geqslant \rho\mathrm{e}^{-\mathrm{L}_\sigma^2\lambda^2\mathrm{e}^{(N+1)\mu}/64}\,;\mathbf{A}\left(\mathrm{e}^{N\mu}-1\,;\lambda\right)\right\} \\ &\quad\leqslant \frac{C_8}{\rho^2}\left[1+\varepsilon^3\mathrm{e}^{N\mu}\right]^{3/2}\exp\left(-\frac{\mathrm{L}_\sigma^2\lambda^2\mathrm{e}^{N\mu}}{64}\right). \end{aligned}$$

Combine this estimate with (6.4) to see that

$$\begin{aligned} &\sum_{N=1}^{\infty}\mathrm{P}\left\{\sup_{x\in\mathbb{T}}\sup_{s\in[\exp(N\mu),\exp([N+1]\mu)]}\left|\frac{u(s\,,x\,;\lambda)}{\mathrm{e}^{-\mathrm{L}_\sigma^2\lambda^2 s/64}}\right| \geqslant \rho\right\} \\ &\quad\leqslant \frac{C_8}{\rho^2}\sum_{N=1}^{\infty}\left[1+\varepsilon^3\mathrm{e}^{N\mu}\right]^{3/2}\exp\left(-\frac{\mathrm{L}_\sigma^2\lambda^2\mathrm{e}^{N\mu}}{64}\right)+\sum_{N=1}^{\infty}\exp\left(-\frac{\lambda^2\mathrm{L}_\sigma^2\left[\mathrm{e}^{N\mu}-1\right]}{64}\right)<\infty. \end{aligned}$$

We can conclude from this and the Borel–Cantelli lemma that

$$\lim_{t\to\infty}\sup_{x\in\mathbb{T}}\left|\frac{u(t\,,x\,;\lambda)}{\mathrm{e}^{-\lambda^2\mathrm{L}_\sigma^2 t/64}}\right| = 0, \tag{7.1}$$

almost surely. This completes the first half of the proof of Theorem 1.2. □

The proof of the lower bound of Theorem 1.2 depends on the following large-deviations bound for sums of dependent Bernoulli random variables. For a proof see Lemma 3.9 of Khoshnevisan, Révész, and Shi [21].

Lemma 7.1 *Suppose $J_1, J_2, \ldots$ are $\{0\,,1\}$-valued random variables that satisfy the following for some non-random constant $q > 0$: $\mathrm{E}(J_{k+1} \mid J_1,\ldots,J_k) \geqslant q$ for all $k \geqslant 1$, a.s. Then,*

$$\mathrm{P}\left\{J_1+\cdots+J_n \leqslant nq(1-\varepsilon)\right\} \leqslant \exp\left(-\frac{nq\varepsilon^2}{2}\right) \qquad \textit{for every } \varepsilon\in(0\,,1) \textit{ and } n\geqslant 1.$$

We now proceed with the derivation of the lower bound of Theorem 1.2.

Proof of Theorem 1.2: Lower bound We appeal to a one-sided adaptation of a method of Mueller [23]. Define $T_0 := 0$ and then iteratively let

$$T_{n+1} := \inf\left\{t > T_n:\ \inf_{x\in\mathbb{T}} u(t\,,x\,;\lambda) < \mathrm{e}^{-1}\inf_{x\in\mathbb{T}} u(T_n\,,x\,;\lambda)\right\} \qquad \text{for all } n\geqslant 0,$$

where $\inf\varnothing := \infty$. We have already proved in (7.1) that $\sup_{x\in\mathbb{T}} u(t\,,x\,;\lambda) \to 0$ a.s. as $t\to\infty$. Therefore, $T_n < \infty$ for all $n \geqslant 0$ almost surely. Moreover, the sample-function continuity of u shows that the T_n's are stopping times with respect to the filtration $\mathscr{F} := \{\mathscr{F}_t\}_{t\geqslant 0}$, defined earlier in (6.1). We may apply the strong Markov property of u, with respect to $\mathscr{F}$, at the stopping time T_n in order to see that for all integers $n \geqslant 0$,

$$\mathrm{P}\left(T_{n+1}-T_n \leqslant \tau \mid \mathscr{F}_{T_n}\right) \leqslant \mathrm{P}\left(\inf_{x\in\mathbb{T}}\inf_{t\in[0,\tau]} v^{(n)}(t\,,x) \leqslant \mathrm{e}^{-1}\inf_{x\in\mathbb{T}} v_0^{(n)}(x)\ \middle|\ v_0^{(n)}\right),$$

a.s., where $v^{(n)} := \{v^{(n)}(t\,,x)\}_{t\geqslant 0, x\in[-1,1]}$ solves (1.9) starting from the random initial profile $v_0^{(n)}(x) := v^{(n)}(0\,,x) = u(T_n\,,x\,;\lambda)$. By the very definition of the stopping time T_n, and since T_n is finite a.s.,

$$v_0^{(n)}(x) = u(T_n\,,x\,;\lambda) \geqslant \mathrm{e}^{-1}\inf_{y\in\mathbb{T}} u(T_{n-1}\,,y\,;\lambda) \geqslant \cdots \geqslant \mathrm{e}^{-n}\inf_{y\in\mathbb{T}} u_0(y) =: \mathrm{e}^{-n}\underline{u}_0,$$

a.s. for every $x\in\mathbb{T}$, and with identity for some $x\in\mathbb{T}$ a.s. Because $\underline{u}_0 > 0$ [see (1.2)], it follows from a comparison theorem [4,23,27] that $v^{(n)}(t\,,x) \geqslant w^{(n)}(t\,,x)$ for all $t\geqslant 0$ and $x\in\mathbb{T}$ a.s., where $w^{(n)}$ solves (1.9) [for a different Brownian sheet] starting from $w_0^{(n)}(x) := w^{(n)}(0\,,x) = \mathrm{e}^{-n}\underline{u}_0$. In particular, for all integers $n\geqslant 0$ and reals $\tau\in(0\,,1)$,

$$\begin{aligned}\mathrm{P}\left(T_{n+1}-T_n \leqslant \tau \mid \mathscr{F}_{T_n}\right) &\leqslant \mathrm{P}\left\{\inf_{x\in\mathbb{T}}\inf_{t\in[0,\tau]} w^{(n)}(t\,,x) \leqslant \mathrm{e}^{-1}w_0^{(n)}(x) \equiv \mathrm{e}^{-1-n}\underline{u}_0\right\}\\ &\leqslant \mathrm{P}\left\{\sup_{x\in\mathbb{T}}\sup_{t\in[0,\tau]}\left|w^{(n)}(t\,,x) - w_0^{(n)}(x)\right| \geqslant \mathrm{e}^{-1-n}\underline{u}_0\right\}\\ &\leqslant \mathrm{e}^2\mathrm{E}\left(\sup_{x\in\mathbb{T}}\sup_{t\in[0,\tau]}\left|\frac{w^{(n)}(t\,,x)}{w_0^{(n)}(0)} - 1\right|^2\right).\end{aligned}$$

Now, $z^{(n)}(t\,,x) := w^{(n)}(t\,,x)/w_0^{(n)}(0)$ solves (1.9), started at $z_0^{(n)}(x) := z^{(n)}(0\,,x) = 1$, with σ replaced by

$$\sigma^{(n)}(a) := \frac{\sigma\left(w_0(0)a\right)}{w_0(0)} = \frac{\mathrm{e}^n}{\underline{u}_0}\sigma\left(\frac{\underline{u}_0 a}{\mathrm{e}^n}\right) \qquad [a\in\mathbb{R}].$$

We may observe that the Lipschitz constant of $\sigma^{(n)}$ is Lip_σ, uniformly for all $n\geqslant 0$. In this way we find that there exists a finite constant C uniformly in n and K such that

$$\sup_{\substack{(t,x),(s,y)\in(0,\tau)\times\mathbb{T}\\ s\neq t, x\neq y}} \mathrm{E}\left(\frac{\left|z^{(n)}(t\,,x) - z^{(n)}(s\,,y)\right|^K}{|(t\,,x)-(s\,,y)|^{K/4}}\right) \leqslant (CK)^{K/2}\mathrm{Lip}_\sigma^K\lambda^K\mathrm{e}^{CK\lambda^4\mathrm{Lip}_\sigma^4\tau},$$

(see, e.g., [28, Corollary 3.4] or [8, Theorem 6.8]). Thanks to this and a quantitative form of the Kolmogorov continuity theorem (see, e.g., Proposition 5.8), there exists a real number c_0 such that, uniformly for all $\tau\in[0\,,1]$,

$$\begin{aligned}\sup_{n\geqslant 1}\mathrm{P}\left(T_{n+1}-T_n \leqslant \tau \mid \mathscr{F}_{T_n}\right) &\leqslant \mathrm{e}^2\sup_{n\geqslant 1}\mathrm{E}\left(\sup_{x\in\mathbb{T}}\sup_{t\in[0,\tau]}\left|z^{(n)}(t\,,x) - z_0^{(n)}(x)\right|^2\right)\\ &\leqslant c_0\lambda^2\mathrm{Lip}_\sigma^2\sqrt{\tau}\mathrm{e}^{c_0\lambda^4\mathrm{Lip}_\sigma^4\tau},\end{aligned}$$

a.s.. Because c_0 does not depend on $\tau\in[0\,,1]$, we may choose a special $\tau=\tau(\lambda\,,\mathrm{Lip}_\sigma, c_0)$ by setting

$$\tau := \frac{\delta^2}{(\lambda\mathrm{Lip}_\sigma)^4}\wedge 1, \tag{7.2}$$

where $\delta=\delta(c_0)$ is the unique strictly-positive solution to $c_0\delta\exp(c_0\delta^2) = \frac12$. This yields

$$\mathrm{P}\left(T_{n+1}-T_n > \tau \mid \mathscr{F}_{T_n}\right) \geqslant \tfrac12 \qquad \text{for all } n\geqslant 1 \text{ a.s.}, \tag{7.3}$$

for the particular choice of τ that is furnished by (7.2). We now apply Lemma 7.1 with

$$J_n := \mathbf{1}_{\{\omega\in\Omega:\, T_{n+1}(\omega)-T_n(\omega)>\tau\}},$$

where τ is given by (7.2) and $q = \varepsilon = 1/2$ in order to deduce from (7.3) that

$$\mathrm{P}\left\{\sum_{i=1}^{n} \mathbf{1}_{\{T_{i+1}-T_i>\tau\}} \leqslant \frac{n}{4}\right\} \leqslant \mathrm{e}^{-n/16} \qquad \text{for all integers } n \geqslant 1.$$

Because

$$T_n \geqslant \sum_{i=0}^{n-1} (T_{i+1} - T_i)\mathbf{1}_{\{T_{i+1}-T_i>\tau\}} \geqslant \tau \sum_{i=1}^{n} \mathbf{1}_{\{T_{i+1}-T_i>\tau\}},$$

we find that

$$\mathrm{P}\left\{\inf_{x\in\mathbb{T}} \inf_{0\leqslant t\leqslant \tau n/4} u(t\,,x\,;\lambda) \leqslant \mathrm{e}^{-n}\underline{u}_0\right\} \leqslant \mathrm{P}\left\{T_n \leqslant \frac{\tau n}{4}\right\} \leqslant \mathrm{e}^{-n/16} \qquad \text{for all } n \geqslant 1. \quad (7.4)$$

Since

$$\inf_{x\in\mathbb{T}} \inf_{\tau(n-1)/4\leqslant t\leqslant \tau n/4} \frac{u(t\,,x\,;\lambda)}{\exp(-n)} \leqslant \mathrm{e} \inf_{x\in\mathbb{T}} \inf_{\tau(n-1)/4\leqslant t\leqslant \tau n/4} \frac{u(t\,,x\,;\lambda)}{\exp\left(-4t/\tau\right)},$$

the Borel–Cantelli lemma implies the remaining half of Theorem 1.2. □

8 Proof of Theorem 1.1

The proof of Theorem 1.1 is based on the proofs of Theorems 1.2 and 1.4.

Proof of Theorem 1.1 Let $t \geqslant 1$. Define $A_1(t)$ as was defined in (6.3) (i.e. $A_1(t) := \mathbf{A}(t\,;\lambda)$ in (6.3)), then (6.4) says

$$\mathrm{P}(A_1(t)) \geqslant 1 - \mathrm{e}^{-\lambda^2 \mathrm{L}_\sigma^2 t/64}. \tag{8.1}$$

We can also get the following variation of (6.7):

$$\mathrm{E}\left(\sup_{x\in\mathbb{T}} |u(t+h\,,x\,;\lambda)|^k\,;\ A_1(t)\right) \leqslant \frac{C_1^k \mathrm{e}^{C_2\lambda^4 k^3[(h/\varepsilon^2)+\varepsilon t]} \|u_0\|_{L^1}^{k(1-\varepsilon/2)} \|u_0\|_{L^\infty}^{k\varepsilon/2}}{h^{k/2} t^{k\varepsilon/2}} \mathrm{e}^{-k\lambda^2 \mathrm{L}_\sigma^2 t/16}$$

for some constants $C_1, C_2 > 0$ which are independent of k and t. We now choose

$$h := \frac{1}{2} \quad \text{and} \quad \varepsilon := \frac{\mathrm{L}_\sigma^2}{32 C_2 k^2 \lambda^2} \wedge \varepsilon_0 \quad (\varepsilon_0 \text{ is defined in Proposition 5.9})$$

to get that for some constant $\tilde{c}_k > 0$ which only depends on k,

$$\mathrm{E}\left(\sup_{x\in\mathbb{T}} |u(t+1/2\,,x\,;\lambda)|^k\,;\ A_1(t)\right) \leqslant \tilde{c}_k \mathrm{e}^{-k\lambda^2 \mathrm{L}_\sigma^2 t/32}.$$

Replacing $t + 1/2$ by t above and redefining $A_1(t) := A_1(t - 1/2)$, we get

$$\mathrm{E}\left(\sup_{x\in\mathbb{T}} |u(t\,,x\,;\lambda)|^k\,;\ A_1(t)\right) \leqslant \tilde{c}_k \mathrm{e}^{k\lambda^2 \mathrm{L}_\sigma^2/64}\, \mathrm{e}^{-k\lambda^2 \mathrm{L}_\sigma^2 t/32}. \tag{8.2}$$

Let us now define

$$A_2(t) := \left\{\omega \in \Omega : \inf_{x\in\mathbb{T}} \inf_{0\leqslant s\leqslant t} u(s\,,x\,;\lambda)(\omega) \geqslant \mathrm{e}^{-4t/\tau}\underline{u}_0\right\}$$

where $\tau = \tau(\lambda, \mathrm{Lip}_\sigma, c_0) > 0$ is the constant defined on (7.2) and $\underline{u}_0 := \inf_{y\in\mathbb{T}} u_0(y)$. By (7.4), we get

$$\mathrm{P}(A_2(t)) \geqslant 1 - \mathrm{e}^{-t/4\tau}. \tag{8.3}$$

Let

$$B(t) := A_1(t) \cap A_2(t). \tag{8.4}$$

By (8.1) and (8.3), we have

$$\begin{aligned}\mathrm{P}(B(t)) &= \mathrm{P}(A_1(t)) + \mathrm{P}(A_2(t)) - \mathrm{P}(A_1(t) \cup A_2(t))\\ &\geqslant 1 - \mathrm{e}^{-t/4\tau} - \mathrm{e}^{\lambda^2 \mathrm{L}_{\mathrm{œ}}^2/128}\mathrm{e}^{-\lambda^2\mathrm{L}_\sigma^2 t/64} \geqslant 1 - b_1\mathrm{e}^{-b_2 t},\end{aligned}$$

where $b_1 := 2\max\left\{1, \mathrm{e}^{\lambda^2\mathrm{L}_{\mathrm{œ}}^2/128}\right\}$ and $b_2 := \min\left\{\frac{1}{4\tau}, \frac{\lambda^2\mathrm{L}_{\mathrm{œ}}^2}{64}\right\}$. Hence, there exists $c > 0$ for some t_0 large, if $t \geqslant t_0$,

$$\mathrm{P}(B(t)) \geqslant 1 - c\mathrm{e}^{-ct} \geqslant 1 - c\mathrm{e}^{-ct_0} > 0.$$

This shows the first statement of Theorem 1.1. For the second statement, the upper bound comes from (8.2) and the lower bound comes from the following:

$$\mathrm{E}\left(\inf_{x\in\mathbb{T}} |u(t\,, x\,; \lambda)|^k;\ B(t)\right) \geqslant \mathrm{e}^{-4t/\tau}\underline{u}_0\,\mathrm{P}(B(t)) \geqslant \mathrm{e}^{-4t/\tau}\underline{u}_0\left(1 - c\mathrm{e}^{-ct_0}\right),$$

which completes the proof. □

A A Real-Variable Inequality

We will have need for the following.

Lemma A.1 *For all $\varepsilon \in (0\,,1)$, $\alpha \in [0\,,1)$, and $\beta \geqslant 1$,*

$$\sup_{t>0}\int_0^t \left(\frac{t}{s}\right)^{1-\varepsilon} \frac{\mathrm{e}^{-\beta(t-s)}}{(t-s)^\alpha}\,\mathrm{d}s \leqslant \frac{2\Gamma(1-\alpha)+1}{(1-\alpha)\varepsilon\beta^{1-\alpha}},$$

where Γ denotes Gamma function.

Proof By scaling, we might as well assume that $\beta = 1$. Now, a change of variables yields

$$\int_0^t \left(\frac{t}{s}\right)^{1-\varepsilon} \frac{\mathrm{e}^{-(t-s)}}{(t-s)^\alpha}\,\mathrm{d}s = t^{1-\alpha}\int_0^1 \frac{\mathrm{e}^{-tr}}{r^\alpha(1-r)^{1-\varepsilon}}\,\mathrm{d}r,$$

whenever $t > 0$ and $0 < \varepsilon < 1$. If $t \leqslant 1$, then we merely bound $t^{1-\alpha}$ and $\exp(-tr)$ by 1 in order to see that the preceding is at most $\mathrm{B}(\varepsilon\,, 1-\alpha)$, where B is beta function. On the other hand, if $t > 1$, then we change variables a few more times in order to see that

$$\int_0^t \left(\frac{t}{s}\right)^{1-\varepsilon} \frac{\mathrm{e}^{-(t-s)}}{(t-s)^\alpha}\,\mathrm{d}s = \int_0^t \frac{\mathrm{e}^{-s}}{\left(1-\frac{s}{t}\right)^{1-\varepsilon} s^\alpha}\,\mathrm{d}s = \int_0^{t/2}(\cdots)\,\mathrm{d}s + \int_{t/2}^t(\cdots)\,\mathrm{d}s,$$

notation being obvious from context. Since $(1-[s/t])^{1-\varepsilon} \geqslant 2^{-1+\varepsilon} \geqslant 1/2$ for all $0 < s < t/2$,

$$\int_0^{t/2}(\cdots)\,\mathrm{d}s \leqslant 2\Gamma(1-\alpha).$$

On the other hand,

$$\int_{t/2}^{t}(\cdots)\,\mathrm{d}s \leqslant \mathrm{e}^{-t/2}\int_{t/2}^{t}\frac{\mathrm{d}s}{s^{\alpha}\left(1-\frac{s}{t}\right)^{1-\varepsilon}} = t^{1-\alpha}\,\mathrm{e}^{-t/2}\int_{1/2}^{1}\frac{\mathrm{d}r}{r^{\alpha}(1-r)^{1-\varepsilon}} \leqslant \mathrm{B}(\varepsilon\,,1-\alpha),$$

where we have used the elementary bound, $x^{1-\alpha}\mathrm{e}^{-x/2}\leqslant 1$, valid for all $x\geqslant 0$.

The preceding argument yields the inequality,

$$\sup_{t>0}\int_0^t\left(\frac{t}{s}\right)^{1-\varepsilon}\frac{\mathrm{e}^{-(t-s)}}{(t-s)^{\alpha}}\,\mathrm{d}s \leqslant 2\Gamma(1-\alpha)+\mathrm{B}(\varepsilon\,,1-\alpha),$$

valid for all $\varepsilon\in(0\,,1)$ and $\alpha\in[0\,,1)$. This implies the lemma since a change of variables yields

$$\mathrm{B}(\varepsilon\,,1-\alpha)=\varepsilon^{-1}\int_0^\infty \mathrm{e}^{-y}\left(1-\mathrm{e}^{-y/\varepsilon}\right)^{-\alpha}\mathrm{d}y \leqslant \varepsilon^{-1}\int_0^\infty \mathrm{e}^{-y}\left(1-\mathrm{e}^{-y}\right)^{-\alpha}\mathrm{d}y = \frac{1}{\varepsilon(1-\alpha)},$$

for all $\varepsilon\in(0\,,1)$ and $\alpha\in[0\,,1)$, which is a well-known fact about the beta integral; see, Dragomir et al. [9, (3.17)] for a different proof of the latter inequality. □

B The Heat Kernel

Recall G and p respectively from (2.2) and (2.1).

Lemma B.1 *For all $x, y\in[-1\,,1]$ and $t>0$,*

$$G_t(x-y)\leqslant p_t(x\,,y)\leqslant 2\max\left(\frac{1}{\sqrt{t}}\,,1\right).$$

Remark B.2 By Lemma B.1, $\sup_{-1<x,y<1}p_t(x\,,y)\geqslant G_t(0)=(4\pi t)^{-1/2}$, pointwise. Also, $2\sup_y p_t(x\,,y)\geqslant\int_{-1}^{1}p_t(x\,,y)\,\mathrm{d}y=1$, for all $t>0$ and $x\in\mathbb{T}$. Therefore, Lemma B.1 has the following consequence:

$$\frac{1}{4}\max\left(\frac{1}{\sqrt{t}}\,,1\right)\leqslant\sup_{x,y\in[-1,1]}p_t(x\,,y)\leqslant 2\max\left(\frac{1}{\sqrt{t}}\,,1\right),$$

for all $t>0$.

Proof of Lemma B.1 The lower bound is immediate; we establish the upper bound.

Consider the summands in (2.1) for $|n|\leqslant 1$ and $|n|\geqslant 2$ separately in order to see that

$$p_t(x\,,y)\leqslant\frac{3}{\sqrt{4\pi t}}+\frac{1}{\sqrt{4\pi t}}\sum_{\substack{n\in\mathbb{Z}:\\|n|\geqslant 2}}\exp\left(-\frac{(x-y-2n)^2}{4t}\right),$$

for all $t>0$ and $x, y\in[-1\,,1]$. Since $(a-b)^2\geqslant\frac{1}{2}a^2-b^2$ for all $a, b\in\mathbb{R}$, the preceding yields $(x-y-2n)^2\geqslant 2n^2-(x-y)^2\geqslant 2(n^2-2)\geqslant n^2$, for all $x, y\in[-1\,,1]$ and integers n with $|n|\geqslant 2$. Thus, we obtain the bound,

$$p_t(x\,,y)\leqslant\frac{1}{\sqrt{t}}+\frac{1}{\sqrt{\pi t}}\sum_{n=2}^{\infty}\exp\left(-\frac{n^2}{4t}\right),\tag{B.1}$$

for all $t > 0$ and $x, y \in [-1\,,1]$. In particular,

$$\sup_{x,y\in[-1,1]} p_t(x\,,y) \leqslant \frac{1}{\sqrt{t}}\left(1+\sum_{n=2}^{\infty} \mathrm{e}^{-n^2/4}\right) \leqslant \frac{2}{\sqrt{t}},$$

uniformly for $t \in (0\,,1]$. If $t > 1$, then use

$$\sum_{n=2}^{\infty} \exp\left(-\frac{n^2}{4t}\right) \leqslant \int_0^{\infty} \exp\left(-\frac{z^2}{4t}\right) \mathrm{d}z = \sqrt{\pi t},$$

in (B.1), to see that

$$\sup_{x,y\in[-1,1]} p_t(x\,,y) \leqslant \frac{1}{\sqrt{t}} + 1,$$

which is at most 2. □

Lemma B.3 *There exists a finite constant C such that*

$$\int_{-1}^{1} |p_t(x\,,w) - p_t(y\,,w)|^2\, \mathrm{d}w \leqslant C\frac{|x-y|}{t\wedge\sqrt{t}},$$

uniformly for all $x, y \in [-1\,,1]$ and $t > 0$.

Proof Choose and fix some $t > 0$ and $x, y \in [-1\,,1]$. Without loss of generality, we assume that $x > y$. By the Chapman–Kolmogorov property, and thanks to the symmetry of p_t,

$$\begin{aligned}\int_{-1}^{1} |p_t(x\,,w) - p_t(y\,,w)|^2\, \mathrm{d}w &= p_{2t}(x\,,x) + p_{2t}(y\,,y) - 2p_{2t}(x\,,y)\\ &= 2\sum_{n=-\infty}^{\infty} [G_{2t}(2n) - G_{2t}(2n+x-y)]\,.\end{aligned}$$

Because $|G_t'(a)| = |a|G_t(a)/2t$,

$$|G_t(2n) - G_t(2n+x-y)| \leqslant \frac{1}{4\sqrt{\pi}t^{3/2}}\int_0^{x-y} |2n+a| \exp\left(-\frac{(2n+a)^2}{4t}\right) \mathrm{d}a, \quad \text{(B.2)}$$

for all $n \in \mathbb{Z}$. In particular,

$$|G_t(2n) - G_t(2n+x-y)| \leqslant \frac{x-y}{4\sqrt{\pi}t^{3/2}} \sup_{r\geqslant 0}\left[r\exp\left(-\frac{r^2}{4t}\right)\right] \leqslant \frac{x-y}{4\sqrt{\pi t}}.$$

The preceding is useful when $|n|$ is not too large, say $|n| \leqslant 2$. On the other hand, if $|n| \geqslant 2$ and $0 \leqslant a \leqslant 2$, then $3|n| \geqslant |2n+a| \geqslant 2(|n|-1) \geqslant |n|$. Therefore, (B.2) implies that

$$|G_t(2n) - G_t(2n+x-y)| \leqslant \frac{3(x-y)}{4\sqrt{\pi}t^{3/2}} \cdot |n| \exp\left(-\frac{n^2}{4t}\right).$$

We combine the preceding two displays to see that

$$\int_{-1}^{1} |p_t(x\,,w) - p_t(y\,,w)|^2\, \mathrm{d}w \leqslant \frac{3(x-y)}{2\sqrt{\pi t}} + \frac{3(x-y)}{2\sqrt{\pi}t^{3/2}} \cdot \sum_{|n|\geqslant 2} |n| \exp\left(-\frac{n^2}{4t}\right).$$

The sum is at most $\int_{-\infty}^{\infty} |w| \exp(-w^2/(8t))\, \mathrm{d}w \propto t$, and this implies the result. □

Lemma B.4 *For each $\delta \in (0\,,1)$ there exists a finite constant C such that*

$$\int_{-1}^{1} |p_t(x\,,w) - p_t(y\,,w)|^2\,\mathrm{d}w \leqslant C\frac{|x-y|^\delta}{t^{(\delta+1)/2} \wedge t^{\delta/2}},$$

uniformly for all $t > 0$ and $x, y \in [-1\,,1]$.

Proof Since $(a+b)^2 \leqslant 2a^2 + 2b^2$ for all $a, b \in \mathbb{R}$, the Chapman–Kolmogorov property yields the following for all $t > 0$ and $x, y \in [-1\,,1]$:

$$\int_{-1}^{1} |p_t(x\,,w) - p_t(y\,,w)|^2\,\mathrm{d}w \leqslant 2p_{2t}(x\,,x) + 2p_{2t}(y\,,y) \leqslant 8\max\left(\frac{1}{\sqrt{t}}, 1\right);$$

see Lemma B.1. Therefore, Lemma B.3 implies that we can find a finite constant C such that, uniformly for all $x, y \in [-1\,,1]$ and $t > 0$,

$$\int_{-1}^{1} |p_t(x\,,w) - p_t(y\,,w)|^2\,\mathrm{d}w \leqslant C\frac{|x-y|}{\sqrt{t}}\max\left\{\frac{1}{\sqrt{t}}, 1\right\},$$

The lemma follows since $\min(A\,,B) \leqslant A^\delta B^{1-\delta}$ for all $A, B \geqslant 0$ and $\delta \in (0\,,1)$. □

Lemma B.5 *There exists a finite constant C such that, uniformly for all $\varepsilon \in (0\,,1)$, $t > 0$, $x, y \in [-1\,,1]$, and $h \in L^1 \cap L^\infty$,*

$$|(P_t h)(x) - (P_t h)(y)| \leqslant C\max\left\{1\,,\frac{1}{\sqrt{t}}\right\}|x-y|^{\varepsilon/2}\|h\|_{L^\infty}^{\varepsilon}\|h\|_{L^1}^{1-\varepsilon}.$$

Proof First, we use Lemma B.1 to get that

$$|(P_t h)(x) - (P_t h)(y)| \leqslant 4\max\left\{1\,,\frac{1}{\sqrt{t}}\right\}\|h\|_{L^1}.$$

We can also apply the Cauchy–Schwarz inequality and then Lemma B.3 to obtain

$$\begin{aligned}
|(P_t h)(x) - (P_t h)(y)|^2 &\leqslant \left(\int_{-1}^{1} |p_t(x\,,w) - p_t(y\,,w)| \cdot |h(w)|\,\mathrm{d}w\right)^2 \\
&\leqslant 2\|h\|_{L^\infty}^2 \int_{-1}^{1} |p_t(x\,,w) - p_t(y\,,w)|^2\,\mathrm{d}w \\
&\leqslant C\|h\|_{L^\infty}^2 \frac{|x-y|}{t \wedge \sqrt{t}} \\
&\leqslant C\|h\|_{L^\infty}^2 |x-y| \max\left\{1\,,\frac{1}{t}\right\}.
\end{aligned}$$

Now the lemma follows since $\min(A\,,B) \leqslant A^\varepsilon B^{1-\varepsilon}$ for all $A, B \geqslant 0$ and $\varepsilon \in (0\,,1)$. □

Lemma B.6 *For each $t, \delta > 0$,*

$$\sup_{x\in\mathbb{T}} \int_{-1}^{1} |p_{t+\delta}(x\,,w) - p_t(x\,,w)|^2\,\mathrm{d}w \leqslant \sqrt{\frac{\pi}{2t}}\min\left(1\,,\frac{\delta}{4t}\right).$$

Proof Choose and fix some $t, \delta > 0$ and $x \in \mathbb{T}$. By the Chapman–Kolmogorov property, and thanks to the symmetry of p_t,

$$\int_{-1}^{1} |p_{t+\delta}(x\,,w) - p_t(x\,,w)|^2\,\mathrm{d}w = p_{2(t+\delta)}(x\,,x) + p_{2t}(x\,,x) - 2p_{2t+\delta}(x\,,x)$$
$$= \sum_{n=-\infty}^{\infty} \left[G_{2t+2\delta}(2n) + G_{2t}(2n) - 2G_{2t+\delta}(2n)\right].$$

Because the Fourier transform of $F(x) := G_\tau(2x)$ is $\widehat{F}(z) = \frac{1}{2}\widehat{G}_\tau(z/2) = \frac{1}{2}\exp(-\tau z^2/4)$, the Poisson summation formula [15, p. 161] implies that

$$\int_{-1}^{1} |p_{t+\delta}(x\,,w) - p_t(x\,,w)|^2\,\mathrm{d}w = \frac{1}{2}\sum_{n=-\infty}^{\infty} \mathrm{e}^{-tn^2/2}\left(\mathrm{e}^{-\delta n^2/2} + 1 - 2\mathrm{e}^{-\delta n^2/4}\right)$$
$$\leqslant \sum_{n=1}^{\infty} \mathrm{e}^{-tn^2/2} \min\left(1\,, \frac{\delta n^2}{4}\right),$$

uniformly for all $t > 0$ and $\delta \in (0\,,1)$. This readily implies the result since

$$\sum_{n=1}^{\infty} \mathrm{e}^{-tn^2/2} \leqslant \int_0^\infty \mathrm{e}^{-tx^2/2}\,\mathrm{d}x = \sqrt{\frac{\pi}{2t}},$$

and

$$\delta\sum_{n=1}^{\infty} \mathrm{e}^{-tn^2/2}n^2 \leqslant \delta\int_0^\infty x^2\mathrm{e}^{-tx^2/2}\,\mathrm{d}x = \frac{\delta}{t}\sqrt{\frac{\pi}{2t}},$$

for all $t > 0$ and $\delta \in (0\,,1)$. □

Lemma B.7 *For every $t, \delta > 0$, $\varepsilon \in (0\,,1)$, and $h \in L^1 \cap L^\infty$,*

$$\sup_{x\in\mathbb{T}} |(P_{t+\delta}h)(x) - (P_t h)(x)| \leqslant 4\left(1 + \frac{1}{\sqrt{t}}\right) \min\left(1\,, \left[\frac{\delta}{4t}\right]^{\varepsilon/2}\right) \cdot \|h\|_{L^\infty}^{\varepsilon} \|h\|_{L^1}^{1-\varepsilon}.$$

Proof We first use Lemma B.1, as we did in the proof of Lemma B.5, to get that

$$|(P_{t+\delta}h)(x) - (P_t h)(x)| \leqslant 4\max\left(1, \frac{1}{\sqrt{t}}\right) \|h\|_{L^1}.$$

We now apply first the Cauchy–Schwarz inequality and then Lemma B.6 in order to deduce

$$|(P_{t+\delta}h)(x) - (P_t h)(x)|^2 \leqslant 2\|h\|_{L^\infty}^2 \int_{-1}^{1} |p_{t+\delta}(x\,,w) - p_t(x\,,w)|^2\,\mathrm{d}w$$
$$\leqslant 2\|h\|_{L^\infty}^2 \sqrt{\frac{\pi}{2t}} \min\left(1\,, \frac{\delta}{4t}\right)$$
$$\leqslant 4\max\left(1\,, \frac{1}{t}\right) \|h\|_{L^\infty}^2 \min\left(1\,, \frac{\delta}{4t}\right).$$

Now the lemma follows since $\max\{A\,, B\} \leqslant A + B$ and $\min(A\,, B) \leqslant A^\varepsilon B^{1-\varepsilon}$ for all $A, B \geqslant 0$ and $\varepsilon \in (0\,,1)$. □

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