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An improved bound for the Shapley–Folkman theorem ☆

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ABSTRACT

We provide an up to 30% improvement in the Shapley–Folkman theorem error-bound, and briefly discuss its consequences for the course allocation problem.

1. Introduction

Budish (2011) considers the indivisible-goods combinatorial assignment problem in the context of an economy in which all goods are in fixed supply and agents are endowed only with fiat money. Budish establishes the existence of an approximate competitive equilibrium in which the agents have nearly equal money endowments and in which markets clear up to an error that is independent of both the number of agents and of the total supplies of all of the goods.[1] That money endowments can be made nearly equal is important for establishing a number of results on the fairness of the final allocation.

A careful look at Budish's proof reveals that it provides an improved error bound for the Shapley–Folkman theorem. The purpose of this note is to provide an explicit statement and proof of this result. We also show that the improved bound can be up to 30% tighter and we provide an application to the course allocation problem in which this maximal improvement is nearly attained.

2. The Shapley–Folkman theorem

Throughout, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^m$, and, for any subset X of $\mathbb{R}^m$, co$X$ denotes its convex hull.

For any nonempty subset S of $\mathbb{R}^m$, define the *diameter* of S by

$$\mathrm{diam}(S) := \sup_{x,y\in S} \|x-y\|,$$

define the *radius* of S by

$$\mathrm{rad}(S) := \inf_{y\in\mathbb{R}^m} \sup_{x\in S} \|x-y\|,$$

define the *inner diameter* of S by

$$\mathrm{indiam}(S) := \sup_{y\in \mathrm{co}S} \inf_{\{T\subseteq S: y\in \mathrm{co}T\}} \mathrm{diam}(T),$$

and define the *inner radius* of S by

$$\mathrm{inrad}(S) := \sup_{y\in \mathrm{co}S} \inf_{\{T\subseteq S: y\in \mathrm{co}T\}} \mathrm{rad}(T).$$

When S is compact, all of the infima and suprema above are attained.

Because T can always be chosen to be equal to S, it is clear that indiam$(S)\le$diam(S) and inrad$(S)\le$ rad(S). It is also not difficult to show that,[2]

$$\frac{\mathrm{diam}(S)}{2} \le \mathrm{rad}(S), \tag{2.1}$$

from which it follows that,[3]

$$\frac{\mathrm{indiam}(S)}{2} \le \mathrm{inrad}(S). \tag{2.2}$$

If S is a sphere or if $\#S = 2$, then (2.1) and (2.2) are equalities. However, the inequalities can be strict, e.g., if $S = \{(1,0,0),(0,1,0),(0,0,1)\}$, then $\mathrm{diam}(S)/2 = \sqrt{2}/2 < \sqrt{6}/3 = \mathrm{rad}(S)$. Our improved bound exploits inequalities (2.1) and (2.2).

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1 Reny (2017) extends Budish's (2011) result to settings with both divisible and indivisible goods.

2 Suppose $rad(S) = \sup_{x\in S}\|x-y^*\|$ and $diam(S) = \|\hat{x}-\hat{y}\|$ for some $y^*\in\mathbb{R}^m$ and $\hat{x},\hat{y}\in S$. Then, by the triangle inequality, $\|\hat{x}-\hat{y}\| \le \|\hat{x}-y^*\| + \|y^*-\hat{y}\|$, and so $rad(S) \ge \max(\|\hat{x}-y^*\|, \|y^*-\hat{y}\|) \ge \|\hat{x}-\hat{y}\|/2$.

3 Indeed, if $diam/2 \le rad$, then the number, $diam(T)$, that appears on the right-hand side of the definition of $indiam(S)$, is less than or equal to $2rad(T)$, implying that the right-hand side is less than or equal to $2inrad(S)$.

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The version of the Shapley–Folkman theorem that has been most useful in the literature is as follows (see Starr, 2008).

Theorem 2.1 (*Shapley–Folkman*). *Suppose that $n \geq m$. If $S_1, \ldots, S_n$ are compact subsets of $\mathbb{R}^m$, if $y \in co(S_1 + \cdots + S_n)$, and if $R = \max(rad(S_1), \ldots, rad(S_n))$, then there exist $x_i \in S_i$ and $y_i \in coS_i$, $i = 1, \ldots, n$, such that $y = \sum_i y_i$, $y_i = x_i$ for all but at most m indices i, and*

$$\left\| y - \sum_{i=1}^{n} x_i \right\| \leq R\sqrt{m}.$$

By using the inner radii of the sets S_i rather than their radii, Starr (1969) obtains the following result with an improved error bound.

Theorem 2.2 (*Starr*). *Suppose that $n \geq m$. If $S_1, \ldots, S_n$ are compact subsets of $\mathbb{R}^m$, if $y \in co(S_1+\cdots+S_n)$, and if $r = \max(inrad(S_1), \ldots, inrad(S_n))$, then there exist $x_i \in S_i$ and $y_i \in co\, S_i$, $i = 1, \ldots, n$, such that $y = \sum_i y_i$, $y_i = x_i$ for all but at most m indices i, and*

$$\left\| y - \sum_{i=1}^{n} x_i \right\| \leq r\sqrt{m}.$$

Remark 1. The Shapley–Folkman and Starr Theorems 2.1 and 2.2 each have more refined versions (see Starr, 1969). The statement of the more refined Shapley–Folkman theorem is as follows: If $S_1, \ldots, S_n$ are compact subsets of $\mathbb{R}^m$ and if $y \in \mathrm{co}(S_1 + \cdots + S_n)$, then there exist $x_i \in S_i$ and $y_i \in \mathrm{co}S_i$, $i = 1, \ldots, n$, such that $y = \sum_i y_i$, $y_i = x_i$ for all but at most $\min(m, n)$ indices i, and

$$\left\| y - \sum_{i=1}^{n} x_i \right\|^2 \leq \sum (\mathrm{rad}(S_i))^2,$$

where the sum on the right-hand side is over the $\min(m, n)$ highest among the n numbers $(\mathrm{rad}(S_1))^2, \ldots, (\mathrm{rad}(S_n))^2$. The statement of the more refined Starr theorem is the same except that $\mathrm{inrad}(S_i)$ everywhere replaces $\mathrm{rad}(S_i)$.

3. An improved bound

We can improve on the error bounds in Theorems 2.1 and 2.2 by using the diameters of the S_i instead of their radii, and by using the inner diameters of the S_i instead of their inner radii. Our results are as follows.

Theorem 3.1. *Suppose that $n \geq m$. If $S_1, \ldots, S_n$ are compact subsets of $\mathbb{R}^m$, if $y \in co(S_1+\cdots+S_n)$, and if $D = \max(diam(S_1), \ldots, diam(S_n))$, then there exist $x_i \in S_i$ and $y_i \in coS_i$, $i = 1, \ldots, n$, such that $y = \sum_i y_i$, $y_i = x_i$ for all but at most m indices i, and*

$$\left\| y - \sum_{i=1}^{n} x_i \right\| \leq D\sqrt{m}/2.$$

Theorem 3.2. *Suppose that $n \geq m$. If $S_1, \ldots, S_n$ are compact subsets of $\mathbb{R}^m$, if $y \in co(S_1+\cdots+S_n)$, and if $d = \max(indiam(S_1), \ldots, indiam(S_n))$, then there exist $x_i \in S_i$ and $y_i \in coS_i$, $i = 1, \ldots, n$, such that $y = \sum_i y_i$, $y_i = x_i$ for all but at most m indices i, and*

$$\left\| y - \sum_{i=1}^{n} x_i \right\| \leq d\sqrt{m}/2.$$

Remark 2. That the inequality bounds in Theorems 2.1, 2.2, 3.1 and 3.2 can all be satisfied with equality for any given dimension m is established by the following example. Let $\hat{z} = \left(\frac{-1}{2(m-1)}, \ldots, \frac{-1}{2(m-1)}\right) \in \mathbb{R}^m$ and for each $i = 1, \ldots, m$, let $S_i = \{(0, \hat{z}_{-i}), (1, \hat{z}_{-i})\}$. Then $\mathrm{diam}(S_i) = \mathrm{indiam}(S_i) = 1$ and $\mathrm{rad}(S_i) = \mathrm{inrad}(S_i) = 1/2$ for every $i = 1, \ldots, m$. Moreover, $0 = \frac{1}{2}\left(\sum_i (0, \hat{z}_{-i})\right) + \frac{1}{2}\left(\sum_i (1, \hat{z}_{-i})\right) \in \mathrm{co}(S_1 + \cdots + S_m)$, and, for every $z \in S_1 + \cdots + S_m$, the ith coordinate of z is $\pm\frac{1}{2}$ for every $i = 1, \ldots, m$. Hence, $\|0 - z\| = \|z\| = \sqrt{m}/2 = R\sqrt{m} = r\sqrt{m} = D\sqrt{m}/2 = d\sqrt{m}/2$, where R, r, D, and d are as in Theorems 2.1, 2.2, 3.1 and 3.2.

3.1. How much better?

Our next result requires the following lemma.

Lemma 3.3. *If S is any nonempty subset of $\mathbb{R}^m$ whose convex hull contains the origin, and $\|x\| = 1$ for every $x \in S$, then $diam(S) \geq \sqrt{2(1 + 1/m)}$.*

With this lemma in hand, we can show that the bounds offered in Theorems 3.1 and 3.2, while always at least as small as the bounds in 2.1 and 2.2, respectively, are never more than a factor of $\sqrt{2}/2$ ($\approx$ .707) smaller, uniformly in the dimension of the ambient space. Indeed, we can show the following.

Proposition 3.4. *For any $m \geq 1$ and for any nonempty compact subset S of $\mathbb{R}^m$,*

$$\mathrm{diam}(S) \geq \sqrt{2(1 + 1/m)}\mathrm{rad}(S).$$

Moreover, for any $m \geq 1$, there are subsets S of $\mathbb{R}^m$ for which equality holds.

Remark 3. As a consequence of Proposition 3.4, if R, r, D, and d are as in Theorems 2.1, 2.2, 3.1, and 3.2, then

$$\frac{\sqrt{2(1 + 1/m)}}{2}R \leq \frac{D}{2} \leq R,$$

and

$$\frac{\sqrt{2(1 + 1/m)}}{2}r \leq \frac{d}{2} \leq r,$$

where the right-hand inequalities in the two displays follow from (2.1) and (2.2), respectively.

Hence, the bounds that we obtain by using D and d are smaller than those obtained by using R and r by a factor of at best $\sqrt{2(1 + 1/m)}/2$, which decreases to $\sqrt{2}/2 = 0.707\ldots$ as $m \to \infty$. So our bound is at most 30% tighter than the Shapley–Folkman bound. In particular therefore, using our tighter bound instead of the Shapley–Folkman bound in results on the rate of convergence to competitive equilibrium (or the core) as the number of consumers grows would leave that rate of convergence unchanged. Nevertheless, as the example in the next section suggests, the 30% reduction can be of some importance in applications for economies with finitely many agents.

To see that, for each m, there are subsets S of $\mathbb{R}^m$ for which the best possible reduction of $\sqrt{2(1 + 1/m)}/2$ is achieved, it is convenient to consider $\mathbb{R}^m$ as a subset of $\mathbb{R}^{m+1}$, specifically, as the (isometrically isomorphic) subset $L_m = \{(x_1, \ldots, x_{m+1}) \in \mathbb{R}^{m+1} : x_1 + \cdots + x_{m+1} = 1\}$ of $\mathbb{R}^{m+1}$. For any $i \in \{1, \ldots, m + 1\}$, let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^{m+1}$, where the 1 appears in the ith coordinate. It is straightforward to verify that the subset $S = \{e_1, \ldots, e_{m+1}\}$ of L_m has both diameter and inner diameter equal to $\sqrt{2}$, and has both radius and inner radius equal to $\sqrt{m/(m + 1)}$, which is the distance of any e_i to the barycenter, $(1/(m + 1), \ldots, 1/(m + 1))$, of S. Hence, the ratio of the diameter to the radius (and of the inner diameter to the inner radius) is $\sqrt{2(1 + 1/m)}$, as desired.

4. An example

As an application, let us consider the problem of allocating courses to students. There are m courses. Feasible consumption bundles consist of any k of the m courses, and we assume that $k < m/2$. For example, if $m = 100$ and $k = 5$, then the set of feasible consumption bundles is the set of vectors with 5 elements equal to $+1$ and 95 elements equal to 0. So $(1, 1, 1, 1, 1, 0, \ldots, 0)$ is a feasible consumption bundle in which the student consumes courses 1 through 5.[4]

Budish (2011) shows that one can allocate courses to students in a manner that is almost competitive but that might not quite be feasible, and he provides a bound on how much excess demand for the m courses there can be. From the analysis in Budish (2011), or Reny (2017), it is evident that either one of the bounds in Theorems 2.1 or 3.1 can serve as a bound on the Euclidean norm of the vector of market-clearing errors in the m courses.[5] Our modest objective here is to compare these two bounds in this practical setting.

What is the Diameter, D, of the set of feasible consumption bundles? It is the maximum distance between any two of them. This is $\sqrt{2k}$ here since $k < m/2$.

What is the Radius, R, of the set of feasible consumption bundles? It is the minimum, among all choices of $x \in \mathbb{R}$, of the distance between $(x, x, \ldots, x)$ and any vector with k one's and $m-k$ zero's. Since this distance is $\sqrt{k(1-x)^2 + (m-k)x^2}$, which is minimized at $x = k/m$, the radius is approximately $\sqrt{k}$ when m is large relative to k.

The ratio between this radius, $\sqrt{k}$, and half the diameter, $\sqrt{2k}/2$, corresponds to the "how much better" discussion in Section 3.1.

Using the Shapley–Folkman bound (Theorem 2.1) gives $R\sqrt{m}$ as an upper bound on the norm of the excess demands across the m courses. Using the approximation for the radius of $\sqrt{k}$, above, this is a bound of $\sqrt{km}$. For $k = 5$ and $m = 100$, this is $\sqrt{500} = 22.36$. The exact radius is $\sqrt{19/2}$, which gives the slightly better bound of $\sqrt{100}\sqrt{19/2} = 21.79$. So an upper bound on the maximum possible excess demand in any one course is 21 students (and then all other courses must have zero excess demand), and an upper bound on the maximum possible average excess demand is 2.1 students per course.[6]

Using our improved bound (Theorem 3.1) gives $D\sqrt{m}/2$, or $\sqrt{km/2}$ as an upper bound on the norm of the vector of excess demands. So with $m = 100$ and $k = 5$, this bound is $\sqrt{1000}/2 = 15.8$. So a better upper bound on the maximum possible excess demand in any one course is 15 students (and then all other courses must have zero excess demand), and a better upper bound on the maximum possible average excess demand is 1.5 students per course.[7]

So the tighter bound produced by Theorem 3.1 on in both the upper bound on the maximum possible excess demand in any one course (from 21 students to 15) and in the upper bound on the maximum possible average excess demand across the 100 courses (from 2.1 students to 1.5). This is close to the maximum possible reduction that can be achieved by our improved bound as discussed in Section 3.1.

In practice, algorithms for finding solutions to the course allocation problem (or any assignment problem) are guided by a variety of criteria. When these criteria include targets for market-clearing errors – as in Budish et al. (2017) – tightening those targets can, at least in principle, provide improvements in the solution that one obtains and so one should employ the tightest theoretical bound that is available (i.e., that given in Theorem 3.1). It would be interesting, though far from trivial, to consider some large collection of problems and determine whether or not the tighter bound given here translates into improved solutions, in some statistical sense, in practice.

5. Proofs

Proof of Theorem 3.1. Since $y \in \text{co}(S_1 + \cdots + S_n)$ and because $\text{co}(S_1 + \cdots + S_n) = \text{co}S_1 + \cdots + \text{co}S_n$, we can write $y = \sum_{i=1}^n \sum_k \alpha_{ik} x_{ik}$, where the sums are finite, all of the finitely many α_{ik} are nonnegative, $\sum_k \alpha_{ik} = 1$ for each i, and $x_{ik} \in S_i$ for each i and k.

For each $i = 1, \ldots, n$, let e_i denote the ith unit vector $(0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n$, and, for any $z \in \mathbb{R}^m$, let $(e_i, z) \in \mathbb{R}^{n+m}$ denote the concatenation of e_i and z. Then $\sum_{i,k} (\alpha_{ik}/n)(e_i, x_{ik}) = (1, \ldots, 1, y)/n$ and therefore, because $\sum_{i,k}(\alpha_{ik}/n) = 1$,

$$\frac{1}{n}(1, \ldots, 1, y) \in \text{co}\left(\cup_{i=1}^n (\{e_i\} \times S_i)\right) \subseteq \Delta_n \times \mathbb{R}^m, \tag{5.1}$$

where Δ_n denotes the $n-1$ dimensional unit simplex.

Because the convex set Δ_n has dimension $n-1$ (even though it is a subset of $\mathbb{R}^n$), the convex set $\text{co}\left(\cup_{i=1}^n (\{e_i\} \times S_i)\right)$, being a subset of $\Delta_n \times \mathbb{R}^m$, has dimension no more than $n-1+m$. Hence, by Caratheodory's theorem (Rockafellar, 1970) $(1, \ldots, 1, y)/n$ can be written as a convex combination of $n+m$ or fewer points belonging to $\cup_{i=1}^n (\{e_i\} \times S_i)$. Thus, for some positive integer K we may write

$$\frac{1}{n}(1, \ldots, 1, y) = \sum_{i=1}^n \sum_{k=1}^K \lambda_{ik}(e_i, x_i^k), \tag{5.2}$$

where the λ_{ik}'s are nonnegative and sum to one, at most $n+m$ of the λ_{ik} are strictly positive and each x_i^k is in S_i.[8]

For each $i = 1, \ldots, n$, let $S_i^+ = \{x_i^k : \lambda_{ik} > 0\}$. Since the first n coordinates of the vector on the left-hand side of (5.2) are equal to 1 and hence are positive, each S_i^+ contains at least one element. Reindexing if necessary, let $S_1^+, \ldots, S_j^+$ denote those S_i^+ that contain two or more elements. So $S_{j+1}^+, \ldots, S_n^+$ are singletons, and, since at most $n+m$ of the λ_{ik} are strictly positive, the union of $S_1^+, \ldots, S_j^+$ contains no more than $m+j$ elements.

Since S_i^+ is a finite subset of S_i, the distance between any point in S_i^+ and the simple average of all of the points in S_i^+ is no greater than $\text{diam}(S_i)(\#S_i^+ - 1)/(\#S_i^+)$. Hence

$$\text{rad}(S_i^+) \leq \text{diam}(S_i)(\#S_i^+ - 1)/(\#S_i^+), \text{ for every } i. \tag{5.3}$$

[4] The choices $m = 100$ and $k = 5$ are reasonable parameters for realistic course allocation problems. Harvard Business School allocates 50 course-sections in a semester, of which students choose 5. Wharton allocates as many as 200 course-sections in a semester, of which students choose 5. So $m = 100$ splits the difference as it were.

[5] The market-clearing error for a course whose price is positive is the absolute value of the excess demand for that course. The market-clearing error for a course whose price is zero is either the excess demand for that course or zero, whichever is larger.

[6] This maximum average is achieved when 86 courses have excess demands of two students and each of the 14 remaining courses have excess demands of three students.

[7] This maximum average is achieved when 49 courses have excess demands of one student and each of the 51 remaining courses have excess demands of two students.

[8] Zhou (1993) makes essentially the same use of an algebraic fact that is closely related to Caratheodory's theorem. Zhou does not consider the implications for the error bound.

The equality in (5.2) for the first n coordinates implies that $\sum_{k=1}^{K} n\lambda_{ik} = 1$ for each i, and the equality for the last m coordinates then implies that y is contained in the sum of the convex hulls of the sets $S_1^+, \ldots, S_n^+$. Hence, y is contained in the convex hull of $S_1^+ + \cdots + S_n^+$.[9] Consequently, by (5.3) and because $\mathrm{rad}(S_i^+) = 0$ for $i > j$, the refined Shapley–Folkman theorem (see Starr, 1969 or Remark 1 in Section 2) implies that there exist $x_i \in S_i^+$ and $y_i \in \mathrm{co}S_i^+$, $i = 1, \ldots, n$, such that $y = \sum_{i=1}^n y_i$, $y_i = x_i$ for all but at most m indices i, and,

$$\left\| y - \sum_{i=1}^n x_i \right\|^2 \le \sum_{i=1}^j \left(\frac{(\#S_i^+ - 1)\mathrm{diam}(S_i)}{\#S_i^+} \right)^2.$$

It is easy to show that $\#S_i^+ \ge 2$ implies that $\left((\#S_i^+ - 1)/(\#S_i^+)\right)^2 \le (\#S_i^+ - 1)/4$. Therefore, since $\#S_i \ge 2$ for every $i = 1, \ldots, j$, we have,

$$\left\| y - \sum_{i=1}^n x_i \right\|^2 \le \sum_{i=1}^j \frac{(\#S_i^+ - 1)(\mathrm{diam}(S_i))^2}{4} \le D^2 m/4,$$

where the second inequality follows because the union of the sets $S_1^+, \ldots, S_j^+$ contains no more than $m + j$ elements and so $\sum_{i=1}^j (\#S_i^+ - 1) \le m$. Hence, we may conclude that

$$\left\| y - \sum_{i=1}^n x_i \right\| \le D\sqrt{m}/2. \quad \square$$

Proof of Theorem 3.2. Since $y \in \mathrm{co}(S_1 + \cdots + S_n) = \mathrm{co}S_1 + \cdots + \mathrm{co}S_n$, there exist $z_i \in \mathrm{co}S_i$, $i = 1, \ldots, n$, such that $y = z_1 + \cdots + z_n$.

Fix any $\varepsilon > 0$. By the definition of the inner diameter, for each $i = 1, \ldots, n$, there is $T_i \subseteq S_i$ such that $z_i \in \mathrm{co}T_i$ and $\mathrm{diam}(T_i) \le \mathrm{indiam}(S_i) + \varepsilon$.

Hence, $y = z_1 + \cdots + z_n \in \mathrm{co}T_1 + \cdots + \mathrm{co}T_n = \mathrm{co}(T_1 + \cdots + T_n)$ and so, letting $D = \max(\mathrm{diam}(T_1), \ldots \mathrm{diam}(T_n))$, Theorem 3.1 implies that there exist $x_i \in T_i \subseteq S_i$ and $y_i \in \mathrm{co}T_i \subseteq \mathrm{co}S_i$, $i = 1, \ldots, n$, such that $y = \sum_i y_i$, $y_i = x_i$ for all but at most m indices i, and

$$\left\| y - \sum_{i=1}^n x_i \right\| \le D\sqrt{m}/2.$$

Setting $d = \max(\mathrm{indiam}(S_1), \ldots, \mathrm{indiam}(S_n))$, we have $D \le d + \varepsilon$ and so,

$$\left\| y - \sum_{i=1}^n x_i \right\| \le (d + \varepsilon)\sqrt{m}/2.$$

Letting $\varepsilon \to 0$ and taking convergent subsequences of the $x_i \in S_i$ and the $y_i \in \mathrm{co}S_i$ completes the proof. $\square$

Proof of Lemma 3.3. Since $0 \in \mathrm{co}S$, Caratheodory's theorem implies that there are $m + 1$ not necessarily distinct points, $x_0, x_1, \ldots, x_m$, in S and there are nonnegative $\lambda_0, \lambda_1, \ldots, \lambda_m$ that sum to 1 such that $\sum_{i=0}^m \lambda_i x_i = 0$. Let $X = \{x_0, \ldots, x_m\}$. Since $S \supseteq X$, we have $\mathrm{diam}(S) \ge \mathrm{diam}(X)$. Hence, it suffices to show that $\mathrm{diam}(X) \ge \sqrt{2(1 + 1/m)}$.

Without loss of generality, we may assume that $\lambda_0 \ge \lambda_i$ for every $i = 0, 1, \ldots, m$. In particular, $\lambda_0 > 0$.

For any $x_i \in X$,

$$\begin{aligned} \|x_i - x_0\|^2 &= \|x_i\|^2 - 2x_i x_0 + \|x_0\|^2 \\ &= 2 - 2x_i x_0, \end{aligned}$$

where the second equality follows because $\|x_i\| = \|x_0\| = 1$.

Since $\sum_{i=0}^m \lambda_i x_i = 0$, we have

$$\sum_{i=1}^m \lambda_i x_i x_0 = -\lambda_0 \|x_0\|^2 = -\lambda_0.$$

Hence there is $j \in \{1, \ldots, m\}$ such that $\lambda_j x_j x_0 \le -\lambda_0/m < 0$, which implies that $-x_j x_0 \ge (\lambda_0/\lambda_j)(1/m) \ge 1/m$, since $\lambda_0 \ge \lambda_j$. Hence,

$$\begin{aligned} (\mathrm{diam}(X))^2 &\ge \left\| x_j - x_0 \right\|^2 = 2 - 2x_j x_0 \\ &\ge 2 + 2/m. \quad \square \end{aligned}$$

Proof of Proposition 3.4. We first prove the lemma for finite subsets S of $\mathbb{R}^m$. So, let S be any nonempty finite subset of $\mathbb{R}^m$ and suppose that $\mathrm{rad}(S) = \rho$. By the definition of $\mathrm{rad}(S)$ and because S is finite, there exists $y^* \in \mathbb{R}^m$ such that $\max_{x \in S} \|y^* - x\| = \rho$. Let $S^* = \{x \in S : \|y^* - x\| = \rho\}$.

We claim that $y^* \in \mathrm{co}S^*$. To prove this claim, let us suppose not. Then, by the separating hyperplane theorem, there is $p \in \mathbb{R}^m$ such that $py^* < px$ for every $x \in S^*$. But then

$$\left.\frac{d}{dt}\right|_{t=0} \left\| y^* + tp - x \right\|^2 = 2(py^* - px) < 0, \text{ for every } x \in S^*,$$

from which we can conclude, since S is finite, that there is a small $t^* > 0$ such that $\max_{x \in S} \|y^* + t^* p - x\| < \rho$ for every $x \in S$. But this contradicts $\mathrm{rad}(S) = \rho$ and proves the claim.

Let $\hat{S} = (1/\rho)(S^* - \{y^*\})$. Then $0 \in \mathrm{co}\hat{S}$ and $\|x\| = 1$ for every $x \in \hat{S}$. Consequently, by Lemma 3.3, $\mathrm{diam}(\hat{S}) \ge \sqrt{2(1 + 1/m)}$. Moreover, since $\mathrm{diam}(\hat{S}) = \mathrm{diam}((1/\rho)(S^* - \{y^*\})) = (1/\rho)\,\mathrm{diam}(S^*)$, this implies that $\mathrm{diam}(S^*) \ge \sqrt{2(1 + 1/m)}\rho = \sqrt{2(1 + 1/m)}\mathrm{rad}(S)$. Finally, since $S \supseteq S^*$ implies that $\mathrm{diam}(S) \ge \mathrm{diam}(S^*)$, we obtain $\mathrm{diam}(S) \ge \sqrt{2(1 + 1/m)}\mathrm{rad}(S)$, proving the lemma for finite S.

To prove the lemma for nonempty compact S, choose any $\varepsilon > 0$ and let S' be a finite ε-dense subset of S.[10] Let us first show that $\mathrm{rad}(S) \le \mathrm{rad}(S') + \varepsilon$.

By the definition of $\mathrm{rad}(S')$, there is $y' \in \mathbb{R}^m$ such that $\sup_{x \in S'} \left\| y' - x \right\| = \mathrm{rad}(S')$. Therefore, by the definition of the radius of S, $\mathrm{rad}(S) \le \sup_{x \in S} \left\| y' - x \right\|$. By the compactness of S, there is $x^* \in S$ such that $\sup_{x \in S} \left\| y' - x \right\| = \left\| y' - x^* \right\|$. And since S' is ε-dense in S, there is $x' \in S'$ such that $\left\| x' - x^* \right\| < \varepsilon$. Hence,

$$\begin{aligned} \mathrm{rad}(S) &\le \sup_{x \in S} \left\| y' - x \right\| = \left\| y' - x^* \right\| \\ &\le \left\| y' - x' \right\| + \left\| x' - x^* \right\| \\ &\le \sup_{x \in S'} \left\| y' - x \right\| + \varepsilon = \mathrm{rad}(S') + \varepsilon, \end{aligned}$$

where the second inequality is the triangle inequality, and the third inequality follows because $x' \in S'$.

By what we have already shown for finite sets, $\mathrm{diam}(S') \ge \sqrt{2(1 + 1/m)}\mathrm{rad}(S')$. Hence, $\mathrm{diam}(S) \ge \mathrm{diam}(S') \ge \sqrt{2(1 + 1/m)}\,\mathrm{rad}(S') \ge \sqrt{2(1 + 1/m)}(\mathrm{rad}(S) - \varepsilon)$, where the first inequality follows because $S \supseteq S'$. Taking the limit as $\varepsilon \to 0$ gives the desired result.

[9] Using once again that the sum of the convex hulls of any finite number of sets is equal to the convex hull of their sum.

[10] Recall that one set is ε-dense in another if for every point in the second set there is a point in the first set that is within distance ε of it.

That the inequality in Proposition 3.4 can be achieved as an equality for any m has already been demonstrated in the main text (see the end of Section 3.1). □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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