Entanglement and the Temperley-Lieb category

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Abstract. We survey some recent results from [**BrCo16b**], where a class of highly entangled subspaces of bipartite quantum systems is described, which arises from unitary fiber functors on the Temperley-Lieb category associated to the representation theory of free orthogonal quantum groups. By exploiting the rich structure of the Temperley-Lieb category and this particular fiber functor, we are able to precisely determine the largest singular values for these subspaces and obtain lower bounds for the minimum output entropy of the corresponding quantum channels. Future research directions and some open problems are also discussed.

1. Introduction

Entanglement is a fundamental notion in quantum mechanics that does not have an analogue in the classical world. Within the framework of quantum computation and quantum information, entanglement in bipartite or multipartite systems produces, on the one hand, many counterintuitive phenomena, while on the other hand, it can be used to design new communications protocols which admit no classical analogues [**Eis06**, **GH96**, **CLSZ95**, **EJ96**].

Throughout this paper we will focus on entanglement in bipartite quantum systems. Within the formalism of quantum mechanics, a quantum mechanical system is described by a complex Hilbert space H. The (pure) states of the system are described by unit norm vectors $\xi \in H$, taken up to a complex phase factor. (In this paper, all Hilbert spaces are taken to be finite-dimensional, unless otherwise specified.) Equivalenty, a pure state of the system can be described by the rank one projector $\rho = |\xi\rangle\langle\xi|$ onto the subspace $\mathbb{C}\xi \subset H$. The (closed) convex hull of pure states (viewed as rank-one projectors on H) is denoted by $\mathcal{D}(H)$, and elements $\rho \in \mathcal{D}(H)$ are called *mixed states*. If we fix a basis of H and identify $\mathcal{B}(H) \cong M_n(\mathbb{C})$ ($n = \dim H$), then the convex set $\mathcal{D}(H)$ is nothing more than the collection of all trace-one positive semidefinite matrices, and the extreme points of $\mathcal{D}(H)$ are the precisely the rank one projectors, i.e., pure states on H.

In the quantum context, one often has to deal with *bipartite systems* AB, built from subsystems A, B. Mathematically, such a bipartitie system is modeled by the

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tensor product Hilbert space $H = H_A \otimes H_B$, where the Hilbert spaces H_A and H_B describe the states of systems A and B, respectively. Given such a bipartite system modeled on $H = H_A \otimes H_B$, a mixed state $\rho \in \mathcal{D}(H)$ is said to be *separable* if it belongs to the convex hull of the set of product states $\rho = \rho_A \otimes \rho_B$, where $\rho_A \in \mathcal{D}(H_A)$ and $\rho_B \in \mathcal{D}(H_B)$. A state ρ is called *entangled* if it is not separable. We shall call a Hilbert subspace $H_0 \subset H_A \otimes H_B$ an *entangled subspace* if all of its associated pure states are entangled. In this paper, we are concerned with studying non-trivial examples of highly-entangled subspaces $H_0 \subset H = H_A \otimes H_B$. By highly-entangled, we shall mean that the set of pure states on H associated to the subspace H_0 are uniformly "far away" from the set of product states $\rho_A \otimes \rho_B \in \mathcal{D}(H)$ with respect to some suitable measure of distance. The choice of "distance" here is not unique, and our choice is based on the largest singular value of pure states - precise details will be given in the next section.

What are some explicit examples of highly entangled subspaces? Well, one trivial example that we always have access to is the one-dimensional subspace $H_0 = \mathbb{C}\xi \subset H_A \otimes H_B$ spanned by a maximally entangled (Bell) state ξ. That is, a state ξ of the form $\xi = d^{-1/2} \sum_{i=1}^d e_i \otimes f_i$, where $d = \min\{\dim H_A, \dim H_B\}$ and $(e_i)_i \subset H_A$, $(f_i)_i \subset H_B$ are orthonormal systems. Another important example that has been well-studied in the literature (see for example [**GHP10**]) is the anti-symmetric subspsace $H \wedge H \subset H \otimes H$ associated to the tensor square of any Hilbert space H. As one might expect, as the relative dimension of the subspace $H_0 \subseteq H$ grows, the more challenging it becomes to find explicit examples of (highly) entangled subspaces. However, in recent years it has become a very important problem in Quantum Information Theory (QIT) and Quantum Computing (QC) to develop means to *construct subspaces H_0 of large relative dimension in a tensor product $H = H_A \otimes H_B$ such that all states are highly entangled.* One major reason for the importance of this problem is for applications to entanglement-assisted communication protocols [**NC00**, **LB16**], and also to the construction of counterexamples to additivity questions related to capacities of quantum channels [**Has09**].

As is the case for many mathematical problems, the question of the *existence* of a rich supply of highly entangled subspaces with large relative dimension can be settled using probabilistic techniques. The idea of studying random subspaces of tensor products dates back to the work of Hayden, Leung, Shor, Winter, Hastings [**HLSW04**, **HW08**, **HLW06**, **Has09**], among others, and it was explored in great detail by Aubrun, Belinschi, Collins, Fukuda, King, Nechita, Szarek, Werner [**ASW11**, **ASY14**, **BCN12**, **FK10**], and others. The whole theory of random subspaces of tensor products is intimately connected with random matrix theory and free probability theory (see [**CN16**] for a good survey on this), and has led to many fruitful interactions between these communities. The general outcome of these works was the conclusion that (at least in certain asymptotic dimension regimes) highly entangled subspaces of large relative dimension are ubiquitous: with high probability, a randomly selected subspace of a tensor product will be highly entangled. These random constructions have had a profound impact on the field, solving several open problems, most notably the minimum output entropy additivity problem [**Has09**, **ASW11**, **BCN16**]. The downside to these highly random techniques is that they provide no information on finding concrete examples that are predicted to exist by these methods. In fact, there seems to be embarrassingly few known examples of such subspaces (beyond the ones already mentioned above). Thus, there is a need for a systematic development of non-random examples of highly entangled subspaces.

The purpose of this survey is to promote our belief that one natural and fruitful place to search for *deterministic* examples of highly entangled subspaces is within the framework of *representation theory*. For example, if we are given a (compact) group G and a pair of (irreducible) unitary representations H_π, H_σ of G, then we can form their tensor product representation $H_\pi \otimes H_\sigma$, and attempt to quantify the entanglement of the irreducible subrepresentations $H_\nu \subset H_\pi \otimes H_\sigma$ that arise in the decomposition of $H_\pi \otimes H_\sigma$ into irreducibles. In this way, we have a natural playground of examples of subspaces of Hilbert space tensor products, and we shall see that understanding the entanglement (=relative position) of these subspaces is essentially equivalent to understanding the representation theory of G.

Our idea here is of course not new. For example, a first attempt was made in this direction by M. Al Nuwairan [**AN13**, **AN14**], by studying the entanglement of subrepresentations of tensor products of irreducible representations of the group $SU(2)$. Here, Al Nuwairan showed that entanglement is always achieved for subrepresentations of tensor products of $SU(2)$-irreducibles (except when one takes the highest weight subrepresentation). However, as is evidenced by the results in [**AN13**, Section 3], a high degree of entanglement is unfortunately not achieved when working with $SU(2)$.

In order to use representation theory to obtain examples of entangled subspaces exhibiting a higher level of entanglement, there are two natural approaches. The first approach would be to consider more complicated examples of compact groups and their representations. The significant downside of this approach is that for most examples of groups G, one lacks the complete understanding of the representation category Rep(G) that one has for $SU(2)$. The second approach, which we follow in this paper, is to instead consider "q-deformations" of the representation category Rep($SU(2)$) arising from certain quantum group constructions.

Perhaps the most well-known examples of such deformations are the canonical realizations of (i.e., unitary fiber functors on) the Temperley-Lieb Categories that are associated to the Drinfeld-Jimbo-Woronowicz q-deformations of $SU(2)$ [**Dri87**, **Jim85**, **Wor88**, **Wor87**]. In this paper, we consider a very different fiber functor on the Temperley-Lieb categories which act on higher dimensional spaces, and come from another class of quantum groups (more closely linked with operator algebra theory and free probability theory), called *free orthogonal quantum groups*.

Given an integer $N \geq 2$, the free orthogonal quatum group O_N^+ is the (compact) quantum group whose Hopf $*$-algebra of polynomial functions $\mathcal{O}(O_N^+)$ is given as a certain natural non-commutative (or free) version of the algebra of polynomial functions on the classical $N \times N$ orthogonal matrix group O_N. In the context of C*-algebraic compact quantum groups, O_N^+ was first introduced and studied by Wang [**Wan95**]. Shortly after Wang's original paper was published, Banica [**Ban96**] studied the representation category Rep(O_N^+) and showed that there is a natural unitary fiber functor on the Temperley-Lieb Category TL(N), which concretely realizes the representation category Rep(O_N^+). In Banica's fiber functor, the generating object of TL(N) is given by the N-dimensional fundamental representation space $\mathbb{C}^N$ (in contrast to $\mathbb{C}^2$ associated to the usual q-deformation of $SU(2)$). It is the entanglement phenomena associated to this "higher dimensional" unitary fiber functor on the Temperley-Lieb category that we study here.

Our motivation to study entanglement in the context of $\mathrm{Rep}(O_N^+) \cong \mathrm{TL}(N)$ comes from the pioneering work of Vergnioux [**Ver07**] on the seemingly unrelated *property of rapid decay (propety RD)* for quantum groups. The property of rapid decay is a geometric-analytic property posessed by certain (quantum) groups and corresponds the existence of polynomial bounds relating non-commutative L^∞-norms of polynomial functions on quantum groups to their (much easier to calculate) L^2-norms. The operator algebraic notion of property RD has its origins in the groundbreaking work of Haagerup [**Haa79**] on approximation properties of free group C*-algebras. Unlike in the case of ordinary groups, where property RD is connected to the combinatorial geometry of a discrete group G, in the quantum world, property RD was observed by Vergnioux to be intrinsically connected to the geometry of the relative position of a subrepresentation of a tensor product of irreducible representations of a given quantum group. More precisely, Vergnioux [**Ver07**, Section 4] points out that property RD for a given quantum group $\mathbb{G}$ is related to the following geometric requirement: *Given any pair of irreducible representations H_A, H_B of $\mathbb{G}$, all multiplicity-free irreducible subrepresentations $H_0 \subset H_A \otimes H_B$ must be asymptotically far from the cone of decomposable tensors in $H_A \otimes H_B$.*

The work [**BC18b**] that we survey here is largely an in-depth exploration of this passing remark of Vergnioux [**Ver07**], and our goal is to show how a rather modest understanding of the structure of the Temperley-Lieb category can be extremely fruitful when analyzing the entanglement problem for $\mathrm{Rep}(O_N^+)$. In this context, we show that one can describe very precisely the largest singular values of states that appear in irreducible subrepresentations of tensor product representations (see Theorem 3.3). This construction produces a new non-random class of subspaces of tensor products with the property of being highly entangled and of large relative dimension. As easy applications of these entanglement results, we establish some interesting properties for a class of quantum channels associated to these subspaces. We compute explicitly the $\mathcal{S}_1 \to \mathcal{S}_\infty$ norms of these channels, and obtain lower bounds on their minimum output entropies (see Section 4). We also show in Section 5 how one can use "planar diagrammatic" arguments to study further properties of our quantum channels, including their entanglement breaking property, and constructing positive maps on matrix algebras which are not completely positive. It is our hope that this survey will inspire others to view quantum groups/symmetries and their associated tensor categories as a new, rich source of entangled subspaces with interesting geometric properties.

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2. Preliminaries

We refer to [**NC00**] for the basics on entangled subspaces and quantum channels. For the reader who is interested in learning more about quantum groups and their representation categories, see [**NT13, Tim08**]. In the following, we use standard notations and conventions from operator algebra theory: Hilbert spaces are typically denoted by the letter H (possibly with some additional subscripts), and are always assumed to be finite-dimensional. The inner product on H is always linear in the right variable, $\mathcal{B}(H)$ denotes the unital $*$-algebra of (automatically bounded) linear operators on H, and $\mathcal{D}(H) \subseteq \mathcal{B}(H)$ denotes the collection of *mixed states on H*. That is, $\rho \in \mathcal{D}(H)$ iff ρ is positive semidefinite (written $\rho \geq 0$) and has unit trace, $\mathrm{Tr}(\rho) = 1$, where Tr denotes the canonical trace on $\mathcal{B}(H)$ satisfying $\mathrm{Tr}(1) = \dim H$. A state $\rho \in \mathcal{D}(H)$ is called a *pure state* if there exists a unit vector $\xi \in H$ so that ρ is given by the rank-one projector $\rho_\xi = |\xi\rangle\langle\xi|$. We denote by $\mathcal{S}_1(H)$ the Banach algebra $\mathcal{B}(H)$, equipped with the *trace norm* $\|\rho\|_{\mathcal{S}_1(H)} = \mathrm{Tr}(|\rho|)$, where $|\rho| := (\rho^*\rho)^{1/2}$. At times, we will also write $\mathcal{S}_\infty(H)$ for the space $\mathcal{B}(H)$ equipped with the operator norm $x \mapsto \|x\|_\infty = \sup_{0\neq\xi\in H} \frac{\|x\xi\|}{\|\xi\|}$.

2.1. Entangled vectors and subspaces. Consider a pair of finite-dimensional complex Hilbert spaces H_A and H_B with $\dim H_A, \dim H_B \geq 2$. Our first goal is to define what it means for a unit vector $\xi \in H_A \otimes H_B$ to be entangled, and how one can quantify the amount of entanglement that ξ has. These goals are best achieved using the *singular value decomposition (SVD)*. Namely, any unit vector $\xi \in H_A \otimes H_B$ admits a representation of the following form:

$$\xi = \sum_{i=1}^{d} \sqrt{\lambda_i} e_i \otimes f_i,$$

where $(e_i)_{i=1}^d \subset H_A$ and $(f_i)_{i=1}^d \subset H_B$ are orthonormal systems, $d = \min\{\dim H_A, \dim H_B\}$, and $\lambda_1 \geq \lambda_2 \geq \dots \lambda_d \geq 0$ satisfy $\sum_{i=1}^d \lambda_i = \|\xi\|^2 = 1$. Although the orthonormal systems defining the singular value decomposition of ξ above are never unique, it turns out the sequence of numbers $(\lambda_i)_i$ is uniquely determined (as a multi-set) by the vector ξ and these numbers are called the *singular values (or Schmidt coefficients)* of ξ.

Thus, to any unit vector $\xi \in H_A \otimes H_B$, we have an essentially unique probability distribution $(\lambda_i)_i$ corresponding to its string of singular values. With this in mind, we shall call $\xi \in H_A \otimes H_B$ *separable* if the corresponding distribution is *deterministic*: $(\lambda_i)_i = (1, 0, 0, \dots, 0)$. Otherwise, we shall call ξ *entangled*. It is easy to see that $\xi \in H_A \otimes H_B$ is entangled if and only if it cannot be expressed as a simple tensor $\xi = \eta \otimes \zeta$ for $\eta \in H_A$, $\zeta \in H_B$. The most fundamental example of an entangled vector is a *Bell vector* (or maximally entangled state), which is a unit vector $\xi_{\mathrm{Bell}} \in H_A \otimes H_B$ with SVD

$$\xi_{\mathrm{Bell}} = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} e_i \otimes f_i.$$

In this case, the singular values of ξ_{Bell} are given by the uniform distribution $\left(\frac{1}{d}, \frac{1}{d}, \dots, \frac{1}{d}\right)$, which is in many senses "far" from the detereministic distribution $(1, 0, 0, \dots, 0)$ associated to a separable vector. In fact, the above remark leads to a rigorous measurement of how entangled a unit vector $\xi \in H_a \otimes H_B$ is, namely the so-called *entanglement entropy*:

$$H(\xi) = -\sum_{i=1}^{d} \lambda_i \log \lambda_i,$$

where $(\lambda_i)_{i=1}^d$ are the singular values of ξ. In other words, $H(\xi)$ is simply the Shannon entropy of the string of singular values associated to ξ, and it follows from the basic properties of the Shannon entropy functional that $0 \leq H(\xi) \leq \log d$, and $H(\xi)$ is minimized (resp. maximized) if and only if ξ is separable (resp. maximally entangled). In general, $H(\xi)$ gives a measure of how entangled ξ is. Another (generally less precise) measure of the entanglement of a unit vector $\xi \in H_a \otimes H_B$

that we shall primarily use in this paper is given by the size of the largest singular value λ_1. Note that geometrically, λ_1 can be interpreted as the square of the cosine of the smallest angle between ξ and any product vector $\eta \otimes \zeta$. In particular, we always have

$$\frac{1}{d} \leq \lambda_1 = \sup_{\|\eta\|_{H_A} = \|\zeta\|_{H_B} = 1} |\langle \xi | \eta \otimes \zeta \rangle|^2 \leq 1,$$

with ξ being entangled (resp. maximally entangled) if and only if $\lambda_1 < 1$ (resp. $\lambda_1 = \frac{1}{d}$).

Suppose now that we are given a linear subspace $H_0 \subseteq H_A \otimes H_B$. We will call H_0 a *separable subspace* (resp. *entangled subspace*) if H_0 contains (resp. does not contain) separable vectors. Using our measure of entanglement coming from the largest singular values for unit vectors in H_0, we can call $H_0 \subseteq H_A \otimes H_B$ *highly entangled* if the supremum of all maximal Schmidt coefficients associated to all unit vectors in H_0 is bounded away from one. That is, the quantity

$$\lambda_1(H_0) := \sup_{\xi \in H_0,\ \|\xi\| = 1} \lambda_1(\xi) = \sup_{\|\xi\|_{H_0} = \|\eta\|_{H_A} = \|\zeta\|_{H_B} = 1} |\langle \xi | \eta \otimes \zeta \rangle|^2 << 1.$$

Of course, in general we have $\frac{1}{d} \leq \lambda_1(H_0) \leq 1$ with $d = \min\{\dim H_a, \dim H_B\}$.

Remark 1. For the sake of comparison, we note here that current random techniques prove the existence of highly entangled subspaces $H_0 \subseteq H_A \otimes H_B$ with $\lambda_1(H_0) \sim t$ for any $t \in (0,1)$, where $t = \frac{\dim H_0}{\dim(H_A \otimes H_B)}$ in certain asymptotically large dimensional regimes. See [**BCN12**, **BCN16**] and the references therein. On the other hand, in Section 3 we shall exhibit *deterministic examples* of $H_0 \subseteq H_A \otimes H_B$ with $\lambda_1(H_0) \sim \sqrt{t}$ where $t = \frac{\dim H_0}{\dim(H_A \otimes H_B)} \in (0,1)$.

2.2. Quantum channels. Given two Hilbert spaces H_A and H_B, a *quantum channel* is a mathematical model that describes the transmission of mixed states on H_A to mixed states on H_B. More precisely, a quantum channel is a linear, completely positive and trace-preserving map (CPTP map) $\Phi : \mathcal{B}(H_A) \to \mathcal{B}(H_B)$ [**NC00**]. By definition, we have $\Phi(\mathcal{D}(H_A)) \subseteq \mathcal{D}(H_B)$ for any quantum channel Φ. The perspective on quantum channels we take here is that they are intimately connected to the geometry subspaces of Hilbert space tensor products. Indeed, suppose we are given a triple of finite dimensional Hilbert spaces (H_A, H_B, H_C) and an isometric linear map $\alpha_A^{B,C} : H_A \to H_B \otimes H_C$, we can then form a pair of quantum channels

$$\Phi_A^{\overline{B},C} : \mathcal{B}(H_A) \to \mathcal{B}(H_C); \quad \Phi_A^{\overline{B},C}(\rho) = (\mathrm{Tr}_{H_B} \otimes \iota)(\alpha_A^{B,C} \rho (\alpha_A^{B,C})^*)$$
$$\Phi_A^{B,\overline{C}} : \mathcal{B}(H_A) \to \mathcal{B}(H_B); \quad \Phi_A^{B,\overline{C}}(\rho) = (\iota \otimes \mathrm{Tr}_{H_C})(\alpha_A^{B,C} \rho (\alpha_A^{B,C})^*).$$

In other words, associated to the subspace $\alpha(H_A) \subseteq H_B \otimes H_C$, we have two channels $\Phi_A^{\overline{B},C}$ and $\Phi_A^{B,\overline{C}}$. Note that in the literature the channel $\Phi_A^{B,\overline{C}}$ is called the *complement* of $\Phi_A^{\overline{B},C}$.

It is a remarkable fact that every quantum channel in fact arises from the above construction. This fact is a special case of the Stinespring dilation theorem for completely positive maps (see [**HW08**]). Stinespring's theorem basically says that if we are given any quantum channel $\Phi : \mathcal{B}(H_A) \to \mathcal{B}(H_B)$, then there exists an essentially unique *Stinespring pair* $(H_C, \alpha_A^{B,C})$, where H_C is an auxiliary *environment* Hilbert space $\alpha_A^{B,C} : H_A \to H_B \otimes H_C$ is a linear isometry, and $\Phi = \Phi_A^{B,\overline{C}}$ in the above notation.

The main quantity associated to quantum channel $\Phi : \mathcal{B}(H_A) \to \mathcal{B}(H_B)$ that we will be interested in this work is called the *minimum output entropy (MOE)*, $H_{\min}(\Phi)$, which is defined by

$$H_{\min}(\Phi) = \min_{\xi \in H_A,\ \|\xi\|=1} H(\Phi(|\xi\rangle\langle\xi|)).$$

where $H(\cdot)$ denotes the *von Neumann entropy* of a state: $H(\rho) = -\mathrm{Tr}(\rho \log \rho)$. Note that by functional calculus, we have $H(\rho) = -\sum_i \lambda_i \log \lambda_i$, where $(\lambda_i)_i \subset [0, \infty)$ denotes the spectrum of ρ. In other words, $H(\rho)$ is nothing but the Shannon entropy of the probability vector $(\lambda_i)_i$ corresponding to the eigenvalues of ρ.

To get a better handle on what exactly $H_{\min}(\Phi)$ is, let us suppose that $\Phi = \Phi_A^{B,\overline{C}} = (\iota \otimes \mathrm{Tr}_{H_C})(\alpha_A^{B,C}(\cdot)(\alpha_A^{B,C})^*)$ is a Stinespring representation for Φ, where $\alpha_A^{B,C} : H_A \to H_B \otimes H_C$ is our Stinespring isometry. We claim that the MOE $H_{\min}(\Phi)$ only depends on the geometry of the set of singular values associated to unit vectors in the subspace $\alpha_A^{B,C}(H_A) \subseteq H_B \otimes H_C$. Indeed, if we fix a unit vector $\xi \in H_A$ and write down the corresponding SVD $\alpha_A^{B,C}(\xi) = \sum_i \sqrt{\lambda_i} e_i \otimes f_i$, then one readily sees that

$$\Phi(|\xi\rangle\langle\xi|) = (\iota \otimes \mathrm{Tr}_{H_C})(|\alpha_A^{B,C}(\xi)\rangle\langle\alpha_A^{B,C}(\xi)|) = \sum_i \lambda_i |e_i\rangle\langle e_i|,$$

and thus

$$H(\Phi(|\xi\rangle\langle\xi|)) = -\sum_i \lambda_i \log \lambda_i = H(\alpha_A^{B,C}(\xi)).$$

In particular, computing $H_{\min}(\Phi)$ amounts to minimizing the entanglement entropy of unit vectors in the subspace $\alpha_A^{B,C}(H_A) \subseteq H_B \otimes H_C$. Namely,

$$H_{\min}(\Phi) = \min_{\xi \in H_A,\ \|\xi\|=1} H(\alpha_A^{B,C}(\xi)).$$

It follows from this calculation that $H_{\min}(\Phi)$ is zero (resp. large) if and only if $\alpha_A^{B,C}(H_A) \subseteq H_B \otimes H_C$ is a separable subspace (resp. highly entangled subspace).

2.3. Free orthogonal quantum groups, their representations, and the Temperley-Lieb Category. We now come to the main algebraic objects of study for us – quantum groups and their representation categories. In fact we will only consider one class of quantum groups here, called the free orthogonal quantum groups, and explain how their representations are connected to the Temperley-Lieb category, which is itself an amazing tensor category that is ubiquitous across many branches of mathematics (e.g., subfactors [**Jon83**], quantum computation [**Abr08**], knot theory [**Jon85**], and mathematical physics [**TL71**].)

2.3.1. *The notion of a quantum group.* The theory of quantum groups was initiated in the second half of the 20th century by several people, including Kac, Vainerman, Enock, Schwartz, Drinfeld, Jimbo, Woronowicz, Kustermans and Vaes [**ES92**, **Dri87**, **Jim85**, **Wor98**, **KV00**]. The perspective taken by each author here varies quite significantly, but the guiding principle is always the same: if we start with, say, a compact matrix group $G \subset U_N$ ($U_N \subseteq M_N(\mathbb{C})$ being the unitary group), then we can encode the entire structure of G in terms of the commutative $*$-algebra

$\mathcal{O}(G)$ of polynomial functions on G, which itself is generated as a $*$-algebra by the N^2 coordinate functions $u_{ij} : G \to \mathbb{C}$ defining the embedding $G \subseteq M_N(\mathbb{C})$. For example, the multiplication $G \times G \to G$, the inversion $G \to G$, the unit $e \in G$ are all encoded at the level of $\mathcal{O}(G)$ in terms of the following unital algebra homomorphisms

$$\begin{aligned} &\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G); \qquad && \Delta(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj} \qquad && \text{(comultiplication)} \\ &S : \mathcal{O}(G) \to \mathcal{O}(G); && S(u_{ij}) = u_{ji}^* && \text{(coinverse)} \\ &\epsilon : \mathcal{O}(G) \to \mathbb{C}; && \epsilon(u_{ij}) = \delta_{ij} && \text{(counit).} \end{aligned}$$

From these defintions, one can readily check that the Hopf identities $(\epsilon \otimes \iota)\Delta = \iota$, $m(S \otimes \iota)\Delta = m(\iota \otimes S)\Delta = \epsilon(\cdot)1$ are satisfied (where $m : \mathcal{O}(G) \otimes \mathcal{O}(G) \to \mathcal{O}(G)$ is the multiplication map), and what one obtains is the structure of a *commutative Hopf $*$-algebra* $(\mathcal{O}(G), \Delta, S, \epsilon)$.

Very loosely speaking, a quantum group is then given by a (possibly) *noncommutative* Hopf $*$-algebra $\mathbb{G} := (\mathcal{O}(\mathbb{G}), \Delta, S, \epsilon)$. Of course, when our $*$-algebra $\mathcal{O}(\mathbb{G})$ is not commutative, there is no longer anything like an underlying group representing $\mathcal{O}(\mathbb{G})$ as an algebra of coordinate functions. Nonetheless, if one places "reasonable" assumptions on $\mathcal{O}(\mathbb{G})$, then one obtains an algebraic/analytic structure that deserves to be considered group like (e.g., the existence of a "Haar measure", a rich finite-dimensional unitary representation theory, a Peter-Weyl theorem, and so on). See [**Tim08**, **Wor98**, **DK94**] for more details. Without going into any of the gory details, we assure the readers that the free orthogonal quantum groups defined below satisfy the reasonable assumptions alluded to above.

First some notation and terminology. Let $N \geq 2$, let A be a unital $*$-algebra over $\mathbb{C}$, and let $u = [u_{ij}]_{1 \leq i,j \leq N} \in M_N(A)$ be a matrix with entries in A. We will write $u^* = [u_{ji}^*] \in M_N(A)$ and $\bar{u} = [u_{ij}^*] \in M_N(A)$. We will call the matrix u an $N \times N$ *quantum orthogonal matrix* if u is invertible in $M_N(A)$, $u^* = u^{-1}$, and $\bar{u} = u$.

Definition (Free Orthogonal Quantum Groups). The *free orthogonal quantum group* (of rank N) is given by the quadruple $O_N^+ := (\mathcal{O}(O_N^+), \Delta, S, \epsilon)$, where

(1) $\mathcal{O}(O_N^+)$ is the universal unital $*$-algebra (over $\mathbb{C}$) generated by the coefficients $(u_{ij})_{1 \leq i,j \leq N}$ of a quantum orthogonal matrix $u = [u_{ij}] \in M_N(\mathcal{O}(O_N^+))$. More precisely, $\mathcal{O}(O_N^+)$ is the universal unital $*$-algebra with generators $(u_{ij})_{1 \leq i,j \leq N}$ satisfing the relations $u_{ij} = u_{ij}^*$ and $\sum_{k=1}^N u_{ik}u_{jk}$ $= \sum_{k=1}^N u_{ki}u_{kj} = \delta_{i,j}1$ for each $1 \leq i,j \leq N$.

(2) $\Delta : \mathcal{O}(O_N^+) \to \mathcal{O}(O_N^+) \otimes \mathcal{O}(O_N^+)$ is the unique unital $*$-algebra homomorphism, called the *coproduct*, given by
$$\Delta(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj} \qquad (1 \leq i,j \leq N).$$

(3) $S : \mathcal{O}(O_N^+) \to \mathcal{O}(O_N^+)$ is the $*$-antiautomorphism given by $S(u_{ij}) = u_{ji}$, $1 \leq i,j \leq N$.

(4) $\epsilon : \mathcal{O}(O_N^+) \to \mathbb{C}$ is the $*$-character given by $\epsilon(u_{ij}) = \delta_{ij}$.

Remark 2. Of course, what we have defined above is a non-commutative Hopf $*$-algebra, which we like to interpret as a noncommutative analogue of the algebra of coordinate functions on O_N. To support this perspective, note that if we quotient $\mathcal{O}(O_N^+)$ by its commutator ideal, we obtain the abelianization of $\mathcal{O}(O_N^+)$, which is isomorphic to $\mathcal{O}(O_N)$, the Hopf $*$-algebra of polynomial functions on the real orthogonal group O_N. The map $\mathcal{O}(O_N^+) \to \mathcal{O}(O_N)$ is given by $u_{ij} \mapsto v_{ij}$, where $v = [v_{ij}] \in M_N(\mathcal{O}(O_N))$ forms the matrix of basic coordinate functions on O_N (a.k.a. the *fundamental representation* of O_N). In this context, the coproduct map Δ on $\mathcal{O}(O_N^+)$ factors through the quotient and induces the corresponding coproduct map Δ on $\mathcal{O}(O_N)$. In this sense, we are justified in calling the quantum group O_N^+ a "free analogue" of the classical orthogonal group O_N, and we can even view O_N as a "quantum subgroup" of O_N^+.

2.3.2. *Unitary representations of* O_N^+. A (finite-dimensional unitary) *representation* of O_N^+ is given by a finite dimensional Hilbert space H_v and unitary matrix $v \in \mathcal{O}(O_N^+) \otimes \mathcal{B}(H_v)$ satisfying

$$(\Delta \otimes \iota)v = v_{13}v_{23} \in \mathcal{O}(O_N^+) \otimes \mathcal{O}(O_N^+) \otimes \mathcal{B}(H_v),$$

where above we use the standard leg numbering notation for linear maps on tensor products. If we fix an orthonormal basis $(e_i)_{i=1}^d \subset H_v$, then we can write v as the matrix $[v_{ij}] \in M_d(\mathcal{O}(O_N^+))$ with respect to this basis and the above formula translates to

$$\Delta v_{ij} = \sum_{k=1}^d v_{ik} \otimes v_{kj} \qquad (1 \le i, j \le d).$$

Observe that the above definition corresponds precisely to our usual notion of a unitary representation of a group if we were to assume that our Hopf $*$-algebra was commutative.

What are some examples of unitary representations of O_N^+? Based on the above formulas, it should be evident to the reader that we have immediate access to at least two distinct representations. The first one is the one-dimensional *trivial representation*, which we denote by $v^0 := 1 \in \mathcal{O}(O_N^+) = M_1(\mathcal{O}(O_N^+))$. The second example is the N-dimensional *fundamental representation* $v^1 := u = [u_{ij}] \in M_N(\mathcal{O}(O_N^+))$ (which is simply the matrix of generators for $\mathcal{O}(O_N^+)$).

In order to generate more examples of unitary representations we use our intuition from group theory and try to build more representations from v^0, v^1 via the operations of direct sum, tensor product, and compression to subrepresentations. Let us recall these notions. Given two representations $v = [v_{ij}]$ and $w = [w_{kl}]$, we can naturally form their *direct sum* $v \oplus w \in \mathcal{O}(O_N^+) \otimes \mathcal{B}(H_v \oplus H_w)$ and their *tensor product* $v \otimes w = v_{12}w_{13} = [v_{ij}w_{kl}] \in \mathcal{O}(O_N^+) \otimes \mathcal{B}(H_v \otimes H_w)$ to obtain new examples of representations from known ones. From a unitary representation $v = [v_{ij}]$, we may also form the *contragredient representation* $\bar{v} := [v_{ij}^*] \in \mathcal{O}(O_N^+) \otimes \mathcal{B}(\overline{H_v})$. Finally, if $p = p^2 = p^* \in \mathcal{B}(H_v)$ satisfies $(1 \otimes p)v = v(1 \otimes p)$, we can form the *subrepresentation* $v' := (1 \otimes p)v(1 \otimes p) \in \mathcal{O}(O_N^+) \otimes \mathcal{B}(pH)$ of v.

In order to fully understand the structure of the unitary representations of O_N^+, we need to study intertwiner spaces between representations. Given two representations u and v of O_N^+, we define the space of *intertwiners* between u and v as

$$\mathrm{Hom}(u, v) = \{T \in \mathcal{B}(H_u, H_v) : (\iota \otimes T)u = v(\iota \otimes T).\}$$

Two representations u, v are called *equivalent* (written $u \cong v$) if $\mathrm{Hom}(u, v)$ contains an invertible operator, and a representation u is called *irreducible* if $\mathrm{Hom}(u, u) = \mathbb{C}1$.

It is a consequence of a general fact about compact quantum groups that every unitary representation of O_N^+ is equivalent to a direct sum of irreducible unitary representations [**Wor87**, **Wor98**].

2.3.3. *Fusion rules for O_N^+-irreducibles.* As is the case for any (compact) quantum group, a problem of fundamental importance concerning O_N^+ following its introduction by Wang [**Wan95**] was to classify its irreducible unitary representations up to unitary equivalence. This problem was solved in the groundbreaking work of Banica [**Ban96**] where he showed that there exists a complete list of irreducible unitary representations of O_N^+, $(v^k)_{k\in\mathbb{N}_0}$, (taken up to unitary equivalence) such that $v^0 = 1$ (the trivial representation), $v^1 = u$ (the fundamental representation), each v^k is unitarily equivalent to its conjugate $\overline{v^k}$, and moreover the following fusion rules hold:

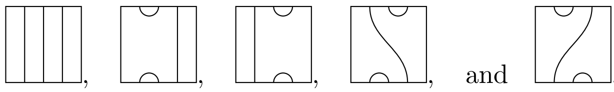
$$v^l \otimes v^m \cong v^{|l-m|} \oplus v^{|l-m|+2} \oplus \ldots \oplus v^{l+m} = \bigoplus_{0\le r\le \min\{l,m\}} v^{l+m-2r}. \tag{1}$$

Note that the above labeling of irreducible representations and fusion rules is exactly the same as those for $SU(2)$. The main difference here is that dimensions of the corresponding representation spaces are larger. More precisely, if we denote by H_k the Hilbert space associated to v^k, then the fusion rules $v^1 \otimes v^k \cong v^{k+1} \oplus v^{k-1}$ dictate that the dimension relation $\dim H_1 \dim H_k = \dim H_{k+1} + \dim H_{k-1}$ must hold for all $k \geq 1$. Taken together with the initial conditions $H_0 = \mathbb{C}$ and $H_1 = \mathbb{C}^N$, one obtains that the dimensions $\dim H_k$ are given by the *quantum integers*

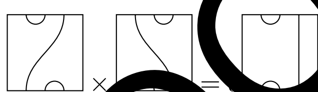
$$\dim H_k = [k+1]_q := q^{-k}\Big(\frac{1-q^{2k+2}}{1-q^2}\Big) \qquad (N \geq 3),$$

where $q \in (0,1)$ is given by $q + q^{-1} = N$. (When $N = 2$, we have $q = 1$, and then $\dim H_k = k+1 = \lim_{q\to 1}[k+1]_q$. However, in the sequel we will always take $N \geq 3$.)

2.3.4. *The connection to Temperley-Lieb.* The striking similarity between the fusion rules for the irreducible representations of O_N^+ to those of $SU(2)$ is no coincidence. This turns out to be a consequence of the fact (observed by Banica) that both representation categories are described in terms of certain unitary fiber functors on Temperley-Lieb categories [**TL71**]. Let $d \geq 2$. Recall that the *Temperley-Lieb Category* $\mathrm{TL}(d)$ is the strict tensor category with duals generated by two simple objects $\{0,1\}$, where 0 denotes the unit object for the tensor category, and $1 \neq 0$ is a self-dual simple object with the property that the morphism spaces $\mathrm{TL}_{k,l}(d) := \mathrm{Hom}(1^{\otimes k}, 1^{\otimes l})$ $(k, l \in \mathbb{N})$ are generated by the identity map $\iota \in \mathrm{Hom}(1,1)$ together with a unique morphism $\cup \in \mathrm{Hom}(0, 1\otimes 1)$ satisfying $\cap \circ \cup = d \in \mathrm{Hom}(0,0) = \mathbb{C}$. Here $\cap := \cup^* \in \mathrm{Hom}(1\otimes 1, 0)$. The Temperley-Lieb category admits a nice diagrammatic presentation [**KL94**] in terms of the so-called Kauffman (or Temperley-Lieb) diagrams. Let $k, l \in \mathbb{N}$ and $d \in \mathbb{C}\backslash\{0\}$ be as above. If $k+l$ is odd, we have $\mathrm{TL}_{k,l}(d) = 0$. Otherwise we plot the set $[k+l] = \{1, \ldots, k+l\}$ on a rectangle clockwise with $\{1, \ldots, k\}$ on the top edge and $\{k+l, \ldots, k+1\}$ on the bottom edge. Next, we consider the set $NC_2(k+l)$ of non-crossing pairings of these $k+l$ points on the boundary of our rectangle (see [**NS06**] for more details). Geometrically, any $p \in NC_2(k+l)$ corresponds to a partition of the set $[k+l]$ into $\frac{k+l}{2}$ pairs with the property that if we connect the $k+l$ boundary points on our rectangle that are paired off by p with smooth curves lying inside of our rectangle, then these curves can all be arranged so that none of them cross. The resulting non-crossing diagram is our Kauffman diagram, and is denoted by D_p. The collection of all Kauffman diagrams $(D_p)_{p\in NC_2(k+l)}$ (taken up to equivalence by planar isotopy) spans a basis for the vector space $\mathrm{TL}_{k,l}(d)$. For example, when $k = l = 3$ there are $|NC_2(6)| = 5$ Kauffman diagrams spanning $\mathrm{TL}_{k,k}(d)$:



In the diagrammatic description of the morphism spaces $\mathrm{TL}_{k,l}(d)$, the composition D_pD_q of diagrams $D_p \in \mathrm{TL}_{k,l}(d)$ and $D_q \in \mathrm{TL}_{m,k}(d)$ is obtained by first stacking the diagram D_p on top of D_q, connecting the bottom row of k points on D_p to the top row of k points on D_q. The result is a new planar diagram, which may have a certain number, c, of internal loops. By removing these loops, we obtain a new Kauffman diagram $D_r \in \mathrm{TL}_{m,l}(d)$, corresponding to some $r \in NC_2(m+l)$ (which is unique up to planar isotopy). The composition D_pD_q is then defined to be d^cD_r. For example, we have



As for the tensor structure on $\mathrm{TL}(d)$, this is simply diagrammatically represented by horizontal concatenation of Kauffman diagrams. We leave it to the reader to verify how each of the above diagrams is obtained from sequences of the the basic operations of tensoring and composing the basic maps $\cup, \cap$, and ι.

Returning now to the connection with $\mathrm{Rep}(O_N^+)$ – observe that we can produce a natural unitary tensor functor $\mathrm{TL}(N) \to \mathrm{Rep}(O_N^+)$ given by $\iota \in \mathrm{TL}_{1,1}(N) \mapsto \mathrm{id}_{\mathbb{C}^N} \in \mathrm{Hom}(u,u)$ and $\cup \in \mathrm{TL}_{0,2}(N) \mapsto \sum_{i=1}^N e_i \otimes e_i \in \mathrm{Hom}(1, u\otimes u)$, where $(e_i)_{i=1}^N$ is an orthonormal basis for $\mathbb{C}^N$. The key point here is that the universal properties of O_N^+ guarantee that this functor is both injective and surjective (in the sense of [**ENO05**]). More precisely, we have the following theorem of Banica.

Theorem 2.1 (Banica [**Ban96**]). *The above functor is in fact a unitary fiber functor $TL(N) \to Rep(O_N^+)$.*

With the above connection between $\mathrm{TL}(N)$ and $\mathrm{Rep}(O_N^+)$, an explicit construction of the irreducible representation spaces $(H_k)_{k\in\mathbb{N}_0}$ of O_N^+ can now proceed as follows [**Ban96, VV07, BDRV06**]. Denote by $(e_i)_{i=1}^N$ a fixed orthonormal basis for $H_1 := \mathbb{C}^N$, and as above, put $\cup = \sum_{i=1}^N e_i \otimes e_i \in \mathrm{Hom}(1, u\otimes u)$. (I.e., $u^{\otimes 2}(1 \otimes \cup) = (1 \otimes \cup)$.) Next, we consider the intertwiner space $\mathrm{Hom}(u^{\otimes k}, u^{\otimes k}) \subseteq \mathcal{B}((\mathbb{C}^N)^{\otimes k})$, which can be shown (using its identification with $\mathrm{TL}_{k,k}(N)$) to contain a unique non-zero self-adjoint projection p_k (*the Jones-Wenzl projection*) [**Wen87**] with the defining property that

$$(\iota_{H_1^{\otimes i-1}} \otimes \cup\cup^* \otimes \iota_{H_1^{\otimes k-i-1}})p_k = 0 \qquad (1 \le i \le k-1).$$

The projections p_k are known to satisfy the *Wenzl recursion*

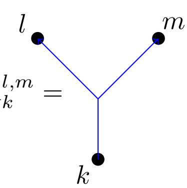
$$p_1 = \iota_{H_1}, \quad p_k = \iota_{H_1} \otimes p_{k-1} - \frac{[k-1]_q}{[k]_q}(\iota_{H_1} \otimes p_{k-1})(\cup \cup^* \otimes \iota_{H_1^{\otimes k-2}})(\iota_{H_1} \otimes p_{k-1}) \qquad (k \geq 2),$$

which can be used to determine p_k. In passing, we point out that the problem of obtaining explicit formulas for Jones-Wenzl projections (beyond the above recursion) has attracted a lot of attention over the years from various mathematical communities. See [**BC18a**, **Mor15**, **FK97**, **Rez07**, **Rez02**] and the references therein.

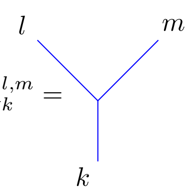
We conclude this section with a description of the non-empty intertwiner spaces $\mathrm{Hom}(v^k, v^l \otimes v^m)$ that arise from the fusion rules (1). To begin, let us call a triple $(k,l,m) \in \mathbb{N}_0^3$ *admissible* if there exists an integer $0 \leq r \leq \min\{l,m\}$ such that $k = l + m - 2r$. In other words, $(k,l,m) \in \mathbb{N}_0^3$ is admissible if and only if the tensor product representation $v^l \otimes v^m$ contains a (multiplicity-free) subrepresentation equivalent to v^k. It is easy to see that the set of admissible triples is invariant under coordinate permutations: (k_1,k_2,k_3) is admissible **iff** $(k_{\sigma(1)}, k_{\sigma(2)}, k_{\sigma(3)})$ is admissible for all $\sigma \in S_3$. Fix an admissible triple $(k,l,m) \in \mathbb{N}_0^3$. Then $\mathrm{Hom}(v^k, v^l \otimes v^m) \subseteq \mathcal{B}(H_k, H_l \otimes H_m) \subseteq \mathcal{B}(H_1^{\otimes k}, H_1^{\otimes l} \otimes H_1^{\otimes m})$ is one-dimensional and is spanned by the following canonical non-zero intertwiner

$$A_k^{l,m} = (p_l \otimes p_m)\Big(\iota_{H_1^{\otimes l-r}} \otimes \cup_r \otimes \iota_{H_1^{\otimes m-r}}\Big)p_k, \tag{2}$$

where $\cup_r \in \mathrm{Hom}(1, u^{\otimes 2r})$ is defined recursively from $\cup_1 := \sum_{i=1}^N e_i \otimes e_i$ via $\cup_r = (\iota_{H_1} \otimes \cup_1 \otimes \iota_{H_1})\cup_{r-1}$. In terms of the planar diagrammatics, $\cup_r$ is simply r nested cups, viewed as an element of $TL_{0,2r}(N)$. The maps $A_k^{l,m}$ are well studied in the Temperley-Lieb recoupling theory [**KL94**], and are known there as *three-vertices*. See also [**EMM17**]. A three-vertex is typically diagrammatically represented as follows:

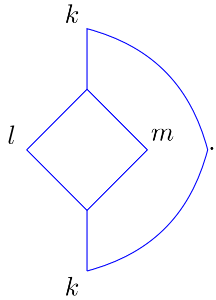
$A_k^{l,m} =$ 

Here, the solid dots at the vertices are meant to depict the Jones-Wenzl projectors at the inputs/outputs. In the following we will simply omit these solid dots in our pictures, and simply draw the three-vertex as

$A_k^{l,m} =$  .

In order to find the unique O_N^+-equivariant *isometry* $\alpha_k^{l,m} : H_k \to H_l \otimes H_m$ (up to multiplication by $\mathbb{T}$), we simply have to renormalize $A_k^{l,m}$, which amounts to computing the norm of $A_k^{l,m}$. To do this, we define (following the terminology and diagrammatics from [**KL94**]) the θ-*net*

$$\theta_q(k,l,m) = \mathrm{Tr}_{H_k}((A_k^{l,m})^* A_k^{l,m}) = $$



Note that the trace on $\mathcal{B}(H_k)$ corresponds to the usual Markov trace on $\mathrm{TL}(N)$ [**KL94**, **Ban96**].

Now, since $A_k^{l,m}$ is a multiple of an isometry, it easily follows that $\|A_k^{l,m}\|^2[k+1]_q = \theta_q(k,l,m)$. θ-net evaluations are well known [**KL94**, **Ver05**, **VV07**], and are given by

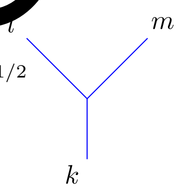
$$\theta_q(k,l,m) := \frac{[r]_q![l-r]_q![m-r]_q![k+r+1]_q!}{[l]_q![m]_q![k]_q!}, \tag{3}$$

where $k = l+m-2r$ and $[x]_q! = [x]_q[x-1]_q \dots [2]_q[1]_q$ denotes the quantum factorial. We thus arrive at the following formula for our isometry $\alpha_k^{l,m}$:

$$\alpha_k^{l,m} = \|A_k^{l,m}\|^{-1} A_k^{l,m} = \Big(\frac{[k+1]_q}{\theta_q(k,l,m)}\Big)^{1/2} A_k^{l,m}. \tag{4}$$

Pictorially, we have

$$\alpha_k^{l,m} = \Big(\frac{[k+1]_q}{\theta_q(k,l,m)}\Big)^{1/2}$$



3. Entanglement analysis

In this section we begin our study of the entanglement geometry of irreducible subrepresentations of tensor products of irreducible representations of O_N^+. The general setup we will consider is a fixed $N \geq 3$ and an admissible triple $(k,l,m) \in \mathbb{N}_0^3$. This corresponds to irreducible representations (v^k, v^l, v^m) of O_N^+ with corresponding representation Hilbert spaces (H_k, H_l, H_m), and a O_N^+-equivariant isometry $\alpha_k^{l,m} : H_k \to H_l \otimes H_m$ as constructed in the previous section. Recall that we set $q = \frac{1}{N}\Big(\frac{2}{1+\sqrt{1-4/N^2}}\Big) \in (0,1)$. Our main interest is to study the entanglement of the subspace $\alpha_k^{l,m}(H_k) \subseteq H_l \otimes H_m$, and the following proposition yields a measure of this.

Proposition 3.1 ([**BC18b**]). *Fix $N \geq 3$ and let $(k,l,m) \in \mathbb{N}_0^3$ be an admissible triple. Then for any unit vectors $\xi \in H_k, \eta \in H_l, \zeta \in H_m$, we have*

$$|\langle \alpha_k^{l,m}(\xi) | \eta \otimes \zeta \rangle| \leq \Big(\frac{[k+1]_q}{\theta_q(k,l,m)}\Big)^{1/2} \leq C(q) q^{\frac{l+m-k}{4}},$$

where

$$C(q) = (1-q^2)^{-1/2}\Big(\prod_{s=1}^{\infty}\frac{1}{1-q^{2s}}\Big)^{3/2}$$

REMARK 3. We note that the bound $C(q)q^{\frac{l+m-k}{4}}$ appearing in Proposition 3.1 is equivalent, as N is large, to the fourth root of the relative dimension, $\Big(\frac{\dim H_k}{\dim H_l \dim H_m}\Big)^{1/4}$.

Proposition 3.1 can be interpreted as giving a general upper bound on the largest Schmidt coefficient of a unit vector belonging to the subspace $\alpha_k^{l,m}(H_k) \subseteq H_l \otimes H_m$. That is, if $\xi \in H_k$ is a unit vector and $\alpha_k^{l,m}(\xi)$ is represented by its singular value decomposition

$$\alpha_k^{l,m}(\xi) = \sum_i \sqrt{\lambda_i} e_i \otimes f_i,$$

with $(e_i)_i \subset H_l, (f_i)_i \subset H_m$ orthonormal systems, and $\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$ satisfy $\sum_i \lambda_i = 1$, then

$$\lambda_1 \leq C(q)^2 q^{\frac{l+m-k}{2}}. \tag{5}$$

Since the above quantity is much smaller than 1 when $k < l+m$, we conclude that $\alpha_k^{l,m}(H_k)$ is "far" from containing separable unit vectors of the form $\eta\otimes\zeta \in H_l \otimes H_m$. That is, $\alpha_k^{l,m}(H_k) \subset H_l \otimes H_m$ is highly entangled. We summarize this in the following theorem.

THEOREM 3.2 ([**BC18b**]). *For k, l, m as above, the subspaces $\alpha_k^{l,m}(H_k) \subseteq H_l \otimes H_m$ are (highly) entangled provided $k < l+m$. When $k = l+m$, the highest weight subspace $\alpha_{l+m}(H_{l+m}) \subset H_l \otimes H_m$ is a separable subspace.*

PROOF. The first statement follows from the previous proposition and the remarks that follow. The second statement follows from the observation that if one considers the elementary (separable) tensors

$$(\ldots \xi\otimes\eta\otimes\xi\otimes\eta)\otimes(\xi\otimes\eta\otimes\xi\otimes\ldots) \in (\mathbb{C}^N)^{\otimes l}\otimes(\mathbb{C}^N)^{\otimes m} \qquad (\xi \perp \eta),$$

then they always lie in the subspace $\alpha_{l+m}^{l,m}(H_{l+m}) \subset H_l \otimes H_m$ (thanks to the algebraic properties of the Jones-Wenzl projections!). See [**BC18b**] for details. □

In fact it turns out that one can say quite a lot more about the largest possible Schmidt coefficients for irreducible subspaces of tensor products than what is said in Proposition 3.1. The following theorem shows that the bound given above is in fact optimal in a very strong sense: For any $d \in \mathbb{N}$, we can find a unit vector $\xi \in H_k$ (provided N is sufficiently large) with the property that $\alpha_k^{l,m}(\xi)$ admits at least d Schmidt coefficients with the same magnitude as that predicted by (5).

THEOREM 3.3 ([**BC18b**]). *Let $(k,l,m) \in \mathbb{N}_0^3$ be an admissible triple, $N \geq 3$, and $d \leq (N-2)(N-1)^{\frac{l+m-k-2}{2}}$. Then there exists a unit vector $\xi \in H_k$ such that $\alpha_k^{l,m}(\xi)$ has a singular value decomposition $\alpha_k^{l,m}(\xi) = \sum_i \sqrt{\lambda_i} e_i \otimes f_i$ with $\lambda_1 \geq \lambda_2 \geq \ldots$ satisfying*

$$\lambda_1 = \lambda_2 = \ldots = \lambda_d = \frac{[k+1]_q}{\theta_q(k,l,m)} \geq q^{\frac{l+m-k}{2}}.$$

Remark 4. For various applications of the above theorem, it is of critical importance to understand if the above result is optimal in the sense that the number d of maximal Schmidt coefficients that is obtainable is indeed given by the above bound. At this stage, we are unable to fully answer this question. However, we can show that the upper bound $d(N) := (N-2)(N-1)^{\frac{m+l-k-2}{2}}$ of maximal Schmidt coefficients $\lambda_{\max} = \frac{[k+1]_q}{\theta_q(k,l,m)}$ is *asymptotically maximal* in the sense that

$$\lim_{N\to\infty} d(N)\frac{[k+1]_q}{\theta_q(k,l,m)} = 1.$$

This shows that in the limit as $N \to \infty$, the vector $\xi \in H_k$ which is asserted to exist by Theorem 3.3 becomes maximally entangled, with the bulk of its Schmidt coefficients equaling the maximal value $\lambda_{\max}$ allowed by Proposition 3.1.

4. O_N^+-equivariant quantum channels and minimum output entropy estimates

In this section we consider some applications of the entanglement results of the preceding section to study the outputs of the canonical quantum channels related to our subspaces.

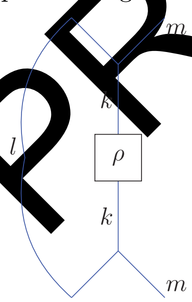
Following Section 2, we form, for any admissible triple $(k,l,m) \in \mathbb{N}_0^3$, the complementary pair of quantum channels

$$\Phi_k^{\bar{l},m} : \mathcal{B}(H_k) \to \mathcal{B}(H_m), \qquad \rho \mapsto (\mathrm{Tr} \otimes \iota)(\alpha_k^{l,m}\rho(\alpha_k^{l,m})^*),$$

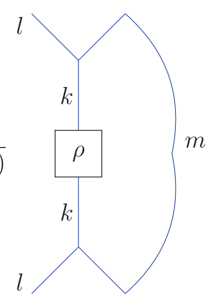
$$\Phi_k^{l,\overline{m}} : \mathcal{B}(H_k) \to \mathcal{B}(H_l), \qquad \rho \mapsto (\iota \otimes \mathrm{Tr})(\alpha_k^{l,m}\rho(\alpha_k^{l,m})^*).$$

In terms of the planar diagrammatics of the Temperley-Lieb category, we have

$\Phi_k^{l,\overline{m}}(\rho) = \frac{[k+1]_q}{\theta_q(k,l,m)}$



and $\Phi_k^{l,\overline{m}}(\rho) = \frac{[k+1]_q}{\theta_q(k,l,m)}$



.

We then have the following proposition concerning the $\mathcal{S}_1 \to \mathcal{S}_\infty$ behavior of these channels.

Proposition 4.1 ([**BC18b**]). *Given any admissible triple $(k,l,m) \in \mathbb{N}_0^3$ and $N \geq 3$, we have*

$$\|\Phi_k^{\bar{l},m}\|_{\mathcal{S}_1(H_k)\to\mathcal{S}_\infty(H_m)} = \|\Phi_k^{l,\overline{m}}\|_{\mathcal{S}_1(H_k)\to\mathcal{S}_\infty(H_l)}$$
$$= \frac{[k+1]_q}{\theta_q(k,l,m)} \in \big[q^{\frac{l+m-k}{2}}, C(q)^2 q^{\frac{l+m-k}{2}}\big].$$

Proof. We shall only consider $\Phi_k^{\bar{l},m}$ as the proof of the other case is identical. To prove the upper bound $\|\Phi_k^{\bar{l},m}\|_{\mathcal{S}_1(H_k)\to\mathcal{S}_\infty(H_m)} \leq \frac{[k+1]_q}{\theta_q(k,l,m)}$, note that by complete positivity, convexity and the triangle inequality, it suffices to consider a pure state

$\rho = |\xi\rangle\langle\xi| \in \mathcal{D}(H_k)$ and show that $\|\Phi_k^{\bar{l},m}(\rho)\|_{\mathcal{S}_\infty(H_m)} \le \frac{[k+1]_q}{\theta_q(k,l,m)}$. But in this case, we have

$$\Phi_k^{\bar{l},m}(\rho) = (\mathrm{Tr}\otimes\iota)(|\alpha_k^{l,m}\xi\rangle\langle\alpha_k^{l,m}\xi|) = \sum_i \lambda_i |f_i\rangle\langle f_i|,$$

where $\alpha_k^{l,m}(\xi) = \sum_i \sqrt{\lambda_i} e_i \otimes f_i$ is the corresponding singular value decomposition. In particular, $\|\Phi_k^{\bar{l},m}(\rho)\|_{\mathcal{S}_\infty(H_m)} = \max_i \lambda_i$, which by Proposition 3.1 is bounded above by $\frac{[k+1]_q}{\theta_q(k,l,m)}$. This upper bound is obtained by taking $\rho = |\xi\rangle\langle\xi|$, where ξ satisfies the hypotheses of Theorem 3.3. □

The preceeding norm computation for the channels $\Phi_k^{\bar{l},m}, \Phi_k^{l,\overline{m}}$ allows for an easy estimate of a lower bound on their minimum output entropies.

Corollary 4.2. [**BC18b**] *Given any admissible triple $(k,l,m) \in \mathbb{N}_0^3$ and $N \ge 3$, we have*

$$H_{\min}(\Phi_k^{\bar{l},m}), H_{\min}(\Phi_k^{l,\overline{m}}) \ge \log\Big(\frac{\theta_q(k,l,m)}{[k+1]_q}\Big) \ge -\Big(\frac{l+m-k}{2}\Big)\log(q) - 2\log(C(q)).$$

Proof. Given a quantum channel $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ and $\rho \in \mathcal{D}(H)$, we note that $H(\Phi(\rho)) = -\sum_i \lambda_i \log\lambda_i$, where $(\lambda_i)_i$ is the spectrum of $\Phi(\rho)$. In particular, we have the estimate

$$H(\Phi(\rho)) \ge -\log\Big(\max_i \lambda_i\Big) = -\log\|\Phi(\rho)\|_{\mathcal{B}(K)} \ge -\log\|\Phi\|_{\mathcal{S}_1(H)\to\mathcal{B}(K)}.$$

The first inequality in the corollary now follows immediately from Proposition 4.1. The second inequality is just a consequence of the inequality $\frac{[k+1]_q}{\theta_q(k,l,m)} \le C(q)^2 q^{\frac{l+m-k}{2}}$. □

Remark 5. The above estimates show that for N large and $k < l+m$ fixed, the minimum output entropy of the channels is quite large and grows logarithmically in N.

On the other hand, if we fix $N \ge 3$ and consider, for example, the sequence of channels $\big(\Phi_{k-1}^{k,\bar{1}} : \mathcal{B}(H_{k-1}) \to \mathcal{B}(H_k)\big)_{k\in\mathbb{N}}$, then Corollary 4.2 yields the uniform positive lower bound

$$H_{\min}(\Phi_{k-1}^{k,\bar{1}}) \ge -\log(q) - 2\log(C(q)) > 0 \qquad (k \in \mathbb{N}).$$

This phenomenon stands in sharp contrast to what happens in the case of the $SU(2)$-equivariant quantum channels studied by Al Nuwairan in [**AN13**, Section 2]. Indeed, in the corresponding $SU(2)$ setting one has $H_{\min}(\Phi_{k-1}^{k,\bar{1}}) \approx \frac{\log(k+1)}{k+1} \to 0$ as $k \to \infty$.

In the case where $k = l+m$ (the highest weight case), we note that

$$H_{\min}(\Phi_k^{\bar{l},m}) = H_{\min}(\Phi_k^{l,\overline{m}}) = 0,$$

which follows from the fact that $\alpha_k^{l,m}(H_k) \subseteq H_l \otimes H_m$ is a separable subspace (cf. Theorem 3.2).

Remark 6. We expect that the lower bound for the minimum output entropies given in Corollary 4.2 to be asymptotically optimal as $N \to \infty$, at least in some cases (e.g. m fixed). Evidence for this is provided by Theorem 3.3 and Remark 4, which shows that $\alpha_k^{l,m}(H_k)$ contains unit vectors which are asymptotically maximally entangled with the bulk of their Schmidt coefficients equal to $\frac{[k+1]_q}{\theta_q(k,l,m)}$.

5. The Choi map and Planar Isotopy

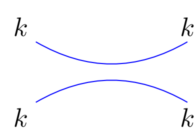
In this final section we indicate how the planar structure of our representation theoretic model for highly entangled subspaces can be used to easily describe the Choi maps associated to our quantum channels. As applications of this description, we construct non-random examples of d-positive maps between matrix algebras that fail to be completely positive, and we also study the entanglement breaking property for our channels.

First we recall the definition of the Choi map associated to a linear map $\Phi : \mathcal{B}(H_A) \to \mathcal{B}(H_B)$. Let $(e_i)_{i\in I}$, $(f_i)_{i\in I}$ be two fixed orthonormal bases for H_A, and let $(e_{ij})_{i,j\in I}$, $(f_{ij})_{i,j\in I}$ be the corresponding matrix units in $\mathcal{B}(H_A)$. Then the Choi map is the operator $C_\Phi \in \mathcal{B}(H_A \otimes H_B)$ given by

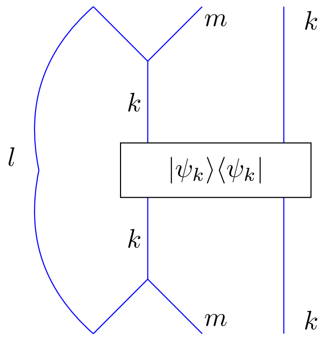
$$C_\Phi = \sum_{i,j\in I} \Phi(e_{ij}) \otimes f_{ij} = (\Phi \otimes \iota)(|\psi\rangle\langle\psi|), \tag{6}$$

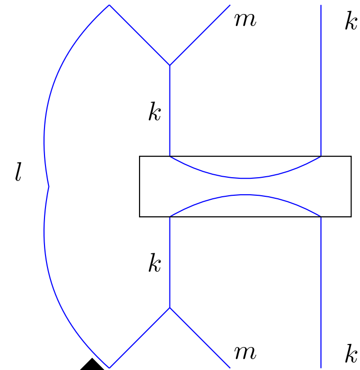
where $\psi = \sum_{i\in I} e_i \otimes f_i \in H_A \otimes H_A$ (which is an unnormalized Bell state in $H_A \otimes H_A$). Of course, C_Φ is only defined uniquely up to the choice of matrix units e_{ij} and f_{ij}. Moreover, one could also define a "right-handed" version of $\tilde{C}_\Phi$ of C_Φ given by $\tilde{C}_\Phi = (\iota \otimes \Phi)|\psi\rangle\langle\psi|$ (i.e., slicing on the right instead of the left). However, for our purposes, the relevant properties of C_Φ (e.g., entanglement, positivity, etc.) do not depend on the choice of matrix units or side of the tensor product on which one slices $|\psi\rangle\langle\psi|$ by Φ. We also note the obvious fact that the map $\Phi \mapsto C_\Phi$ is linear in Φ.

Turning back to our representation category $\mathrm{Rep}(O_N^+)$ and our quantum channels $\Phi_k^{\bar{l},m} : \mathcal{B}(H_k) \to \mathcal{B}(H_m)$ ($(k,l,m) \in \mathbb{N}_0^3$ admissible), we judiciously choose orthonormal bases $(e_i)_i$ and $(f_i)_i$ of H_k so that the unnormalized Bell vector $\psi_k = \sum_i e_i \otimes f_i \in H_k \otimes H_k$ belongs to the one-dimensional Hom-space $\mathrm{Hom}(u^0, u^k \otimes u^k)$ (this is always possible, thanks to the fact that O_N^+ is a compact quantum group of Kac type. See for example [**Ver07**]). Using our identification $\mathrm{Rep}(O_N^+) \cong \mathrm{TL}(N)$, we can depict ψ_k (in terms of planar diagrams) as a three-vertex corresponding to the admissible triple $(0,k,k)$, which is explicitly given by $(p_k \otimes p_k) \circ \cup_k \in \mathrm{TL}_{0,2k}(N)$, where p_k is the kth Jones-Wenzl projector. Considering the projection $|\psi_k\rangle\langle\psi_k|$, we have

$|\psi_k\rangle\langle\psi_k| =$ 

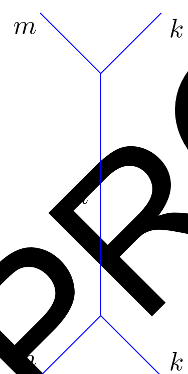
Then we can compute the corresponding Choi map $C_{\Phi_k^{\bar{l},m}} = \big(\Phi_k^{\bar{l},m} \otimes \iota\big)(|\psi_k\rangle\langle\psi_k|)$ diagrammatically by

$$\frac{\theta_q(k,l,m)}{[k+1]_q} C_{\Phi_k^{\bar{l},m}} = $$ 

$$=$$ 

,

Since the linear map defined by the above planar tangle is invariant under planar isotopy (by construction it belongs to the Temperley-Lieb category!), we see that $\frac{\theta_q(k,l,m)}{[k+1]_q} C_{\Phi_k^{\bar{l},m}}$ also corresponds to the following planar tangle:



$$= \frac{\theta_q(k,l,m)}{[l+1]_q} \alpha_l^{m,k} (\alpha_l^{m,k})^*,$$

Note here that $\alpha_l^{m,k}(\alpha_l^{m,k})^*$ is simply the orthogonal equivariant projection from $H_m \otimes H_k$ onto the unique subspace equivalent to H_l. We have therefore arrived at the following theorem.

THEOREM 5.1. *For the O_N^+-equivariant quantum channel $\Phi_k^{\bar{l},m} : \mathcal{B}(H_k) \to \mathcal{B}(H_m)$, we have*

$$C_{\Phi_k^{\bar{l},m}} = \frac{[k+1]_q}{[l+1]_q} \alpha_l^{m,k} (\alpha_l^{m,k})^*. \tag{7}$$

A similar argument for the complementary channel $\Phi_k^{l,\overline{m}} : \mathcal{B}(H_k) \to \mathcal{B}(H_l)$, yields

$$\tilde{C}_{\Phi_k^{l,\overline{m}}} = \frac{[k+1]_q}{[m+1]_q} \alpha_m^{k,l} (\alpha_m^{k,l})^*. \tag{8}$$

In the following subsections, we show the utility of Theorem 5.1.

5.1. Examples of positive but not completely positive maps. A crucial property of the Choi map C_Φ associated to a linear map $\Phi : \mathcal{B}(H_A) \to \mathcal{B}(H_B)$ is that it can be used to detect positivity properties of Φ. More precisely, we have that Φ is completely positive if and only if C_Φ is positive semidefinite [**Cho75**]. More generally, C_Φ can be used to detect whether or not Φ is d-positive for any $d \in \mathbb{N}$ [**HLP+12**]: Φ is d-positive if and only if

$$\langle C_\Phi x | x\rangle \geq 0$$

for all $x \in H_A \otimes H_B$ with a *Schmidt rank* of at most d. (That is, x admits a singular value decomposition $x = \sum_{i=1}^s \sqrt{\lambda_i} e_i \otimes f_i$ with $\min_i \lambda_i > 0$ and $s \leq d$).

Let us now return to our usual setup of an admissible triple $(k, l, m) \in \mathbb{N}_0^3$ corresponding to a non-highest-weight inclusion $\alpha_k^{l,m} : H_k \hookrightarrow H_l \otimes H_m$ of irreducible representations of O_N^+, $N \geq 3$. For each $t \geq 0$, we can consider the linear map $\Phi_t : \mathcal{B}(H_k) \to \mathcal{B}(H_m)$ given by

$$\Phi_t = \mathrm{Tr}_{H_k}(\cdot) 1_{\mathcal{B}(H_l)} - t \frac{[l+1]_q}{[k+1]_q} \Phi_k^{l,m}. \tag{9}$$

Using Theorem 5.1 together with the simple fact that the Choi map associated to $\mathcal{B}(H_k) \ni \rho \mapsto \mathrm{Tr}_{H_k}(\rho) 1_{\mathcal{B}(H_l)}$ is given by $1_{\mathcal{B}(H_l \otimes H_k)}$, we conclude that the Choi map of Φ_t is given by

$$C_{\Phi_t} = 1_{\mathcal{B}(H_m \otimes H_k)} - t\alpha_l^{m,k} (\alpha_l^{m,k})^*. \tag{10}$$

From this expression for C_{Φ_t}, it is clear that Φ_t is completely positive iff $C_{\Phi_t} \geq 0$ iff $t \leq 1$. On the other hand, we can prove the following result on d-positivity of Φ_t.

Theorem 5.2 ([**BC18b**]). *Fix $N \geq 3$ and $(k, l, m) \in \mathbb{N}_0^3$, and fix a natural number $d \leq (N-2)(N-1)^{\frac{k+m-l-2}{2}}$. Then the map $\Phi_t : \mathcal{B}(H_k) \to \mathcal{B}(H_m)$ is d-positive (but not completely positive) if and only if*

$$1 < t \leq \frac{\theta_q(k,l,m)}{d[l+1]_q} \leq C(q)^{-2} q^{-\frac{k+m-l}{2}} d^{-1}.$$

Proof. We have already observed that Φ_t is not completely positive when $t > 1$. Now fix $d \in \mathbb{N}$ and $x = \sum_{i=1}^s \sqrt{\lambda_i} e_i \otimes f_i \in H_m \otimes H_k$ with Schmidt-rank at most d. Using the inequality of Proposition 3.1, the triangle inequality, and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\langle C_{\Phi_t} x | x \rangle &= \|x\|^2 - t \langle \alpha_l^{m,k} (\alpha_l^{m,k})^*(x) | x\rangle \\
&\geq \|x\|^2 - t \frac{[l+1]_q}{\theta_q(k,l,m)} \Big(\sum_{1 \leq i \leq s} \sqrt{\lambda_i} \|e_i\| \|f_i\| \Big)^2 \\
&\geq \|x\|^2 - t \frac{[l+1]_q}{\theta_q(k,l,m)} \Big(\sum_{1 \leq i \leq s} \sqrt{\lambda_i} \Big)^2 \\
&\geq \|x\|^2 - t \frac{[l+1]_q}{\theta_q(k,l,m)} s \|x\|^2 \\
&\geq \|x\|^2 \Big(1 - td \frac{[l+1]_q}{\theta_q(k,l,m)} \Big).
\end{aligned}$$

From this inequality, we obtain d-positivity of Φ_t provided $1 - td \frac{[l+1]_q}{\theta_q(k,l,m)} \geq 0$, as claimed.

To show failure of d-positivity when $t > \frac{\theta_q(k,l,m)}{d[l+1]_q}$, one has to find $x = \sum_{i=1}^d \eta_i \otimes \zeta_i \in H_l \otimes H_m$ with Schmidt rank d satisfying $\langle C_{\Phi_t} x | x \rangle < 0$. It turns out that such an x can be canonically constructed – see [**BC18b**] for details. □

Remark 7. The above theorem can readily be used to construct maps on matrix algebras that are d positive but not $d+1$ positive. Indeed, one just has to choose $t > 1$, $N \geq 3$ and an admissible triple $(k,l,m) \in \mathbb{N}_0^3$ so that

$$\frac{\theta_q(k,l,m)}{(d+1)[l+1]_q} < t \leq \frac{\theta_q(k,l,m)}{d[l+1]_q}.$$

Then the corresponding Φ_t will do the job.

5.2. Entanglement breaking channels. We now turn to another application of Theorem 5.1, to the entanglement breaking property of our quantum channels $\Phi_k^{\bar{l},m}$.

Definition. A quantum channel $\Phi : \mathcal{B}(H_A) \to \mathcal{B}(H_B)$ is called *entanglement breaking (or EBT)* if for any finite-dimensional auxiliary Hilbert space H_0, and any state $\rho \in \mathcal{D}(H_0 \otimes H_A)$, we have that $(\iota \otimes \Phi)(\rho) \in \mathcal{D}(H_0 \otimes H_B)$ is a separable state.

The class of EBT channels are precisely those which eliminate entanglement between the input states of composite systems. These channels form an important class which are amenable to analysis. For example, it is known that for EBT channels, both the minimum output entropy and the Holevo capacity (i.e., the capacity of a quantum channel used for classical communication with product inputs) is additive [**Hol01**, **Sho02**].

In order to detect whether or not a given quantum channel is EBT, it suffices to check whether or not the corresponding Choi map is a multiple of an entangled state. The following result is well known: see for example [**AN13**, Proposition 3.4].

Proposition 5.3. *For a quantum channel $\Phi : \mathcal{B}(H_A) \to \mathcal{B}(H_B)$, the following conditions are equivalent.*

(1) *Φ is EBT.*
(2) *The state $\rho = \frac{1}{\dim H_A} C_\Phi \in \mathcal{D}(H_B \otimes H_A)$ is separable.*

Before coming to our main result of this section characterizing the EBT property for the channels $\Phi_k^{\bar{l},m}$, we first need an elementary lemma.

Lemma 5.4. *Let H_A and H_B be finite dimensional Hilbert spaces, let $0 \neq p \in \mathcal{B}(H_B \otimes H_A)$ be an orthogonal projection, and let $H_0 \subseteq H_B \otimes H_A$ denote the range of p. If H_0 is an entangled subspace of $H_B \otimes H_A$, then the state $\rho := \frac{1}{\dim H_0} p$ is entangled.*

Proof. We prove the contrapositive. If ρ is separable, then we can write

$$p = \sum_i |\xi_i\rangle\langle\xi_i| \otimes |\eta_i\rangle\langle\eta_i| \qquad (0 \neq \xi_i \in H_B,\ 0 \neq \eta_i \in H_A).$$

For each i put $x_i = |\xi_i\rangle\langle\xi_i| \otimes |\eta_i\rangle\langle\eta_i|$. Then since $x_i \leq p$ and p is a projection, it follows that $x_i = px_ip$, which implies that the range of x_i is contained in the range of p. In particular, $\xi_i \otimes \eta_i \in H_0$, so H_0 is separable. □

Theorem 5.5. *Let $(k,l,m) \in \mathbb{N}_0^3$ be an admissible triple. If $k \neq l - m$, then the quantum channel $\Phi_k^{\bar{l},m}$ is not EBT.*

Proof. We have from Theorem 5.1 that $C_{\Phi_k^{\bar{l},m}} = \frac{[k+1]_q}{[l+1]_q}\alpha_l^{m,k}(\alpha_l^{m,k})^* \in \mathcal{B}(H_m \otimes H_k)$. Consider the orthogonal projection $p = \alpha_l^{m,k}(\alpha_l^{m,k})^*$. The range of p is the subrepresentation of $H_m \otimes H_k$ equivalent to H_l, and by Theorem 3.2 this subspace is entangled iff $l \neq k+m$. Applying Lemma 5.4 and Proposition 5.3, we conclude that $\Phi_k^{\bar{l},m}$ is not EBT whenever $k \neq l-m$. □

Remark 8. We note that Theorem 5.5 leaves open whether or not the channels $\Phi_{l-m}^{\bar{l},m}$ are EBT. In this case, the corresponding Choi map is a multiple of a projection onto a separable subspace, and we do not know if this projection is a multiple of an entangled state.

6. Future work and open problems

We conclude this survey with a list of open problems and directions for future work.

(1) A major problem in QIT is to find explicit examples of quantum channels Φ, Ψ which are *strictly MOE-subbaditive*: $H_{\min}(\Phi \otimes \Psi) < H_{\min}(\Phi) + H_{\min}(\Psi)$. Such channels are known to exist with high probability [**Has09**, **ASW11**, **BCN16**], but no explicit examples are known. It is therefore tempting to wonder whether or not the channels considered in this work might be MOE subadditive. The first step in considering this question is to have an effective means of estimating the MOE of tensor products of our channels. In this context some computations are actually possible. In particular, if one takes one of our Temperley-Lieb chanels Φ, then it is always possible to explicitly compute the von Neumann entropy $H(\Phi \otimes \Phi^c)(\rho)$ of the output of a Bell state ρ, where Φ^c denotes the so-called *complementary channel* associated to Φ. It turns out that this computation involves the *quantum 6j symbols* associated to the Temperley-Lieb category. This particular calculation is the topic of work in preparation [**BCLY18**]. At the present time, it seems that in order to have any hope of whitnessing strict MOE subadditivity in our channels, more tensor products beyond simply channels and their complements need to be studied, and at this time, a new idea is needed.

(2) Another important question related to our class of quantum channels is the problem of computing their classical and quantum capacities. This is another completely open and important research direction.

(3) As we have seen in this work, the Temperley-Lieb category provides a tractable concrete model for highly entangled subspaces. It is natural to wonder what other nice tensor categories or related structures give nice models of entanglement. Perhaps certain examples coming from planar algebras [**Jon99**] might give some interesting results?

(4) It would be interesting to make a further study of the family of d-positive maps Φ_t given here. The importance of such maps in QIT is for *entanglement detection* in bipartite systems: Positive maps that are not completely positive can be used to distinguish entangled states from separable ones. Of particular interest is the problem of detecting entangled states from the *positive partial transpose (PPT) states*. In this context, the relevant maps for entanglement detection are the *indecomposable maps*. I.e., positive maps Φ which are not of the form $\Phi = \Phi_1 + \Phi_2 \circ t$, where $\Phi_{1,2}$ are

completely positive, and t denotes the transpose map. In this context, we ask: *Are our families of maps Φ_t indecomposable?*

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