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Temperley–Lieb Quantum Channels

Michael Brannan[1], **Benoît Collins**[2], **Hun Hee Lee**[3], **Sang-Gyun Youn**[4]

[1] Department of Mathematics, Texas A&M University, Mailstop 3368, College Station, TX 77843-3368, USA. E-mail: mbrannan@math.tamu.edu
[2] Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan. E-mail: collins@math.kyoto-u.ac.jp
[3] Department of Mathematical Sciences and the Research Institute of Mathematics, Seoul National University, Gwanak-ro 1, Gwanak-gu, Seoul 08826, Republic of Korea. E-mail: hunheelee@snu.ac.kr
[4] Department of Mathematics Education, Seoul National University, Gwanak-ro 1, Gwanak-gu, Seoul 08826, Republic of Korea. E-mail: s.youn@snu.ac.kr

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Abstract: We study a class of quantum channels arising from the representation theory of compact quantum groups that we call Temperley–Lieb quantum channels. These channels simultaneously extend those introduced by Brannan and Collins (Commun Math Phys 358(3):1007–1025, 2018), Nuwairan (Int J Math 25(6):1450048, 2014) and Lieb and Solovej (Acta Math 212(2):379–398, 2014). (Quantum) Symmetries in quantum information theory arise naturally from many points of view, providing an important source of new examples of quantum phenomena, and also serve as useful tools to simplify or solve important problems. This work provides new applications of quantum symmetries in quantum information theory. Among others, we study entropies and capacitites of Temperley–Lieb channels, their (anti-) degradability, PPT and entanglement breaking properties, as well as the behaviour of their tensor products with respect to entangled inpurs. Finally we compare the Tempereley–Lieb channels with the (modified) TRO-channels recently introduced by Gao et al. (Commun Math Phys 364(1):83–121, 2018)).

1. Introduction

A fundamental problem in (quantum) information theory is to understand the capacity of a noisy communications channel. In the quantum world, this is harder, because there are many notions of capacities, many of which are defined through a regularization limit whose computation and mathematical understanding is much more subtle and mathematically involved than in the classical case. The non-trivial channels for which many entropic or capacity related quantities can be computed and be of non-trivial value or interest are rather scarce. One reason for this paucity is that many quantities are defined with minimizers, and many properties (e.g. PPT, entanglement breaking property (shortly, EBT), degradability and so on) rely on the existence of auxiliary objects or computations of tensors that are close to impossible to describe effectively without additional conceptual assumptions on the quantum channel.

One of the most natural (and to our mind, underrated) properties of a quantum channel is for it to possess some sort of group symmetry. In this paper, we will focus on quantum channels which feature symmetries with respect to structures which are more general than groups: compact quantum groups. For example, the notion of a covariant quantum channel with respect to a compact group action was introduced in many contexts [WH02, DFH06, MSD17, AN14, LS14, Rit05] but these properties have not been extensively used from the analysis point of view of quantum information theory (shortly, QIT) such as estimating capacities and related quantities. In addition, most of the time, the covariance under consideration is with respect to the most elementary group representations, e.g., the basic representation of a matrix group $G \subset M_n(\mathbb{C})$ on $\mathbb{C}^n$. The principal reason behind the restriction to the basic representations so far is that the symmetries involved and the analysis behind many aspects of representation theory are not well-understood to the degree required to estimate important quantities. Nonetheless, it was observed in many places that such symmetries can be useful (e.g. [MHRW16, HM15, Sch05, DFH06, KW09, SWPGC09, MSD17], etc). See also [COS18] for a covariant characterization of k-positive maps.

The first systematic attempt to remedy this limitation was conducted by Al Nuwairan [AN14] in the context of $SU(2)$ symmetries. Here, Al Nuwairan investigated quantum channels arising from the intertwining isometries of the irreducible decomposition of the tensor product of two irreducible representations of $SU(2)$, which we will call $SU(2)$-Temperley–Lieb quantum channels (shortly, $SU(2)$-TL-channels). Thanks to the well-known SU(2)-Clebsch–Gordan formulas, explicit results could be obtained and it turned out that SU(2)-TL-channels play an important role in describing general SU(2)-covariant quantum channels. However, from the perspective of entanglement theory, the performance of SU(2)-TL-channels was not spectacular: although a complete description was given of which irreducible representation appearing in a tensor product of representations is entangled, the minimum entanglement entropy was computed explicitly, and allowed to deduce that such quantum channels were not suitable candidates for the violation of additivity for the minimum output entropy [BC18, Remark6]. Subsequently, [BC18] considered a quantum extension of $SU(2)$-TL-channels using irreducible representations of free orthogonal quantum groups, which we call O_N^+-TL-channels in this paper, and noticed that a notion of rapid decay was exactly the concept needed to quantify the high level of entanglement in this new setup. The main theme of [BC18] was to replace group symmetries by quantum group symmetries, especially for the free orthogonal quantum group O_N^+ case, whose main advantage is that it allows one to work with a well-understood C*-tensor category (the Temperley–Lieb category) which facilitates very explicit computations and estimates.

The present work undertakes a much more systematic study of $SU(2)$-TL-channels and O_N^+-TL-channels, and compares their various information theoretic properties. One important achievement of this paper is that the minimum output entropy (shortly, MOE) $H_{\min}$, the coherent information $Q^{(1)}$ and the Holevo information χ can be estimated, and that these estimates are asymptotically sharp as N becomes big, in the case of O_N^+-TL-channels. More generally, the main results of this paper are summarized in Table 1.

Table 1. Summary of results

Properties\channels	O_N^+-TL-ch. [Sects. 4, 5]	$SU(2)$-TL-ch. [Sect. 5]
$H_{\min}$	Asympt. sharp	[AN13]
$Q^{(1)}$ and χ	Asympt. sharp	Rough estimates
EBT	No except for the lowest weight	Complete
PPT	No except for the lowest weight with $N \gg 1$	Complete
(Anti-)degradability	No except for the lowest weight with $N \gg 1$	Partial results
C (classical capacity)	$C \leq (2+\varepsilon)\chi$ with $N \gg 1$	? (open)
Equivalence to TRO ch.	? (open)	No in general [Sect. 7]

The term TRO in the above will be clarified later in the introduction and in Sect. 7 with more details. As it appears from the above table, many interesting and unexpected phenomena are unveiled, which we find counterintuitive, and whose proof boils down to an extensive case analysis. Let us highlight a few points:

- Many non-trivial results can be obtained about the degradability and anti-degradability of the covariant quantum channels. To the best of our knowledge, although these notions are really important to estimate capacities (and we use such results), there are almost no non-trivial examples in the literature of quantum channels for which one can assess the degradability and anti-degradability. Our computation is possible thanks to averaging methods stemming from (quantum) group invariance.
- In most cases, O_N^+-TL-channels with large N have a highly non-trivial structure. Indeed, they are not PPT, not degradable, not anti-degradable except for the possibility of lowest weight subrepresentations, which we still have not settled. Moreover, we present a complete list for EBT and PPT for $SU(2)$-TL-channels and it turns out that the notions of PPT and EBT are actually equivalent in the case of $SU(2)$. One important ingredient here is the diagrammatic calculus for Temperley–Lieb category covered in Sect. 3.3.
- On the other hand, we reveal unexpected results on (anti-)degradability of $SU(2)$-TL-channels. We show that they are degradable for extremal cases such as lowest or highest weight, whereas it is not true for other intermediate cases. Indeed, we provide an example of a non-degradable $SU(2)$-TL-channel in low dimensions (see Example 5.9).
- One notable result is that we have exhibited infinitely many quantum channels $\Phi : B(H_A) \to B(H_B)$ satisfying the extremal condition "$Q(\Phi) = 0$ and $C(\Phi) = \log(\dim(H_B))$" among $SU(2)$-TL-channels, which are anti-degradable and non-PPT. (See Sect. 5.4).

One crucial point in QIT is that it is often unavoidable to consider tensor products of quantum channels, and in general, computations in tensor products become very involved. However when the channels have nice symmetries, as we show in this paper, computations can remain tractable, even in non-trivial cases. In particular, we can fully describe the output of the maximally entangled state under tensor products of certain Temperley–Lieb channels. The main technical tool is an application of diagrammatic calculus explained in Sect. 3.3, which can be applied to O_N^+-TL-channels, see Sect. 6 for the details.

Finally, TL-channels bear some resemblance to another important family of channels introduced in [GJL18], called TRO-channels and their modified versions. Here, TRO refers to ternary ring of operators and the name "TRO-channel" comes from the fact that its Stinespring space, i.e. the range of the Stinespring isometry actually has a TRO structure. Examples of TRO-channels include random unitary channels from regular representations of finite (quantum) groups and generalized dephasing channels [GJL18]. While the authors were preparing this manuscript and discussing it for the first time publicly, the question of how our TL-channels compare to TRO channels was posed (and, in particular, whether or not TL implies TRO). The answer is that these classes of channels bear important differences, as explained in Sect. 7.

This paper is organized as follows. After this introduction, Sect. 2 provides some background and reminders about quantum channels and compact quantum groups. Section 3 recalls some details on free orthogonal quantum groups and their associated representation theory. Then, we introduce Tempereley-Lieb quantum channels (shortly, TL-channels) and collect some details on their associated diagrammatic calculus. Section 4 contains results about the entropies and capacities of TL-channels. Then, Sect. 5 addresses the property of entanglement breaking and PPT for TL-channels. Section 6 shows that O_N^+-TL-channels (unlike most 'structureless' quantum channels) behave very well under tensor products. Finally, Sect. 7 addresses the question of comparing the class of Kac type TL-channels with other previously well-studied classes of quantum channels such as quantum erasure channels, amplitude damping channels, dephasing channels and depolarizing channels. We end the final section with an example of Kac type TL-channel not belonging the above mentioned class of (modified) TRO-channels.

2. Preliminaries

2.1. Quantum channels and their information theoretic quantities. Here, we are only interested in quantum channels based on finite dimensional Hilbert spaces. Recall that a quantum channel is a linear completely positive trace-preserving (shortly, CPTP) map $\Phi : B(H_A) \to B(H_B)$. It is well-known that there is a so called *Stinespring isometry* $V : H_A \to H_B \otimes H_E$ such that

$$\Phi(\rho) = (\iota \otimes \mathrm{Tr}_E)(V\rho V^*), \quad \rho \in B(H_A),$$

where Tr_E refers to the trace on $B(H_E)$. For a given Stinespring isometry V we can consider the complementary channel $\tilde{\Phi} : B(H_A) \to B(H_E)$ of Φ given by

$$\tilde{\Phi}(\rho) = (\mathrm{Tr}_B \otimes \iota)(V\rho V^*), \quad \rho \in B(H_A).$$

For each quantum channel there are several important information theoretic quantities, which we recall in the following.

Definition 2.1. Let $\Phi : B(H_A) \to B(H_B)$ be a quantum channel.

(1) The Holevo information $\chi(\Phi)$ is defined by

$$\chi(\Phi) := \max \Big\{ H\big(\Phi\big(\sum_x p_x \rho_x\big)\big) - \sum_x p_x H(\Phi(\rho_x)) \Big\},$$

where the maximum runs over all possible choice of ensemble of quantum states $\{(p_x), (\rho_x)\}$ on H_A and $H(\cdot)$ refers to the von Neumann entropy of a state $\rho \in B(H_A)$.

(2) The coherent information $Q^{(1)}(\Phi)$ is defined by

$$Q^{(1)}(\Phi) := \max\{H(\Phi(\rho)) - H(\tilde{\Phi}(\rho))\}$$

where the maximum runs over all quantum states ρ in $B(H_A)$. Note that the definition is independent of the choice of Stinespring isometry which determines the complementary channel $\tilde{\Phi}$.

(3) The classical capacity $C(\Phi)$ and the quantum capacity $Q(\Phi)$ are obtained by the regularizations of the Holevo information and the coherent information, respectively, as follows.

$$C(\Phi) = \lim_{n\to\infty} \frac{\chi(\Phi^{\otimes n})}{n}, \quad Q(\Phi) = \lim_{n\to\infty} \frac{Q^{(1)}(\Phi^{\otimes n})}{n}.$$

(4) The minimum output entropy (MOE) $H_{\min}(\Phi)$ is given by

$$H_{\min}(\Phi) := \min_{\rho} H(\Phi(\rho)),$$

where the minimum runs over all quantum states ρ in $B(H_A)$.

Remark 2.2. The two quantities χ and $H_{\min}$ are closely related. In general, we have the following for a quantum channel $\Phi : B(H_A) \to B(H_B)$.

$$\chi(\Phi) \leq \log d_B - H_{\min}(\Phi), \tag{2.1}$$

where d_B refers to the dimension of H_B [Hol12].

The regularization precedure for the classical capacity and the quantum capacity causes serious difficulties for the calculations of capacities in general. There are, however, some properties of channels that allow us to simplify the calculation, which we present below.

Definition 2.3. Let $\Phi : B(H_A) \to B(H_B)$ be a quantum channel with the complementary channel $\tilde{\Phi} : B(H_A) \to B(H_E)$.

(1) We say that Φ is degradable (resp. anti-degradable) if there exists a channel $\Psi : B(H_B) \to B(H_E)$ (resp. $\Psi : B(H_E) \to B(H_B)$) such that $\tilde{\Phi} = \Psi \circ \Phi$ (resp. $\Phi = \Psi \circ \tilde{\Phi}$).
(2) We say that Φ is entanglement-breaking (shortly, EBT) if there exist a probability distribution $(p_x)_x$ and product states $\rho_x^B \otimes \rho_x^A \in B(H_B \otimes H_A)$ such that the Choi matrix of Φ, $C_\Phi := \frac{1}{d_A} \sum_{i,j=1}^{d_A} \Phi(e_{ij}) \otimes e_{ij}$ is given by $C_\Phi = \sum_x p_x \rho_x^B \otimes \rho_x^A$.
(3) We say that Φ is PPT (positive partial transpose) if $(T_B \otimes \iota)C_\Phi$ is a positive matrix in $B(H_B \otimes H_A)$, equivalently if $T_B \circ \Phi$ is also a channel where T_B is the transpose map on $B(H_B)$.
(4) We say that Φ is *bistochastic* if $\Phi(\frac{1_A}{d_A}) = \frac{1_B}{d_B}$.

From the definition it is clear that EBT channels are PPT and by [Hol12, Corollary 10.28] they are also anti-degradable. Note that we have the following consequences of the above properties.

Proposition 2.4. *Let $\Phi : B(H_A) \to B(H_B)$ be a quantum channel.*

(1) [DS05] *If Φ is degradable, then $Q(\Phi) = Q^{(1)}(\Phi)$.*
(2) [HHH96,Per96,Hol12] *If Φ is PPT or anti-degradable, then $Q(\Phi) = Q^{(1)}(\Phi) = 0$.*
(3) [Sho02] *If Φ is EBT, then $C(\Phi) = \chi(\Phi)$.*

Some bistochastic channels have the following straightforward capacity estimates.

Proposition 2.5. *Let $\Phi : B(H_A) \to B(H_B)$ be a bistochastic quantum channel with a Stinespring isometry $V : H_A \to H_B \otimes H_E$. Suppose further that its complementary channel $\tilde{\Phi}$ is also bistochastic, then we have*

$$\log \frac{d_B}{d_E} \leq Q^{(1)}(\Phi) \leq C(\Phi) \leq \min\{\log d_A, \log d_B, \log \frac{d_A d_B}{d_E}\}. \tag{2.2}$$

Proof. We first observe that positivity of Φ tells us

$$||\Phi||_{S^1(H_A)\to B(H_B)} \leq ||\Phi||_{B(H_A)\to B(H_B)} = ||\Phi(1_A)||_{B(H_B)} = \frac{d_A}{d_B}.$$

Since $\Phi^{\otimes n}$ is also bistochastic, we also have $||\Phi^{\otimes n}||_{S^1(H_A^{\otimes n})\to B(H_B^{\otimes n})} \leq \left(\frac{d_A}{d_B}\right)^n$. Thus, we have

$$H_{\min}(\Phi^{\otimes n}) = \min_\rho H(\Phi^{\otimes n}(\rho)) \geq -\log ||\Phi^{\otimes n}||_{S^1(H_A^{\otimes n})\to B(H_B^{\otimes n})} \geq n \log \frac{d_B}{d_A}.$$

Note also that $H_{\min}(\Phi^{\otimes n}) = H_{\min}(\widetilde{\Phi^{\otimes n}}) = H_{\min}(\tilde{\Phi}^{\otimes n}) \geq n \log \frac{d_E}{d_A}$ so that we have

$$\begin{aligned}\chi(\Phi^{\otimes n}) &\leq \log d_B^n - H_{\min}(\Phi^{\otimes n})\\ &\leq n \log d_B - n \cdot \max\{\log \frac{d_B}{d_A}, \log \frac{d_E}{d_A}\}\\ &= n \cdot \min\{\log d_A, \log \frac{d_A d_B}{d_E}\}.\end{aligned}$$

Thus, we have

$$C(\Phi) = \lim_{n\to\infty} \frac{\chi(\Phi^{\otimes n})}{n} \leq \min\{\log d_A, \log d_B, \log \frac{d_A d_B}{d_E}\}$$

together with the obvious estimate $\chi(\Phi^{\otimes n}) \leq n \cdot \log d_B$.

The lower bound is direct from the definition of the coherent information.

$$Q^{(1)}(\Phi) \geq H(\Phi(\frac{1_A}{d_A})) - H(\tilde{\Phi}(\frac{1_A}{d_A})) \geq H(\frac{1_B}{d_B}) - \log d_E = \log \frac{d_B}{d_E}.$$

□

2.2. Compact quantum groups and their representations. A *compact quantum group* is a pair $\mathbb{G} = (C(\mathbb{G}), \Delta)$ where $C(\mathbb{G})$ is a unital C^*-algebra and $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes_{\min} C(\mathbb{G})$ is a unital $*$-homomorphism satisfying that (1) $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ and (2) each of the spaces span $\{\Delta(a)(1 \otimes b) : a, b \in C(\mathbb{G})\}$ and span $\{\Delta(a)(b \otimes 1) : a, b \in C(\mathbb{G})\}$ are dense in $C(\mathbb{G}) \otimes_{\min} C(\mathbb{G})$. It is well known that every compact quantum group has the (unique) *Haar state* h, which is a state on $C(\mathbb{G})$ such that $(\iota \otimes h)\Delta = h(\cdot)1 = (h \otimes \iota)\Delta$. If the Haar state h is tracial, i.e. $h(ab) = h(ba)$ for all $a, b \in C(\mathbb{G})$, then $\mathbb{G}$ is said to be of *Kac type*.

A (finite dimensional) *representation* of $\mathbb{G}$ is a pair (u, H_u) where H_u is a finite dimensional Hilbert space and $u = (u_{i,j})_{1\leq i,j\leq d_u} \in B(H_u) \otimes C(\mathbb{G})$ such that $\Delta(u_{i,j}) = \sum_{k=1}^{d_u} u_{i,k} \otimes u_{k,j}$ for all $1 \leq i, j \leq d_u$. Here, d_u refers to the dimension of u. The representation u is called *unitary* if it further satisfies $u^*u = 1_u \otimes 1 = uu^*$. Whenever we have a unitary representation (u, H_u) of $\mathbb{G}$ we obtain a so-called $\mathbb{G}$-*action* on $B(H_u)$

$$\beta_u : B(H_u) \to B(H_u) \otimes C(\mathbb{G}), \quad x \mapsto u(x \otimes 1)u^*. \tag{2.3}$$

For given unitary representations v and w, we say that a linear map $T : B(H_v) \to B(H_w)$ *intertwines* v and w if

$$(T \otimes 1)v = w(T \otimes 1)$$

and denote by $\mathrm{Hom}_{\mathbb{G}}(v, w)$ (simply, $\mathrm{Hom}(v, w)$) the space of *intertwiners*. If $\mathrm{Hom}(v, w)$ contains an invertible intertwiner, then v and w are said to be *equivalent*. A unitary representation (v, H_v) is called *irreducible* if $\mathrm{Hom}(v) = \mathrm{Hom}(v, v) = \mathbb{C} \cdot 1_v$ and we denote by $\mathrm{Irr}(\mathbb{G})$ the set of all irreducible unitary representations of $\mathbb{G}$ up to equivalence.

When we fix a representative $u^\alpha = [u^\alpha_{ij}]_{i,j=1}^{d_\alpha} \in M_{d_\alpha}(C(\mathbb{G}))$ for each $\alpha \in \mathrm{Irr}(\mathbb{G})$, the Peter-Weyl theory for compact quantum groups says the space $\mathrm{Pol}(\mathbb{G}) := \mathrm{span}\{u^\alpha_{ij} : \alpha \in \mathrm{Irr}(\mathbb{G}), 1 \leq i, j \leq d_\alpha\}$ is a subalgebra of $C(\mathbb{G})$ containing all the information on the quantum group $\mathbb{G}$. In particular, it hosts the map S called the *antipode* determined by the formula

$$S(u^\alpha_{ij}) = (u^\alpha_{ji})^*, \quad \alpha \in \mathrm{Irr}(\mathbb{G}), \quad 1 \leq i, j \leq d_\alpha.$$

For representations $v = (v_{ij})$ and $w = (w_{kl})$ we define its *tensor product* $v \textcircled{\scriptsize T} w$ by

$$v \textcircled{\scriptsize T} w = \sum_{i,j=1}^{d_v} \sum_{k,l=1}^{d_w} e_{ij} \otimes e_{kl} \otimes v_{ij} w_{kl} \in B(H_v) \otimes B(H_w) \otimes C(\mathbb{G}).$$

Then *the representation category* consisting of unitary representations as objects and intertwiners as morphisms is a *strict* C^**-tensor category* under the natural adjoint operation $Hom(v, w) \to Hom(w, v)$, $T \mapsto T^*$, and the tensor product $\textcircled{\scriptsize T}$. It is well known that any finite dimensional representation decomposes into a direct sum of irreducible representations, so that we have

$$v \textcircled{\scriptsize T} w \cong \oplus_{i=1}^N u_i.$$

In case u is a component of the irreducible decomposition of $v \textcircled{\scriptsize T} w$ we write $u \subset v \textcircled{\scriptsize T} w$.

For a given unitary representation (v, H_v) we consider the map $j : B(H) \to B(\overline{H})$ defined by $j(T)\overline{\xi} = \overline{T^*\xi}$. Then the *contragredient representation* of v is given by

$$v^c = (v^*_{ij})_{1 \leq i,j \leq d_v} = (j \otimes \iota)(v^{-1}) \in B(\overline{H}) \otimes C(\mathbb{G}).$$

The contragredient representation v^c is unitary if $\mathbb{G}$ is of Kac type.

For each compact quantum group $\mathbb{G}$ we have its opposite version $\mathbb{G}^{\mathrm{op}}$ with the same algebra $C(\mathbb{G}^{\mathrm{op}}) = C(\mathbb{G})$, but with the flipped co-multiplication $\Delta_{\mathrm{op}} = \Sigma \circ \Delta$, where Σ is the flip map on $C(\mathbb{G}) \otimes_{\min} C(\mathbb{G})$. Then, for any unitrary representation $u = (u_{ij}) \in B(H_u) \otimes C(\mathbb{G})$ of $\mathbb{G}$ we have an associated representation $u^* = (u^*_{ji}) \in B(H_u) \otimes C(\mathbb{G})$ of $\mathbb{G}^{\mathrm{op}}$.

2.3. Clebsch–Gordan channels. Let $\mathbb{G}$ be a compact quantum group and $(u, H_u), (v, H_v)$ and (w, H_w) be unitary irreducible representations of $\mathbb{G}$ such that $u \subset v \textcircled{\scriptsize T} w$, which gives us its intertwining isometry $\alpha_u^{v,w} : H_u \to H_v \otimes H_w$. By using $\alpha_u^{v,w}$ as the Stinespring isometry we get the following complementary pair of quantum channels:

$$\Phi_u^{\bar{v},w} : B(H_u) \to B(H_w); \quad \rho \mapsto \mathrm{Tr}_v(\alpha_u^{v,w} \rho (\alpha_u^{v,w})^*)$$
$$\Phi_u^{v,\bar{w}} : B(H_u) \to B(H_v); \quad \rho \mapsto \mathrm{Tr}_w(\alpha_u^{v,w} \rho (\alpha_u^{v,w})^*).$$

We name the above channels as Clebsch–Gordan channels (shortly, CG-channels) since the isometry $\alpha_u^{v,w}$ reflects the Clebsch–Gordan coefficients directly. Note that the symbol $\bar{v}$ does not refer to the conjugate representation, instead it means that we trace out the

H_v part. These channels have been studied by Al-Nuwairan [AN14], Brannan–Collins [BC18], and also Lieb–Solovej [LS14]. It turns out that CG-channels preserve certain "quantum symmetries". Recall that groups provide a certain symmetry on quantum channels through their (projective) unitary representations, namely covariance of channels. This concept naturally extends to the case of quantum groups as follows.

Definition 2.6. Let $\Phi : B(H_A) \to B(H_B)$ be a quantum channel. Suppose that there are unitary representations (u, H_A) and (w, H_B) of a compact quantum group $\mathbb{G}$ such that

$$(\iota \otimes \Phi)(\beta_u(\rho)) = \beta_w(\Phi(\rho)), \quad \rho \in B(H_A),$$

where β_u and β_w are $\mathbb{G}$-actions from (2.3). Then we say that the channel Φ is $\mathbb{G}$-covariant with respect to (u, w). In case we have no possibility of confusion we simply say $\mathbb{G}$-covariant.

Note that the covariance with respect to group representations has been studied in various contexts and has provided useful tools to handle information-theoretic problems [Sch05,DFH06,KW09,MW09,SWPGC09,MS14,NU17,MSD17].

We show that with mild assumptions, CG-channels are also $\mathbb{G}$-*covariant*.

Proposition 2.7. *Let u, v and w be irreducible unitary representations of a compact quantum group $\mathbb{G}$ such that $u \subset v \circledT w$. Then the CG-channel $\Phi_u^{v,\bar{w}}$ is $\mathbb{G}$-covariant with respect to (u, v) if the conjugate representation w^c is also unitary. Similarly, $\Phi_u^{\bar{v},w}$ is $\mathbb{G}^{\mathrm{op}}$-covariant with respect to (u^*, w^*) if v^c is unitary.*

Proof. We first check the case of $\Phi_u^{v,\bar{w}}$. For any quantum state $\rho \in B(H_u)$ we have

$$\begin{aligned}
&(\Phi_u^{v,\bar{w}} \otimes \iota)(u(\rho \otimes 1)u^*)\\
&= \iota \otimes \mathrm{Tr} \otimes \iota[(\alpha_u^{v,w} \otimes \iota)u(\rho \otimes 1)u^*((\alpha_u^{v,w})^* \otimes \iota)]\\
&= \iota \otimes \mathrm{Tr} \otimes \iota[(v \circledT w)(\alpha_u^{v,w} \otimes \iota)(\rho \otimes 1)((\alpha_u^{v,w})^* \otimes \iota)(v \circledT w)^*]\\
&= \sum_{i,j,i',j'=1}^{d_v} \sum_{k,l,k',l'=1}^{d_w} \iota \otimes \mathrm{Tr}[(|i\rangle\langle j| \otimes |k\rangle\langle l|)\alpha_u^{v,w} \rho (\alpha_u^{v,w})^*(|j'\rangle\langle i'| \otimes |l'\rangle\langle k'|)]\\
&\quad \otimes v_{ij} w_{kl} w^*_{k'l'} v^*_{i'j'}\\
&= \sum_{i,j,i',j'=1}^{d_v} \sum_{l,l'=1}^{d_w} \iota \otimes \mathrm{Tr}[(|i\rangle\langle j| \otimes |l'\rangle\langle l|)\alpha_u^{v,w} \rho (\alpha_u^{v,w})^*(|j'\rangle\langle i'| \otimes 1)]\\
&\quad \otimes v_{ij} (\sum_{k=1}^{d_w} w_{kl} w^*_{kl'}) v^*_{i'j'}\\
&= \sum_{i,j,i',j'=1}^{d_v} \sum_{l,l'=1}^{d_w} \iota \otimes \mathrm{Tr}[(|i\rangle\langle j| \otimes |l'\rangle\langle l|)\alpha_u^{v,w} \rho (\alpha_u^{v,w})^*(|j'\rangle\langle i'| \otimes 1)]\\
&\quad \otimes v_{ij} (w^t w^c)_{ll'} v^*_{i'j'}\\
&= \sum_{i,j,i',j'=1}^{d_v} \iota \otimes \mathrm{Tr}[(|i\rangle\langle j| \otimes 1)\alpha_u^{v,w} \rho (\alpha_u^{v,w})^*(|j'\rangle\langle i'| \otimes 1)] \otimes v_{ij} v^*_{i'j'}\\
&= v(\Phi_u^{v,\bar{w}}(\rho) \otimes 1)v^*,
\end{aligned}$$

where we use tracial property for the fourth equality and the assumption that w^c is unitary for $(w^t w^c)_{ll'} = \delta_{ll'}$.

For $\Phi_u^{\bar{v},w}$ we observe that

$$(\alpha_u^{v,w} \otimes \iota)u^*(|\xi\rangle \otimes 1) = (\alpha_u^{v,w} \otimes S)u(|\xi\rangle \otimes 1) = (\iota \otimes S)[(v \mathbin{\textcircled{\scriptsize T}} w)(\alpha_u^{v,w}|\xi\rangle \otimes 1)],$$

where S is the antipode of the quantum group $\mathbb{G}$. Thus, we get

$$\begin{aligned}&(\Phi_u^{\bar{v},w} \otimes \iota)(u^*(\rho \otimes 1)u)\\ &= \sum_{i,j,i',j'=1}^{d_v} \sum_{k,l,k',l'=1}^{d_w} \mathrm{Tr} \otimes \iota[(|i\rangle\langle j| \otimes |k\rangle\langle l|)\alpha_u^{v,w}\rho(\alpha_u^{v,w})^*(|j'\rangle\langle i'|\\ &\quad \otimes |l'\rangle\langle k'|)] \otimes w_{lk}^* v_{ji}^* v_{j'i'} w_{l'k'}.\end{aligned}$$

Then, we get the wanted conclusion by the same argument. □

The $\mathbb{G}$-covariance property has the following useful consequence.

Proposition 2.8. *Let $\Phi : B(H_u) \to B(H_v)$ be a quantum channel which is $\mathbb{G}$-covariant with respect to a pair of unitary representations (u, v) of a compact quantum group $\mathbb{G}$. If, in addition, v is assumed to be irreducible, then Φ is bistochastic. In particular, all CG-channels associated to a Kac type compact quantum group are bistochastic.*

Proof. Since Φ is $\mathbb{G}$-covariant and $\frac{1_u}{d_u} \in \mathrm{Hom}(u, u)$, we get $\Phi(\frac{1_u}{d_u}) \in \mathrm{Hom}(v, v)$. But irreducibility and Schur's lemma then give $\Phi(\frac{1_u}{d_u}) \in \mathbb{C}I$, which implies $\Phi(\frac{1_u}{d_u}) = \frac{1_v}{d_v}$. □

The following Proposition tells us that, under the assumption that $\mathbb{G}$ is of Kac type and $u \subseteq v \mathbin{\textcircled{\scriptsize T}} w$, the orthogonal projection from $H_v \otimes H_w$ onto H_u can be obtained by applying an averaging technique using the Haar state, for each unit vector $\xi \in H_u$. Moreover, together with Theorem 3.3, the following Proposition will be used to characterize EBT for TL-channels.

Proposition 2.9. *Let $\mathbb{G}$ be a compact quantum group of Kac type and $u, v, w \in \mathrm{Irr}(\mathbb{G})$ with $u \subset v \mathbin{\textcircled{\scriptsize T}} w$. Then for any unit vector $\xi \in \alpha_u^{v,w}(H_u) \subseteq H_v \otimes H_w$ we have*

$$\frac{1}{d_u}\alpha_u^{v,w}(\alpha_u^{v,w})^* = (\iota \otimes \iota \otimes h)((v \mathbin{\textcircled{\scriptsize T}} w)^*(|\xi\rangle\langle\xi| \otimes 1)(v \mathbin{\textcircled{\scriptsize T}} w)).$$

Proof. Let $A = (\iota \otimes \iota \otimes h)((v \mathbin{\textcircled{\scriptsize T}} w)^*(1 \otimes |\xi\rangle\langle\xi|)(v \mathbin{\textcircled{\scriptsize T}} w))$. Then, in order to reach the conclusion, it is enough to show that

$$\langle\eta|(\alpha_{u'}^{v,w})^* A \alpha_{u'}^{v,w}|\eta\rangle = \frac{\delta_{u,u'}}{d_u} 1_u$$

for any irreducible components u' of $v \mathbin{\textcircled{\scriptsize T}} w$ and any $\eta \in H_{u'}$. Indeed,

$$\begin{aligned}\langle\eta|(\alpha_{u'}^{v,w})^* A \alpha_{u'}^{v,w}|\eta\rangle &= h([(\langle\eta|(\alpha_{u'}^{v,w})^* \otimes 1)(v \mathbin{\textcircled{\scriptsize T}} w)^*](|\xi\rangle\langle\xi| \otimes 1)\\ &\qquad [(v \mathbin{\textcircled{\scriptsize T}} w)(\alpha_{u'}^{v,w}|\eta\rangle \otimes 1)])\\ &= h((\langle\eta| \otimes 1)(u')^*((\alpha_{u'}^{v,w})^*|\xi\rangle\langle\xi|\alpha_{u'}^{v,w} \otimes 1)u'(|\eta\rangle \otimes 1)).\end{aligned}$$

Then the facts that $(\iota \otimes h)((u')^*(B \otimes 1)u') = \dfrac{\mathrm{Tr}(B)}{d_{u'}} 1_{u'}$ and

$$\mathrm{Tr}((\alpha_{u'}^{v,w})^*|\xi\rangle\langle\xi|\alpha_{u'}^{v,w}) = \langle\xi|\alpha_{u'}^{v,w}(\alpha_{u'}^{v,w})^*|\xi\rangle = \delta_{u,u'}$$

complete the proof. □

3. Temperley–Lieb Channels

3.1. Free orthogonal quantum groups O_F^+. Let us fix an integer $N \geq 2$, $F \in \mathrm{GL}_N(\mathbb{C})$ satisfying $F\bar{F} = \pm 1$. We define $C(O_F^+)$ as the universal C^*-algebra generated by u_{ij} $(1 \leq i, j \leq N)$ with the defining relations (1) $u^*u = 1_N \otimes 1 = uu^*$ and (2) $u = (F \otimes 1)u^c(F^{-1} \otimes 1)$ where $u = (u_{ij})_{1\leq i,j\leq N} \in B(\mathbb{C}^N) \otimes C(O_F^+)$ that is called the *fundamental representation*. Then, together with a unital $*$-homomorphism $\Delta : C(O_F^+) \to C(O_F^+) \otimes_{\min} C(O_F^+)$ determined by

$$\Delta(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj},$$

$O_F^+ = (C(O_F^+), \Delta)$ forms a compact quantum group, which is called the free orthogonal quantum group with parameter matrix F [VDW96,Ban96,Ban97]. In particular, $O_F^+ = SU(2)$ if $F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and we denote by O_N^+ if $F = 1_N$. Note that O_F^+ is of Kac type if and only if F is unitary [Ban97], which covers both of the above cases.

3.2. Representations of O_F^+. It is known from [Ban96] that the irreducible representations of O_F^+ can be labelled $(v^k)_{k\in\mathbb{N}_0}$ (up to unitary equivalence) in such a way that $v^0 = 1$, $v^1 = u$, the fundamental representation, $v^l \cong \overline{v^l}$, and the following fusion rule holds:

$$v^l \, ⓣ \, v^m \cong \bigoplus_{0\leq r\leq \min\{l,m\}} v^{l+m-2r}. \tag{3.1}$$

Denote by H_k the Hilbert space associated to v^k. Then $H_0 = \mathbb{C}$, $H_1 = \mathbb{C}^N$, and (3.1) shows that the dimensions $\dim H_k$ satisfy the recursion relations $\dim H_1 \dim H_k = \dim H_{k+1} + \dim H_{k-1}$. Defining the quantum parameter

$$q_0 := \frac{1}{N}\Big(\frac{2}{1+\sqrt{1-4/N^2}}\Big) \in (0, 1],$$

then one has $q_0 + q_0^{-1} = N$, and it can be shown by induction that the dimensions $\dim H_k$ are given by the *quantum integers*

$$\dim H_k = [k+1]_{q_0} := {q_0}^{-k}\Big(\frac{1-{q_0}^{2k+2}}{1-{q_0}^2}\Big) \quad (N \geq 3).$$

When $N = 2$, we have $q_0 = 1$, and then $\dim H_k = k+1 = \lim_{q_0\to 1^-}[k+1]_{q_0}$. Note that for $N \geq 3$, we have the exponential growth asymptotic $[k+1]_{q_0} \sim N^k$ (as $N \to \infty$).

We now describe the explicit construction of the representations v^k and their corresponding Hilbert spaces H_k due to Banica [Ban96]. (See also the description in [VV07, Sect. 7]). The idea is that according to the fusion rules (3.1), the kth tensor power $u^{ⓣ k}$ of the fundamental representation contains exactly one irreducible subrepresentation equivalent to v^k. In particular, if we agree to explicitly identify v^k as a subrepresentation of $u^{ⓣ k}$, then there exists a unique projection $0 \neq p_k \in \mathrm{Hom}_{O_F^+}(u^{ⓣ k}, u^{ⓣ k}) \subset B(H_1^{\otimes k})$ called the *Jones–Wenzl projection* [Jon83,Wen87] satisfying $H_k = p_k(H_1^{\otimes k})$ and

$$v^k = (p_k \otimes 1)u^{ⓣ k}(p_k \otimes 1) \in B(H_k) \otimes C(O_F^+).$$

Thus, we are left with the problem of describing the projection p_k. To this end, fix an orthonormal basis $(e_i)_{i=1}^N$ for $H_1 = \mathbb{C}^N$, and put

$$\cup_F = \sum_{i=1}^{N} e_i \otimes F e_i. \tag{3.2}$$

It is then a simple matter to check that $\cup_F \in \mathrm{Hom}_{O_F^+}(1, u \,\textcircled{\tiny T}\, u)$, i.e. $u^{\textcircled{\tiny T}2}(\cup_F \otimes 1) = (\cup_F \otimes 1)$. In particular, $\iota_{H_1^{\otimes i-1}} \otimes \cup_F \otimes \iota_{H_1^{\otimes k-i-1}} \in \mathrm{Hom}_{O_F^+}(u^{\textcircled{\tiny T}(k-2)}, u^{\textcircled{\tiny T}k})$ for each $1 \le i \le k-1$. Using these observations, we inductively define $(p_k)_{k\ge1}$ using $p_1 = \iota_{H_1}$ together with the so-called *Wenzl recursion*

$$p_k = \iota_{H_1} \otimes p_{k-1} - \frac{[k-1]_q}{[k]_q}(\iota_{H_1} \otimes p_{k-1})(\cup_F \cup_F^* \otimes \iota_{H_1^{\otimes k-2}})(\iota_{H_1} \otimes p_{k-1}) \qquad (k \ge 2), \tag{3.3}$$

where $q = q(F) \in (0, q_0]$ is another quantum parameter defined so that $q + q^{-1} = \mathrm{Tr}(F^*F)$.

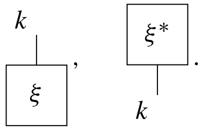
The Jones–Wenzl projections first appeared in the context of II_1-subfactors [Jon83]. The shared connection between subfactor theory and the representation theory of O_F^+ is through the famous *Temperley–Lieb category*. Indeed, as explained for example in [Ban96,BC18,BC17], given $d \in (-\infty, -2] \cup [2, \infty)$ the Temperley–Lieb Category $\mathrm{TL}(d)$ is defined to be the strict C*-tensor category generated by two simple objects $\{0, 1\}$, where 0 denotes the unit object for the tensor category, and $1 \neq 0$ is a self-dual simple object with the property that the morphism spaces $\mathrm{TL}_{k,l}(d) := \mathrm{Hom}(1^{\otimes k}, 1^{\otimes l})$ ($k, l \in \mathbb{N}$) are generated by the identity map $\iota \in \mathrm{Hom}(1, 1)$ together with a unique morphism $\cup \in \mathrm{Hom}(0, 1 \otimes 1)$ satisfying $\cap \circ \cup = |d| \in \mathrm{Hom}(0, 0) = \mathbb{C}$ and the "snake equation" $(\iota \otimes \cap)(\cup \otimes \iota) = (\cap \otimes \iota)(\iota \otimes \cup) = \mathrm{sgn}(d)\iota$. Here, the "cap" $\cap$ is simply the adjoint $\cup^* \in \mathrm{Hom}(1 \otimes 1, 0)$ of the "cup" $\cup$. On the other hand, we have the concrete C*-tensor category $\mathrm{Rep}(O_F^+)$ of finite dimensional unitary representations of O_F^+, and it was shown by Banica [Ban96] that if $d = \mathrm{Tr}((F\bar{F})(F^*F))$, then there exists a *unitary fiber functor* $\mathrm{TL}(d) \to \mathrm{Rep}(O_F^+)$ which is determined by mapping the simple objects $0, 1 \in \mathrm{TL}(d)$ to $v^0, v^1 \in \mathrm{Rep}(O_F^+)$, respectively, and by mapping the generating morphisms as follows

$$\iota \in \mathrm{TL}_{1,1}(d) \mapsto \iota_{H_1} \in \mathrm{Hom}_{O_F^+}(v^1, v^1) \;\&\; \cup \in \mathrm{TL}_{0,2}(d) \mapsto \cup_F \in \mathrm{Hom}_{O_F^+}(v^0, v^1 \,\textcircled{\tiny T}\, v^1).$$

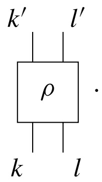
In other words, with d and F as above, we can concretely realize $\mathrm{TL}(d)$ in terms of the subcategory of finite dimensional Hilbert spaces $\mathrm{Rep}(O_F^+)$. In particular, for calculations involving morphisms and objects in $\mathrm{Rep}(O_F^+)$, one can perform these calculations using the well-known planar diagrammatic calculus in the Temperley–Lieb category $\mathrm{TL}(d)$ [BC17,KL94,CFS95], which we now briefly review.

3.3. Diagrammatic calculus for Rep(O_F^+). In the following, we continue to use the notations (e.g. $H_k = p_k(H_1^{\otimes k})$, $\cup_F$, etc.) defined above. We use the standard string diagram calculus to depict linear transformations between Hilbert spaces. That is, a linear operator $\rho \in B(H_k, H_l)$ will be diagrammatically represented as a string diagram

with the input Hilbert space at the bottom of the diagram, and the output at the top. The string corresponding to H_l will be labeled by l. We will generally omit the string corresponding to $H_0 = \mathbb{C}$, so a vector $\xi \in H_k \cong B(\mathbb{C}, H_k)$ and a covector $\xi^* \in H_k^* \cong B(H_k, \mathbb{C})$ will be drawn, respectively, as



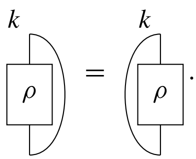
Similarly, $\rho \in B(H_k \otimes H_l, H_{k'} \otimes H_{l'})$ is denoted using parallel input/output strings



We define (for later use) the (kth) *quantum trace*[1] functional

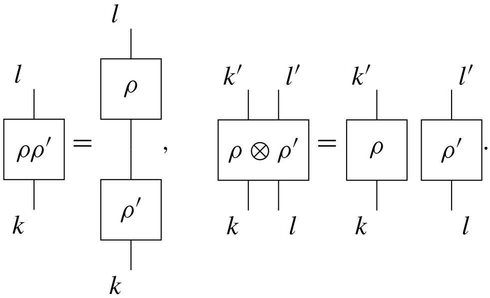
$$\tau_k : \mathcal{B}(H_1^{\otimes k}) \to \mathbb{C}, \qquad \tau_k(\rho) := \mathrm{Tr}_{H_1}^{\otimes k}((F^t \bar{F})^{\otimes k} \rho) \qquad (k \in \mathbb{N}),$$

which is depicted by the closure of a string diagram as follows:

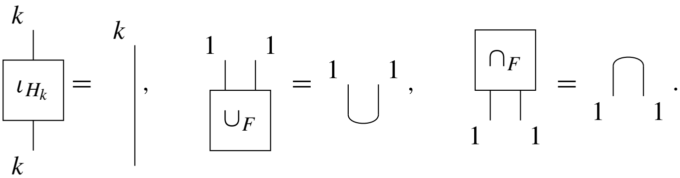


[1] The term "trace" comes from the fact that under the fiber functor $\mathrm{TL}(d) \to \mathrm{Rep}(O_F^+)$, τ_k corresponds to the well-known Markov trace $\tau_k : \mathrm{TL}_{k,k}(d) \to \mathbb{C}$ obtained by tracial closure of Temperley–Lieb diagrams [KL94].

Composition of linear maps is depicted by vertical concatenation of string diagrams and tensoring is depicted by placing them in parallel, respectively.

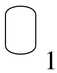


Let us end this subsection by describing the string-diagrammatic representation of the maps specific to the representation category Rep(O_F^+). Recall that for Rep(O_F^+), we have the fundamental generating morphisms $\iota_{H_k}, \cup_F, \cap_F := \cup_F^*$. We depict these maps as follows:



Then one has that the fundamental Temperley–Lieb relations are graphically depicted. For example, the value of a closed loop is $|d|$:

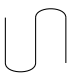
$$\| \cup_F \|^2 = \cap_F \circ \cup_F =$$



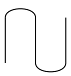
$$= \mathrm{Tr}(F^* F) = |d|,$$

and the snake equations are given by

$$(\iota_{H_1} \otimes \cap_F)(\cup_F \otimes \iota_{H_1}) =$$



$$= F\bar{F} = \mathrm{sgn}(d)\,\Big| =$$



$$= (\cap_F \otimes \iota_{H_1})(\iota_{H_1} \otimes \cup_F).$$

3.4. Temperley–Lieb channels. We now come to our main objects of study, which are the CG-channels associated to the irreducible representations of the quantum groups O_F^+, which, in view of the above connection with the Temperley–Lieb category, we redub "Temperley–Lieb channels":

Definition 3.1. A triple $(k, l, m) \in \mathbb{N}_0^3$ is called *admissible* if there exists an integer $0 \leq r \leq \min\{l, m\}$ such that $k = l + m - 2r$. For an admissible triple $(k, l, m) \in \mathbb{N}_0^3$ we have $v^k \subset v^l \textcircled{\scriptsize T} v^m$ with the intertwining isometry $\alpha_k^{l,m} : H_k \to H_l \otimes H_m$ and the corresponding CG-channels $\Phi_{v^k}^{\overline{v^l}, v^m}$ and $\Phi_{v^k}^{v^l, \overline{v^m}}$ (shortly, $\Phi_k^{\bar{l},m}$ and $\Phi_k^{l,\bar{m}}$) are called (O_F^+-)*Temperley–Lieb channels*.

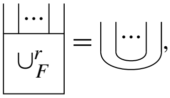
Let us now give a string-diagrammatic description of the covariant isometries $\alpha_k^{l,m}$ which define the TL-channels above. We begin by fixing an admissible triple $(k, l, m) \in \mathbb{N}_0^3$ and define

$$A_k^{l,m} = (p_l \otimes p_m)\Big(\iota_{H_{l-r}} \otimes \cup_F^r \otimes \iota_{H_{m-r}}\Big) p_k \in \mathrm{Hom}_{O_F^+}(v^k, v^l \textcircled{\top} v^m), \tag{3.4}$$

where $\cup_F^r \in \mathrm{Hom}_{O_F^+}(v^0, v^{\textcircled{\top}2r})$ is defined recursively from

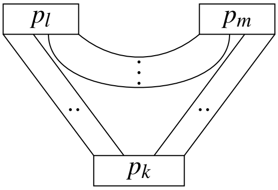
$$\cup_F^1 := \cup_F, \quad \cup_F^r := (\iota_{H_1^{\otimes r-1}} \otimes \cup_F \otimes \iota_{H_1^{\otimes r-1}}) \cup_F^{r-1} .$$

In terms of our string diagram formalism, $\cup_F^r$ is given by r nested cups



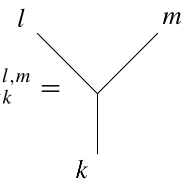
and $A_k^{l,m}$ is given by

$A_k^{l,m} =$



The (non-zero) map $A_k^{l,m}$ is often called a *three-vertex* in the context of tensor category theory and Temperley–Lieb recoupling theory [KL94], and (following standard conventions) the above string diagram for $A_k^{l,m}$ is simply drawn as a trivalent vertex:

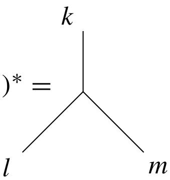
$A_k^{l,m} =$



.

We then have that the the adjoint $(A_k^{l,m})^*$ is obtained by rotating 180 degrees about the horizontal axis.

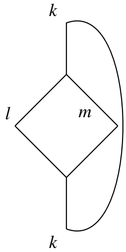
$(A_k^{l,m})^* =$



.

From Schur's Lemma and irreducibility, it follows that our required isometry $\alpha_k^{l,m}$ must be a scalar multiple of the three-vertex $A_k^{l,m}$, and this scaling factor is given in terms of the so-called *theta-net* $\theta_q(k, l, m)$ [KL94].

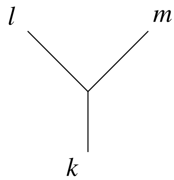
$$\theta_q(k,l,m) := \tau_k((A_k^{l,m})^* A_k^{l,m}) =$$



$$= \frac{[r]_q![l-r]_q![m-r]_q![k+r+1]_q!}{[l]_q![m]_q![k]_q!},$$

where $q = q(F)$, $k = l + m - 2r$, and $[x]_q! = [x]_q[x-1]_q \ldots [2]_q[1]_q$ denotes the quantum factorial. Then one has

$$\alpha_k^{l,m} = \left(\frac{\tau_k(\iota_{H_k})}{\tau_k((A_k^{l,m})^* A_k^{l,m})}\right)^{1/2} A_k^{l,m} = \left(\frac{[k+1]_q}{\theta_q(k,l,m)}\right)^{1/2}$$



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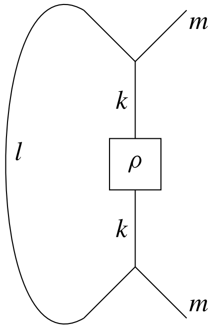
3.5. Kac type Temperley–Lieb channels. Throughout the rest of the paper we make the standing assumption that all free orthogonal quantum groups O_F^+ under consideration are of Kac type, which is equivalent to the unitarity of F [Ban97]. (In fact, for the most part we just consider O_N^+, however this slightly higher level of generality is useful at times, allowing us for example to prove results for $SU(2)$ simultaneously). The main reason for making the Kac assumption is that for the calculations that follow, it is essential for us to have that the "physical operations" of taking partial traces in tensor product spaces such as $B(H_l \otimes H_m)$ agree with the "quantum operations" coming from taking (partial) quantum traces using the functionals τ_k described above. In this case, we also have the handy feature that the O_F^+-covariant unit vectors $\alpha_0^{k,k} \in H_k \otimes H_k$ are all maximally entangled states.

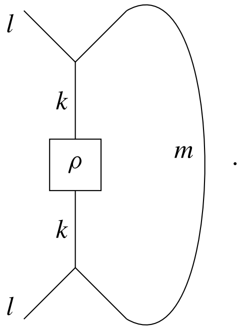
Remark 3.2. Note that when O_F^+ is of Kac type, we have that both the quantum parameters q_0 and q defined above are equal (since $N = \mathrm{Tr}(F^*F)$ when F is unitary). From now on we simply use the letter q to denote the quantum parameter.

Of course, since in the Kac case the quantum traces and ordinary traces agree, we have the following diagrammatic representations for the Temperley–Lieb quantum channels $\Phi_k^{\bar{l},m}$, $\Phi_k^{l,\bar{m}}$:

$$\Phi_k^{\bar{l},m}(\rho) = \frac{[k+1]_q}{\theta_q(k,l,m)}$$



$$\Phi_k^{l,\overline{m}}(\rho) = \frac{[k+1]_q}{\theta_q(k,l,m)}$$



.

Let us finish this section with an application of our string diagram formalism to the Choi matrices associated to the TL-channels. The result below was proved for the cases of $SU(2)$ by Al-Nuwairan [AN14] and O_N^+ in [BC17]. The following general case follows by the exact same planar isotopy arguments used in [BC17].

Theorem 3.3. *For any admissible triple $(k,l,m) \in \mathbb{N}_0^3$ the Choi matrices associated to any Kac type O_F^+-TL-channels $\Phi_k^{\bar{l},m}$ and $\Phi_k^{l,\bar{m}}$ are given by*

$$C_{\Phi_k^{\bar{l},m}} = \sum_{i,j} \Phi_k^{\bar{l},m}(|e_i\rangle\langle e_j|) \otimes |e_i\rangle\langle e_j| = \frac{[k+1]_q}{[l+1]_q} \alpha_l^{m,k} (\alpha_l^{m,k})^* \text{ and} \tag{3.5}$$

$$\sigma(C_{\Phi_k^{l,\overline{m}}}) = \sum_{i,j} |e_i\rangle\langle e_j| \otimes \Phi_k^{l,\overline{m}}(|e_i\rangle\langle e_j|) = \frac{[k+1]_q}{[m+1]_q} \alpha_m^{k,l} (\alpha_m^{k,l})^* \tag{3.6}$$

respectively, where $(e_i)_i$ is the canonical orthonormal basis of H_k and $\sigma : B(H_k) \otimes B(H_l) \mapsto B(H_l) \otimes B(H_k)$, $A \otimes B \mapsto B \otimes A$ is the canonical flip-map. In particular, these Choi matrices are scalar multiples of O_F^+-covariant projections onto irreducible subrepresentations.

Example 3.4. In low dimensions we can write down explicit formulas of the Kac type TL-channels.

(1) ($\mathbb{G} = O_N^+$ with $N \geq 2$) From (3.4), (3.5) and the last formula of Sect. 3.4 we can readily check

$$\sigma(C_{\Phi_1^{k,\overline{k+1}}}) = \frac{[2]_q}{[k+2]_q} p_{k+1}, \tag{3.7}$$

where $p_{k+1} \in B(H_1 \otimes H_k) \subset B(H_1^{\otimes(k+1)})$ is the Jones–Wenzl projection. Then the recursive formula (3.3) tells us that

$$\Phi_1^{k,\overline{k+1}}(\rho) = \frac{[2]_q}{[k+2]_q}\left(\mathrm{Tr}(\rho)1_{H_k} - \frac{[k]_q}{[k+1]_q} p_k(\rho \otimes 1_{H_{k-1}})p_k\right), \quad \rho \in B(H_k).$$

In particular, when $k = 1$ we clearly have $p_1 = 1_{H_1}$, and thus

$$\Phi_1^{1,\bar{2}}(\rho) = \frac{N}{N^2-1}\mathrm{Tr}(\rho)1_{H_1} - \frac{1}{N^2-1}\rho, \quad \rho \in B(H_1). \tag{3.8}$$

Moreover, we have $\Phi_1^{\bar{2},1} = \Phi_1^{1,\bar{2}}$ thanks to (3.7) and the fact that $\Sigma \circ p_2 \circ \Sigma = p_2$, where $\Sigma : H_1 \otimes H_1 \to H_1 \otimes H_1$, $|ij\rangle \mapsto |ji\rangle$ is the canonical flip-map. Note that $\Sigma \circ T \circ \Sigma = \sigma(T)$ for any $T \in B(H_1 \otimes H_1)$.

(2) ($\mathbb{G} = SU(2)$) We record some low dimensional $SU(2)$-TL-channels, which follows from the explicit formulae of Clebsch–Gordan coefficients (See [Boh01, Sect. V.2] for details):

(a) $\Phi_1^{\bar{2},1} : M_2(\mathbb{C}) \to M_2(\mathbb{C})$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} \frac{1}{3}(a+2d) & -\frac{1}{3}b \\ -\frac{1}{3}c & \frac{1}{3}(2a+d) \end{bmatrix}$,

(b) $\Phi_1^{2,\bar{1}} : M_2(\mathbb{C}) \to M_3(\mathbb{C})$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} \frac{2}{3}a & \frac{\sqrt{2}}{3}b & 0 \\ \frac{\sqrt{2}}{3}c & \frac{1}{3}(a+d) & \frac{\sqrt{2}}{3}b \\ 0 & \frac{\sqrt{2}}{3}c & \frac{2}{3}d \end{bmatrix}$,

(c) $\Phi_2^{\bar{1},1} : M_3(\mathbb{C}) \to M_2(\mathbb{C})$, $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \mapsto \begin{bmatrix} a_{11} + \frac{1}{2}a_{22} & \frac{1}{\sqrt{2}}(a_{12}+a_{23}) \\ \frac{1}{\sqrt{2}}(a_{21}+a_{32}) & \frac{1}{2}a_{22}+a_{33} \end{bmatrix}$,

(d) $\Phi_2^{\bar{3},1} : M_3(\mathbb{C}) \to M_2(\mathbb{C})$,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \mapsto \begin{bmatrix} \frac{1}{4}(a_{11}+2a_{22}+3a_{33}) & \frac{-1}{2\sqrt{2}}(a_{12}+a_{23}) \\ \frac{-1}{2\sqrt{2}}(a_{21}+a_{32}) & \frac{1}{4}(3a_{11}+2a_{22}+a_{33}) \end{bmatrix}.$$

4. The Minimum Output Entropy and Holevo/Coherent Information of O_N^+-Temperley–Lieb Channels

In this section we establish asymptotically sharp estimates on the minimum output entropy, the Holevo information and the coherent information of O_N^+-TL-channels for large enough N. The estimate begins with the following result of [BC18, Corollary 4.2].

$$H_{\min}(\Phi_k^{l,\bar{m}}) = H_{\min}(\Phi_k^{\bar{l},m}) \geq \log\left(\frac{\theta_q(k,l,m)}{[k+1]_q}\right) \geq \frac{l+m-k}{2}\cdot \log N - C(N) \tag{4.1}$$

with $C(N) \to 0$ as $N \to \infty$. The above estimate was conjectured to be asymptotically optimal as $N \to \infty$ in [BC18], which will be confirmed to be true below.

Before we dig into the above conjecture we prepare several elementary estimates. Let $f(t) = -t \log t$, $0 < t < 1$ be the function we use for the entropy. Then it is straightforwad to see that $f(t) \lesssim t^{1/2}$ and $f(t) \lesssim 1 - t$, where $a \lesssim b$ means that there is a universal constant $C > 0$ such that $a \leq C \cdot b$. The Fannes–Audenaert inequality [Aud07] says that for any quantum states $X, Y \in B(H)$ with $\dim H = n$

$$|H(X) - H(Y)| \leq \delta \log(n-1) + f(\delta) + f(1-\delta), \quad \delta = \frac{1}{2}||X - Y||_1,$$

where $||\cdot||_1$ is the trace norm, so that we have

$$|H(X) - H(Y)| \lesssim \log n \cdot ||X - Y||_1 + ||X - Y||_1^{1/2}. \tag{4.2}$$

Lemma 4.1. *Let $X, Y \in B(H)_+$ with $\dim H = n$. Suppose further that $\mathrm{Tr}(X) = 1 \geq \mathrm{Tr}(Y) > 0$. Then we still have*

$$|H(X) - H(Y)| \lesssim \log n \cdot ||X - Y||_1 + ||X - Y||_1^{1/2}. \tag{4.3}$$

Proof. First we observe that

$$\begin{aligned} H(X) - H(Y) &= H(X) + \mathrm{Tr}(Y) \log \mathrm{Tr}(Y) - \mathrm{Tr}(Y) H(\frac{Y}{\mathrm{Tr}(Y)}) \\ &= \mathrm{Tr}(Y) \log \mathrm{Tr}(Y) + (1 - \mathrm{Tr}(Y)) H(X) + \mathrm{Tr}(Y)(H(X) - H(\frac{Y}{\mathrm{Tr}(Y)})) \\ &= A + B + C, \end{aligned}$$

where $A = \mathrm{Tr}(Y) \log \mathrm{Tr}(Y)$, $B = (1 - \mathrm{Tr}(Y)) H(X)$ and $C = \mathrm{Tr}(Y)(H(X) - H(\frac{Y}{\mathrm{Tr}(Y)}))$. Since we have $1 - \mathrm{Tr}(Y) = \mathrm{Tr}(X - Y) \leq ||X - Y||_1$ we know

$$|A| \lesssim ||X - Y||_1, \quad |B| \lesssim \log n \cdot ||X - Y||_1.$$

For the third term we have

$$|C| \leq |H(X) - H(\frac{Y}{\mathrm{Tr}(Y)})| \lesssim \log n \cdot ||X - \frac{Y}{\mathrm{Tr}(Y)}||_1 + ||X - \frac{Y}{\mathrm{Tr}(Y)}||_1^{1/2}.$$

Finally we observe that

$$\begin{aligned} ||X - \frac{Y}{\mathrm{Tr}(Y)}||_1 &\leq ||X - Y||_1 + (\frac{1}{\mathrm{Tr}(Y)} - 1)||Y||_1 \\ &= ||X - Y||_1 + 1 - \mathrm{Tr}(Y) \leq 2||X - Y||_1, \end{aligned}$$

which leads us to the conclusion we wanted. □

Lemma 4.2. *For any admissible $(l, m, k) \in \mathbb{N}_0^3$ with $k = l + m - 2r$ we have*

$$\frac{N^r [k+1]_q}{\theta_q(k, l, m)} = 1 + O(\frac{1}{N^2}).$$

Proof. We first observe for any $k \geq 1$ that

$$\begin{aligned}\frac{[k+1]_q}{[k]_q N} &= \frac{1}{2}(1+\sqrt{1-4/N^2})\frac{1-q^{2k+2}}{1-q^{2k}} \\ &= \frac{1}{2}(1+\sqrt{1-4/N^2})(1+\frac{q^{2k}-q^{2k+2}}{1-q^{2k}}) \\ &= \frac{1}{2}(2+O(\frac{1}{N^2}))(1+O(\frac{1}{N^2})) = 1+O(\frac{1}{N^2}).\end{aligned}$$

Then, we can easily see for all $a > b \in \mathbb{N}$ that

$$\frac{[a]_q}{[b]_q N^{a-b}} = \frac{[a]_q}{[a-1]_q N} \cdots \frac{[b+1]_q}{[b]_q N} = (1+O(\frac{1}{N^2}))^{a-b} = 1+O(\frac{1}{N^2}),$$

which can be extended to the following

$$\frac{[a]_q!}{[b]_q![a-b]_q!N^{b(a-b)}} = 1+O(\frac{1}{N^2}) = \frac{[b]_q![a-b]_q!N^{b(a-b)}}{[a]_q!}.$$

Finally, we have

$$\begin{aligned}\frac{N^r[k+1]_q}{\theta_q(k,l,m)} &= N^r\frac{[l]_q![m]_q![k+1]_q!}{[r]_q![l-r]_q![m-r]_q![k+r+1]_q!} \\ &= \frac{[l]_q!}{[r]_q![l-r]_q!N^{r(l-r)}} \cdot \frac{[m]_q!}{[m-r]_q![r]_q!N^{r(m-r)}} \cdot \frac{[k+1]_q![r]_q!N^{r(k+1)}}{[k+r+1]_q!} \\ &= 1+O(\frac{1}{N^2})\end{aligned}$$

since $-r(l-r) - r(m-r) + r(k+1) = r(2r-l-m+k+1) = r$. □

Here, we introduce some notations. For $N \geq 2$ we write the index set $I = \{1, 2, \ldots, N\}$. We also need multi-index sets

$$I^n = \{\mathbf{i} = (i_1, \ldots, i_n) : i_k \in I,\ 1 \leq k \leq n\}$$

and

$$I^n_{\neq} := \{\mathbf{i} = (i_1, \ldots, i_n) \in I^n : i_k \neq i_{k+1},\ 1 \leq k \leq n-1\}.$$

We sometimes need to aviod particular indices as follows.

$$(s,t)/I^n_{\neq} := \{\mathbf{i} = (i_1, \ldots, i_n) \in I^n_{\neq} : i_1 \neq s, i_1 \neq t\}$$

and

$$I^n_{\neq}\backslash(t) := \{\mathbf{i} = (i_1, \ldots, i_n) \in I^n_{\neq} : i_n \neq t\}$$

for $n \in \mathbb{N}$, $s \neq t \in I$. Note that we have $|(s,t)/I^n_{\neq}| = (N-2)(N-1)^{n-1}$ and $|I^n_{\neq}\backslash(t)| = (N-1)^n$.

For each $\mathbf{i} \in I^n_{\neq}$ we can easily see that $|\mathbf{i}\rangle = e_{i_1} \otimes \cdots \otimes e_{i_n} \in H_n$ so that $p_n|\mathbf{i}\rangle = |\mathbf{i}\rangle$ from the Jones–Wenzl recursion.

For $\mathbf{i} \in I^n$ and $\mathbf{j} \in I^m$ the vector $|\mathbf{i}\rangle \otimes |\mathbf{j}\rangle \in \mathbb{C}^{N(n+m)}$ will simply be denoted by $|\mathbf{ij}\rangle$. We will use a very specific index $\mathbf{m}^k := (1, 2, 1, \ldots) \in I^k, k \geq 1$. For $\mathbf{i} = (i_1, \ldots, i_n) \in I^n$, its order reversed multi-index $\check{\mathbf{i}} = (i_n, \ldots, i_1) \in I^n$ will be considered.

Theorem 4.3. *For each admissible triple* $(l, m, k) \in \mathbb{N}_0^3$ *we have*

$$\frac{l+m-k}{2} \cdot \log N - C(N) \leq H_{\min}(\Phi_k^{l,\bar{m}}) = H_{\min}(\Phi_k^{\bar{l},m}) \leq \frac{l+m-k}{2} \cdot \log N + D(N) \tag{4.4}$$

with $C(N), D(N) \to 0$ *as* $N \to \infty$*. When* $k = l+m$*, we actually recover the following result ([BC18, Remark 7]).*

$$H_{\min}(\Phi_{l+m}^{l,\bar{m}}) = H_{\min}(\Phi_{l+m}^{\bar{l},m}) = 0$$

for any $N \geq 2$*.*

Proof. We set $r = \frac{l+m-k}{2}$. We will use a very specific index $\mathbf{m} := (1, 2, 1, \ldots) \in H_k \subseteq H_1^{\otimes k}$, which splits into $(m_1, \ldots, m_k) = \mathbf{m} = \mathbf{m}'\mathbf{m}''$, where $\mathbf{m}' = (m_1, \ldots, m_{l-r}) \in H_{l-r} \subseteq H_1^{\otimes l-r}$ and $\mathbf{m}'' = (m_{l-r+1}, \ldots, m_k) \in H_{m-r} \subseteq H_1^{\otimes m-r}$. Then, we have

$$\begin{aligned}
&\frac{\theta_q(k,l,m)}{[k+1]_q}\Phi_k^{\bar{l},m}(|\mathbf{m}\rangle\langle\mathbf{m}|) = \mathrm{Tr}\otimes\iota(A_k^{l,m}|\mathbf{m}\rangle\langle\mathbf{m}|(A_k^{l,m})^*) \\
&\quad = \mathrm{Tr}\otimes\iota(A_k^{l,m}|\mathbf{m}'\mathbf{m}''\rangle\langle\mathbf{m}'\mathbf{m}''|(A_k^{l,m})^*) \\
&\quad = \sum_{\mathbf{i},\mathbf{i}'\in I^r} \mathrm{Tr}\otimes\iota[(p_l\otimes p_m)(|\mathbf{m}'\mathbf{i}\rangle\langle\mathbf{m}'\mathbf{i}'|\otimes|\check{\mathbf{i}}\mathbf{m}''\rangle\langle\check{\mathbf{i}}'\mathbf{m}''|)(p_l\otimes p_m)] \\
&\quad = \sum_{\mathbf{i},\mathbf{i}'\in I^r} \langle\mathbf{m}'\mathbf{i}'|p_l|\mathbf{m}'\mathbf{i}\rangle\cdot p_m|\check{\mathbf{i}}\mathbf{m}''\rangle\langle\check{\mathbf{i}}'\mathbf{m}''|p_m \\
&\quad = \sum_{\mathbf{i}\in(1,2)/I^r_{\neq}} |\check{\mathbf{i}}\mathbf{m}''\rangle\langle\check{\mathbf{i}}\mathbf{m}''| + \sum_{\mathbf{i},\mathbf{i}'\notin(1,2)/I^r_{\neq}} \langle\mathbf{m}'\mathbf{i}'|p_l|\mathbf{m}'\mathbf{i}\rangle\cdot p_m|\check{\mathbf{i}}\mathbf{m}''\rangle\langle\check{\mathbf{i}}'\mathbf{m}''|p_m \\
&\quad = \frac{\theta_q(k,l,m)}{[k+1]_q}(Z(1)+Z(2)),
\end{aligned} \tag{4.5}$$

where $\frac{\theta_q(k,l,m)}{[k+1]_q}Z(1) = \sum_{\mathbf{i}\in(1,2)/I^r_{\neq}} |\check{\mathbf{i}}\mathbf{m}''\rangle\langle\check{\mathbf{i}}\mathbf{m}''|$ and $\frac{\theta_q(k,l,m)}{[k+1]_q}Z(2) = \sum_{\mathbf{i},\mathbf{i}'\notin(1,2)/I^r_{\neq}} \langle\mathbf{m}'\mathbf{i}'|p_l|\mathbf{m}'\mathbf{i}\rangle\cdot p_m|\check{\mathbf{i}}\mathbf{m}''\rangle\langle\check{\mathbf{i}}'\mathbf{m}''|p_m$. Here, we used the fact that for $\mathbf{i} \in (1,2)/I^r_{\neq}$ we have $\mathbf{m}'\mathbf{i} \in H_l$ and $\check{\mathbf{i}}\mathbf{m}'' \in H_m$. Note that

$$\frac{\theta_q(k,l,m)}{[k+1]_q}Z(2) = \mathrm{Tr}\otimes\iota((p_l\otimes p_m)|\xi\rangle\langle\xi|(p_l\otimes p_m)) \geq 0,$$

where $|\xi\rangle = \sum_{\mathbf{i}\notin(1,2)/I^r_{\neq}} |\mathbf{m}'\mathbf{i}\rangle\otimes|\check{\mathbf{i}}\mathbf{m}''\rangle$. The term $Z(1)$ is the dominant one with entropy

$$\begin{aligned}
H(Z(1)) &= (N-2)(N-1)^{r-1}\frac{[k+1]_q}{\theta_q(k,l,m)}\log\frac{\theta_q(k,l,m)}{[k+1]_q} \\
&= (1-\frac{2}{N})(1-\frac{1}{N})^{r-1}\frac{N^r[k+1]_q}{\theta_q(k,l,m)}\log\frac{\theta_q(k,l,m)}{[k+1]_q} \\
&= (1+O(\frac{1}{N}))\log[(1+O(\frac{1}{N^2}))N^r]
\end{aligned}$$

by Lemma 4.2. For the second term $Z(2)$ we have

$$\mathrm{Tr}(Z(2)) = 1 - \mathrm{Tr}(Z(1)) = 1 - (N-2)(N-1)^{r-1}\frac{[k+1]_q}{\theta_q(k,l,m)} = O(\frac{1}{N}).$$

By Lemma 4.1 we have

$$|H(\Phi_k^{\bar{l},m}(|\mathbf{m}\rangle\langle\mathbf{m}|)) - H(Z(1))| \lesssim m\log N\,\mathrm{Tr}(Z(2)) + \mathrm{Tr}(Z(2))^{1/2} \lesssim O(\frac{1}{\sqrt{N}}),$$

which leads us to the conclusion we wanted.

If $k = l + m$, then we have $r = 0$ and

$$\Phi_{l+m}^{\bar{l},m}(|\mathbf{m}\rangle\langle\mathbf{m}|) = |\mathbf{m}'\rangle\langle\mathbf{m}'|,$$

which is a pure state. Thus, we get the conclusion we wanted. □

We record examples of TL-channels with a precise formula for their MOEs.

Example 4.4 (MOE formula for $\Phi_1^{1,\bar{2}}$). From Example 3.4 (1) we can see that the O_N^+-channel $\Phi_1^{1,\bar{2}}$ satisfies the following covariance property:

$$\Phi_1^{1,\bar{2}}(UXU^*) = U\Phi_1^{1,\bar{2}}(X)U^* \tag{4.6}$$

for any unitary $U \in U(N)$. Since any pure state $|\xi\rangle\langle\xi|$ in $B(H_1)$ can be written as of the form $U|\xi_0\rangle\langle\xi_0|U^*$ with $U \in U(N)$ and a fixed unit vector $\xi_0 \in H_1$, we have

$$\begin{aligned} H(\Phi_1^{1,\bar{2}}(|\xi\rangle\langle\xi|)) &= H(\Phi_1^{1,\bar{2}}(|\xi_0\rangle\langle\xi_0|)) \\ &= H\left(\frac{N}{N^2-1}\mathrm{Id}_N - \frac{1}{N^2-1}|\xi_0\rangle\langle\xi_0|\right) \\ &= (N-1)\cdot\frac{N}{N^2-1}\log\left(\frac{N^2-1}{N}\right) + \frac{1}{N+1}\log(N+1), \end{aligned} \tag{4.7}$$

and the above demonstrates that

$$\log(N) - H_{min}(\Phi_1^{1,\bar{2}}) = \frac{N}{N+1}\log\left(\frac{N}{N-1}\right) - \log\left(\frac{N+1}{N}\right) \to 0 \text{ as } N \to \infty,$$

which is the conclusion in Theorem 4.3.

Now we move to the case of other quantities. We will apply a similar argument for the lower bound on the coherent information.

Theorem 4.5. *For each admissible triple $(k, l, m) \in \mathbb{N}_0^3$ we have*

$$\begin{cases} \frac{l+k-m}{2}\cdot\log N - C(N) \le Q^{(1)}(\Phi_k^{l,\bar{m}}) \\ \frac{m+k-l}{2}\cdot\log N - D(N) \le Q^{(1)}(\Phi_k^{\bar{l},m}) \end{cases} \tag{4.8}$$

with constants $C(N), D(N) \to 0$ as $N \to \infty$. When $k = l + m$, we actually have the following.

$$\begin{cases} l\cdot\log(N-1) \le Q^{(1)}(\Phi_{l+m}^{l,\bar{m}}) \\ m\cdot\log(N-1) \le Q^{(1)}(\Phi_{l+m}^{\bar{l},m}). \end{cases} \tag{4.9}$$

Proof. We set $r = \frac{l+m-k}{2}$ and fix a specific index $\mathbf{n} := (1, 2, 1, \ldots) \in H_{m-r} \subseteq H_1^{\otimes m-r}$. We first consider the estimates of $Q^{(1)}(\Phi_k^{\bar{l},m})$. For any $\mathbf{j} \in I_{\neq}^{l-r}\backslash(1)$ we use the same argument as in the proof of Theorem 4.3 to get

$$\begin{aligned}
&\frac{\theta_q(k,l,m)}{[k+1]_q}\Phi_k^{\bar{l},m}(|\mathbf{jn}\rangle\langle\mathbf{jn}|) = \mathrm{Tr}\otimes\iota(A_k^{l,m}|\mathbf{jn}\rangle\langle\mathbf{jn}|(A_k^{l,m})^*)\\
&= \sum_{\mathbf{i},\mathbf{i}'\in I^r}\langle\mathbf{ji}'|p_l|\mathbf{ji}\rangle\cdot p_m|\check{\mathbf{i}}\,\mathbf{n}\rangle\langle\check{\mathbf{i}}'\mathbf{n}|p_m\\
&= \sum_{\mathbf{i}\in(1,j_{l-r})/I_{\neq}^r}|\check{\mathbf{i}}\mathbf{n}\rangle\langle\check{\mathbf{i}}\mathbf{n}| + \sum_{\mathbf{i},\mathbf{i}'\notin(1,j_{l-r})/I_{\neq}^r}\langle\mathbf{ji}'|p_l|\mathbf{ji}\rangle\cdot p_m|\check{\mathbf{i}}\,\mathbf{n}\rangle\langle\check{\mathbf{i}}'\mathbf{n}|p_m\\
&= \frac{\theta_q(k,l,m)}{[k+1]_q}(Z(1,\mathbf{j}) + Z(2,\mathbf{j})).
\end{aligned}$$

Now we set $\rho = \dfrac{1}{(N-1)^{l-r}}\displaystyle\sum_{\mathbf{j}\in I_{\neq}^{l-r}\backslash(1)}|\mathbf{jn}\rangle\langle\mathbf{jn}|$ and we get

$$\Phi_k^{\bar{l},m}(\rho) = \frac{1}{(N-1)^{l-r}}\sum_{\mathbf{j}\in I_{\neq}^{l-r}\backslash(1)}(Z(1,\mathbf{j}) + Z(2,\mathbf{j})) = Z(1) + Z(2),$$

where $Z(1) = \frac{1}{(N-1)^{l-r}}\sum_{\mathbf{j}\in I_{\neq}^{l-r}\backslash(1)} Z(1,\mathbf{j})$ and $Z(2) = \frac{1}{(N-1)^{l-r}}\sum_{\mathbf{j}\in I_{\neq}^{l-r}\backslash(1)} Z(2,\mathbf{j})$. Then

$$\begin{aligned}
Z(1) &= \frac{[k+1]_q}{(N-1)^{l-r}\theta_q(k,l,m)}\sum_{\mathbf{j}\in I_{\neq}^{l-r}\backslash(1)}\sum_{\mathbf{i}\in(1,j_{l-r})/I_{\neq}^r}|\check{\mathbf{i}}\mathbf{n}\rangle\langle\check{\mathbf{i}}\mathbf{n}|\\
&= \frac{[k+1]_q}{(N-1)\theta_q(k,l,m)}\sum_{j_{l-r}=2}^{N}\sum_{\mathbf{i}\in(1,j_{l-r})/I_{\neq}^r}|\check{\mathbf{i}}\mathbf{n}\rangle\langle\check{\mathbf{i}}\mathbf{n}|\\
&= \frac{(N-2)[k+1]_q}{(N-1)\theta_q(k,l,m)}\sum_{\mathbf{i}\in(1)/I_{\neq}^r}|\check{\mathbf{i}}\mathbf{n}\rangle\langle\check{\mathbf{i}}\mathbf{n}|.
\end{aligned}$$

As before we use Lemma 4.2 to get

$$\begin{aligned}
H(Z(1)) &= (N-1)^r\frac{(N-2)[k+1]_q}{(N-1)\theta_q(k,l,m)}\log\frac{(N-1)\theta_q(k,l,m)}{(N-2)[k+1]_q}\\
&= (1-\frac{1}{N})^r\frac{N-2}{N-1}\frac{N^r[k+1]_q}{\theta_q(k,l,m)}\log\frac{(N-1)\theta_q(k,l,m)}{(N-2)[k+1]_q}\\
&= (1+O(\frac{1}{N}))\log[(1+O(\frac{1}{N}))N^r]
\end{aligned}$$

and

$$\mathrm{Tr}(Z(2)) = 1 - \mathrm{Tr}(Z(1)) = 1 - (N-1)^r\frac{(N-2)[k+1]_q}{(N-1)\theta_q(k,l,m)} = O(\frac{1}{N}).$$

By Lemma 4.1 again we still have

$$|H(\Phi_k^{\bar{l},m}(\rho)) - H(Z(1))| \lesssim m \log N \operatorname{Tr}(Z(2)) + \operatorname{Tr}(Z(2))^{1/2} \lesssim O(\frac{1}{\sqrt{N}}).$$

For the complementary channel we similarly have

$$\begin{aligned}
\frac{\theta_q(k,l,m)}{[k+1]_q}\Phi_k^{l,\bar{m}}(|\mathbf{jn}\rangle\langle\mathbf{jn}|) &= \iota \otimes \operatorname{Tr}(A_k^{l,m}|\mathbf{jn}\rangle\langle\mathbf{jn}|(A_k^{l,m})^*) \\
&= \sum_{\mathbf{i},\mathbf{i}'\in I^r} p_l|\mathbf{ji}\rangle\langle\mathbf{ji}'|p_l \cdot \langle\check{\mathbf{i}}'\mathbf{n}|p_m|\check{\mathbf{i}}\,\mathbf{n}\rangle \\
&= \sum_{\mathbf{i}\in(1,j_{l-r})/I^r_{\neq}} |\mathbf{ji}\rangle\langle\mathbf{ji}| + \sum_{\mathbf{i},\mathbf{i}'\notin(1,j_{l-r})/I^r_{\neq}} \langle\check{\mathbf{i}}'\mathbf{n}|p_m|\check{\mathbf{i}}\,\mathbf{n}\rangle \cdot p_l|\mathbf{ji}\rangle\langle\mathbf{ji}'|p_l \\
&= \frac{\theta_q(k,l,m)}{[k+1]_q}(Y(1,\mathbf{j}) + Y(2,\mathbf{j})).
\end{aligned}$$

Thus, we have

$$\Phi_k^{l,\bar{m}}(\rho) = \frac{1}{(N-1)^{l-r}} \sum_{\mathbf{j}\in I^{l-r}_{\neq}\setminus(1)} (Y(1,\mathbf{j}) + Y(2,\mathbf{j})) = Y(1) + Y(2),$$

which means

$$Y(1) = \frac{[k+1]_q}{(N-1)^{l-r}\theta_q(k,l,m)} \sum_{\mathbf{j}\in I^{l-r}_{\neq}\setminus(1)} \sum_{\mathbf{i}\in(1,j_{l-r})/I^r_{\neq}} |\mathbf{ji}\rangle\langle\mathbf{ji}|.$$

Now we have

$$\begin{aligned}
H(Y(1)) &= (N-2)(N-1)^{r-1}\frac{[k+1]_q}{\theta_q(k,l,m)} \log \frac{(N-1)^{l-r}\theta_q(k,l,m)}{[k+1]_q} \\
&= (1-\frac{2}{N})(1-\frac{1}{N})^{r-1}\frac{N^r[k+1]_q}{\theta_q(k,l,m)} \log \frac{(N-1)^{l-r}\theta_q(k,l,m)}{[k+1]_q} \\
&= (1+O(\frac{1}{N}))\log[(1+O(\frac{1}{N}))N^l]
\end{aligned}$$

and

$$\operatorname{Tr}(Y(2)) = 1 - \operatorname{Tr}(Y(1)) = 1 - (1-\frac{2}{N})(1-\frac{1}{N})^{r-1}\frac{N^r[k+1]_q}{\theta_q(k,l,m)} = O(\frac{1}{N}).$$

Thus, we similarly get, by Lemma 4.1, that $|H(\Phi_k^{l,\bar{m}}(\rho)) - H(Y(1))| \lesssim O(\frac{1}{\sqrt{N}})$.

Combining all the above estimates we get

$$\lim_{N\to\infty} |H(\Phi_k^{l,\bar{m}}(\rho)) - H(\Phi_k^{l,\bar{m}}(\rho)) - \frac{l+k-m}{2} \cdot \log N| = 0,$$

which gives us the desired lower estimate for $Q^{(1)}(\Phi_k^{l,\bar{m}})$ as $N \to \infty$.

For the case $k = l + m$ we actually have the following exact formulae.

$$\Phi_{l+m}^{l,\bar{m}}\left(\frac{1}{(N-1)^l}\sum_{\mathbf{j}\in I^l_{\neq}/(1)} |\mathbf{jn}\rangle\langle\mathbf{jn}|\right) = \frac{1}{(N-1)^l}\sum_{\mathbf{j}\in I^l_{\neq}/(1)} |\mathbf{j}\rangle\langle\mathbf{j}|$$

and

$$\Phi_{l+m}^{\bar{l},m}\left(\frac{1}{(N-1)^l}\sum_{\mathbf{j}\in I^l_{\neq}/(1)} |\mathbf{jn}\rangle\langle\mathbf{jn}|\right) = |\mathbf{n}\rangle\langle\mathbf{n}|,$$

which tells us that $Q^{(1)}(\Phi_{l+m}^{l,\bar{m}}) \geq l \cdot \log(N-1)$.

The estimates for $Q^{(1)}(\Phi_k^{\bar{l},m})$ can be obtained in a similar way. □

Combining Theorems 4.3 and 4.5, we obtain the following asymptotically sharp Holevo information and coherent information:

Corollary 4.6. *For each admissible triple* $(k, l, m) \in \mathbb{N}_0^3$ *we have*

$$\frac{l+k-m}{2}\log(N) - C_1(N) \leq Q^{(1)}(\Phi_k^{l,\bar{m}}) \leq \chi(\Phi_k^{l,\bar{m}}) \leq \frac{l+k-m}{2}\log(N) + C_2(N)$$

and

$$\frac{m+k-l}{2}\log(N) - D_1(N) \leq Q^{(1)}(\Phi_k^{\bar{l},m}) \leq \chi(\Phi_k^{\bar{l},m}) \leq \frac{m+k-l}{2}\log(N) + D_2(N)$$

with constants $C_1(N), C_2(N), D_1(N), D_2(N) \to 0$ *as* $N \to \infty$.

Proof. Theorem 4.5 directly gives us the wanted lower bounds, and Theorem 4.3 together with a general fact (2.1) completes the conclusion. □

Remark 4.7. We note that Corollary 4.6 gives us asymptotically sharp private information $P^{(1)}(\Phi_k^{l,\bar{m}})$ and $P^{(1)}(\Phi_k^{\bar{l},m})$ since

$$Q^{(1)} \leq P^{(1)} \leq \chi$$

in general. The private information $P^{(1)}$ is defined as

$$\max\left\{H\Big(\sum_x p_x\Phi(\rho_x)\Big) - \sum_x p_x H(\Phi(\rho_x)) - H\Big(\sum_x p_x\widetilde{\Phi}(\rho_x)\Big) + \sum_x p_x H(\widetilde{\Phi}(\rho_x))\right\}$$

where the maximum runs over all ensembles of quantum states $\{(p_x), (\rho_x)\}$. See [Wil17, Sect. 13.6] for details.

5. EBT/PPT and (Anti-)degradability of Kac Type Temperly–Lieb Channels

Since we have studied the coherent information $Q^{(1)}$ and the Holevo information χ for O_N^+-TL-channels in previous section, it is very natural to investigate their regularized quantities Q and C. Since our O_N^+-TL-channels are bistochastic, we know that the classical capacity C is smaller than 2χ asymptotically by Proposition 2.5:

$$C(\Phi_k^{l,\bar{m}}) \leq (l+k-m)\log(N), \quad C(\Phi_k^{\bar{l},m}) \leq (m+k-l)\log(N).$$

Although the regularized quantities Q and C are computationally intractible for many channels, some structural properties such as EBT/PPT/(anti-)degradability enable us to handle the regularization issues (See Proposition 2.4). However, we will show that our TL-channels associated with O_N^+ and $SU(2)$ have no such structural properties in most cases.

5.1. The case of O_N^+.

5.1.1. EBT property We now apply Theorem 3.3 to investigate EBT property for our O_N^+-TL-channels $\Phi_k^{\bar{l},m}$. Before coming to our result characterizing the EBT property for the channels $\Phi_k^{\bar{l},m}$, we first need an elementary lemma. We say that a subspace of $H_A \otimes H_B$ is *entangled* if it does not contain a product vector $\xi_A \otimes \xi_B$.

Lemma 5.1. *Let H_A and H_B be finite dimensional Hilbert spaces, let $0 \neq p \in B(H_A \otimes H_B)$ be an orthogonal projection, and let $H_0 \subseteq H_A \otimes H_B$ denote the range of p. If H_0 is an entangled subspace of $H_A \otimes H_B$, then the state $\rho := \frac{1}{\dim H_0} p$ is entangled.*

Proof. We prove the contrapositive. If ρ is separable, then we can write

$$p = \sum_i |\xi_i\rangle\langle\xi_i| \otimes |\eta_i\rangle\langle\eta_i| \qquad (0 \neq \xi_i \in H_A,\ 0 \neq \eta_i \in H_B).$$

For each i put $x_i = |\xi_i\rangle\langle\xi_i| \otimes |\eta_i\rangle\langle\eta_i|$. Then since $x_i \leq p$ and p is a projection, it follows that $x_i = px_i p$, which implies that the range of x_i is contained in the range of p. In particular, $\xi_i \otimes \eta_i \in H_0$, so H_0 is separable. □

Theorem 5.2. *Let $(k, l, m) \in \mathbb{N}_0^3$ be an admissible triple. If $k \neq l - m$, then the O_N^+-TL-channel $\Phi_k^{\bar{l},m}$ is not EBT. Also, if $k \neq m - l$, then the O_N^+-TL-channel $\Phi_k^{l,\bar{m}}$ is not EBT.*

Proof. We have from Theorem 3.3 that $C_{\Phi_k^{\bar{l},m}} = \frac{[k+1]_q}{[l+1]_q} \alpha_l^{m,k} (\alpha_l^{m,k})^* \in B(H_m \otimes H_k)$. Consider the orthogonal projection $p = \alpha_l^{m,k}(\alpha_l^{m,k})^*$. The range of p is the subrepresentation of $H_m \otimes H_k$ equivalent to H_l, and by [Theorem 3.2, [BC18]] this subspace is entangled iff $l \neq k + m$. Applying Lemma 5.1, we conclude that $\Phi_k^{\bar{l},m}$ is not EBT whenever $k \neq l - m$. □

Remark 5.3. We note that Theorem 5.2 leaves open whether or not the channels $\Phi_{l-m}^{\bar{l},m}$ are EBT. In this case, the corresponding Choi matrix is a multiple of the orthogonal projection onto a separable subspace, and we do not know if this projection is a multiple of an entangled state. One exception is the case of $\Phi_1^{\bar{2},1} = \Phi_1^{1,\bar{2}}$, which can be shown to be EBT for all $N \geq 2$. Indeed, the covariance property (4.6) combined with the fact $C_{\Phi_1^{\bar{2},1}} = \dfrac{N}{N^2 - 1} p_2$ tells us that

$$\frac{1}{N^2 - 1} p_2 = \int_{U(N)} U|1\rangle\langle 1|U^* \otimes \overline{U}|2\rangle\langle 2|U^t dU,$$

which explains that $\Phi_1^{\bar{2},1} = \Phi_1^{1,\bar{2}}$ is EBT.

5.1.2. PPT/(anti-)degradability As the next step, one might naturally ask if O_N^+-TL-channels have the PPT property or are (anti-)degradable. In fact, Theorem 4.5 provides a strong partial answer on these structural questions for large N as follows:

Corollary 5.4. *(1) The channel $\Phi_k^{l,\bar{m}}$ is not PPT if $k > m - l$ and $\Phi_k^{\bar{l},m}$ is not PPT if $k > l - m$ for sufficiently large N. In particular, the channels $\Phi_{l+m}^{l,\bar{m}}$ and $\Phi_{l+m}^{\bar{l},m}$ are not PPT for all $N \geq 3$.*

(2) *The channels* $\Phi_k^{l,\bar{m}}$ *and* $\Phi_k^{\bar{l},m}$ *are neither degradable nor anti-degradable if* $k > |l-m|$ *for sufficiently large* N.

Proof. (1) Note that every PPT channel should have zero quantum capacity and that $Q(\Phi_k^{l,\bar{m}}) > 0$ if $k > m-l$ for sufficiently large N. Similar arguments are valid for $\Phi_k^{\bar{l},m}$.

(2) Note that every anti-degradable channel must have zero quantum capacity, while on the other hand both $\Phi_k^{l,\bar{m}}$ and $\Phi_k^{\bar{l},m}$ have strictly positive quantum capacities for sufficiently large N if $k > |l-m|$.

5.2. The case of $SU(2)$. We have a much better understanding about the TL-channels associated with $SU(2)$ than the ones from O_N^+ based on the following concrete description of Clebsch–Gordan coefficients. For an admissible triple $(k,l,m) \in \mathbb{N}_0^3$ we consider the associated isometry

$$\alpha_k^{l,m}|i\rangle = \sum_{j=0}^{l}\sum_{j'=0}^{m} C_{j,j',i}^{l,m,k}|jj'\rangle.$$

We actually have a precise but complicated formula (e.g. [VK95, page 510]) for the constant $C_{j,j',i}^{l,m,k}$, which is a sum with multiple terms. Thus, the general constant $C_{j,j',i}^{l,m,k}$ is difficult to handle, but they satisfy several symmetries and some extremal cases can be written in a simpler form.

Proposition 5.5. *For any admissible triples* $(k,l,m), (i,j,j') \in \mathbb{N}_0^3$ *we have*

(1) $C_{j,j',i}^{l,m,k} = 0$ *if* $i + \frac{l+m-k}{2} \neq j + j'$,

(2) $\begin{cases} \langle i_1|\Phi_k^{l,\bar{m}}(|i\rangle\langle j|)|j_1\rangle = 0, & i_1 - j_1 \neq i - j \\ \langle i_2|\Phi_k^{\bar{l},m}(|i\rangle\langle j|)|j_2\rangle = 0, & i_2 - j_2 \neq i - j \end{cases}$ *for* $\begin{cases} 0 \le i_1, j_1 \le l,\ 0 \le i, j \le k \\ 0 \le i_2, j_2 \le m,\ 0 \le i, j \le k \end{cases}$,

(3) $C_{j,j',i}^{l,m,k} = (-1)^{\frac{l+m-k}{2}} C_{j',j,i}^{m,l,k}$,

(4) $C_{j,j',i}^{l,m,k} = (-1)^{\frac{l+m-k}{2}} C_{l-j,m-j',k-i}^{l,m,k}$,

(5) $C_{j,j',i}^{l,m,k} \neq 0$ *if* $i + \dfrac{l+m-k}{2} = j + j'$ *and if one of the following is true:* $\begin{cases} j = 0, l \\ j' = 0, m \\ i = 0, k \end{cases}$.

Proof. (2) We have $\langle i_1|\Phi_k^{l,\bar{m}}(|i\rangle\langle j|)|j_1\rangle = \sum_{i_2=0}^{m} C_{i_1,i_2,i}^{l,m,k}\overline{C_{j_1,i_2,j}^{l,m,k}} = 0$ if $i_1 - i \neq j_1 - j$ by (1) and a similar argument holds for $\Phi_k^{\bar{l},m}$.

(5) If one of the parameters i, j, j' becomes extremal, then the constant $C_{j,j',i}^{l,m,k}$ can be expressed in a single term, which is a ratio of several factorials by [VK95, Sect. 8.2.6] and the above symmetries (3) and (4). □

The $SU(2)$-TL-channel $\Phi_k^{l,\bar{m}}$ is of the following form.

$$\Phi_k^{l,\bar{m}}(|i\rangle\langle\tilde{i}|) = (\iota \otimes \mathrm{Tr})(\alpha_k^{l,m}|i\rangle\langle\tilde{i}|(\alpha_k^{l,m})^*)$$

$$= (\iota \otimes \mathrm{Tr})\Big(\sum_{j,\tilde{j}=0}^{l} \sum_{j',\tilde{j}'=0}^{m} C^{l,m,k}_{j,j',i} \overline{C^{l,m,k}_{\tilde{j},\tilde{j}',\tilde{i}}} |jj'\rangle\langle \tilde{j}\tilde{j}'|\Big)$$

$$= \sum_{j,\tilde{j}=0}^{l} \sum_{j'=0}^{m} C^{l,m,k}_{j,j',i} \overline{C^{l,m,k}_{\tilde{j},j',\tilde{i}}} \, |j\rangle\langle \tilde{j}|$$

$$= \sum_{j'=0}^{m} \sum_{j,\tilde{j}=0}^{l} C^{m,l,k}_{j',j,i} \overline{C^{m,l,k}_{j',\tilde{j},\tilde{i}}} \, |j\rangle\langle \tilde{j}| = \Phi^{\bar{m},l}_k(|i\rangle\langle \tilde{i}|). \qquad (5.1)$$

The fourth equality is due to (3) of Proposition 5.5.

Proposition 5.6. *For any admissible triple* $(k, l, m) \in \mathbb{N}_0^3$ *we have* $\Phi^{l,\bar{m}}_k = \Phi^{\bar{m},l}_k$*. In particular, we have* $\Phi^{l,\bar{l}}_k = \Phi^{\bar{l},l}_k$*, so that the channel* $\Phi^{l,\bar{l}}_k$ *is always degradable and anti-degradable.*

This allows us to restrict our attention to the case of $l \geq m$.

5.2.1. EBT/PPT properties In this subsection, we completely characterize when the $SU(2)$-TL-channels $\Phi^{l,\overline{m}}_k$ and $\Phi^{\bar{l},m}_k$ are EBT or PPT. The main result of this subsection is as follows.

Theorem 5.7. *Let* $(k, l, m) \in \mathbb{N}_0^3$ *be an admissible triple with* $l \geq m$*.*

(1) The channel $\Phi^{l,\bar{m}}_k$ *is EBT if and only if it is PPT if and only if* $k = 0$*.*

(2) The channel $\Phi^{\bar{l},m}_k$ *is EBT if and only if it is PPT if and only if* $k = l - m$*.*

Proof. (1) If the channel $\Phi^{l,\bar{m}}_k$ is PPT, then its Choi matrix

$$C_{T\circ\Phi} = (T \circ \Phi \otimes \iota)\Big(\sum_{i,j=1}^{d_A} |i\rangle\langle j| \otimes |i\rangle\langle j|\Big) = \sum_{i,j=1}^{d_A} T \circ \Phi(|i\rangle\langle j|) \otimes |i\rangle\langle j|$$

should be a positive definite matrix. In particular, for any orthogonal unit vectors $v_1, v_2 \in H_B \otimes H_A$ we should have

$$\begin{bmatrix} \langle v_1|C_{T\circ\Phi}|v_1\rangle & \langle v_1|C_{T\circ\Phi}|v_2\rangle \\ \langle v_2|C_{T\circ\Phi}|v_1\rangle & \langle v_2|C_{T\circ\Phi}|v_2\rangle \end{bmatrix} = \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix} \geq 0.$$

We take a particular choice of v_1, v_2 as follows.

$$\begin{cases} |v_1\rangle = |l0\rangle, |v_2\rangle = |0l\rangle & \text{if } k > l \\ |v_1\rangle = |l0\rangle, |v_2\rangle = |l-k, k\rangle & \text{if } k \leq l \end{cases}.$$

Now we have $a = \langle v_1|C_{T\circ\Phi}|v_1\rangle = \sum_{j'=0}^{m} C^{l,m,k}_{l,j',0}\overline{C^{l,m,k}_{l,j',0}}$. Since the channel $\Phi^{l,\bar{m}}_0$ is trivially EBT (and PPT) we may assume $k > 0$, then $l + j' \neq \frac{l+m-k}{2}$ from the restriction that $l \geq m$. Thus, we get $a = 0$ by (1) of Proposition 5.5. Similarly, we can check that $b = C^{l,m,k}_{0,\frac{l+m-k}{2},0}\overline{C^{l,m,k}_{l,\frac{l+m-k}{2},l}}$ for $k > l$. By (5) of Proposition 5.5 we know that $b \neq 0$, so that $\det \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix} = -|b|^2 < 0$, which is a contradition. The case $k \leq l$ can be done by the same argument.

(2) We apply a similar argument as before. By taking

$$\begin{cases} |v_1\rangle = |m0\rangle, |v_2\rangle = |m-k,k\rangle & \text{if } l-m < k \le m \\ |v_1\rangle = |m0\rangle, |v_2\rangle = |0m\rangle & \text{if } k > m \vee l-m \end{cases}$$

we can similarly check that the matrix $\begin{bmatrix} \langle v_1|C_{T\circ\Phi}|v_1\rangle & \langle v_1|C_{T\circ\Phi}|v_2\rangle \\ \langle v_2|C_{T\circ\Phi}|v_1\rangle & \langle v_2|C_{T\circ\Phi}|v_2\rangle \end{bmatrix}$ is not positive definite, so that the channels $\Phi_k^{\bar{l},m}$ is not PPT if $k > l-m$.

But the case $k = l - m$ is no longer trivial. Note that we can pick a product vector $e \otimes f \in H_l \subseteq H_m \otimes H_{l-m}$ with $e \in H_m$ and $f \in H_{l-m}$. Then, by Theorem 3.3 and Proposition 2.9, we have

$$\begin{aligned} \frac{1}{l-m+1} C_{\Phi_{l-m}^{\bar{l},m}} &= \frac{1}{l+1} \alpha_l^{m,l-m} (\alpha_l^{m,l-m})^* \\ &= \int_{SU(2)} \pi_m(x^{-1})|e\rangle\langle e|\pi_m(x) \otimes \pi_{l-m}(x^{-1})|f\rangle\langle f|\pi_{l-m}(x)dx, \end{aligned}$$

where dx implies the normalized Haar measure on $SU(2)$. This implies that the normalized Choi matrix of $\Phi_{l-m}^{\bar{l},m}$ is a separable state since the set of separable states are closed. □

5.3. (Anti-)degradability. We first present the following cases when $SU(2)$-TL-channels are (anti-)degradable.

Theorem 5.8. *Let $(k,l,m) \in \mathbb{N}_0^3$ be an admissible triple with $l \ge m$.*

(1) The channel $\Phi_k^{l,\bar{m}}$ is degradable if (a) $l = m$ or (b) $k = l + m$ or (c) $k = l - m$. Moreover, we have a degrading channel for the case (b) as follows.

$$\Phi_l^{m,\overline{l-m}} \circ \Phi_{l+m}^{l,\bar{m}} = \Phi_{l+m}^{m,\bar{l}}. \tag{5.2}$$

(2) The channel $\Phi_k^{l,\bar{m}}$ is not anti-degradable for $l > m$. Equivalently, $\Phi_k^{\bar{l},m}$ is not degradable for $l > m$.

Proof. (1) The first two cases (a) and (c) follow from Proposition 5.6 and Theorem 5.7, respectively. For the identity (5.2) we need to show that for any $0 \le i, j \le l+m$ and for any s_2 such that $\max\{0, i-j\} \le s_2 \le \min\{m, m+i-j\}$,

$$(\Phi_l^{\overline{l-m},m} \circ \Phi_{l+m}^{l,\bar{m}}(|i\rangle\langle j|))_{s_2,s_2+j-i} = (\Phi_{l+m}^{\bar{l},m}(|i\rangle\langle j|))_{s_2,s_2+j-i}$$

by (2) of Proposition 5.5. Equivalently, let us show that for any $\max\{0, i-j\} \le s_2 \le \min\{m, m+i-j\}$

$$\sum_{i_2} C_{i-i_2,i_2,i}^{l,m,l+m} \overline{C_{j-i_2,i_2,j}^{l,m,l+m}} C_{i-i_2-s_2,s_2,i-i_2}^{l-m,m,l} \overline{C_{i-i_2-s_2,s_2+j-i,j-i_2}^{l-m,m,l}} = C_{i-s_2,s_2,i}^{l,m,l+m} \overline{C_{i-s_2,s_2+j-i,j}^{l,m,l+m}},$$

where i_2 runs over $\max\{0, i-s_2-l+m\} \le i_2 \le \min\{m, i-s_2\}$. We use the following explicit formula for Clebsch–Gordan coefficients to the highest weight case, namely for any l, m

$$C_{j_1,j_2,j}^{l,m,l+m} = \delta_{j_1+j_2,j} \sqrt{\frac{l!m!}{(l+m)!}} \sqrt{\frac{j!(l+m-j)!}{j_1!j_2!(l-j_1)!(m-j_2)!}}.$$

Now, we have

$$\sum_{i_2} C^{l,m,l+m}_{i-i_2,i_2,i}\overline{C^{l,m,l+m}_{j-i_2,i_2,j}}C^{l-m,m,l}_{i-i_2-s_2,s_2,i-i_2}\overline{C^{l-m,m,l}_{i-i_2-s_2,s_2+j-i,j-i_2}}$$

$$= \frac{l!m!}{(l+m)!}\frac{(l-m)!m!}{l!}\sqrt{\frac{i!(l+m-i)!j!(l+m-j)!}{s_2!(m-s_2)!(s_2+j-i)!(m-s_2-j+i)!}}$$

$$\times \sum_{i_2}\frac{1}{i_2!(m-i_2)!(i-i_2-s_2)!(l-m+s_2+i_2-i)!}$$

$$= \frac{l!m!}{(l+m)!l!}\sqrt{\frac{i!(l+m-i)!j!(l+m-j)!}{s_2!(m-s_2)!(s_2+j-i)!(m-s_2-j+i)!}}\sum_{i_2}\binom{m}{i_2}\binom{l-m}{i-s_2-i_2}$$

$$= \frac{l!m!}{(l+m)!l!}\sqrt{\frac{i!(l+m-i)!j!(l+m-j)!}{s_2!(m-s_2)!(s_2+j-i)!(m-s_2-j+i)!}}\binom{l}{i-s_2}$$

$$= \frac{l!m!}{(l+m)!}\frac{1}{(i-s_2)!(l+s_2-i)!}\sqrt{\frac{i!(l+m-i)!j!(l+m-j)!}{s_2!(m-s_2)!(s_2+j-i)!(m-s_2-j+i)!}}$$

$$= \sqrt{\frac{l!m!}{(l+m)!}}\sqrt{\frac{i!(l+m-i)!}{(i-s_2)!s_2!(l-i+s_2)!(m-s_2)!}}$$

$$\times \sqrt{\frac{l!m!}{(l+m)!}}\sqrt{\frac{j!(l+m-j)!}{(i-s_2)!(s_2+j-i)!(l-i+s_2)!(m-s_2-j+i)!}}$$

$$= C^{l,m,l+m}_{i-s_2,s_2,i}\overline{C^{l,m,l+m}_{i-s_2,s_2+j-i,j}}.$$

The third equality in the above is from the following fact

$$\binom{l}{i-s_2} = \sum_{\max\{0,i-s_2-l+m\}\le i_2\le \min\{m,i-s_2\}}\binom{m}{i_2}\binom{l-m}{i-s_2-i_2}.$$

(2) By Proposition 2.5 and Proposition 2.8 we know that

$$0 < \log(\frac{l+1}{m+1}) \le Q^{(1)}(\Phi_k^{l,\bar{m}}),$$

which leads us to the conlusion we wanted. □

Remark 5.9. We remark that in the case of $SU(2)$-TL-channels, it is possible to show that the channels $\Phi_k^{l,\overline{m}}$ can be non-degradable for intermediate $l-m<k<l+m$ at least in some low dimensional examples. The general strategy is to find an explicit state $\rho \in M_{k+1}$ such that

$$0 < H(\Phi_k^{\bar{l},m}(\rho)) - H(\Phi_k^{l,\overline{m}}(\rho)) \le Q^{(1)}(\Phi_k^{\bar{l},m}).$$

The inequality above implies that $\Phi_k^{\bar{l},m}$ is not anti-degradable, or equivalently $\Phi_k^{l,\bar{m}}$ is not degradable. For a concrete example, let us consider the channel $\Phi_3^{3,\bar{2}}$. For $\rho =$

$\begin{bmatrix} 0.25 & 0 & 0 & 0 \\ 0 & 0.75 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, we have

$$H(\Phi_3^{\bar{3},2}(\rho)) - H(\Phi_3^{3,\bar{2}}(\rho)) = H(\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}) - H(\begin{bmatrix} 0.45 & 0 & 0 & 0 \\ 0 & 0.15 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix})$$
$$\approx 0.0192,$$

where the first equality is obtained by the precise description of the associated isometry

$$\alpha_3^{3,2} : \mathbb{C}^4 \to \mathbb{C}^4 \otimes \mathbb{C}^3, \quad \begin{aligned} |1\rangle &\mapsto -\sqrt{\tfrac{3}{5}}|12\rangle + \sqrt{\tfrac{2}{5}}|21\rangle, \\ |2\rangle &\mapsto -\sqrt{\tfrac{2}{5}}|13\rangle - \sqrt{\tfrac{1}{15}}|22\rangle + \sqrt{\tfrac{8}{15}}|31\rangle \end{aligned}$$

using the known formula of $SU(2)$-Clebsch–Gordan coefficients [Boh01]. Here, $\{|j\rangle\}_{j=1}^{n+1}$ refers to the canonical orthonormal basis of $H_n \cong \mathbb{C}^{n+1}$ and we have written the image of $\alpha_3^{3,2}$ for the first two basis elements since they are the only relevant entries.

5.4. Remarks on associated classical and quantum capacities. In this subsection we collect some immediate consequences of the already obtained results on associated classical and quantum capacities.

Proposition 5.10. *Let Φ be one of the following TL-channels:*

$$SU(2)\text{-}TL\text{-}channels\ \Phi_{l-m}^{l,\overline{m}}, \Phi_{l-m}^{\bar{l},m}, \Phi_{l+m}^{\bar{l},m}, \Phi_{l+m}^{l,\overline{m}} (l \geq m),$$
$$O_N^+\text{-}TL\text{-}channels\ \Phi_1^{\bar{2},1}.$$

Then we have

$$C(\Phi) = \chi(\Phi).$$

Moreover, for the $SU(2)$-TL-channels $\Phi_{l+m}^{l,\overline{m}}, \Phi_{l+m}^{\bar{l},m}$ we have

$$C(\Phi_{l+m}^{l,\overline{m}}) = \chi(\Phi_{l+m}^{l,\overline{m}}) = \log(l+1) \quad and \quad C(\Phi_{l+m}^{\bar{l},m}) = \chi(\Phi_{l+m}^{\bar{l},m}) = \log(m+1),$$

and for the O_N^+-TL channel $\Phi_1^{\bar{2},1} = \Phi_1^{1,\bar{2}}$ we have

$$C(\Phi_1^{1,\bar{2}}) = \chi(\Phi_1^{1,\bar{2}}) = \frac{N}{N+1}\log\left(\frac{N}{N-1}\right) - \log\left(\frac{N+1}{N}\right).$$

Proof. For the $SU(2)$-TL channel $\Phi_{l-m}^{\bar{l},m}$, $l \geq m$ and the O_N^+-TL-channel $\Phi_1^{\bar{2},1}$ we get the conclusion by Proposition 2.4 since they are EBT by Theorem 5.7 and Remark 5.3. Moreover for the O_N^+-TL channel $\Phi_1^{\bar{2},1}$ we have

$$C(\Phi_1^{\bar{2},1}) = \chi(\Phi_1^{\bar{2},1}) \leq \log(N) - H_{\min}(\Phi_1^{1,\bar{2}}) = \frac{N}{N+1}\log\left(\frac{N}{N-1}\right) - \log\left(\frac{N+1}{N}\right)$$

by Remark 2.2 and Remark 5.3. Then equality follows from the ensemble $\{|i\rangle\langle i|\}_{1\le i\le N}$ with the uniform distribution.

Now we consider the case of the $SU(2)$-TL-channels $\Phi^{l,\overline{m}}_{l+m}$ and $\Phi^{\overline{l},m}_{l+m}$, $l \ge m$. By combining the following two results [AN13, Theorem 2.16] and [Hol06], we have

$$\log(m+1) - 0 = \log(m+1) - H_{\min}(\Phi^{\overline{l},m}_{l+m}) = \chi(\Phi^{\overline{l},m}_{l+m}) \le C(\Phi^{\overline{l},m}_{l+m}) \le \log(m+1) \text{ and}$$
$$\log(l+1) - 0 = \log(l+1) - H_{\min}(\Phi^{l,\overline{m}}_{l+m}) = \chi(\Phi^{l,\overline{m}}_{l+m}) \le C(\Phi^{l,\overline{m}}_{l+m}) \le \log(l+1).$$

For the remaining case of $SU(2)$-TL-channels $\Phi^{l,\overline{m}}_{l-m}$ we have

$$\begin{aligned} C(\Phi^{l,\overline{m}}_{l-m}) &= \log(l+1) - \lim_{n\to\infty}\frac{1}{n}H_{\min}((\Phi^{l,\overline{m}}_{l-m})^{\otimes n}) \\ &= \log\left(\frac{l+1}{m+1}\right) + \log(m+1) - \lim_{n\to\infty}\frac{1}{n}H_{\min}((\Phi^{\overline{l},m}_{l-m})^{\otimes n}) \\ &= \log\left(\frac{l+1}{m+1}\right) + C(\Phi^{\overline{l},m}_{l-m}) \\ &= \log(l+1) + \chi(\Phi^{\overline{l},m}_{l-m}) - \log(m+1) \\ &= \log(l+1) - H_{\min}(\Phi^{\overline{l},m}_{l-m}) \\ &= \log(l+1) - H_{\min}(\Phi^{l,\overline{m}}_{l-m}) = \chi(\Phi^{l,\overline{m}}_{l-m}). \end{aligned}$$

Here, the 1st, 3rd, 5th and the last equalities come from [Hol06], the 2nd and 6th equalities are due to the fact that $H_{\min}(\Psi) = H_{\min}(\widetilde{\Psi})$ for any quantum channel, and the 4th equality is thanks to EBT and [Sho02]. □

Proposition 5.11. *Let Φ be one of the following TL-channels:*

$$\begin{cases} SU(2)\text{-}TL\text{-}channels\ \Phi^{l,\overline{m}}_{k}\ with\ (a)\ k = l+m,\ (b)\ k = |l-m|\ or\ (c) l = m \\ O^+_N\text{-}TL\text{-}channels\ \Phi^{\overline{1},2}_{1},\ \Phi^{2,\overline{1}}_{1}. \end{cases}$$

Then we have

$$Q(\Phi) = Q^{(1)}(\Phi).$$

Let Ψ be one of the following TL-channels:

$$\begin{cases} SU(2)\text{-}TL\text{-}channels\ \Phi^{\overline{l},m}_{k}\ with\ (a)\ k = l+m,\ (b)\ k = |l-m|\ or\ (c) l = m \\ O^+_N\text{-}TL\text{-}channels\ \Phi^{\overline{2},1}_{1},\ \Phi^{1,\overline{2}}_{1}. \end{cases}$$

Then we have

$$Q(\Psi) = Q^{(1)}(\Psi) = 0.$$

Proof. By Theorem 5.8 the above listed $SU(2)$-TL-channels $\Phi^{l,\overline{m}}_{k}$ and $\Phi^{\overline{l},m}_{k}$ are degradable and anti-degradable, respectively, which leads us to the conlusion by Proposition 2.4. In the case of O^+_N-TL-channels we can appeal to the fact that $\Phi^{1,\overline{2}}_{1} = \Phi^{\overline{2},1}_{1}$ is EBT (and consequently anti-degradable) from Remark 5.3. □

Proposition 5.12. *Let $\varepsilon > 0$ be any fixed constant and Φ be a O_N^+-TL channel $\Phi_k^{l,\overline{m}}$ or $\Phi_k^{\overline{l},m}$. Then we have*

$$C(\Phi) \leq (2+\varepsilon)\chi(\Phi)$$

for sufficiently large N.

Proof. A proof follows by combining the bistochastic property of TL-channels, Proposition 2.5 and Corollary 4.6. Indeed, we have

$$\begin{aligned}&\liminf_N \left\{2\chi(\Phi_k^{l,\overline{m}}) - C(\Phi_k^{l,\overline{m}})\right\}\\&= \liminf_N \left\{2\left(\chi(\Phi_k^{l,\overline{m}}) - \frac{l+k-m}{2}\log(N)\right)\right.\\&\quad \left. - (C(\Phi_k^{l,\overline{m}}) - (l+k-m)\log(N))\right\} \geq 0,\end{aligned}$$

which gives us the conclusion. Note that

$$\begin{aligned}&\lim_{N\to\infty}\left\{\log\left(\frac{[l+1]_q[k+1]_q}{[m+1]_q}\right) - (l+k-m)\log(N)\right\}\\&= \lim_{N\to\infty}\left\{\log\left(\frac{[l+1]_q}{N^l}\right) + \log\left(\frac{[k+1]_q}{N^k}\right) - \log\left(\frac{[m+1]_q}{N^m}\right)\right\} = 0.\end{aligned}$$

And we can apply a similar argument for $\Phi_k^{\overline{l},m}$. □

Example 5.13. Combining Propositions 5.10 and 5.11 we know that the $SU(2)$-TL channel $\Phi_{l+m}^{\overline{l},m}$ $(l \geq m)$ satisfy the following extremal condition

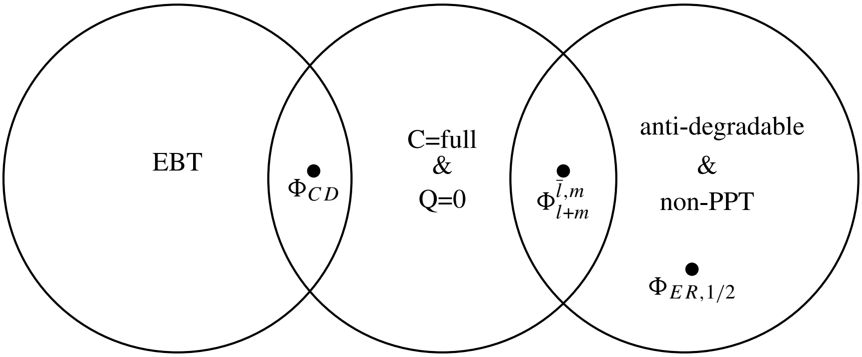
$$Q(\Phi_{l+m}^{\overline{l},m}) = 0 \ \& \ C(\Phi_{l+m}^{\overline{l},m}) = \log(m+1),$$

where the quantum capacity is smallest possible, namely zero and the classical capacity is the largest possible in the sense that any quantum channel $\Phi : B(H_A) \to B(H_B)$ satisfies $0 \leq Q(\Phi) \leq C(\Phi) \leq \log d_B$. Of course, it is rather straightforward to find quantum channels satisfying the above extremal condition. An easy example would be the completely dephasing channel

$$\Phi_{CD} : M_n(\mathbb{C}) \to M_n(\mathbb{C}),\ |i\rangle\langle j| \mapsto \delta_{i,j}|i\rangle\langle j|.$$

Since Φ_{CD} is clearly EBT (and consequently anti-degradable) we have $Q(\Phi_{CD}) = 0$ and the fact that $C(\Phi_{CD}) = \log n$ is trivial by considering the canonical classical-quantum encoding to diagonal elements and the associated canonical quantum-classical decoding. One big difference between the above two examples of channels is that Φ_{CD} belongs to the class of EBT channels, while the $SU(2)$-TL channel $\Phi_{l+m}^{\overline{l},m}$ $(l \geq m)$ is non-PPT by Theorem 5.7. Thus, we found a family of channels in the class of "anti-degradable & non-PPT" channels exhibiting the extremal property "C =full & $Q = 0$".

We believe this to be the first such example, to the best of our knowledge.



Note that the class of "anti-degradable & non-PPT" channels contains interesting examples of channels, which have played crucial roles in QIT. For example, the 50%-erasure channel $\Phi_{ER,1/2} : M_n(\mathbb{C}) \to M_{n+1}(\mathbb{C})$ given by $\rho \mapsto \frac{1}{2}\begin{bmatrix} \mathrm{Tr}(\rho) & 0 \\ 0 & \rho \end{bmatrix}$ belongs to the class of "anti-degradable & non-PPT" channels and was used to discover superactivation of quantum capacity [SY08]. Note that the qubit 50%-erasure channel $\Phi_{ER,1/2}$ does not have the full classical capacity since $C(\Phi_{ER,1/2}) = \frac{1}{2} < 1$ [BDS97].

6. Tensor Products of Temperley–Lieb Channels and Outputs of Entangled Covariant States

It is well known that additivity of Holevo information is equivalent to additivity of minimum output entropies [Sho04] and Hastings [Has09] established non-additivity of the minimum output entropy by exhibiting the existence of random unitary channels Φ such that

$$H_{\min}(\Phi \otimes \overline{\Phi}) < H_{\min}(\Phi) + H_{\min}(\overline{\Phi}), \tag{6.1}$$

where $\overline{\Phi}$ is the conjugate channel of Φ. In the proof of (6.1), the maximally entangled state was used to estimate an upper bound of $H_{\min}(\Phi\otimes\overline{\Phi})$. Since we know the minimum output entropies for single O_N^+-TL-channels in an asymptotic sense, it is natural to try to evaluate the minimum output entropies for tensor products of O_N^+-TL-channels. Although we are unable to fully evaluate such minimum output entropies for all tensor products, we do establish upper bounds for the minimum output entropies $H_{\min}(\Phi^{\bar{l}_1,m_1}_{k_1} \otimes \Phi^{l_2,\bar{m}_2}_{k_2})$. This is achieved by evaluating the entropies $H((\Phi^{\bar{l}_1,m_1}_{k_1} \otimes \Phi^{l_2,\bar{m}_2}_{k_2})(\rho))$ for certain entangled states ρ. More precisely, we will present explicit formulae for

$$H((\Phi^{\bar{l}_1,m_1}_{k_1} \otimes \Phi^{l_2,\bar{m}_2}_{k_2})(\frac{1}{[i+1]_q}\alpha_i^{k_1,k_2}(\alpha_i^{k_1,k_2})^*))$$

for all admissible triples $(i, k_1, k_2) \in \mathbb{N}_0^3$.

In this section we use all the notation and planar string diagram formalism for $\mathrm{Rep}(O_F^+)$ introduced in Sect. 3.

6.1. Tetrahedral nets and the quantum 6j-symbols. Following [KL94], let $\mathcal{A} \subset \mathbb{N}_0^6$ be the set of all sextuples $\begin{bmatrix} a & b & i \\ c & d & j \end{bmatrix}$ with the property that each of the following triples

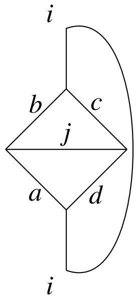
$$(a, d, i),\ (b, c, i),\ (a, b, j),\ (d, c, j)$$

is admissible. We define the *tetrahedral net* to be the function $\mathrm{Tet}_q : \mathcal{A} \to \mathbb{C}$ given by

$$\mathrm{Tet}_q \begin{bmatrix} a & b & i \\ c & d & j \end{bmatrix} = \tau_i((A_i^{b,c})^*(\iota_{H_b} \otimes (A_0^{j,j})^* \otimes \iota_{H_c})(A_a^{b,j} \otimes A_d^{j,c})A_i^{a,d}).$$

In terms of planar string diagrams, the Tet_q functions are given by

$$\mathrm{Tet}_q \begin{bmatrix} a & b & i \\ c & d & j \end{bmatrix} =$$



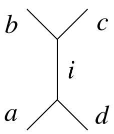
.

Next, we introduce the *quantum* $6j$*-symbols* $\{\cdot\}_q : \mathcal{A} \to \mathbb{C}$, which are defined in terms of the tetrahedral nets as follows:

$$\begin{Bmatrix} a & b & i \\ c & d & j \end{Bmatrix}_q = \frac{\mathrm{Tet}_q \begin{bmatrix} a & b & i \\ c & d & j \end{bmatrix} [i+1]_q}{\theta_q(a, d, i)\theta_q(b, c, i)}.$$

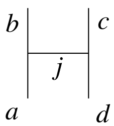
Remark 6.1. We note that there exist simple algebraic formulae that allow one to numerically evaluate the tetrahedral nets (and hence also the quantum $6j$-symbols). See [KL94, Sect. 9.11] for example.

The most important geometric-algebraic feature of the quantum $6j$-symbols $\begin{Bmatrix} a & b & i \\ c & d & j \end{Bmatrix}_q$ is that they arise as the basis change coefficients for two canonical bases for the Hom-space $\mathrm{Hom}_{O_F^+}(H_a \otimes H_d, H_b \otimes H_c)$. More precisely, $\mathrm{Hom}_{O_F^+}(H_a \otimes H_d, H_b \otimes H_c)$ has one linear basis given by the string diagrams



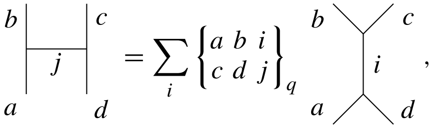
($i \in \mathbb{N}_0$ such that (i, a, d), (i, b, c) admissible),

and another linear basis given by



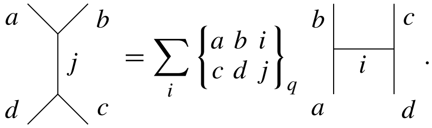
($j \in \mathbb{N}_0$ such that (j, a, b), (j, c, d) admissible).

We then have that the quantum $6j$-symbols are the basis change coefficients between these two bases [KL94, Proposition 11]:



(6.2)

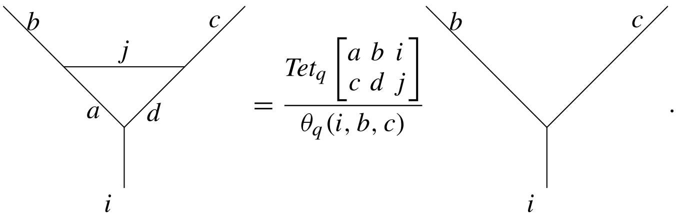
and similarly by a rotational symmetry argument,



(6.3)

The following formula involving three-vertices and tetrahedral nets will be handy in the next subsection.

Lemma 6.2. *Let* $\begin{bmatrix} a & b & i \\ c & d & j \end{bmatrix} \in \mathcal{A}$*. Then*



Proof. Denote the quantity on the left hand side by B. Then $B \in \mathrm{Hom}_{O_F^+}(H_i, H_b \otimes H_c) = \mathbb{C}A_i^{b,c}$, and so there exists $\lambda \in \mathbb{C}$ such that $B = \lambda A_i^{b,c}$ (i.e., B is a multiple of a three-vertex). But then we have

$$\mathrm{Tet}_q \begin{bmatrix} a & b & i \\ c & d & j \end{bmatrix} = \tau_i((A_i^{b,c})^* B) = \tau_i((A_i^{b,c})^* \lambda A_i^{b,c}) = \lambda \theta_q(i,b,c).$$

□

6.2. Tensor products of TL-channels and outputs of entangled states. Here we address tensor products of the form $\Phi_{k_1}^{\bar{l}_1,m_1} \otimes \Phi_{k_2}^{l_2,\bar{m}_2}$, and compute explicitly the outputs of O_F^+-covariant states of the form $\rho_i^{k_1,k_2} = \frac{1}{[i+1]_q} \alpha_i^{k_1,k_2} (\alpha_i^{k_1,k_2})^*$, for all admissible triples (i, k_1, k_2). Note that in the special case of $i = 0$ and $k_1 = k_2$, we have that $\rho_0^{k,k}$ is a maximally entangled state, and in general, $\rho_i^{k_1,k_2}$ is an entangled state [BC17, Theorem 5.5] if $k_1, k_2 > 0$.

In order to ease the notational burden on the following theorem, let us fix once and for all admissible triples (i, k_1, k_2), $(k_j, l_j, m_j) \in \mathbb{N}_0^3$ $(j = 1, 2)$, and let $X_i = \big(\Phi_{k_1}^{\bar{l}_1,m_1} \otimes \Phi_{k_2}^{l_2,\bar{m}_2}\big)(\rho_i^{k_1,k_2})$

Theorem 6.3. *We have the following spectral decomposition for X_i:*

$$X_i = \sum_{\substack{l=m_1+l_2-2r\\ 0\le r\le \min\{m_1,l_2\}}} \lambda_{i,l}^{m_1,l_2}\alpha_l^{m_1,l_2}(\alpha_l^{m_1,l_2})^*,$$

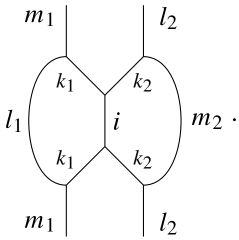
where

$$\lambda_{i,l}^{m_1,l_2} = \left(\frac{[i+1]_q[k_1+1]_q[k_2+1]_q\theta_q(l,m_1,l_2)}{[l+1]_q\theta_q(k_1,l_1,m_1)\theta_q(k_2,l_2,m_2)\theta_q(i,k_1,k_2)}\right) \times \sum_{\substack{j=2t\\ 0\le t\le \min\{k_1,k_2\}}} \frac{\begin{Bmatrix} k_1 & k_2 & j\\ k_2 & k_1 & i\end{Bmatrix}_q Tet_q\begin{bmatrix} l_1 & m_1 & m_1\\ j & k_1 & k_1\end{bmatrix} Tet_q\begin{bmatrix} k_2 & j & l_2\\ l_2 & m_2 & k_2\end{bmatrix}\begin{Bmatrix} m_1 & m_1 & l\\ l_2 & l_2 & j\end{Bmatrix}_q}{\theta_q(m_1,m_1,j)\theta_q(l_2,j,l_2)},$$

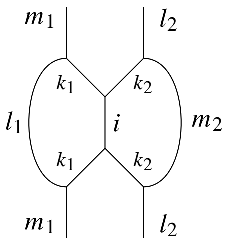
and occurs with multiplicity $[l+1]_q$.

Proof. We have that, up to planar isotopy, the planar tangle representing X_i is given by:

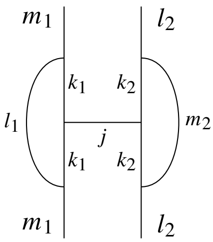
$$X_i = \frac{[i+1]_q[k_1+1]_q[k_2+1]_q}{\theta_q(k_1,k_2,i)\theta_q(l_1,m_1,k_1)\theta_q(l_2,m_2,k_2)}$$



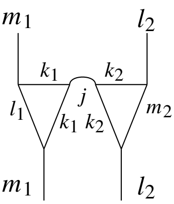
Using the formulae (6.2)–(6.3) for the quantum $6j$-symbols together with Lemma 6.2, we have



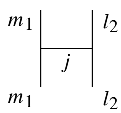
$$= \sum_j \begin{Bmatrix} k_1 & k_2 & j\\ k_2 & k_1 & i\end{Bmatrix}_q$$



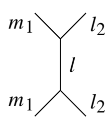
$$= \sum_j \begin{Bmatrix} k_1 & k_2 & j\\ k_2 & k_1 & i\end{Bmatrix}_q$$



$$= \sum_j \begin{Bmatrix} k_1 & k_2 & j\\ k_2 & k_1 & i\end{Bmatrix}_q \frac{\mathrm{Tet}_q\begin{bmatrix} l_1 & m_1 & m_1\\ j & k_1 & k_1\end{bmatrix}\mathrm{Tet}_q\begin{bmatrix} k_2 & j & l_2\\ l_2 & m_2 & k_2\end{bmatrix}}{\theta_q(m_1,m_1,j)\theta_q(l_2,j,l_2)}$$



$$= \sum_{l} \sum_{j} \begin{Bmatrix} k_1 & k_2 & j \\ k_2 & k_1 & i \end{Bmatrix}_q \frac{\mathrm{Tet}_q \begin{bmatrix} l_1 & m_1 & m_1 \\ j & k_1 & k_1 \end{bmatrix} \mathrm{Tet}_q \begin{bmatrix} k_2 & j & l_2 \\ l_2 & m_2 & k_2 \end{bmatrix}}{\theta_q(m_1, m_1, j)\theta_q(l_2, j, l_2)} \begin{Bmatrix} m_1 & m_1 & l \\ l_2 & l_2 & j \end{Bmatrix}_q$$



$$= \sum_{l} \Big(\sum_{j} \begin{Bmatrix} k_1 & k_2 & j \\ k_2 & k_1 & i \end{Bmatrix}_q \frac{\mathrm{Tet}_q \begin{bmatrix} l_1 & m_1 & m_1 \\ j & k_1 & k_1 \end{bmatrix} \mathrm{Tet}_q \begin{bmatrix} k_2 & j & l_2 \\ l_2 & m_2 & k_2 \end{bmatrix}}{\theta_q(m_1, m_1, j)\theta_q(l_2, j, l_2)} \begin{Bmatrix} m_1 & m_1 & l \\ l_2 & l_2 & j \end{Bmatrix}_q \Big) \frac{\theta_q(l, m_1, l_2)}{[l+1]_q} \alpha_l^{m_1,l_2} \alpha_l^{m_1,l_2 *}.$$

In the above, the summands run over l such that (l, m_1, l_2) is admissible, and j such that both (j, k_1, k_1) and (j, k_2, k_2) are admissible. This corresponds exactly to $l = m_1 + l_2 - 2r$ with $0 \le r \le \min\{m_1, l_2\}$ and $j = 2t$ with $0 \le t \le \min\{k_1, k_2\}$. The claimed formula for the eigenvalue $\lambda_{i,l}^{m_1,l_2}$ is now immediate. Note also that the multiplicity of $\lambda_{i,l}^{m_1,l_2}$ is $\mathrm{rank}(\alpha_l^{m_1,l_2}(\alpha_l^{m_1,l_2})^*) = \dim H_l = [l+1]_q$. $\square$

Remark 6.4. As remarked above, the element $X_0 \in B(H_{m_1} \otimes H_{l_2})$ is the output of the O_F^+-covariant Bell state $\rho_0^{k,k} \in B(H_k \otimes H_k)$. In this situation, the eigenvalue formula for X_0 simplifies greatly. This can be seen by using similar arguments to those in the proof given above, or by directly using algebraic relations satisfied by the quantum $6j$-symbols. In any case, we get

$$X_0 = \sum_{\substack{l = m_1 + l_2 - 2r \\ 0 \le r \le \min\{m_1, l_2\}}} \lambda_{0,l}^{m_1,l_2} \alpha_l^{m_1,l_2} \alpha_l^{m_1,l_2 *},$$

with

$$\lambda_{0,l}^{m_1,l_2} = \frac{[k+1]_q \mathrm{Tet}_q \begin{bmatrix} m_1 & l_1 & l \\ m_2 & l_2 & k \end{bmatrix}^2}{\theta_q(l_1, m_1, k)\theta_q(l_2, m_2, k)\theta_q(m_1, l_2, l)\theta_q(l_1, m_2, l)}$$
$$= \frac{[k+1]_q \begin{Bmatrix} m_1 & l_1 & l \\ m_2 & l_2 & k \end{Bmatrix}_q^2 \theta_q(l, l_1, m_2)\theta_q(l, m_1, l_2)}{\theta_q(l_1, m_1, k)\theta_q(l_2, m_2, k)[l+1]_q^2},$$

occurring with multiplicity $[l+1]_q$.

6.3. Remarks on the MOE additivity problem for certain O_N^+-TL-channels. Given that we have, on the one hand, asymptotically sharp estimates on the MOE of the O_N^+-TL-channels $\Phi_k^{\bar{l},m}$, $\Phi_k^{l,\bar{m}}$ (given by $H_{\min}(\Phi_k^{\bar{l},m}), H_{\min}(\Phi_k^{l,\bar{m}}) \sim \left(\frac{l+m-k}{2}\right) \log N$ - cf. Theorem 4.3), and on the other hand, we have exact formulae for the outputs $X_i = \left(\Phi_{k_1}^{\bar{l}_1,m_1} \otimes \Phi_{k_2}^{l_2,\bar{m}_2}\right)(\rho_i^{k_1,k_2})$ of entangled states under the tensor products of certain TL-channels, it is natural to ask whether one can obtain a strict inequality of the form

$$H(X_i) < \left(\frac{l_1 + m_1 - k_1}{2}\right) \log N + \left(\frac{l_2 + m_2 - k_2}{2}\right) \log N \qquad \text{(for suitable } i, k_j, l_j, m_j).$$

If this were the case, we would have obtained deterministic examples of pairs of quantum channels which witness the non-additivity of their minimum output entropy.

Unfortunately, however, extensive numerical evaluations of $H(X_i)$ for suitable parameter choices always yield inequalities of the form $H(X_i) - \left(\frac{l_1+m_1-k_1}{2}\right)\log N - \left(\frac{l_2+m_2-k_2}{2}\right)\log N > 0$ with the difference going to zero as $N \to \infty$. We see this as strong evidence that the pairs of quantum channels $\Phi_{k_1}^{\bar{l}_1,m_1}$, $\Phi_{k_2}^{l_2,\bar{m}_2}$ are not MOE strictly subadditive.

7. Comparing the Class of Kac Type TL-Channels with Other Classes of Quantum Channels

In this final section we would like to compare the class of $SU(2)$-TL-channels and O_N^+-TL-channels ($N \geq 3$) with other previously well-known classes of non-trivial quantum channels, which we recall as follows.

(1) (Quantum erasure channels) For $0 < p \leq 1$ and $H_A \subseteq H_B$, $|e\rangle \in H_B \ominus H_A$ we set

$$\Phi_{ER,p} : B(H_A) \to B(H_B), \quad \rho \mapsto (1-p)\rho + p\mathrm{Tr}(\rho)|e\rangle\langle e|.$$

(2) (Amplitude damping channels) For $0 < \gamma \leq 1$ we set $A_1 = |1\rangle\langle 1| + \sqrt{1-\gamma}|2\rangle\langle 2|$, $A_2 = \sqrt{\gamma}|1\rangle\langle 2|$ and

$$\Phi_{AD,\gamma} : B(\mathbb{C}^2) \to B(\mathbb{C}^2), \quad \rho \mapsto A_1\rho A_1^\dagger + A_2\rho A_2^\dagger.$$

(3) (Dephasing channels) For $0 < p \leq 1$ we set

$$\Phi_{DePh,p} : B(\mathbb{C}^2) \to B(\mathbb{C}^2), \quad \rho \mapsto (1-\frac{p}{2})\rho + \frac{p}{2}Z\rho Z,$$

where $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(4) (Depolarizing channels) For $0 < p \leq 1$ and $H_A = H_B$ we set

$$\Phi_{DePo,p} : B(H_A) \to B(H_B), \quad \rho \mapsto (1-p)\rho + \frac{p}{d_A}\mathrm{Tr}(\rho)I_A.$$

We will consider 3 different types of comparision for quantum channels.

Definition 7.1. We say that two quantum channels $\Phi : B(H_A) \to B(H_B)$ and $\Psi : B(H_{A'}) \to B(H_{B'})$ are "**identical**" if $H_A = H_{A'}$, $H_B = H_{B'}$ and $\Phi(\rho) = \Psi(\rho)$ for any $\rho \in B(H_A)$. We also say that Φ and Ψ are "**unitarily equivalent**" if $H_A = H_{A'}$, $H_B = H_{B'}$ and there are unitaries $U \in B(H_A)$ and $V \in B(H_B)$ such that $V\Phi(U^*\rho U)V^* = \Psi(\rho)$ for any $\rho \in B(H_A)$. We say that " Φ **can be transformed into** Ψ" if $H_A = H_{A'}$, $H_B \subseteq H_{B'}$ and there are a unitary $U \in B(H_A)$ and an isometry $V \in B(H_B, H_{B'})$ such that $V\Phi(U^*\rho U)V^* = \Psi(\rho)$ for any $\rho \in B(H_A)$.

Proposition 7.2. *None of the above mentioned channels can be transformed into $SU(2)$-TL-channels or O_N^+-TL-channels with $N \geq 3$.*

Proof. First of all, note that any CG-channels are bistochastic. Thus, we get the conclusions immediately for non-bistochastic channels, which include quantum erasure channels and amplitude damping channels except the channels $\Phi_{ER,p}$, $p = \frac{1}{d_A+1}$, which we postpone the proof to the end.

Note also that a bistochastic channel can be transformed into another bistochastic channel only when they are unitarily equivalent. Thus, for dephasing channels we need to compare them with the only possible qubit channel, namely $SU(2)$-TL-channel $\Phi_1^{\bar{2},1} = \Phi_1^{1,\bar{2}} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} \frac{1}{3}(a+2d) & -\frac{1}{3}b \\ -\frac{1}{3}c & \frac{1}{3}(2a+d) \end{bmatrix}$. For any unitaries $U, V \in B(\mathbb{C}^2)$, we can readily check the state $\rho = V^*\Phi_1^{\bar{2},1}(U^*|1\rangle\langle 1|U)V$ satisfies the condition

$$\mathrm{Tr}(\rho^2) = \mathrm{Tr}((\Phi_1^{\bar{2},1}(U^*|1\rangle\langle 1|U))^2) = \frac{5}{9}. \tag{7.1}$$

On the other hand we can easily see that $\mathrm{Tr}(\Phi_{DePh,p}(|1\rangle\langle 1|)^2) = \mathrm{Tr}((|1\rangle\langle 1|)^2) = \mathrm{Tr}(|1\rangle\langle 1|) = 1$, which means that so that the channel $\Phi_1^{\bar{2},1}$ is not unitarily equivalent to the channel $\Phi_{DePh,p}$.

The case of depolarizing channels are a bit more complicated. Since they are bistochastic, we should compare them with Kac type TL-channels $\Phi_k^{\bar{l},k}$, $\Phi_k^{k,\bar{l}}$ with $k \geq 1$. When $k \geq 2$, we can use the same index $\mathbf{m} = (1, 2, 1, \ldots)$ with the associated state $|\mathbf{m}\rangle \in H_k \subseteq H_1^{\otimes k}$ as in the proof of Theorem 4.3. We also consider a companion state $|\mathbf{n}\rangle \in H_k$ with the index $\mathbf{n} = (2, 1, 2, \ldots)$. For splittings $\mathbf{m} = \mathbf{m}'\mathbf{m}''$ and $\mathbf{n} = \mathbf{n}'\mathbf{n}''$ we note that $\mathbf{m}'' \neq \mathbf{n}''$. Since $p_k|\mathbf{n}\rangle = |\mathbf{n}\rangle \in H_k$ we can see that

$$\langle \mathbf{n}|\Phi_k^{\bar{l},k}(|\mathbf{m}\rangle\langle \mathbf{m}|)|\mathbf{n}\rangle = 0$$

from (4.5). On the other hand, for any unitaries U, V with appropriate size, we have

$$V\Phi_{DePo,p}(U|\mathbf{m}\rangle\langle \mathbf{m}|U^*)V^* = (1-p)VU|\mathbf{m}\rangle\langle \mathbf{m}|U^*V^* + \frac{p}{d_A}I_A,$$

which we get $\langle \mathbf{n}|V\Phi_{DePo,p}(U|\mathbf{m}\rangle\langle \mathbf{m}|U^*)V^*|\mathbf{n}\rangle > 0$ since $p > 0$. The case of $\Phi_k^{k,\bar{l}}$, $k \geq 2$ is the same. Now we focus on TL-channels $\Phi_1^{\bar{l},1}$, especially the channel $\Phi_1^{\bar{2},1} = \Phi_1^{1,\bar{2}}$ since $k = 1$ forces $l = 2$. Note that for any unitaries U, V with appropriate size we have

$$V\Phi_{DePo,p}(U|j\rangle\langle j|U^*)V^* = (1-p)VU|j\rangle\langle j|U^*V^* + \frac{p}{N}I_N,$$

which is clearly different from $\Phi_1^{\bar{2},1}(|j\rangle\langle j|)$ for the case $\mathbb{G} = O_N^+$ by (3.8). For $SU(2)$-TL channel $\Phi_1^{\bar{2},1}$ we compare the image for $|j\rangle\langle j|$, $j = 1, 2$ to get $p = \frac{2}{3}$ and $VU = I_2$, which gives us $V\Phi_{DePo,p}(U \cdot U^*)V^* : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} \frac{1}{3}(a+2d) & \frac{1}{3}b \\ \frac{1}{3}c & \frac{1}{3}(2a+d) \end{bmatrix}$, which is different from $SU(2)$-TL channel $\Phi_1^{\bar{2},1}$.

Now we focus on the only remaining bistochastic channel $\Phi_{ER,p}$, $p = \frac{1}{d_A+1}$, which should be compared with TL-channels $\Phi_k^{\bar{l},k+1}$ or $\Phi_k^{k+1,\bar{l}}$. Recall that $\Phi_k^{\bar{l},k+1}$ is $\mathbb{G}$-covariant with respect to $(u^{(k)}, u^{(k+1)})$. We suppose that the channel $\Psi(\cdot) = V\Phi_{ER,p}(U \cdot U^*)V^*$ is $\mathbb{G}$-covariant with respect to $(u^{(k)}, u^{(k+1)})$ for any unitaries U, V with appropriate size. Let us denote $V = [V_{ab}]$ and $VU = W = [w_{ab}]$, then for $\rho = |m\rangle\langle m'| \in B(H_A)$ we have

$$
\begin{aligned}
&(\Psi \otimes \iota)[u^{(k)}(\rho \otimes 1)(u^{(k)})^*] \\
&\quad = (\Psi \otimes \iota)[\sum_{i,i'=1}^{n_k} |i\rangle\langle i'| \otimes u^{(k)}_{im}(u^{(k)}_{i'm'})^*] \\
&\quad = (1-p)\sum_{i,i'=1}^{n_k} W|i\rangle\langle i'|W^* \otimes u^{(k)}_{im}(u^{(k)}_{i'm'})^* + p\sum_{i=1}^{n_k} V|e\rangle\langle e|V^* \otimes u^{(k)}_{im}(u^{(k)}_{i'm'})^* \\
&\quad = (1-p)\sum_{i,i'=1}^{n_k}\sum_{a,a'=1}^{n_{k+1}} w_{ai}\overline{w_{a'i'}}|a\rangle\langle a'| \otimes u^{(k)}_{im}(u^{(k)}_{i'm'})^* \\
&\qquad + p\sum_{i=1}^{n_k}\sum_{a,a'=1}^{n_{k+1}} v_{ae}\overline{v_{a'e}}|a\rangle\langle a'| \otimes u^{(k)}_{im}(u^{(k)}_{i'm'})^* \\
&\quad = \sum_{a,a'=1}^{n_{k+1}} |a\rangle\langle a'| \otimes [(1-p)\sum_{i,i'=1}^{n_k} w_{ai}\overline{w_{a'i'}}u^{(k)}_{im}(u^{(k)}_{i'm'})^* + p\sum_{i=1}^{n_k} v_{ae}\overline{v_{a'e}}u^{(k)}_{im}(u^{(k)}_{i'm'})^*] \\
&\quad = \sum_{a,a'=1}^{n_{k+1}} |a\rangle\langle a'| \otimes [(1-p)[Wu^{(k)}]_{am}[(u^{(k)})^*W^*]_{m'a'} + p \cdot \delta_{m,m'}v_{ae}\overline{v_{a'e}}].
\end{aligned}
$$

We also have

$$
\begin{aligned}
u^{(k+1)}[\Psi(\rho) \otimes 1](u^{(k+1)})^* &= u^{(k+1)}[(1-p)W\rho W^* + p \cdot \delta_{m,m'}V|e\rangle\langle e|V^*](u^{(k+1)})^* \\
&= (1-p)\sum_{a,a',b,b'=1}^{n_k+1} w_{bm}\overline{w_{b'm'}}|a\rangle\langle a'| \otimes u^{(k+1)}_{ab}(u^{(k+1)}_{a'b'})^* \\
&\quad + p \cdot \delta_{m,m'}\sum_{a,a',b,b'=1}^{n_k+1} v_{be}\overline{v_{b'e}}|a\rangle\langle a'| \otimes u^{(k+1)}_{ab}(u^{(k+1)}_{a'b'})^* \\
&= \sum_{a,a'=1}^{n_{k+1}} |a\rangle\langle a'| \otimes [(1-p)[u^{(k+1)}W]_{am}[W^*(u^{(k+1)})^*]_{m'a'} \\
&\quad + p \cdot \delta_{m,m'}[u^{(k+1)}V]_{ae}[V^*(u^{(k+1)})^*]_{ea'}].
\end{aligned}
$$

Comparing coefficients of $|a\rangle\langle a'|$, $1 \le a, a' \le n_{k+1}$ we get

$$
\begin{aligned}
&(1-p)[Wu^{(k)}]_{am}[(u^{(k)})^*W^*]_{m'a'} + p \cdot \delta_{m,m'}v_{ae}\overline{v_{a'e}} \\
&\quad = (1-p)[u^{(k+1)}W]_{am}[W^*(u^{(k+1)})^*]_{m'a'} + p \cdot \delta_{m,m'}[u^{(k+1)}V]_{ae}[V^*(u^{(k+1)})^*]_{ea'}
\end{aligned}
$$

for any $1 \le m, m' \le n_k$. Summation over all $m = m'$ we have

$$
(1-p)\delta_{a,a'} + p \cdot n_k v_{ae}\overline{v_{a'e}} = (1-p)\delta_{a,a'} + p \cdot n_k[u^{(k+1)}V]_{ae}[V^*(u^{(k+1)})^*]_{ea'}.
$$

Since $p \neq 0$ we actually have $v_{ae}\overline{v_{a'e}} = [u^{(k+1)}V]_{ae}[V^*(u^{(k+1)})^*]_{ea'}$, which, in turn, gives us

$$[Wu^{(k)}]_{am}[(u^{(k)})^*W^*]_{m'a'} = [u^{(k+1)}W]_{am}[W^*(u^{(k+1)})^*]_{m'a'}$$

for any $1 \leq m, m' \leq n_k$ and $1 \leq a, a' \leq n_{k+1}$. Finally, we take the Haar state on both sides to get $\delta_{a,a'}\delta_{m,m'}n_k^{-1} = \delta_{a,a'}\delta_{m,m'}n_{k+1}^{-1}$, which is a contradiction, so that we can conculude that the channel $\Phi_{ER,p}$, $p = \frac{1}{d_A+1}$ can not be transformed into a Kac type TL-channels. □

We close this section by comparing Kac type TL-channels with a class of channels called *TRO-channels* and their modifications recently introduced in [GJL18]. For a quantum channel $\Phi : B(H_A) \to B(H_B)$ with a Stinespring isometry $V : H_A \to H_B \otimes H_E$ the range space $\mathrm{Ran}\, V \subseteq H_B \otimes H_E$ is called a *Stinespring space* of Φ. Note that the choice of isometry V is not unique, but any associated Stinespring space is known to determine the channel Φ. For this reason we will fix a Stinespring isometry V and refer to the range $\mathrm{Ran}\, V$ as *the Stinespring space*. We say that the channel Φ is a *TRO-channel* if its Stinespring space is a *TRO*, i.e. a *ternary ring of operators*. Recall that a TRO is a subspace X of $B(H, K)$ for some Hilbert spaces H, K such that "$x, y, z \in X \Rightarrow xy^*z \in X$", i.e. closed under triple product. It is well-known that finite dimensional TRO's are direct sums of rectangular matrix spaces with mutiplicity. Since the Stinespring space determines the channel it has been observed in [GJL18] that a TRO-channel $\Phi : B(H_A) \to B(H_B)$ is always of the following form: the channel Φ has a Stinespring space X given by

$$X = \oplus_{i=1}^M B(\mathbb{C}^{m_i}, \mathbb{C}^{n_i}) \otimes 1_{l_i} \subseteq B(H_E, H_B),$$

where

$$H_E = \oplus_{i=1}^M \mathbb{C}^{m_i} \otimes \mathbb{C}^{l_i} \;\text{ and }\; H_B = \oplus_{i=1}^M \mathbb{C}^{n_i} \otimes \mathbb{C}^{l_i}.$$

Moreover, we have $H_A = (X, \langle \cdot, \cdot \rangle_{H_A})$, where the inner product is given by $\langle x, y \rangle_{H_A} := \mathrm{Tr}_E(y^*x)$, $x, y \in X \subseteq B(H_E, H_B)$. Finally, the channel Φ is given by

$$\Phi(|x\rangle\langle y|) = xy^*, \;\; x, y \in H_A = X \subseteq B(H_E, H_B).$$

Based on the above description we can define a variant of TRO-channels. We first fix a *symbol* $f \in B(H_E)$, i.e. a positive matrix with $\tau(f) := \frac{\mathrm{Tr}_E(f)}{d_E} = 1$ and *strongly independent* of the right algebra $\mathcal{R}(X) = \mathrm{span}\{x^*y : x, y \in X\}$. Here, we say that $x \in B(H_E)$ is *independent* of $\mathcal{R}(X)$ if $\tau(xy) = \tau(x)\tau(y)$ for all $y \in \mathcal{R}(X)$ and *strongly independent* of $\mathcal{R}(X)$ if x^n is indepedent of $\mathcal{R}(X)$ for every $n \geq 1$. Then the *modified TRO-channel* Φ_f with the symbol f is defined by

$$\Phi_f : B(H_A) \to B(H_B), \;\; |x\rangle\langle y| \mapsto xfy^*.$$

The original TRO-channel Φ corresponds to the case of Φ_f with $f = 1_E$. It has been proved in [GJL18] that we have exact calculations for various quantities of Φ as follows.

$$Q^{(1)}(\Phi) = P^{(1)}(\Phi) = Q(\Phi) = P(\Phi) = \log(\max_i n_i), \;\; \chi(\Phi) = C(\Phi) = \log(\sum_i n_i). \tag{7.2}$$

Moreover, we also have the following estimates for modified TRO-channels.

$$Q^{(1)}(\Phi) \leq Q^{(1)}(\Phi_f) \leq Q^{(1)}(\Phi) + \tau(f \log f).$$

The same estimates hold for other capacities, i.e. we may replace $Q^{(1)}$ with $P^{(1)}$, Q, P, χ and C. Important examples of (modified) TRO-channels include random unitary channels using projective unitary representations of finite (quantum) groups and generalized dephasing channels [GJL18].

In this section we prove that some TL-channels do not belong to the class of modified TRO-channels. We can even find an example with minimal non-trivial dimensions.

Proposition 7.3. *The $SU(2)$-TL-channel $\Phi_1^{\bar{2},1}$ can not be transformed into any modified TRO-channel.*

Proof. We first observe that $\mathrm{Ran}\Phi_1^{\bar{2},1} = B(\mathbb{C}^2)$, which is a full matrix algebra. Let Φ_f be a modified TRO-channel with the parameters n_i, m_i, l_i, $1 \le i \le M$ as above. Since we need to match the dimensions of the sender's Hilbert spaces we only have the following 3 possible cases. (1) $M = 1, n_1 = 2, m_1 = 1$, (2) $M = 1, n_1 = 1, m_1 = 2$ and (3) $M = 2, n_1 = n_2 = m_1 = m_2 = 1$.

Case (1): The corresponding modified TRO-channel becomes (after identifying the orthonormal basis in a suitable way)

$$\Phi_f : B(\mathbb{C}^2) \to B(\mathbb{C}^2) \otimes B(\mathbb{C}^{l_1}),\ |i\rangle\langle j| \mapsto |i\rangle\langle j| \otimes \frac{f}{l_1}.$$

If we assume that $\Phi_1^{\bar{2},1}$ is equivalent to Φ_f, then there are a unitary $U : \mathbb{C}^2 \to \mathbb{C}^2$ and an isometry $V : \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^{l_1}$ such that

$$V\Phi_1^{\bar{2},1}(U^*\rho U)V^* = \Phi_f(\rho), \quad \rho \in B(H_A).$$

Since $\mathrm{Ran}\Phi_1^{\bar{2},1} = B(\mathbb{C}^2)$ we also have $\mathrm{Ran}\Phi_f \cong B(\mathbb{C}^2)$ as a subalgebra of $B(\mathbb{C}^2) \otimes B(\mathbb{C}^{l_1})$, which forces $g := \frac{f}{l_1}$ to be a pure state. This implies that $g^2 = g$, so that $\mathrm{Tr}((|1\rangle\langle 1| \otimes g)^2) = \mathrm{Tr}(|1\rangle\langle 1| \otimes \frac{f}{l_1}) = 1$. However, the state $\rho' = \Phi_1^{\bar{2},1}(U^*|1\rangle\langle 1|U)$ satisfies $\mathrm{Tr}((\rho')^2) = 5/9 \neq 1$ as we have seen before in (7.1). Since $X \mapsto VXV^*$ is a trace preserving map, we get a contradiction.

Case (2): The corresponding modified TRO-channel becomes

$$\Phi_f : B(\mathbb{C}^2) \to B(\mathbb{C}^{l_1}),\ |i\rangle\langle j| \mapsto \frac{f_{ij}}{l_1},$$

where $f = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \in B(\mathbb{C}^2) \otimes B(\mathbb{C}^{l_1})$ with $f_{ij} \in B(\mathbb{C}^{l_1})$, $1 \le i, j \le 2$. Since $\mathrm{Ran}\Phi_1^{\bar{2},1} = B(\mathbb{C}^2)$ we know that $l_1 \ge 2$. We assume that there are a unitary $U : \mathbb{C}^2 \to \mathbb{C}^2$ and an isometry $V : \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^{l_1}$ such that $V\Phi_1^{\bar{2},1}(U^*\rho U)V^* = \Phi_f(\rho), \quad \rho \in B(H_A)$ as before. In this case we have $\mathcal{R}(X) = B(\mathbb{C}^2) \otimes \mathbb{C}1_{l_1}$. It is straightforward to check that independence of f with respect to $\mathcal{R}(X)$ implies that $\mathrm{Tr}(f_{11}) = l_1$. We also know that f^2 is independent of $\mathcal{R}(X)$, which means that $\mathrm{Tr}((f^2)_{11}) = l_1$. However, we have

$$l_1 = \mathrm{Tr}((f^2)_{11}) = \mathrm{Tr}(f_{11}^2 + f_{12}f_{21}) \ge \mathrm{Tr}(f_{11}^2) = \frac{5}{9}l_1^2,$$

which is a contradiction. The above inequality is from $f_{12}^* = f_{21}$ and the last equality is from the fact that

$$\mathrm{Tr}((\frac{f_{11}}{l_1})^2) = \mathrm{Tr}((\rho')^2) = 5/9.$$

Case (3): The corresponding modified TRO-channel becomes

$$\Phi_f : B(\mathbb{C}^2) \to B(\mathbb{C}^{l_1+l_2}),\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mapsto \left[\frac{a_{ij} f_{ij}}{\sqrt{l_i l_j}}\right]_{1\le i,j\le 2},$$

where $f = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \in B(\mathbb{C}^{l_1+l_2})$ with $f_{ij} \in B(\mathbb{C}^{l_j}, \mathbb{C}^{l_i})$, $1 \le i, j \le 2$. Since $\mathrm{Ran}\Phi_1^{\bar{2},1} = B(\mathbb{C}^2)$ we know that $l_1 \ge 2$. We assume that there are a unitary $U : \mathbb{C}^2 \to \mathbb{C}^2$ and an isometry $V : \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^{l_1}$ such that $V\Phi_1^{\bar{2},1}(U^*\rho U)V^* = \Phi_f(\rho)$, $\rho \in B(H_A)$ as before. In this case we have $\mathcal{R}(X) = \mathbb{C}1_{l_1} \oplus \mathbb{C}1_{l_2} \subseteq B(\mathbb{C}^{l_1+l_2})$. It is also straightforward to check that independence of f with respect to $\mathcal{R}(X)$ implies that $\mathrm{Tr}(f_{11}) = l_1$. We also know that f^2 is independent of $\mathcal{R}(X)$, which means that $\mathrm{Tr}((f^2)_{11}) = l_1$. However, we have

$$l_1 = \mathrm{Tr}((f^2)_{11}) = \mathrm{Tr}(f_{11}^2 + f_{12} f_{21}) \ge \mathrm{Tr}(f_{11}^2) = \frac{5}{9} l_1 l_2,$$

where the last identity is from the fact that

$$\mathrm{Tr}((\frac{f_{11}}{l_1})^2) = \mathrm{Tr}(\begin{bmatrix} \frac{f_{11}}{l_1} & 0 \\ 0 & 0 \end{bmatrix}^2) = \mathrm{Tr}((\rho')^2) = 5/9.$$

Thus, we can conclude that $l_1 = 1$, which actually means that $f_{11} = \mathrm{Tr}(f_{11}) = l_1 = 1$. Thus, we have $\mathrm{Tr}((\frac{f_{ij}}{l_1})^2) = 1 \ne 5/9$, so that we get a contradiction.

Remark 7.4. (1) The canonical complementary channel $\tilde{\Phi}_f$ of a modified TRO-channel Φ_f can be written as follows.

$$\tilde{\Phi}_f : B(H_A) \to B(H_E),\ \ |x\rangle\langle y| \mapsto \sqrt{f} y^* x \sqrt{f}.$$

Then, we can also show that the Temperley–Lieb channel $\Phi_1^{\bar{2},1}$ for $G = SU(2)$ is not equivalent to any canonical complementary channel $\tilde{\Phi}_f$ of a modified TRO-channel Φ_f. This time the argument is easier since we only need to observe that $\mathrm{rank}(\tilde{\Phi}_f) \le 2$ in all the 3 possible cases in the proof of Proposition 7.3.

(2) Note that TRO-channels provides examples with the extremal property "C = full & $Q = 0$" considered in Example 5.13 by setting $n_1 = \cdots = n_M = 1 = l_1 = \cdots = l_M$ by (7.2). However, one can readily check that corresponding TRO-channels is nothing but a completely dephasing channel.

□

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