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Property RD and Hypercontractivity for Orthogonal Free Quantum Groups

Michael Brannan[1], Roland Vergnioux[2] and Sang-Gyun Youn[3,*]

[1]Department of Mathematics, Mailstop 3368, Texas A & M University, College Station, TX 77843-3368, USA, [2]Normandie Univ, UNICAEN,CNRS, LMNO, 14000 Caen, France, and [3]Department of Mathematics Education, Seoul National University, Gwanak-ro 1, Gwanak-gu, Seoul, 08826, Republic of Korea

**Correspondence to be sent to: e-mail: s.youn@snu.ac.kr*

We prove that the twisted property RD introduced in [2] fails to hold for all non-Kac type, non-amenable orthogonal free quantum groups. In the Kac case we revisit property RD, proving an analogue of the $L_p - L_2$ non-commutative Khintchine inequality for free groups from [29]. As an application, we give new and improved hypercontractivity and ultracontractivity estimates for the generalized heat semigroups on free orthogonal quantum groups, both in the Kac and non-Kac cases.

1 Introduction

The property of rapid decay (RD), also called Haagerup's inequality is a fundamental tool in the study of the reduced C^*-algebra of discrete groups, allowing one to control the operator norm of convolution operators by means of the much simpler ℓ^2-norm (see Section 2.4 for more details). It appeared in the seminal paper [19] where it was used in conjunction with Haagerup's approximation property (HAP) to establish the metric approximation property (MAP) for reduced C^*-algebras of free groups.

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The definition of property RD was extended to discrete quantum groups in [32] and was proved there to be satisfied by Kac-type (unimodular) orthogonal and unitary free quantum groups. In [5], a quantum analogue of the HAP was established for these free quantum groups, thus yielding a proof of the MAP for the corresponding reduced C^*-algebras. Property RD was, moreover, used for the study of other aspects of discrete quantum group operator algebras, see for example [6, 31, 33, 35]. Interesting connections to quantum information theory, specific to the quantum framework, were also unveiled in [7].

The definition of property RD used in [32] can only be satisfied by Kac-type discrete quantum groups. In [2], the authors give a ″twisted″ version of the definition that holds for all (duals of) q-deformations of connected compact semi-simple Lie groups and give applications to noncommutative geometry.

Hypercontractivity describes the regularization effect, in terms of L_p-norms, of a given Markov semigroup. It has been studied extensively since the early 70's, starting with the work of Nelson and Gross [17, 26] and has found surprising applications in harmonic analysis, information theory, and statistical mechanics. In the case of the Ornstein–Uhlenbeck semigroup on the Clifford algebra with one generator, the two-point inequality of Bonami, rediscovered by Gross [4, 18], already has deep applications to (quantum) information theory [9, 16, 23, 24].

In the noncommutative framework, hypercontractivity problems for Orstein–Uhlenbeck-like semigroups emerged from quantum field theory and optimal times have been obtained in the fermionic case in [10, 26], using noncommutative L_p-theory. Moving further away from the commutative situation, hypercontractivity results for free group algebras were obtained in [3, 21] (with respect to different semigroups). Note that the connection between hypercontractivity and property RD in that case was already noticed by Biane [3].

The study of hypercontractivity for discrete quantum group algebras was initiated in [14], where a natural analogue of the heat semigroup on the reduced C^*-algebra of orthogonal free quantum groups was studied. In the Kac case, the authors of [14] obtain the ultracontractivity of these semigroups (at all times), as well as hypercontractivity with explicit upper bounds for the optimal time to contractivity.

In the present article we pursue the study of property RD for non-Kac-type discrete quantum groups. We prove that non-Kac and non-amenable orthogonal free quantum groups do not satisfy the property RD introduced in [2] (Theorem 3.3). Then we state and prove a weaker RD inequality (Proposition 3.4), which holds for all orthogonal free quantum groups and which was already used without proof in [31] in a slightly less precise form.

In the 2nd part of the article we continue the study of ultra- and hypercontractivity for the heat semigroup on free orthogonal quantum groups. We obtain in particular the 1st known results in the non-Kac case, namely ultracontractivity with a *strictly positive* optimal time (Proposition 4.1) and hypercontractivity for large time (Proposition 4.2). In the Kac case we sharpen the upper bound of [14] for the optimal time to hypercontractivity (Theorems 4.5, 4.6, and 4.7), using a non-commutative Khintchine-type inequality (Theorem 4.4). We give as well a lower bound for the optimal time to hypercontractivity (Lemma 4.3). Motivated by these results, we end the article with a conjectural formula for the asymptotical behavior of the optimal time to hypercontractivity when the rank of the free orthogonal quantum group tends to infinity.

The article is organized as follows. In Section 2 we recall the necessary preliminaries about compact quantum groups and property RD on their duals. Section 3 is devoted to the study of property RD on non-Kac-type orthogonal free quantum groups. Finally, in Section 4 we produce applications to hypercontractivity as described above.

2 Preliminaries

We assume that the reader is familiar with the basic notation and terminology on compact and discrete quantum groups. For details, we refer the reader to the standard references [25, 27, 30, 34]. In this paper we will mainly be concerned with the class of free orthogonal quantum groups and their associated dual discrete quantum groups. We now recall these objects.

2.1 Compact quantum groups

A compact quantum group $\mathbb{G}$ is given by a Woronowicz C^*-algebra $C(\mathbb{G})$, which is in particular a unital Hopf-C^*-algebra with co-associative coproduct $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$. We denote by h the Haar state on $C(\mathbb{G})$, which is the unique state on $C(\mathbb{G})$ satisfying

$$(h \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes h)\Delta = h(\cdot)1.$$

The Haar state induces the inner product $\langle f, g\rangle = h(f^*g)$ and the norm $\|f\|_2 = h(f^*f)^{1/2}$ for $f, g \in C(\mathbb{G})$. By completion we obtain the Gelfand-Naimark-Segal (GNS) space $L_2(\mathbb{G})$ with canonical cyclic vector ξ_0, and we denote $\pi_h : C(\mathbb{G}) \to B(L_2(\mathbb{G}))$ the associated representation. The image of π_h is the reduced Woronowicz C^*-algebra denoted $C_r(\mathbb{G})$ and the associated von Neumann algebra is $L_\infty(\mathbb{G}) = C_r(\mathbb{G})'' \subset B(L_2(\mathbb{G}))$.

We then define $L_1(\mathbb{G})$ as the predual of $L_\infty(\mathbb{G})$ and consider the natural embedding $L_\infty(\mathbb{G}) \hookrightarrow L_1(\mathbb{G})$ given by $x \mapsto h(\cdot x)$. Then $(L_\infty(\mathbb{G}), L_1(\mathbb{G}))$ is a compatible pair of Banach spaces, which allows one to define the non-commutative L_p-spaces $L_p(\mathbb{G}) = (L_\infty(\mathbb{G}), L_1(\mathbb{G}))_{1/p}$ by the complex interpolation method [28]. When the Haar state is tracial we have $\|a\|_{L_p(\mathbb{G})} = h(|a|^p)^{1/p}$ for any $1 \leq p < \infty$ and $a \in L_\infty(\mathbb{G})$.

A representation of $\mathbb{G}$ on a Hilbert space H_v is an invertible element $v \in M(K(H_v) \otimes C(\mathbb{G}))$ such that $(\mathrm{id} \otimes \Delta)(v) = v_{12}v_{13}$, using the leg-numbering notation. Here, $v_{12} = v \otimes 1$ and $v_{13} = \sigma_{23}(v_{12})$ in $M(K(H_v) \otimes C(\mathbb{G}) \otimes C(\mathbb{G}))$, where σ_{23} is the unique extension of the $*$-homomorphism on $K(H_v) \otimes C(\mathbb{G}) \otimes C(\mathbb{G})$ given by $T \otimes a \otimes b \mapsto T \otimes b \otimes a$.

Furthermore, v is called a unitary representation if $v^*v = \mathrm{Id}_{H_v} \otimes 1_{C(\mathbb{G})} = vv^*$. If H_v is finite-dimensional and equipped with an orthonormal basis $(e_i)_i$, the associated matrix elements of v are $v_{i,j} = (e_i^* \otimes \mathrm{id})v(e_j \otimes \mathrm{id})$. Then we have $v = \sum e_i e_j^* \otimes v_{i,j}$ and $\Delta(v_{i,j}) = \sum v_{i,j} \otimes v_{k,j}$. For two unitary representations $v \in M(K(H_v) \otimes C(\mathbb{G}))$ and $w \in M(K(H_w) \otimes C(\mathbb{G}))$, the tensor product representation is $v \mathbin{\bigcirc\!\!\!\!\top} w = v_{13}w_{23} \in M(K(H_v \otimes H_w) \otimes C(\mathbb{G}))$.

Furthermore, we say that v is *irreducible* if $\mathrm{Mor}(v, v) := \{T \in B(H_v) : v(T \otimes 1) = (T \otimes 1)v\} = \mathbb{C}\cdot\mathrm{id}_{H_v}$. We denote by $\mathrm{Irr}(\mathbb{G})$ the set of all irreducible unitary representations of $\mathbb{G}$ up to unitary equivalence. For each $\alpha \in \mathrm{Irr}(\mathbb{G})$ we choose $u = u^\alpha \in \alpha$ and denote $H_\alpha = H_u$ (which is always finite-dimensional). The coefficients of u^α with respect to some orthonormal basis $(e_i)_i \subset H_\alpha$ are denoted $u^\alpha_{i,j}$. The multiplicity of an irreducible representation u in another representation v is $\mathrm{mult}(u \subset v) = \dim \mathrm{Mor}(u, v)$.

There is, for each irreducible unitary representation u, a uniquely defined positive element $Q_u \in B(H_u)$ such that $d_u := \mathrm{Tr}(Q_u) = \mathrm{Tr}(Q_u^{-1})$ and such that the following orthogonality relations hold

$$\begin{aligned} h(u^*_{i,j}u_{k,l}) &= d_u^{-1}\delta_{jl}(e_k \mid Q_u^{-1}e_i), \\ h(u_{k,l}u^*_{i,j}) &= d_u^{-1}\delta_{ik}(e_j \mid Q_u e_l). \end{aligned} \tag{2.1}$$

The number d_u is called the *quantum dimension* of u, as opposed to the classical dimension $n_u = \dim H_u$. The compact quantum group $\mathbb{G}$ is said to be of *Kac type* if $Q_\alpha = \mathrm{id}_{H_\alpha}$ for all $\alpha \in \mathrm{Irr}(\mathbb{G})$. This is equivalent to the Haar state h being tracial.

The coefficients $u^\alpha_{i,j}$ of irreducible unitary representations span a dense subalgebra $\mathcal{O}(\mathbb{G}) \subset C(\mathbb{G})$, which is a Hopf algebra with respect to the restriction of the coproduct Δ. We recall that h is faithful on $\mathcal{O}(\mathbb{G})$ and we shall identify $\mathcal{O}(\mathbb{G})$ with its image in $C_r(\mathbb{G})$ through the GNS representation π_h—in particular we identify a representation v and its image $(\mathrm{id}\otimes\pi_h)(v) \in B(H_v)\otimes C_r(\mathbb{G})$. Note also that $\mathcal{O}(\mathbb{G})$ is dense in $L_p(\mathbb{G})$ for any $1 \le p < \infty$.

A compact quantum group $\mathbb{G}$ is said to be a compact matrix quantum group if there exists a finite generating subset $\{\alpha_1, \cdots, \alpha_n\}$ of $\mathrm{Irr}(\mathbb{G})$ in the sense that any irreducible unitary representation u^α appears as an irreducible component of a tensor product representation $u^{\alpha_{m_1}} ⓣ u^{\alpha_{m_2}} ⓣ \cdots ⓣ u^{\alpha_{m_k}}$ for some $k \in \mathbb{N}$ and $1 \le m_1, m_2, \cdots, m_k \le n$. In this case, for any α, the minimal number $k \in \mathbb{N}_0$ required to generate u^α as a subrepresentation as above is called the *length of* α and denoted $|\alpha| = k$. the length of the trivial representation is 0. We say that a non-zero element $f \in C(\mathbb{G})$ or $C_r(\mathbb{G})$ has length k if it can be written as a linear combination of coefficients $u^\alpha_{i,j}$ with irreducible representations α of length k. We denote $p_k \in B(L_2(\mathbb{G}))$ the orthogonal projection onto the subspace of $L_2(\mathbb{G})$ spanned by elements of length k.

2.2 Dual algebras

Associated to each compact quantum group $\mathbb{G}$ is its dual discrete quantum group $\widehat{\mathbb{G}}$. For us the main object of interest will be the algebra

$$\ell_\infty(\widehat{\mathbb{G}}) = \{a \in \textstyle\prod_{\alpha\in\mathrm{Irr}(\mathbb{G})} B(H_\alpha) : (\|a_\alpha\|)_\alpha \text{ bounded}\}$$

and the subalgebras $c_{00}(\widehat{\mathbb{G}})$, $c_0(\widehat{\mathbb{G}})$ of sequences with finite support, resp. converging to 0. For each $\alpha \in \mathrm{Irr}(\mathbb{G})$ we denote p_α the corresponding minimal central projection in any of these algebras. We use the same notation p_α for the orthogonal projection onto the subspace of $L_2(\mathbb{G})$ spanned by the GNS images of the coefficients $u^\alpha_{i,j}$—indeed there is a natural representation of $c_0(\widehat{\mathbb{G}})$ on $L_2(\mathbb{G})$ which realizes this identification.

The algebras $c_0(\widehat{\mathbb{G}})$ and $C(\mathbb{G})$ are related through the "multiplicative unitary" $V = \bigoplus_\alpha u^\alpha \in M(c_0(\widehat{\mathbb{G}}) \otimes C(\mathbb{G}))$. We endow $c_0(\widehat{\mathbb{G}})$ and $\ell_\infty(\widehat{\mathbb{G}})$ with the coproduct $\hat\Delta$ such that $(\hat\Delta\otimes\mathrm{id})(V) = V_{13}V_{23}$. By definition this coproduct is related to the tensor product construction for representations, more precisely we have, for all α, β, $\gamma \in \mathrm{Irr}(\mathbb{G})$, $a \in B(H_\gamma)$ and $T \in \mathrm{Mor}(\gamma, \alpha ⓣ \beta)$, the following identity in $B(H_\gamma, H_\alpha \otimes H_\beta)$:

$$(p_\alpha \otimes p_\beta)\hat\Delta(a)T = Ta.$$

There is a distinguished weight $\hat h$ on $\ell_\infty(\widehat{\mathbb{G}})$, called the *left Haar weight*, given by

$$\hat h(a) = \sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} d_\alpha \mathrm{Tr}(Q_\alpha a_\alpha) \qquad (a = (a_\alpha)_\alpha \in c_{00}(\widehat{\mathbb{G}})).$$

We denote again $\|a\|_2 = \hat{h}(a^*a)^{1/2}$ the norm on $c_{00}(\hat{\mathbb{G}})$ associated with this weight. By restriction and tensor product one obtains as well norms, still denoted $\|\cdot\|_2$, on $B(H_\alpha)$ and $B(H_\beta \otimes H_\gamma)$, associated to the inner products $\langle a_1, a_2\rangle = d_\alpha \mathrm{Tr}(Q_\alpha a_1^* a_2)$ for all $a_1, a_2 \in B(H_\alpha)$ and $\langle x_1, x_2\rangle = d_\beta d_\gamma \mathrm{Tr}((Q_\beta \otimes Q_\gamma) x_1^* x_2)$ for all $x_1, x_2 \in B(H_\beta \otimes H_\gamma)$. Note that the collection of matrices Q_α defines an algebraic (in general unbounded) multiplier $Q = (Q_\alpha)_\alpha$ of $c_{00}(\hat{\mathbb{G}})$, the *modular element*.

The analogue of the classical Fourier transform is the linear map $\mathcal{F} : c_{00}(\widehat{\mathbb{G}}) \to C(\mathbb{G})$ given by $\mathcal{F}(a) = (\hat{h} \otimes \mathrm{id})(V(a \otimes 1))$. Explicitly, we have

$$\mathcal{F}(a) = \sum_{\alpha \in \mathrm{Irr}(\mathbb{G})} \sum_{i,j=1}^{n_\alpha} d_\alpha (a_\alpha Q_\alpha)_{j,i} u_{i,j}^\alpha \in C_r(\mathbb{G}).$$

The Haar state h on $\mathbb{G}$ and the left Haar weight $\hat{h}$ on $\widehat{\mathbb{G}}$ are related through the Plancherel theorem, which asserts that for any $a = (a_\alpha)_{\alpha \in \mathrm{Irr}(\mathbb{G})} \in c_{00}(\widehat{\mathbb{G}})$, we have $\hat{h}(a^*a) = h(\mathcal{F}(a)^*\mathcal{F}(a))$.

Let us note the following algebraic properties of the Fourier transform. Recall that for $f \in \mathcal{O}(\mathbb{G})$, $\varphi \in \mathcal{O}(\mathbb{G})^*$ we denote $f * \varphi = (\varphi \otimes \mathrm{id})\Delta(f)$ and $\varphi * f = (\mathrm{id} \otimes \varphi)\Delta(f)$. Then we have, for $a \in c_{00}(\widehat{\mathbb{G}})$, $\varphi \in \mathcal{O}(\mathbb{G})^*$:

$$\varphi * \mathcal{F}(a) = \mathcal{F}(ba) \qquad \text{and} \qquad \mathcal{F}(a) * \varphi = \mathcal{F}(ab^Q),$$

where $b = (\mathrm{id} \otimes \varphi)(V)$ is an algebraic multiplier of $c_{00}(\widehat{\mathbb{G}})$ and $b^Q = QbQ^{-1}$. On the other hand for $a, b \in c_{00}(\widehat{\mathbb{G}})$ we have $\mathcal{F}(a)\mathcal{F}(b) = \mathcal{F}(a \star b)$ where $a \star b$ is the unique element of $c_{00}(\widehat{\mathbb{G}})$ such that $(\hat{h} \otimes \hat{h})(\hat{\Delta}(c)(a \otimes b)) = \hat{h}(c(a \star b))$ for all $c \in c_{00}(\widehat{\mathbb{G}})$. The map $a \otimes b \mapsto a \star b$ defined above is referred to as the *convolution product* on $c_{00}(\widehat{\mathbb{G}})$.

We say that $\widehat{\mathbb{G}}$ is finitely generated when $\mathbb{G}$ is a compact matrix quantum group. Having fixed a generating subset in $\mathrm{Irr}(\mathbb{G})$, we put $p_n = \sum_{|\alpha|=n} p_\alpha \in c_{00}(\hat{\mathbb{G}})$. This is compatible with the notation $p_n \in B(L_2(\mathbb{G}))$ introduced previously, in the sense that we have $\mathcal{F}(p_n a)\xi_0 = p_n \mathcal{F}(a)\xi_0$ for any $n \in \mathbb{N}_0$ and $a \in c_{00}(\hat{\mathbb{G}})$.

2.3 The free orthogonal quantum groups

We now come to the main objects of study in this paper. Let $N \in \mathbb{N}$, $N \geq 2$ and $F \in GL_N(\mathbb{C})$ such that $F\overline{F} = \pm 1$. The *free orthogonal quantum group* is the compact quantum group $O_F^+ = (C(O_F^+), \Delta)$, where

(1) $C(O_F^+)$ is the universal unital C^*-algebra generated by N^2 elements $u_{i,j}$, $1 \leq i,j \leq N$, satisfying the relations making u unitary and $u = (F \otimes 1)u^c(F^{-1} \otimes 1)$, where $u = \left(u_{i,j}\right)_{1 \leq i,j \leq N} \in M_N(\mathbb{C}) \otimes C(O_F^+)$ and $u^c = \left(u_{i,j}^*\right)_{1 \leq i,j \leq N}$.

(2) $\Delta : C(O_F^+) \to C(O_F^+) \otimes C(O_F^+)$ is the unital $*$-homomorphism determined by $\Delta(u_{i,j}) = \sum_{k=1}^N u_{i,k} \otimes u_{k,j}$.

The compact quantum group O_F^+ is a compact matrix quantum group and we choose the *fundamental representation* $u = (u_{i,j})_{i,j} \in B(\mathbb{C}^N) \otimes C(O_F^+)$, coming from the canonical generators of $C(O_F^+)$, as the (unique) generating representation. Then it is known from [1] that for each $k \in \mathbb{N}_0$ there is a unique irreducible representation (up to equivalence) of length k, which is equivalent to its conjugate. We denote this class k, yielding an identification of $\mathrm{Irr}(O_F^+)$ with $\mathbb{N}_0$. We have in particular $u^0 = 1_{C(\mathbb{G})}$ (the *trivial representation*), and $u^1 = u = (u_{i,j}) \in B(H_1) \otimes C(\mathbb{G})$ with $H_1 = \mathbb{C}^N$.

One can check that $Q_1 = F^{\mathrm{tr}}\bar{F}$, so that $d_1 = \mathrm{Tr}(F^*F)$. There exists a unique $q \in (0,1]$ such that $d_1 = q + q^{-1}$ and we denote also $N_q = d_1 = q + q^{-1}$. On the other hand one can see that $\|Q_k\| = \|Q_1\|^k = \|F\|^{2k}$ for all $k \in \mathbb{N}$, and that O_F^+ is of Kac type iff F is unitary. This is typically the case of $F = I_N$ and we denote in this case $O_N^+ := O_{I_N}^+$.

It is, moreover, known that $u^m \mathbin{\bar{\otimes}} u^n$ is unitarily equivalent to $u^{|m-n|} \oplus u^{|m-n+2|} \oplus \cdots \oplus u^{m+n}$. We denote by $P_l = P_l^{m,n}$ the orthogonal projection from $H_m \otimes H_n$ onto H_l for any one of $l = |m-n|, |m-n|+2, \cdots, m+n$. We have in particular $n_0 = 1, n_1 = N$, $n_1 n_{k+1} = n_{k+2} + n_k$ for all $k \in \mathbb{N}_0$ and $d_0 = 1$, $d_1 = N_q := \mathrm{Tr}(F^*F)$, $d_1 d_{k+1} = d_{k+2} + d_k$ for all $k \in \mathbb{N}_0$. Finally, it was also shown by Banica [1] that the *fundamental character* $\chi_1 = \sum_{i=1}^N u_{i,i}$ is a semicircular element (on $[-2,2]$) with respect to the Haar state.

2.4 Property RD and its generalizations

In the case when $\widehat{\mathbb{G}}$ is a classical discrete group Γ, the property of rapid decay amounts to controlling the norm of $C_r(\mathbb{G}) = C_r^*(\Gamma)$ from above by the 2-norm. More precisely a discrete group Γ has Property RD if there exists a polynomial P such that

$$\|x\|_{C_r^*(\Gamma)} \leq P(k)\|x\|_2 \tag{2.2}$$

for all $k \in \mathbb{N}_0$ and all $x \in C_r^*(\Gamma)$ supported on elements of length k in Γ, with respect to some fixed length (for instance a word length if Γ is finitely generated). Note that the reverse inequality $\|x\|_2 \leq \|x\|_{C_r^*(\Gamma)}$ is always true.

A quantum generalization of property RD was introduced in [32] by means of the same inequality (2.2), with appropriate notions of length and support as introduced above. It was shown in the same article that property RD holds for the dual of O_N^+ but fails for the dual of any compact quantum group $\mathbb{G}$ which is not of Kac type. Later, a modification of the quantum definition was proposed in [2] so as to accommodate non-Kac examples such as $SU_q(2)$ and more generally quantum groups $\mathbb{G}$ with (classical) polynomial growth. This modification is obtained by replacing the 2-norm on the right-hand side of (2.2) by a still "easily computable" twisted 2-norm.

In this setting, "easily computable" means a norm of the form $\|f\|_\varphi = \|\varphi \star f\|_2$ or $\|f \star \varphi\|_2$ for $f \in \mathcal{O}(\mathbb{G})$, with $\varphi \in \mathcal{O}(\mathbb{G})^*$ fixed. Using the fact that the Fourier transform is isometric, this can also be written $\|\mathcal{F}(a)\|_\varphi = \|Da\|_2$ or $\|aD^Q\|_2$ for $a \in c_{00}(\hat{\mathbb{G}})$, where $D = (\mathrm{id} \otimes \varphi)(V)$, and these norms can indeed be computed by multiplying matrices and summing their traces. In this picture the twisted property RD takes the form $\|\mathcal{F}(a)\|_{C_r(\mathbb{G})} \leq P(k)\|Da\|_2$ or $\|\mathcal{F}(a)\|_{C_r(\mathbb{G})} \leq P(k)\|aD'\|_2$ if $\mathcal{F}(a)$ is of length k, for some fixed algebraic multiplier D or D' of $c_{00}(\widehat{\mathbb{G}})$. Observe that by polar decomposition one can assume $D > 0$ (resp. $D'\sqrt{Q} > 0$) without changing the associated twisted norm.

Of course one could always achieve such inequalities by taking a central multiplier $D = (b_\alpha I_{H_\alpha})_\alpha$ with weights b_α growing sufficiently rapidly (see, e.g., [31] and the discussion at the beginning of Section 3.1). However, for some applications (e.g., to the metric approximation property [5], and to non-commutative geometry [2]), it is desirable to use "natural" or "optimal" elements D, D'.

We note that the authors of [2] choose the twisted 2-norm in such a way that $\left\{\sqrt{n_\alpha}\, u^\alpha_{i,j}\right\}_{\substack{1 \leq i,j \leq n_\alpha \\ \alpha \in \mathrm{Irr}(\mathbb{G})}}$ forms an orthonormal basis, as it is in the case of Kac-type compact quantum groups. An easy inspection with our conventions shows that the only twisted norm with this property is $\|\mathcal{F}(a)\|_\varphi := \|a\sqrt{C}\|_2$, where

$$C = \left(\frac{d_\alpha}{n_\alpha} Q_\alpha\right)_\alpha \tag{2.3}$$

is the canonical element used in [2] to define their twisted 2-norms. In the following definition we fix a multiplier $D = (D_\alpha)_{\alpha \in \mathrm{Irr}(\mathbb{G})}$ of $c_{00}(\widehat{\mathbb{G}})$, we consider the associated twisted norms $\|a\|_{2,D} := \|aD\|_2$ for $a \in c_{00}(\widehat{\mathbb{G}})$, and we put

$$\|f\|_{2,D} := \|\mathcal{F}^{-1}(f)\|_{2,D} = \|\mathcal{F}^{-1}(f)D\|_2$$

for $f \in \mathcal{O}(\mathbb{G})$. Observe that D is uniquely determined by $\|\cdot\|_{2,D}$ if we assume $D\sqrt{Q} \geq 0$.

Definition 2.1. Let $\mathbb{G}$ be a compact matrix quantum group with a fixed family of generating irreducible representations and D a multiplier of $c_{00}(\widehat{\mathbb{G}})$. We say that $\widehat{\mathbb{G}}$ has

Property RD_D if there exists a polynomial $P \in \mathbb{R}_+[X]$ such that for all $k \in \mathbb{N}_0$ and $f \in \mathcal{O}(\mathbb{G})$ of length k, we have

$$\|f\|_{C_r(\mathbb{G})} \leq P(k)\|f\|_{2,D}.$$

The property above can also be written $\|\mathcal{F}(a)\|_{C_r(\mathbb{G})} \leq P(k)\|a\|_{2,D}$ for all $k \in \mathbb{N}_0$ and $a \in p_k c_{00}(\widehat{\mathbb{G}})$. Explicitly, property RD_D asks that

$$\left\| \sum_{|\alpha|=k} \sum_{i,j=1}^{n_\alpha} d_\alpha (a_\alpha Q_\alpha)_{j,i} u^\alpha_{i,j} \right\|^2_{C_r(\mathbb{G})} \leq P(k)^2 \sum_{|\alpha|=k} d_\alpha \operatorname{Tr}(D_\alpha Q_\alpha D^*_\alpha a^*_\alpha a_\alpha). \tag{2.4}$$

The property RD considered in [2] corresponds to the case $D = \sqrt{C}$, which satisfies $D\sqrt{Q} \geq 0$ since C commutes with Q. If $\mathbb{G}$ is of Kac type, the property $RD_{\sqrt{C}}$ coincides with the property RD in [32]. In particular, if $\widehat{\mathbb{G}}$ is a discrete group Γ, then Property $RD_{\sqrt{C}}$ is exactly same with the property RD of Γ.

We now restate [32, Lemma 4.6] in a slightly more general form. Note that in the case $\mathbb{G} = O^+_F$ equipped with the canonical generating representation, there is only one irreducible representation $\alpha = k \in \mathrm{Irr}(\mathbb{G})$ for each given length k, and the inclusions $u^l \subset u^k \textcircled{\scriptsize T} u^n$ are multiplicity-free.

Lemma 2.2. Let $\mathbb{G}$ be a compact matrix quantum group with a fixed family of generating irreducible representations. For $k, n \in \mathbb{N}_0$ and $\gamma \in \mathrm{Irr}(\mathbb{G})$ we denote

$$\nu^\gamma_{k,n} = \sum_{|\alpha|=k, |\beta|=n} \frac{d_\alpha d_\beta}{d_\gamma} \operatorname{mult}(u^\gamma \subset u^\alpha \textcircled{\scriptsize T} u^\beta). \tag{2.5}$$

Then the discrete quantum group $\widehat{\mathbb{G}}$ has property RD_D with respect to a multiplier D **iff** there exists a polynomial P such that we have, for any $k, l, n \in \mathbb{N}_0$ and for every $a \in p_k c_0(\widehat{\mathbb{G}})$, $b \in p_n c_0(\widehat{\mathbb{G}})$:

$$\sum_{|\gamma|=l} \nu^\gamma_{k,n} \left\| \hat{\Delta}(p_\gamma)(a \otimes b)\hat{\Delta}(p_\gamma) \right\|^2_2 \leq P(k)^2 \left\| aD \otimes b \right\|^2_2. \tag{2.6}$$

Proof. The proof is a straightforward extension of the ideas in the proof of [32, Lemma 4.6] using our notation. Let us recall the main ideas for the convenience of the reader.

First of all, property RD_D is equivalent to the fact that $\|p_l f p_n\|_{C_r(\mathbb{G})} \leq P(k)\|f\|_{2,D}$ for all $k, l, n \in \mathbb{N}_0$ and $f \in \mathcal{O}(\mathbb{G})$ of length k, see [32, Proposition 3.5] and [2, Proposition

3.4]. Using the Fourier transform, this means that we require

$$\| p_l \mathcal{F}(a)\mathcal{F}(b)\xi_0\|_2 \le P(k)\|a\|_{2,D}\|b\|_2 = P(k)\|aD\otimes b\|_2, \tag{2.7}$$

for all a of length k and b of length n. Moreover, we have $\|p_l\mathcal{F}(a)\mathcal{F}(b)\xi_0\|_2 = \|p_l\mathcal{F}(a\star b)\xi_0\|_2 = \|\mathcal{F}(p_l(a\star b))\|_2 = \|p_l(a\star b)\|_2$ — indeed by definition of p_l (in $c_0(\widehat{\mathbb{G}})$ and $B(L_2(\mathbb{G}))$) and of V we have $(1\otimes p_l)V(1\otimes\xi_0) = (p_l\otimes \mathrm{id})V(1\otimes\xi_0)$. Then we can decompose into orthogonal components: $\|p_l(a\star b)\|_2^2 = \sum_{|\gamma|=l}\|p_\gamma(a\star b)\|_2^2$.

Then by definition of the convolution product we can write, for any $c\in c_0(\widehat{\mathbb{G}})$:

$$\begin{aligned}\hat h(c^*p_\gamma(a\star b)) &= (\hat h\otimes\hat h)(\hat\Delta(c)^*\hat\Delta(p_\gamma)(a\otimes b))\\ &= (\hat h\otimes\hat h)((p_k\otimes p_n)\hat\Delta(c)^*\hat\Delta(p_\gamma)(a\otimes b)\hat\Delta(p_\gamma)).\end{aligned}$$

Note that p_γ is central in $c_0(\widehat{\mathbb{G}})$ and that $\hat\Delta(p_\gamma)$ is $(\hat h\otimes\hat h)$-central. To obtain the expression of $\|p_\gamma(a\star b)\|_2^2$, which appears in the left-hand side of (2.6), it remains to take the supremum over $c\in p_\gamma c_0(\widehat{\mathbb{G}})$, with $\|c\|_2\le 1$. We show below that we have in fact $\|(p_k\otimes p_n)\hat\Delta(c)\|_2^2 = \nu_{k,n}^\gamma\|c\|_2^2$, which yields the correct expression $\|p_\gamma(a\star b)\|_2 = (\nu_{k,n}^\gamma)^{1/2}\|\hat\Delta(p_\gamma)(a\otimes b)\hat\Delta(p_\gamma)\|_2$ so that (2.6) results from (2.7).

Indeed we have $\hat\Delta(Q) = Q\otimes Q$ in the multiplier algebra of $c_{00}(\widehat{\mathbb{G}})\otimes c_{00}(\widehat{\mathbb{G}})$, and on the matrix algebra $p_\gamma c_0(\widehat{\mathbb{G}}) = B(H_\gamma)$ the $*$-homomorphism $(p_\alpha\otimes p_\beta)\hat\Delta$ is an amplification with the same multiplicity as the inclusion $u^\gamma\subset u^\alpha \oplus\!\!\!\!\top u^\beta$. Thus, we have

$$\mathrm{Tr}\left((Q\otimes Q)(p_\alpha\otimes p_\beta)\hat\Delta(d)\right) = \mathrm{mult}(u^\gamma\subseteq u^\alpha \oplus\!\!\!\!\top u^\beta)\,\mathrm{Tr}(Qd) \tag{2.8}$$

for any $d\in B(H_\gamma)$. As a result we can write

$$\begin{aligned}(\hat h\otimes\hat h)\left((p_k\otimes p_n)\hat\Delta(p_\gamma d)\right) &= \sum_{|\alpha|=k,|\beta|=n} d_\alpha d_\beta(\mathrm{Tr}\otimes\mathrm{Tr})[(p_\alpha\otimes p_\beta)(Q\otimes Q)\hat\Delta(p_\gamma d)]\\ &= \sum_{|\alpha|=k,|\beta|=n} d_\alpha d_\beta \mathrm{mult}(u^\gamma\subset u^\alpha \oplus\!\!\!\!\top u^\beta)\,\mathrm{Tr}(Qp_\gamma d) = \nu_{k,n}^\gamma\hat h(p_\gamma d).\end{aligned}$$

Taking $d = c^*c$ we obtain $\|(p_k\otimes p_n)\hat\Delta(c)\|_2^2 = \nu_{k,n}^\gamma\|c\|_2^2$ as claimed. ■

3 On Property RD_D for $\widehat{O_F^+}$

In this section we turn our attention to the duals of the free orthogonal quantum groups O_F^+, establishing some necessary conditions for property RD_D to hold for a given multiplier D. In this case property RD_D with respect to a multiplier D and a polynomial P is characterized by the following multiplicity-free version of (2.6):

$$\|\hat{\Delta}(p_l)(a \otimes b)\hat{\Delta}(p_l)\|_2 \leq \sqrt{\frac{d_l}{d_k d_n}} P(k)\|aD_k \otimes b\|_2, \tag{3.1}$$

for all k, l, n such that $u^l \subset u^k \mathbin{⊤} u^n$, $a \in B(H_k)$ and $b \in B(H_n)$.

Here the 2-norms are the ones coming from the weight $\hat{h}$, but one can use as well the twisted Hilbert-Schmidt norms, for example, $\|a\|^2_{\mathrm{HS}} = \mathrm{Tr}(Q_k a^* a)$ for $a \in B(H_k)$, since these two norms only differ by a scalar factor $\sqrt{d_k}$. Moreover, if we fix an isometric intertwiner (unique up to a phase) $v = v_l^{k,n} \in \mathrm{Mor}(u^l, u^k \otimes u^n)$ we have $\|\hat{\Delta}(p_l)(a \otimes b)\hat{\Delta}(p_l)\|_{\mathrm{HS}} = \|vv^*(a \otimes b)vv^*\|_{\mathrm{HS}} = \|v^*(a \otimes b)v\|_{\mathrm{HS}}$ — notice that the last norm is the twisted Hilbert–Schmidt norm on $B(H_l)$.

We can moreover give an explicit form to the intertwiners $v_l^{k,n}$ as follows. For each $n \in \mathbb{N}_0$ the tensor power representation $u^1 \mathbin{⊤} \cdots \mathbin{⊤} u^1$ contains a unique copy of u^n, we choose for H_n the corresponding subspace of $H_1^{\otimes n}$ and we denote $P_n = p_1^{\otimes n}\hat{\Delta}^{n-1}(p_n) \in B(H_1^{\otimes n})$ the corresponding orthogonal projection. We further fix an intertwiner (unique up to a phase) $t_n \in \mathrm{Mor}(1, u^n \mathbin{⊤} u^n)$ such that $\|t_n\| = \sqrt{d_n}$ and we consider the intertwiners $A_l^{k,n} = (P_k \otimes P_n)(\mathrm{id} \otimes t_r \otimes \mathrm{id})P_l \in \mathrm{Mor}(u^l, u^k \otimes u^n)$, where $r = (k + n - l)/2$. One can then take $v_l^{k,n} = \|A_l^{k,n}\|^{-1} A_l^{k,n}$.

The (operator) norm of $A_l^{k,n}$ can be explicitly computed, see for example [32, Lemma 4.8] or [7, Equation (6) and Proposition 3.1]. This norm happens to be controlled from below and above, up to factors depending only the parameter $0 < q < 1$ given by $q + q^{-1} = \mathrm{Tr}(F^*F)$, as follows:

$$\frac{1}{d_r} \leq \|A_l^{k,n}\|^{-2} = \frac{1}{d_r}\prod_{s=1}^{r} \frac{(1-q^{2+2s})(1-q^{2l-2r+2s})(1-q^{2m-2r+2s})}{(1-q^{2k+2+2s})(1-q^{2s})^2}$$
$$\leq \frac{1}{[r+1]_q}\Big(\prod_{s=1}^{r}\frac{1}{1-q^{2s}}\Big)^3 \leq \frac{1}{d_r}\Big(\prod_{s=1}^{\infty}\frac{1}{1-q^{2s}}\Big)^3,$$

where $l = k + n - 2r$. If we put

$$1 < C(q) = \frac{1}{(1-q^2)}\Big(\prod_{s=1}^{\infty}\frac{1}{1-q^{2s}}\Big)^3, \tag{3.2}$$

and use the inequality

$$(1-q^2)^3 \le \frac{d_k d_n}{d_l d_r^2} \le (1-q^2)^{-2},$$

which follows from the dimension formula $d_n = \frac{q^{n+1}-q^{-n-1}}{q-q^{-1}}$, we get

$$(1-q^2)^{3/2}\Big(\frac{d_l}{d_k d_n}\Big)^{1/2} \le \|A_k^{l,m}\|^{-2} \le C(q)\Big(\frac{d_l}{d_k d_n}\Big)^{1/2}. \tag{3.3}$$

Inequality (3.3) shows that $\|A_l^{k,n}\|^{-2}$ compensates exactly for the analogous factor of the right-hand side of the RD inequality (3.1), which can thus be rewritten in the equivalent formulation

$$\|(A_l^{k,n})^*(a\otimes b)A_l^{k,n}\|_{\mathrm{HS}} \le P(k)\|aD_k \otimes b\|_{\mathrm{HS}}. \tag{3.4}$$

In (3.4), it is important to use the twisted Hilbert–Schmidt norms since the matrix spaces are no longer the same on both sides.

Remark. The universal constant $C(q)$ defined in (3.2) will make several appearances in the remainder of the paper.

Note that we have $u^l \subset u^k \mathbin{\textcircled{\scriptsize T}} u^n$ **iff** $l \in \{|n-k|, |n-k+2|, \dots, n+k\}$. One can obtain necessary conditions for property RD_D by fixing the value of l. More specifically, we say that the dual of O_F^+ satisfies property RD_D^0 (resp. $RD_D^{\max}$) for some polynomial P if the above inequality is satisfied for all $k = n \in \mathbb{N}_0$ and $l = 0$ (resp. for all $k, n \in \mathbb{N}_0$ and for $l = k+n$).

In the case of RD_D^0 we have simply $A_l^{k,n} = A_0^{n,n} = t_n : \mathbb{C} \to H_n \otimes H_n$ so that Property RD_D^0, with respect to the polynomial P, is equivalent to the fact that

$$|t_n^*(a\otimes b)t_n| \le P(n)\|aD_n \otimes b\|_{\mathrm{HS}}. \tag{3.5}$$

for all $n \in \mathbb{N}_0$ and $a, b \in B(H_n)$. Note that t_n can be written uniquely as $t_n(1) = \sum_i e_i \otimes j_n(e_i)$, where the anti-linear map $j_n : H_n \to H_n$ does not depend on the chosen orthonormal basis $(e_i)_i$ of H_n, and recall that we have $Q_n = j_n^* j_n$ and $j_n^2 = \pm\,\mathrm{id}$. Then one can compute (recalling that $(\zeta|j\xi) = (\xi|j^*\zeta)$ for an anti-linear map j):

$$t_n^*(a\otimes b)t_n = \mathrm{Tr}(j_n^* b^* j_n a) \tag{3.6}$$

Summing everything up, one can reformulate property RD_D^0 as follows:

Definition 3.1. We say that $\widehat{O_F^+}$ has RD_D^0 if there exists a polynomial P such that

$$\left|t_n^*(a \otimes b)t_n\right| = \mathrm{Tr}(j_n^* b^* j_n a) \leq P(n)\|aD_n \otimes b\|_{\mathrm{HS}} \tag{3.7}$$

for all $a, b \in B(H_n)$ and $n \in \mathbb{N}_0$.

It turns out that property RD_D^0 for the dual of O_F^+ can be explicitly characterized in terms of the matrices D_n (and Q_n), as follows.

Proposition 3.2. The discrete quantum group $\widehat{O_F^+}$ has RD_D^0 with respect to a polynomial P if and only if

$$\|Q_n^{-1/2} D_n^{-1} Q_n\| \|Q_n^{1/2}\| \leq P(n) \textit{ for all } n \in \mathbb{N}. \tag{3.8}$$

Proof. Note that (3.7) can be written as

$$|\,\mathrm{Tr}(b^* a)|^2 \leq P(n)^2 \|aD_n \otimes j_n^* b j_n\|_{\mathrm{HS}}^2 = P(n)^2\,\mathrm{Tr}(D_n^* Q_n D_n a^* a)\,\mathrm{Tr}(Q_n j_n^* b^* j_n j_n^* b j_n).$$

We have, moreover, $D_n^* Q_n D_n = \tilde{D}_n^2$, where $\tilde{D}_n = D_n \sqrt{Q_n}$ is positive, and $\mathrm{Tr}(Q_n j_n^* b^* j_n j_n^* b j_n) = \mathrm{Tr}(j_n j_n^* j_n j_n^* b^* j_n j_n^* b) = \mathrm{Tr}(Q_n^{-1} b^* Q_n^{-1} b Q_n^{-1})$. Now we note that, by the Cauchy–Schwarz inequality (for the untwisted Hilbert–Schmidt scalar product), the maximum of $|\,\mathrm{Tr}(b^* a)|^2 / \mathrm{Tr}(\tilde{D}_n^2 a^* a)$ equals $\mathrm{Tr}(\tilde{D}_n^{-2} b^* b)$, attained at $a = b\tilde{D}_n^{-2}$, so that RD_D^0 is equivalent to

$$\mathrm{Tr}(\tilde{D}_n^{-2} b^* b) \leq P(n)^2\,\mathrm{Tr}(Q_n^{-1} b^* Q_n^{-1} b Q_n^{-1}).$$

Replacing b with $Q_n^{\frac{1}{2}} b Q_n$, the above can be written as

$$\mathrm{Tr}(Q_n \tilde{D}_n^{-2} Q_n b^* Q_n b) \leq P(n)^2\,\mathrm{Tr}(b^* b).$$

We note that, for positive matrices $M, N \in B(H_n)_+$,

$$\max_{b \neq 0} \frac{\mathrm{Tr}(Mb^*Nb)}{\mathrm{Tr}(b^*b)} = \left(\max_{b \neq 0} \frac{\|N^{\frac{1}{2}} b M^{\frac{1}{2}}\|_{HS}}{\|b\|_{HS}}\right)^2$$

equals $\|M\|\|N\|$, attained at $b = \xi\eta^* \in B(H_n)$, where ξ, η are unit vectors chosen to satisfy $\|N^{\frac{1}{2}}\xi\| = \|N^{\frac{1}{2}}\|$ and $\|M^{\frac{1}{2}}\eta\| = \|M^{\frac{1}{2}}\|$. Therefore, RD_D^0 is equivalent to

$$\|Q_n\tilde{D}_n^{-2}Q_n\|\|Q_n\| = \|\tilde{D}_n^{-1}Q_n\|^2\|Q_n^{1/2}\|^2 \leq P(n)^2 \text{ for all } n \in \mathbb{N},$$

which gives us the desired conclusion. ■

Remark. The same techniques apply if one tries to twist the 2-norm from the other side, that is, if one considers inequalities of the form $|t_n^*(a \otimes b)t_n| \leq P(n)\|D_n a \otimes b\|_{\mathrm{HS}}$. Then one arrives at the condition $\|D_n^{-1}Q_n^{1/2}\|\|Q_n^{1/2}\| \leq P(n)$, which is equivalent to the condition of Proposition 3.2 if D and Q commute.

Recall that [2] take $D_k^2 = C_k = \frac{d_k}{n_k}Q_k$ to verify RD_D for $SU_q(2)$ (more generally, the Drinfeld–Jimbo q-deformations G_q). It seems reasonable to consider a continuous family of variants of this multiplier by taking $D_k = (\frac{d_k}{n_k})^{|s|/2}Q_k^{s/2}$ with $s \in \mathbb{R}$. Recalling that $\|Q_k\| = \|Q_k^{-1}\| = \|Q_1\|^k$, we see that in that case Proposition 3.2 reads

$$\left(\frac{n_k}{d_k}\right)^{|s|}\|Q_1\|^{(|1-s|+1)k} \leq P(k)^2 \text{ for all } k \geq 0.$$

However, the following theorem shows that this inequality is not satisfied for any non-Kac O_F^+ as soon as $N \geq 3$.

Theorem 3.3. Let $N \geq 3$, $s \in \mathbb{R}$ and consider the multiplier $D(s) = (D_k)_{k\in\mathbb{N}_0}$, with $D_k = (d_k/n_k)^{|s|/2}\, Q_k^{s/2}$. Then $\widehat{O_F^+}$ has property $RD_{D(s)}^0$ if and only if O_F^+ is of Kac type. In particular, all non-Kac O_F^+ do not have property $RD_{D(s)}$.

Proof. We only need to consider the case where O_F^+ is not of Kac type. That is, we assume $Q_1 \neq I$. Since $\left(\frac{n_k}{d_k}\right)^{|s|}\|Q_1\|^{(|1-s|+1)k} \geq \left(\frac{n_k}{d_k}\|Q_1\|^k\right)^{|s|}$ for all $k \in \mathbb{N}$, it suffices to show that $\frac{n_k}{d_k}\|Q_1\|^k$ has exponential growth, that is,

$$\liminf_{k\to\infty}\left(\frac{n_k\|Q_1\|^k}{d_k}\right)^{\frac{1}{k}} > 1.$$

First of all, we have $\lim_{k\to\infty} d_k^{\frac{1}{k}} = f(d_1)$ and $\lim_{k\to\infty} n_k^{\frac{1}{k}} = f(n_1)$ where $f(t) = \frac{t+\sqrt{t^2-4}}{2}$. Let us denote by $\lambda_1 \leq \cdots \leq \lambda_N$ the eigenvalues of Q_1. Then, since the spectrum of Q_1 is symmetric under inversion, we have $\lambda_1 < 1$ and $\lambda_2 \leq 1$.

Then, in the expansion of $(\lambda_1+\cdots+\lambda_N)^2=\sum_{i,j=1}^N \lambda_i\lambda_j$, we have $\lambda_1^2, \lambda_1\lambda_2, \lambda_2\lambda_1 < 1$, $\lambda_2^2 \le 1$ and the other terms are smaller than λ_N^2. From this observation we obtain

$$d_1^2 < 4 + (N^2-4)\lambda_N^2,$$

which together with the obvious estimate $d_1 < N\lambda_N$ yields

$$f(d_1) = \frac{d_1+\sqrt{d_1^2-4}}{2} < \frac{N\lambda_N+\sqrt{(N^2-4)\lambda_N^2}}{2} = f(n_1)\|Q_1\|.$$

Hence, we have $\liminf_{k\to\infty}\left(\frac{n_k\|Q_k\|}{d_k}\right)^{\frac{1}{k}} = \frac{f(n_1)\|Q_1\|}{f(d_1)} > 1$. ■

Remark. On the other hand, one might try to consider the "opposite" case of $RD_D^{\max}$, which is satisfied iff we have $\|P_{k+n}(a\otimes b)P_{k+n}\|_{\mathrm{HS}} \le P(k)\|aD_k\otimes b\|_{\mathrm{HS}}$ for all $k,n\in\mathbb{N}_0$ and $a\in B(H_k)$, $b\in B(H_n)$. In fact, when D commutes with Q it turns out that $RD_D^{\max}$ is a consequence of RD_D^0, thank to Proposition 3.2.

Indeed we always have $\|P_{k+n}(a\otimes b)P_{k+n}\|_{\mathrm{HS}} \le \|a\otimes b\|_{\mathrm{HS}}$ (using the fact that P_{k+n} commutes with $Q_k\otimes Q_n$), so that $\|a\|_{\mathrm{HS}} \le P(k)\|aD_k\|_{\mathrm{HS}}$ implies $RD_C^{\max}$. Performing the same analysis as in the proof of Proposition 3.2 we see that this stronger condition is equivalent to $\|D_k^{-1}\| \le P(k)$ for all k. On the other hand when D commutes with Q we can write $\|D_k^{-1}\| = \|Q_k^{-1/2}D_k^{-1}Q_kQ_k^{-1/2}\| \le \|Q_k^{-1/2}D_k^{-1}Q_k\|\|Q_k^{-1/2}\|$, which makes the connection with the characterization of RD_D^0 given at Proposition 3.2 since $\|Q_k^{-1/2}\| = \|Q_k^{1/2}\|$.

Remark. The analysis of the above two subcases of Property RD_D leads us to ask whether property RD_D is equivalent to RD_D^0 for $\widehat{O_F^+}$? That is, is property RD_D equivalent to the inequalities $\|Q_k^{-1/2}D_k^{-1}Q_k\|\|Q_k^{1/2}\| \le P(k)$, at least when D and Q commute?

3.1 A weaker variant of property RD

Despite the failure of $RD_{\sqrt{C}}$ for non-amenable, non-Kac-type orthogonal free quantum groups, one can prove a weaker RD inequality (corresponding to a larger multiplier D), which holds for all orthogonal free quantum groups, and also for all discrete quantum groups with polynomial growth. This inequality was already stated (without proof) and used in [31], see Remark 7.6 therein. We provide below a slightly more precise statement and a proof. In the following section, we will see how this weakened property RD is applicable to find *almost* sharp optimal time estimates for ultracontractivity of heat semigroups on O_F^+.

Proposition 3.4. Let $F \in GL_N(\mathbb{C})$ be such that $F\bar{F} = \pm I_N$, $N \geq 2$. Then for any $k, l, n \in \mathbb{N}_0$ and $a \in B(H_k) \subset \ell_\infty(\hat{O}_F^+)$ we have

$$\|p_l \mathcal{F}(a) p_n\| \leq C(q)\|F\|^{2k}\|a\|_2 \quad \text{and} \quad \|\mathcal{F}(a)\| \leq C(q)(k+1)\|F\|^{2k}\|a\|_2,$$

where $C(q) > 1$ is the constant defined by (3.2) for $0 < q < 1$ such that $\mathrm{Tr}(F^*F) = q + q^{-1}$.

Proof. We follow quite closely the proof of [32, Theorem 4.9], taking into account the twisting of Hilbert–Schmidt norms. Starting again from (3.1) and taking into account (3.3) as in the beginning of Section 3 the 1st inequality will follow if we prove

$$\|(A_l^{k,n})^*(a \otimes b)A_l^{k,n}\|_{\mathrm{HS}} \leq \|F\|^{2k}\|a \otimes b\|_{\mathrm{HS}} \tag{3.9}$$

for any $k, l, n \in \mathbb{N}_0$ such that $u^l \subset u^k \textcircled{\scriptsize T} u^n$, $a \in B(H_k)$ and $b \in B(H_n)$. Since P_l is an orthogonal projection, the left-hand side admits $\|(\mathrm{id} \otimes t_r^* \otimes \mathrm{id})(a \otimes b)(\mathrm{id} \otimes t_r \otimes \mathrm{id})\|_{\mathrm{HS}}$ as an evident upper bound, where $r = (k + n - l)/2$ and we are using the twisted Hilbert–Schmidt norm on $B(H_{k-r} \otimes H_{n-r})$. (Note that the projection P_l commutes with the matrix $Q_k \otimes Q_n$ defining the twisting of the Hilbert–Schmidt norm.)

We decompose $a = \sum_i a_i \otimes E_i$ and $b = \sum_i j_r^* E_i j_r \otimes b_i$, where $a_i \in B(H_{k-r})$, $b_i \in B(H_{n-r})$ and $(E_i)_i$ is the basis of matrix units in $B(H_r)$ corresponding to an orthonormal basis of eigenvectors of Q_r in H_r. With this choice we have in particular that $(E_i)_i$ and $(j_r^* E_i j_r)_i$ are orthogonal bases with respect to the twisted Hilbert–Schmidt scalar product on $B(H_r)$, and one can moreover compute, if $E_i = e_p e_q^*$ and e_p, e_q are eigenvectors of Q_r with respect to eigenvalues λ_p, λ_q: $\|E_i\|_{\mathrm{HS}}^2 = \lambda_q$, $\|j_r^* E_i j_r\|_{\mathrm{HS}}^2 = \lambda_q^{-2}\lambda_p^{-1}$. In particular, we note that

$$\|E_i\|_{\mathrm{HS}}^{-2} = \lambda_q \lambda_p \|j_r^* E_i j_r\|_{\mathrm{HS}}^2 \leq \|Q_r\|^2 \|j_r^* E_i j_r\|_{\mathrm{HS}}^2. \tag{3.10}$$

According to (3.6), we can then write

$$\|(\mathrm{id} \otimes t_r^* \otimes \mathrm{id})(a \otimes b)(\mathrm{id} \otimes t_r \otimes \mathrm{id})\|_{\mathrm{HS}} = \|\textstyle\sum_{i,j} \mathrm{Tr}(E_j^* E_i)(a_i \otimes b_j)\|_{\mathrm{HS}} = \|\textstyle\sum_i a_i \otimes b_i\|_{\mathrm{HS}}.$$

Now we apply the triangle inequality and Cauchy–Schwartz inequality:

$$\|(A_l^{k,n})^*(a\otimes b)A_l^{k,n}\|_{\mathrm{HS}}^2 \le \left(\sum_i \|a_i\otimes b_i\|_{\mathrm{HS}}\right)^2 = \left(\sum_i \|a_i\|_{\mathrm{HS}}\|b_i\|_{\mathrm{HS}}\right)^2$$
$$\le \sum_i \|a_i\|_{\mathrm{HS}}^2\|E_i\|_{\mathrm{HS}}^2 \sum_i \|b_i\|_{\mathrm{HS}}^2\|E_i\|_{\mathrm{HS}}^{-2} = \|a\|_{\mathrm{HS}}^2 \sum_i \|b_i\|_{\mathrm{HS}}^2\|E_i\|_{\mathrm{HS}}^{-2}.$$

Finally, we have $\sum_i \|b_i\|_{\mathrm{HS}}^2\|E_i\|_{\mathrm{HS}}^{-2} \le \|Q_r\|^2 \sum_i \|b_i\|_{\mathrm{HS}}^2\|j_r^* E_i j_r\|_{\mathrm{HS}}^2 = \|Q_r\|^2\|b\|_{\mathrm{HS}}^2$ by (3.10). Since $\|Q_r\| = \|Q_1\|^r = \|F\|^{2r} \le \|F\|^{2k}$ we have proved (3.9).

The 2nd inequality in the statement follows from the 1st one by a standard argument, see [32, Proposition 3.5], using the fact that for any n the tensor product $u^k \mathbin{\top} u^n$ has at most $k+1$ irreducible subobjects. ■

Remark. The property above can be interpreted as property RD_D with respect to the *central* multiplier $D = \sum_{k\in\mathbb{N}_0} \|F\|^{2k} p_k = \sum_{k\in\mathbb{N}_0} \|Q_k\| p_k$ and the (constant) polynomial $P = C(q)$. Note that the element C in [2] satisfies $\sqrt{C} \le D$ and thanks to [2, Proposition 4.2] this implies that property RD_D, for this element D, is also satisfied by all discrete quantum groups of polynomial growth, and still reduces to the usual property RD for (classical) discrete groups.

4 Applications: Ultracontractivity and Hypercontractivity of the Heat Semigroup on O_F^+

In this section of the paper, we are interested in studying hypercontractivity and ultracontractivity properties of the heat semigroup $(T_t)_{t>0}$ on the free orthogonal quantum groups O_F^+. This heat semigroup was introduced and studied in [12, 14] in the Kac-type setting (i.e., $F = I_N$), but a standard argument using results from [13, 15] on monoidal equivalences and transference properties of central multipliers allows one to define an appropriate heat semigroup on all free orthogonal quantum groups O_F^+'s. The details of this are spelled out, for example, in [11, Section 6.1].

Let M be a von Neumann algebra equipped with a fixed faithful normal state φ. In the following, a *φ-Markov semigroup on M* will mean a σ-weakly continuous semigroup $(T_t)_{t\ge 0}$ of normal unital completely positive φ-preserving maps $T_t : M \to M$. With a slight abuse of notation, we will identify $M \subset L_2(M)$ as a dense subspace (via the GNS map associated to φ) also denote by $T_t : L_2(M) \to L_2(M)$ the canonical extension of T_t to a contraction on the GNS space $L_2(M)$. The semigroup $(T_t)_{t\ge 0}$ is called *ultracontractive* if there exists some $t_\infty \ge 0$ such that $T_t(L_2(M)) \subset M$ for all $t > t_\infty$. By the closed graph theorem, ultracontractivity is equivalent to that $T_t : L_2(M) \to M$ is bounded for all $t > t_\infty$. We call $(T_t)_{t\geq 0}$ *hypercontractive* if for each $2 < p < \infty$, there exists a $t_p > 0$ such that for all $t \geq t_p$, we have:

$$\|T_t\|_{L_2(M)\to L_p(M)} \leq 1.$$

(In the above, we have used the standard fact [22] that the contractions T_t admit canonical extensions to contractions $T_t : L_p(M) \to L_p(M)$ on the associated non-commutative L_p-spaces for all $p \in [1,\infty)$. We omit the precise details regarding these extensions here because in the following we only consider hypercontractivity in the tracial setting.) In the case of ultracontractive (resp. hypercontractive) semigroups $(T_t)_t$ the *optimal time* t^o_∞ (resp. t^o_p) for ultracontractivity (resp. hypercontractivity) is given by $t^o_\infty = \inf\{t_\infty\}$ (resp. $t^o_p = \inf\{t_p\}$).

Let us now consider the heat semigroup on O^+_F.

4.1 The heat semigroup on O^+_F

Fix $N \in \mathbb{N}$ and $F \in \mathrm{GL}_N(\mathbb{C})$ with $F\bar{F} = \pm 1$. Let $0 < q < 1$ be such that $N_q := \mathrm{Tr}(F^*F) = q + q^{-1}$, and define

$$\lambda(k) = \lambda_q(k) = \frac{U'_k(N_q)}{U_k(N_q)} \qquad (k \in \mathbb{N}_0), \tag{4.1}$$

where U_k is the k-th type-II Chebychev polynomial (defined by $U_0(x) = 1$, $U_1(x) = x$, and $xU_k(x) = U_{k+1}(x) + U_{k-1}(x)$). The *heat semigroup on* O^+_F [11, 14] is the h-Markov semigroup $(T_t)_{t\geq 0}$ on $L_\infty(O^+_F)$ given by

$$T_t\left(u^k_{i,j}\right) = e^{-t\lambda(k)}u^k_{i,j},$$

for all $1 \leq i, j \leq n_k$, and $k \in \mathbb{N}_0$.

Note that we have $\lambda(0) = 0$, $\lambda(1) = 1/N_q$ and, moreover, from the estimates in [14] we have $\frac{k}{N_q} \leq \lambda(k) \leq \frac{k}{N_q-2}$ for all $k \in \mathbb{N}$.

4.2 Ultracontractivity of the heat semigroup on O^+_F

We first consider the ultracontractivity of the heat semigroups. In the tracial case, the ultracontractivity of the heat semigroup *for all time* (with $t_\infty = 0$) is well known and follows from standard tracial property RD estimates. See [14, Theorem 2.1]. In the case of general O^+_F, we show below that ultracontractivity still holds, but generally not for all time.

Proposition 4.1. The heat semigroup $(T_t)_{t\geq 0}$ is ultracontractive for every free orthogonal quantum group O_F^+. Moreover, if t_F is the optimal time for ultracontractivity of the heat semigroup of O_F^+ we have

$$2(N_q - 2)\log \|F\| \leq t_F \leq 2N_q \log \|F\|. \tag{4.2}$$

Proof. First of all, let us suppose that $t > 2N_q \log \|F\|$ and take $f \in L_2(O_F^+)$. Then we can write $f = \sum_{k\geq 0} f_k$, with $f_k \in \text{span}\left\{u_{i,j}^k : 1 \leq i, j \leq n_k\right\}$. Using the exponential form of property RD given by Proposition 3.4, we then have

$$\begin{aligned}
\|T_t(f)\|_\infty &\leq \sum_{k\geq 0} e^{-\lambda(k)t}\|f_k\|_\infty \\
&\leq \sum_{k\geq 0} e^{-\lambda(k)t} C(q)(k+1)\|F\|^{2k}\|f_k\|_2 \\
&\leq C(q)\Big(\sum_{k\geq 0} e^{-2\lambda(k)t}(k+1)^2\|F\|^{4k}\Big)^{1/2}\|f\|_2.
\end{aligned}$$

Hence, the conclusion follows if

$$e^{-2\lambda(k)t}\|F\|^{4k} \leq e^{-Mk}$$

for all $k \in \mathbb{N}_0$ by a universal constant $M > 0$. Indeed, let $M = \frac{2}{N_q}t - 4\log\|F\| > 0$. Then

$$\inf_{k\in\mathbb{N}_0}\left\{\frac{\lambda(k)}{k}t - 2\log\|F\|\right\} \geq \frac{1}{N_q}t - 2\log\|F\| = \frac{M}{2} > 0,$$

which completes the proof.

To prove the stated lower bound, let us assume that $\|T_t(f)\|_{L_\infty(O_F^+)} \leq K\|f\|_{L_2(O_F^+)}$ for a universal constant $K > 0$. Then for any $k \geq 0$ and $1 \leq i, j \leq n_k$ we have

$$\begin{aligned}
e^{-t\lambda(k)}\|(u_{i,j}^k)^*\|_{L_2(O_F^+)} &\leq e^{-t\lambda(k)}\|(u_{i,j}^k)^*\|_{L_\infty(O_F^+)} \\
&= e^{-t\lambda(k)}\|u_{i,j}^k\|_{L_\infty(O_F^+)} \leq K\|u_{i,j}^k\|_{L_2(O_F^+)}.
\end{aligned}$$

On the other hand, using (2.1) it is easy to compute $\|u_{i,j}^k\|_{L_2(O_F^+)}^2 = d_k^{-1}(Q_k^{-1})_{ii}$ and $\|(u_{i,j}^k)^*\|_{L_2(O_F^+)}^2 = d_k^{-1}(Q_k)_{jj}$. Therefore, using a basis for H_k in which Q_k is diagonal we obtain

$$\|F\|^{2k} = \sqrt{\|Q_k\|\|Q_k\|} = \max_{i,j} \frac{\|(u^k_{i,j})^*\|_{L_2(O_F^+)}}{\|u^k_{i,j}\|_{L_2(O_F^+)}} \le K \cdot e^{t\lambda(k)} \le K \cdot e^{\frac{tk}{N_q-2}},$$

which implies $t \ge 2(N_q - 2)\log\|F\| - \frac{(N_q-2)\log(K)}{k}$ for all k. Then, taking the limit $k \to \infty$ gives the desired conclusion. ∎

Remark. A closer examination of the above proof actually shows that $T_t(L_2(O_F^+)) \subset C_r(O_F^+)$ for all $t > t_F$. That is, the heat semigroup on O_F^+ has some additional "smoothing" properties beyond what is guaranteed by ultracontractivity.

Remark. Of course, it is natural to wonder if hypercontractivity holds for the heat semigroups of *all* free orthogonal quantum groups O_F^+. Actually we can show that hypercontractivity is always obtained, although at this time we have no clue for optimal estimates for the time to contraction.

Proposition 4.2. Let $2 < p < \infty$. For sufficiently large t (depending on p), $T_t : L_2(O_F^+) \to L_p(O_F^+)$ is a contraction.

Proof. For any $f \in L_2(O_F^+)$, we have from [29, Theorem 1],

$$\begin{aligned}
\|T_t(f)\|_p^2 &\le \|h(T_t(f))1\|_p^2 + (p-1)\|T_t(f) - h(T_t(f))\|_p^2 \\
&\le |h(f)|^2 + (p-1)\Big(\sum_{n\ge1} e^{-\lambda(n)t}\|f_n\|_p\Big)^2 \\
&\le |h(f)|^2 + (p-1)\Big(\sum_{n\ge1} e^{-\lambda(n)t}\|f_n\|_\infty\Big)^2 \\
&\le |h(f)|^2 + (p-1)\Big(\sum_{n\ge1} e^{-\lambda(n)t}C(q)(n+1)\|F\|^{2n}\|f_n\|_2\Big)^2 \\
&\le |h(f)|^2 + (p-1)\Big(\sum_{n\ge1} e^{-2\lambda(n)t}C(q)^2(n+1)^2\|F\|^{4n}\Big)\|f - h(f)1\|_2^2 \le \|f\|_2^2,
\end{aligned}$$

for all t large enough so that

$$\sum_{n\ge1}(p-1)e^{-2\lambda(n)t}C(q)^2(n+1)^2\|F\|^{4n} \le 1.$$

∎

4.3 Improved hypercontractivity results for O_N^+

For the remainder of the paper we turn our attention to the Kac setting and consider O_N^+. Our aim is to revisit the hypercontractivity results of [14], and obtain some improved estimates (from above and below) on the optimal time to contraction for the heat semigroup $(T_t)_{t\geq 0}$. In the following we let $t_{N,p}$ be the optimal time for $L_2 \to L_p$ hypercontractivity of the heat semigroup on O_N^+.

We begin with a necessary lower bound for $t_{N,p}$.

Lemma 4.3. For each $N \geq 2$ and $2 < p < \infty$, we have

$$t_{N,p} \geq \frac{N}{2}\log(p-1).$$

Proof. Let χ_1 denote the character of the fundamental representation of O_N^+. With $f_a = 1 + a\chi_1 \in L_2(O_N^+)$ and sufficiently small $a > 0$, we have

$$\begin{aligned}(1+a^2)^{\frac{p}{2}} = \|f_a\|^p_{L_2(O_N^+)} \geq \|T_{t_{N,p}}(f_a)\|^p_{L_p(O_N^+)} &= \frac{1}{2\pi}\int_{-2}^{2}(1+ae^{-\frac{t_{N,p}}{N}}x)^p\sqrt{4-x^2}\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+2ae^{-\frac{t_{N,p}}{N}}\sin(\theta))^p\cos^2(\theta)\,\mathrm{d}\theta.\end{aligned}$$

Then, using Taylor expansion up to 2nd order, we can obtain

$$\frac{p}{2} = \lim_{a\searrow 0}\frac{(1+a^2)^{\frac{p}{2}}-1}{a^2} \geq \lim_{a\searrow 0}\frac{\frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+2ae^{-\frac{t_{N,p}}{N}}\sin(\theta))^p\cos^2(\theta)\,\mathrm{d}\theta - 1}{a^2} = \frac{p(p-1)e^{-\frac{2t_{N,p}}{N}}}{2}.$$

Equivalently, we have $t_{N,p} \geq \frac{N}{2}\log(p-1)$. ■

4.3.1 *Khintchine inequalities for $L_p(O_N^+)$*

Our next goal is to establish upper bounds for the optimal time to contraction $t_{N,p}$. To do this, we follow along the lines of [29, Lemma 7], establishing and then exploiting a certain non-commutative Khintchine-type inequality over O_N^+. More precisely, we are interested in finding the optimal constants $K_{m,p} > 1$ such that for all $m \in \mathbb{N}$, $p > 2$, and $f \in \mathcal{O}(\mathbb{G})$ of length m we have

$$\|f\|_{L_p(O_N^+)} \leq K_{m,p}\|f\|_{L_2(O_N^+)}.$$

Theorem 4.4. For O_N^+ we have the following estimates for $K_{m,p}$:

(1) $K_{m,p} \leq (C(q)^2(m+1))^{1-\frac{3}{p}}$ $(4 < p \leq \infty)$,

(2) $K_{m,p} \leq (C(q)^2(m+1))^{\frac{1}{2}-\frac{1}{p}}$ $(2 \leq p \leq 4)$.

Proof. For any admissible triple $(l, m, n) \in \mathbb{N}_0^3$ let $v_l^{m,n} = \|A_l^{m,n}\|^{-1} A_l^{m,n} \in \mathrm{Mor}_{O_N^+}(H_l, H_m \otimes H_n)$ be the isometric intertwiner considered in Section 3. If we repeat the usual RD-type calculations for O_N^+ (e.g., [32, Section 4] or [8, Section 5]), one obtains the following general inequality for the (untwisted) Hilbert–Schmidt norms

$$\left\|(v_l^{m,n})^*(y \otimes z) v_l^{m,n}\right\|_{HS} \leq \|A_l^{m,n}\|^{-2} \|y\|_{HS} \|z\|_{HS} \leq C(q) \Big(\frac{d_l}{d_m d_n}\Big)^{1/2} \|y\|_{HS} \|z\|_{HS}$$

for any $y \in B(H_m)$, $z \in B(H_n)$. Note that the 2nd inequality above follows from (3.3).

We now consider the case $p = \infty$. In this case, we note that the above inequality is exactly the required estimate (3.1) for property RD to hold: it says that $\|p_l f p_n\| \leq C(q)\|f\|_2$ for each $f \in \mathrm{span}\left\{u_{i,j}^m : 1 \leq i, j \leq n_m\right\}$. This implies that $\|f\|_{C_r(O_N^+)} \leq C(q)(m+1)\|f\|_2$ for all $m \in \mathbb{N}_0$ and all $f \in \mathrm{span}\left\{u_{i,j}^m : 1 \leq i, j \leq n_m\right\}$. That is, we have $K_{m,\infty} \leq C(q)(m+1) \leq C(q)^2(m+1)$.

Next, we consider $p = 4$. Now, we define an involution structure $\sharp$ on $B(H_m)$ by $a^\sharp = J_m^{-1} \bar{a} J_m$ for all $a \in B(H_m)$, where J_m is the unique anti-unitary satisfying $(u^m)^c = (J_m \otimes 1) u^m (J_m^{-1} \otimes 1)$. Then, for any $f = \sum_{i,j=1}^{n_m} a_{j,i} u_{i,j}^m \in C_r(O_N^+)$, we have

$$f^* f = \sum_{s=0}^{m} \sum_{i,j=1}^{n_m} [(v_{2s}^{m,m})^* (a^\sharp \otimes a) v_{2s}^{m,m}]_{j,i} u_{i,j}^{2s}.$$

Thus,

$$\begin{aligned}
\|f\|_{L_4(O_N^+)}^4 = \|f^* f\|_{L_2(O_N^+)}^2 &= \sum_{s=0}^{m} \frac{1}{d_{2s}} \left\|(v_{2s}^{m,m})^* (a^\sharp \otimes a) v_{2s}^{m,m}\right\|_{HS}^2 \\
&\leq \sum_{s=0}^{m} \frac{C(q)^2 d_{2s}}{d_{2s} d_m^2} \|a^\sharp\|_{HS}^2 \|a\|_{HS}^2 \\
&= C(q)^2 \frac{\|a\|_{HS}^4}{d_m^2} (m+1) \\
&= C(q)^2 (m+1) \|f\|_2^4.
\end{aligned}$$

Thus, $K_{m,4} \leq \sqrt{C(q)}(m+1)^{1/4}$. The rest of the proof now follows from complex interpolation theorem and our estimates for $K_{m,\infty}, K_{m,4}$. ■

Remark. The above bound for $K_{m,4}$ is essentially optimal, since $\|\chi_m\|_4 = (m+1)^{1/4}$ (the 4th moment of the mth type II Chebychev polynomial).

4.3.2 *Applications to improved optimal time estimates*

Theorem 4.5. Let $p \geq 4$ and $c_p = 1 + \frac{4}{\log(p-1)}$. Then we have

$$t_{N,p} \leq \frac{c_p N}{2}\log(p-1) + \left(1 - \frac{3}{p}\right) \cdot 2N\log(C(q)).$$

Proof. By the non-commutative martingale convexity inequality of [29, Theorem 1] and our Theorem 4.4, we have

$$\begin{aligned}
\|T_t(f)\|^2_{L_p(O_N^+)} &\leq h(f)^2 + (p-1)\|\sum_{k\geq 1} e^{-t\lambda(k)} f_k\|^2_{L_p(O_N^+)} \\
&\leq h(f)^2 + (p-1)\left(\sum_{k\geq 1} e^{-t\lambda(k)} C(q)^{2(1-\frac{3}{p})}(1+k)^{1-\frac{3}{p}}\|f_k\|_{L_2(O_N^+)}\right)^2 \\
&\leq h(f)^2 + (p-1)\left(\sum_{k\geq 1} e^{-2t\lambda(k)} C(q)^{4(1-\frac{3}{p})}(1+k)^{2-\frac{6}{p}}\right)\left(\sum_{k\geq 1}\|f_k\|^2_{L_2(O_N^+)}\right)
\end{aligned}$$

for any $f \in L_p(O_N^+)$ and $p \geq 4$.

Note that, for any $c \geq 1$, the assumption $t \geq \frac{cN}{2}\log(p-1) + 2N(1-\frac{3}{p})\log(C(q))$ implies

$$\begin{aligned}
t &\geq \frac{cN}{2}\log(p-1) + \frac{2N}{k}\left(1-\frac{3}{p}\right)\log(C(q)) \\
&\geq \frac{ck}{2\lambda(k)}\log(p-1) + \frac{2}{\lambda(k)} \cdot \left(1-\frac{3}{p}\right)\log(C(q)).
\end{aligned}$$

Here, the 2nd inequality results from the estimate $\lambda(k) \geq \frac{k}{N}$, where the $\lambda(k)$ are the coefficients (4.1) used in the definition of the generalized heat semigroup. Thus, we can write $e^{-2t\lambda(k)}C(q)^{4(1-\frac{3}{p})} \leq (p-1)^{-ck}$. Now, let us try to find $c \geq 1$ satisfying

$$\phi(c) := \sum_{k\geq 1}(p-1)^{1-ck}(1+k)^{2-\frac{6}{p}} \leq 1.$$

To do this, we will use the following estimation

$$\begin{aligned}\phi(c) &\leq \sum_{k\geq 1}(p-1)^{1-ck}(1+k)^2 \\ &= (1-(p-1)^{-c})^{-3}(p-1)(4(p-1)^{-c}-3(p-1)^{-2c}+(p-1)^{-3c}) =: \psi(c).\end{aligned}$$

By setting $t = \frac{1}{1-(p-1)^{-c}}$, the problem to find $c \geq 1$ satisfying $\psi(c) \leq 1$ becomes equivalent to solve the following inequality

$$2t^3 - t^2 - 1 \leq \frac{1}{p-1} \Leftrightarrow t^2(2t-1) \leq \frac{p}{p-1}.$$

Now, our claim is that the above inequality holds at $t = 1 + \frac{a}{p-1}$ with $a = \frac{1}{24}$. Indeed, since $(1+x)^3 \leq 1 + 3x(1+x)^2$ for all $x > 0$, we have

$$\begin{aligned}\left(1+\frac{a}{p-1}\right)^2 \cdot \left(1+\frac{2a}{p-1}\right) &\leq \left(1+\frac{2a}{p-1}\right)^3 \\ &\leq 1 + \frac{6a}{p-1}\cdot\left(1+\frac{2a}{p-1}\right)^2 \leq 1 + \frac{24a}{p-1} = \frac{p}{p-1}.\end{aligned}$$

Therefore, we can see that $\phi(c) \leq \psi(c) \leq 1$ at $c = 1 + \frac{\log(24-\frac{23}{p})}{\log(p-1)}$. Lastly, since $c_p = 1 + \frac{4}{\log(p-1)} \geq 1 + \frac{\log(24)}{\log(p-1)} \geq c$ and ϕ is decreasing, we have $\phi(c_p) \leq \phi(c) \leq 1$. ■

Theorem 4.5 sharpens [14, Theorem 2.6] in the case when

$$1 + \frac{4}{\log(p-1)} \leq \frac{2\log(1+\sqrt{3})}{\log(3)} \approx 1.8297,$$

that is, when $p \geq 125.1085$ approximately. However, even for $2 \leq p \leq 125.1085$, we can obtain an improved time to contractivity:

Theorem 4.6. Let $c = \frac{9}{8}\log(2) + 1 \approx 1.7798$ and $p \geq 4$. Then

$$\|T_t(f)\|_{L_p(O_N^+)} \leq \|f\|_{L_2(O_N^+)}$$

for all $t \geq \frac{cN}{2}\log(p-1) + (1-\frac{3}{p})\cdot 2N\log(C(q))$.

Proof. In the proof of Theorem 4.5, let $\phi_k(p) = (p-1)^{1-ck}(1+k)^{2-\frac{6}{p}}$. Then

$$\phi_k'(p) = (p-1)^{-ck}(1+k)^{2-\frac{6}{p}}((1-ck) + \frac{6(p-1)}{p^2}\log(1+k)).$$

Let us suppose that $1 \le c \le 2$ and consider functions $f(p) = \frac{6(p-1)}{p^2}$ and $g(k) = \frac{ck-1}{\log(1+k)}$. Then it is easy to check that $g'(k) \ge 0$ for all $k \ge 1$ and $f'(p) = 6p^{-3}(2-p)$.

Since $f(4) = \frac{9}{8} \le g(1) = \frac{c-1}{\log(2)}$ for all $c \ge \frac{9}{8}\log(2) + 1 \approx 1.7798$, the function ϕ_k is decreasing on $[4, \infty)$ for each $k \ge 1$. Therefore, $\phi = \sum_{k\ge1} \phi_k$ is decreasing on $[4,\infty)$ and

$$\begin{aligned}\phi(4) &= 3\sum_{k\ge1} 3^{-ck}(1+k)^{\frac{1}{2}} \\ &\le 3\left(\sum_{k\ge1} 3^{-ck}(1+k)\right)^{\frac{1}{2}}\left(\sum_{k\ge1} 3^{-ck}\right)^{\frac{1}{2}} = 3(1-3^{-c})^{-\frac{3}{2}}3^{-c}(2-3^{-c})^{\frac{1}{2}} \le 1\end{aligned}$$

if and only if $3^{-c} \le X_0$, where X_0 is the 2nd largest solution of the equation $8X^3 - 15X^2 - 3X + 1 \ge 0$. Hence, $\phi(p) \le 1$ for all $p \ge 4$ whenever $c \ge -\log_3(X_0) \approx 1.547326$. ■

In [29], their $L_p - L_2$ Khintchine inequalities were used in conjunction with a clever choice of conditional expectation onto the subalgebra generated by a semicircular system to find the optimal time $t_{N,p}$ for heat semigroups of free groups. However, it is not clear what would be the right choice of subalgebra to play the same game for the free orthogonal quantum groups O_N^+. Nevertheless, our Khintchine inequalities (Theorem 4.4) enable us to get an almost optimal time to contraction under the additional assumption that $h(fu_{i,j}) = 0$ for all $1 \le i, j \le N$:

Theorem 4.7. Let $N \ge 3$ and $p \ge 4$. Then the following inequality

$$\|T_t(f)\|_{L_p(O_N^+)} \le \|f\|_{L_2(O_N^+)}$$

holds

(1) if $f \in L_2(O_N^+)$ satisfies $h(fu_{i,j}) = 0$ for all $1 \le i, j \le N$ and
(2) if $t \ge \frac{N}{2}\log(p-1) + (1-\frac{3}{p}) \cdot 2N\log(C(q))$.

Proof. By repeating the proof of Theorem 4.5, since $c = 1$ and $f_1 = 0$, the calculation can be distilled to show

$$\phi(p) = \sum_{k\geq 2}(p-1)^{1-k}(1+k)^{2-\frac{6}{p}} \leq 1 \text{ for all} p \geq 4.$$

It is easy to check that $\phi_k(p) = (p-1)^{1-k}(1+k)^{2-\frac{6}{p}}$ is a decreasing function for any $k \geq 3$ and $\sup_{p\geq 4}\phi_2(p) = \sup_{p\geq 4}\left\{(p-1)^{-1}3^{2-\frac{6}{p}}\right\} \approx 0.60348$. Thus,

$$\begin{aligned}\phi(p) &\leq \sup_{p\geq 4}\phi_2(p) + \sum_{k\geq 3} 3^{1-k}\sqrt{1+k} \\ &= \sup_{p\geq 4}\phi_2(p) + 9\sum_{k\geq 4} 3^{-k}\sqrt{k} \\ &\leq \sup_{p\geq 4}\phi_2(p) + 9\left(\sum_{k=4}^{6} 3^{-k}(\sqrt{k}-k) + \sum_{k\geq 4} k3^{-k}\right) \\ &\approx 0.60348 + 0.38158 < 1.\end{aligned}$$

■

Based on the above results, we are led to make the following conjecture on the asymptotic behavior of the optimal time-to-contraction for the heat semigroups:

Conjecture 4.8. The optimal time to L_p-hypercontractivity for O_N^+ should be of the form

$$t_{N,p} = \frac{N}{2}\log(p-1) + \epsilon_N \quad \text{with} \quad \lim_{N\to\infty} \epsilon_N = 0. \tag{4.3}$$

Remark. The conjecture above is motivated by the following observations:

(1) We have $t_{N,p} \geq \frac{N}{2}\log(p-1)$ (Lemma 4.3).

(2) There exists $c \approx 1.83297$ such that $t_{N,p} \leq \frac{cN}{2}\log(p-1) + \epsilon_N$ for all $p \geq 4$, with $\epsilon_N \to 0$ [14, Theorem 2.6].

(3) The above c can be sharpened to $\frac{9}{8}\log(2) + 1 \approx 1.7798$ for all $p \geq 4$ (Theorem 4.6).

(4) Let c_p be the best constant c for fixed $p \geq 4$. Then $\lim_{p\to\infty} c_p = 1$ (Theorem 4.5).

(5) The constant c can be chosen to be 1 under the additional condition that $h(fu_{i,j}) = 0$ for all $1 \leq i, j \leq N$ (Theorem 4.7).

In the case of duals of discrete groups we have $t_p = \frac{1}{2}\log(p-1)$ for the Poisson semigroup on $\mathbb{T}^N$ (Weissler and Bonami's induction trick), on the dual of $\mathbb{Z}_2^{*N}$ [21] and on the dual of $\mathbb{F}_N = \mathbb{Z}^{*N}$ [20, 29].

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