Mechanical Systems and Signal Processing 134 (2019) 106340

Contents lists available at ScienceDirect

Mechanical Systems and Signal Processing

journal homepage: www.elsevier.com/locate/ymssp

An assessment and extension of geometrically nonlinear beam theories

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ARTICLE INFO

Article history:
Received 1 May 2019
Received in revised form 19 July 2019
Accepted 29 August 2019
Available online 14 September 2019

Keywords:
Nonlinear structures
Strain energy
Variational mechanics
Beam theories

ABSTRACT

Equations of motion for beams assuming only longitudinal, normal stress are derived for large-amplitude beam deflection. The formulation includes various physical extensibility conditions without expansion or truncation of their nonlinear nature. The Lagrange multiplier for an inextensible beam model is explicitly written. The limits of the applicability of various beams models with purely longitudinal, normal stress are explored, and the validity of a pure transverse displacement for a beam with fixed ends is quantified.

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1. Introduction

Modeling beam dynamics requires context for efficient analysis. The geometric and interfacial constraints on the system in question as well as the severity of its excitation inform decisions about which internal stresses to consider. This is the reason for various beam theories (including Euler-Bernoulli and Timoshenko) and modeling techniques (such as strain energy formulations in Ref. [22] and corotational approaches in Ref. [3]). The need to model different forms of shear stress distributions through the thickness of a beam presents even more options, as discussed in Ref. [19]. For example, the transverse deflection of a thin and long, uniform beam experiencing light excitation may be modeled by normal stress in the longitudinal direction and neglecting nonlinear terms (see Refs. [17,8,1]). A cantilever beam excited such that its transverse response amplitude is moderate may be modeled with normal stress including third-order nonlinearities, such as in Refs. [22,14,20,9,4,6] or even fifth order nonlinearities as shown in Ref. [12]. Beams with fixed ends experience a tension nonlinearity, particularly prevalent in the buckling literature (e.g. Refs. [20,13,16,24]), that may also be captured within the limitations of normal stress. At large amplitudes and for thick beams, neglecting shear (the condition which allows for the strain energy to be written exclusively in terms of the normal stress) is no longer a viable approximation. In this more general case, the deformation and motion of a differential element must include shear effects (see, for example Refs. [23,10]).

The common thread among all of these cases is approximation. Modeling the bending of beams inherently generates intricate terms in the system strain energy, and distilling those into equations of motion can be a challenge. In most cases, these terms are expanded relative to the first spatial derivative of displacement variables (usually assumed small compared to 1). Then generating equations of motion through energy methods is fairly straightforward. See Refs. [11,10] for thorough summaries of in-plane motions with and without shear including third order nonlinearities.

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https://doi.org/10.1016/j.ymssp.2019.106340
0888-3270/Published by Elsevier Ltd.

Here we simplify the procedure for arriving at governing equations for in-plane motions even further and show that, without assumptions or neglecting terms, Hamilton's principle may be used to derive the equations of motion for a large-amplitude beam response to any order, at least in principle. Note that the model included here applies to scenarios where normal stress is dominant, and does not consider shear. See Refs. [2,19,5] for studies showing how beam thickness drives the necessity of including shear, and how thin beams ($\frac{2h}{l} \ll 1$ where h is thickness and l is length) incur little error by neglecting shear effects. In the following sections we will (1) illustrate that PDEs predicting the motion of beams modeled with only normal stress require no truncation and (2) show how this provides opportunities to write explicit expressions for Lagrange multipliers and constraint forces through different formulations of the problem using inextensibility as an example. (3) We discuss scenarios where the ends of a beam are fixed, and inaccuracies associated with assuming that longitudinal displacement is zero.

2. Modeling

Consider a differential element of length dx, height dy, and depth dz in a straight, uniform beam. Further assume that only longitudinal stresses influence the beam's motion. The strain energy density from normal stress for this element may be written as

$$u_E = \frac{1}{2}\sigma_{xx}\epsilon_{xx} \tag{1}$$

where σ_{xx} is the normal stress in the axial direction, and ϵ_{xx} is the corresponding strain. Integrating throughout the volume of the beam and noting $\sigma_{xx} = E\epsilon_{xx}$, the total strain energy is

$$U = \frac{1}{2}\int_V \frac{\sigma_{xx}^2}{E} dV \tag{2}$$

where E is the elastic modulus. Note σ_{xx} can be written in terms of internal tensile force N_x, moment M, cross-sectional area A and area moment of inertia I,

$$\sigma_{xx} = \frac{N_x}{A} - \frac{My}{I} \tag{3}$$

where

$$N_x(x) = EA\epsilon \tag{4}$$

$$M(x) = EI\frac{d\theta}{ds} \tag{5}$$

Now, from Eq. (2), the expression for strain energy may be written

$$U = \frac{1}{2}\int_0^l \int_A \frac{1}{E}\left(\frac{N_x}{A}\right)^2 - 2\frac{N_x My}{EAI} + \frac{1}{E}\left(\frac{My}{I}\right)^2 dA\ dx \tag{6}$$

where l is the unstretched length of the beam. Due to the definition of the neutral axis where $\int_A y dA = 0$, this reduces to

$$U = \frac{1}{2}\int_0^l \frac{M^2}{EI} dx + \frac{1}{2}\int_0^l \frac{N_x^2}{EA} dx \tag{7}$$

and then from Eqs. (4) and (5),

$$U = \frac{1}{2}\int_0^l EI\left(\frac{d\theta}{ds}\right)^2 dx + \frac{1}{2}\int_0^l EA\epsilon^2\ dx \tag{8}$$

To proceed, we require expressions for the curvature and strain of the neutral axis, as the moment and tension depend on these characteristics directly. Consider a differential length of the neutral axis of the beam, shown in Fig. 1. From this figure, we may infer that

$$\epsilon = \frac{\Delta s - \Delta x}{\Delta x} = \frac{\sqrt{\Delta w^2 + (\Delta x + \Delta u)^2} - \Delta x}{\Delta x} \tag{9}$$

$$\theta = \arctan\frac{\Delta w}{\Delta x + \Delta u} \tag{10}$$

and in the limit where $\Delta x \to 0$,

$$\epsilon = \frac{ds}{dx} - 1 = \sqrt{(w')^2 + (1 + u')^2} - 1 \tag{11}$$

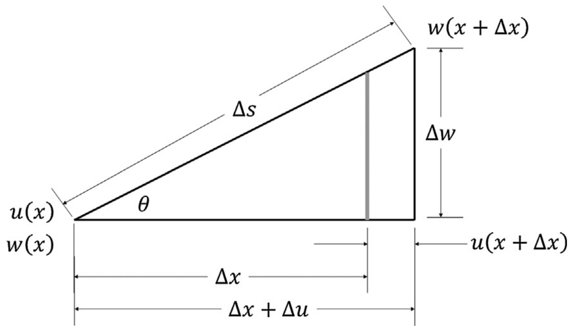

Fig. 1. Differential element of a beam's neutral axis experiencing bending and tension.

$$\theta = \arctan \frac{w'}{1+u'} \tag{12}$$

where a prime indicates a derivative with respect to x. Consequently, the curvature may be written as

$$\frac{d\theta}{ds} = \frac{d\theta}{dx}\frac{dx}{ds} = \frac{(1+u')w'' - w'u''}{\left[(w')^2 + (1+u')^2\right]^{\frac{3}{2}}} \tag{13}$$

2.1. Energy

We nondimensionalize the energy in the system according to the procedure in Appendix A. Note that a prime now indicates a derivative with respect to η, where η is the nondimensional independent space variable $\eta = \frac{x}{l}$. Omitting the tilde notation for simplicity,

$$V = \frac{1}{2}\int_0^1 \frac{[(1+u')w'' - w'u'']^2}{\left[(w')^2 + (1+u')^2\right]^3} d\eta + \frac{1}{2}\int_0^1 \gamma\left[\sqrt{(w')^2 + (1+u')^2} - 1\right]^2 d\eta \tag{14}$$

where $\gamma = \frac{Al^2}{I}$. It is important to recognize that γ is typically a very large value. For our prismatic rectangular beam, $\gamma = 12\left(\frac{l}{h}\right)^2$. The ratio of the length of the beam l to its thickness h is already a large value, and γ is an order larger than the square of that ratio. Thus the potential energy stored in the longitudinal strain should not be underestimated.

The arguments of these integrals are quite intricate, but we may leverage that both u' and w' are small relative to 1 outside of extreme cases. Rearrange the arguments of these integrals so they read

$$V = \frac{1}{2}\int_0^1 \frac{[(1+u')w'' - w'u'']^2}{\left[\frac{(w')^2}{(1+u')^2} + 1\right]^3 (1+u')^6} d\eta + \frac{1}{2}\int_0^1 \gamma\left[(1+u')\sqrt{\frac{(w')^2}{(1+u')^2} + 1} - 1\right]^2 d\eta \tag{15}$$

The potential energy may now be written (via binomial expansions[1]) as polynomials, where

$$\begin{aligned} V = \tfrac{1}{2}\textstyle\int_0^1 \quad & [(1+u')w'' - w'u'']^2\left[(1+u')^4 - 3(w')^2(1+u')^2 + 6(w')^4 + O\left(\epsilon_w^5\right)\right] \\ & \cdot\left[1 - 10u' + 55(u')^2 - 220(u')^3 + 715(u')^4 + O\left(\epsilon_u^5\right)\right] d\eta \\ & + \tfrac{1}{2}\textstyle\int_0^1 \gamma\Big(\left[1 - 3u' + 6(u')^2 - 10(u')^3 + 15(u')^4 + O\left(\epsilon_u^5\right)\right] \\ & \cdot\left[(1+u')^4 + \tfrac{1}{2}(w')^2(1+u')^2 - \tfrac{1}{8}(w')^4 + O\left(\epsilon_w^5\right)\right] - 1\Big)^2 d\eta \end{aligned} \tag{16}$$

where ϵ_w is the order of $\frac{w'}{1+u'}$ and ϵ_u is the order of u'. Though elaborate, this expansion maintains polynomial nonlinearities in the energy expression, and that offers truncation and error estimation early in the analysis process. It also lends itself to Galerkin decomposition later, where polynomial nonlinearities allow for the exponential forms of the linear modes to be combined and reduced.

[1] See Appendix B for the intermediate steps of the expansion.

2.1.1. Scaling approximation

Note that in many cases u' is regarded as on the order of $(w')^2$, which would allow for a greater simplification of the expanded potential energy, as in Ref. [22,6,15]. To demonstrate this approximation, consider an expansion of Eq. (11) as follows.

$$\begin{aligned}\epsilon &= \sqrt{(w')^2 + (1+u')^2} - 1 \\ &= \left(1 + (w')^2 + 2u' + (u')^2\right)^{1/2} - 1 \\ &= \tfrac{1}{2}\left((w')^2 + 2u' + (u')^2\right) - \tfrac{1}{4}\left((w')^2 + 2u' + (u')^2\right)^2 + O\left[\left((w')^2 + 2u' + (u')^2\right)^3\right]\end{aligned}$$

Note that w' and u' are, in most cases, both considered small with respect to 1, and only the lowest order terms need to be retained. Therefore, u' is regarded as on the order of $(w')^2$ and any terms of order $(w')^3, (u')^2$, or higher are discarded. As a result, ϵ may be approximated as

$$\epsilon \approx \frac{1}{2}(w')^2 + u' \tag{17}$$

This approximation will be used in Section 2.4 to discuss the validity of assuming $u = 0$ for fixed ends. However, it is important to note that neglecting the strain contribution from the $\frac{1}{2}(u')^2$ incurs error in higher-order nonlinear terms. Quantifying this error for various extensibility conditions is left to future work.

2.2. General extensibility

Note that Eq. (14) can be used in Hamilton's principle explicitly. Using the kinetic energy in the system,

$$T = \frac{1}{2}\int_0^1 \dot{u}^2 + \dot{w}^2 d\eta \tag{18}$$

and constructing the Lagrangian

$$L = T - V \tag{19}$$

Applying Hamilton's Principle,

$$\begin{aligned}\delta \int L dt = \quad & \int \int_0^1 \dot{u}\delta\dot{u} + \dot{w}\delta\dot{w} \\ & -\Bigg(\tfrac{(1+u')w'' - w'u''}{\left[(w')^2 + (1+u')^2\right]^4}\left[3w'(1+u')u'' - \left(2(1+u')^2 - (w')^2\right)w''\right] \\ & +\gamma\left[1 - \tfrac{1}{\sqrt{(w')^2+(1+u')^2}}\right][1+u']\Bigg)\delta u' \\ & +\Bigg(\tfrac{(1+u')w'' - w'u''}{\left[(w')^2 + (1+u')^2\right]^4}\left[3w'(1+u')w'' + \left((1+u')^2 - 2(w')^2\right)u''\right] \\ & -\gamma\left[1 - \tfrac{1}{\sqrt{(w')^2+(1+u')^2}}\right]w'\Bigg)\delta w' \\ & +\tfrac{(1+u')w'' - w'u''}{\left[(w')^2 + (1+u')^2\right]^3}w'\delta u'' \\ & -\tfrac{(1+u')w'' - w'u''}{\left[(w')^2 + (1+u')^2\right]^3}[1+u']\delta w'' d\eta dt\end{aligned} \tag{20}$$

For simplicity, we rewrite this as

$$\delta \int L dt = \int \int_0^1 \dot{u}\delta\dot{u} + \dot{w}\delta\dot{w} - \chi_{u'}\delta u' + \chi_{w'}\delta w' + \chi_{u''}\delta u'' - \chi_{w''}\delta w'' d\eta dt \tag{21}$$

Integrating by parts and collecting terms, we find the system of equations of motion

$$\ddot{u} - \left[\chi_{u'} + \chi'_{u''}\right]' = 0 \tag{22}$$

$$\ddot{w} + \left[\chi_{w'} + \chi'_{w''}\right]' = 0 \tag{23}$$

subject to the boundary conditions

$$\chi_{u''}\delta u'\big|_0^1 = 0 \tag{24}$$

$$\chi_{w''}\delta w'\big|_0^1 = 0 \tag{25}$$

$$(\chi_{u'} + \chi'_{u''})\delta u\big|_0^1 = 0 \tag{26}$$

$$(\chi_{w'} + \chi'_{w''})\delta w\big|_0^1 = 0 \tag{27}$$

where

$$\chi_{u''} = \frac{(1+u')w'' - w'u''}{\left[(w')^2 + (1+u')^2\right]^3} w' \tag{28}$$

$$\chi_{w''} = \frac{(1+u')w'' - w'u''}{\left[(w')^2 + (1+u')^2\right]^3}[1+u'] \tag{29}$$

$$\chi_{u'} = \frac{(1+u')w''-w'u''}{\left[(w')^2+(1+u')^2\right]^4}\left[3w'(1+u')u'' - \left(2(1+u')^2 - (w')^2\right)w''\right] + \gamma\left[1 - \frac{1}{\sqrt{(w')^2+(1+u')^2}}\right][1+u'] \tag{30}$$

$$\chi_{w'} = \frac{(1+u')w''-w'u''}{\left[(w')^2+(1+u')^2\right]^4}\left[3w'(1+u')w'' + \left((1+u')^2 - 2(w')^2\right)u''\right] - \gamma\left[1 - \frac{1}{\sqrt{(w')^2+(1+u')^2}}\right]w' \tag{31}$$

2.3. Inextensibility

Let us consider the condition of inextensibility, used when modeling beams with one or more free boundary conditions. For inextensible beams, the normal strain is identically zero, meaning

$$\epsilon = \sqrt{(w')^2 + (1+u')^2} - 1 \equiv 0 \tag{32}$$

$$1 = (w')^2 + (1+u')^2 \tag{33}$$

where Eq. (33) is known as the *inextensibility constraint equation*.

Eq. (14) may be significantly simplified, as the only remaining strain energy is in bending (there is no internal tensile force N_x). Note that by taking the derivative of the constraint with respect to η,

$$u'' = -\frac{w'w''}{1+u'} \tag{34}$$

Potential energy may now be written as

$$V = \frac{1}{2}\int_0^1 \left[(1+u')w'' + \frac{(w')^2 w''}{1+u'}\right]^2 d\eta \tag{35}$$

Finding a common denominator and recognizing that $(1+u')^2 = 1-(w')^2$,

$$V = \frac{1}{2}\int_0^1 \frac{(w'')^2}{1-(w')^2} d\eta \tag{36}$$

This equation (prevalent in the literature, e.g. Ref. [22]) is particularly convenient as the potential energy no longer depends explicitly on u or its derivatives. Including the inextensibility constraint in the kinetic energy directly is more of a challenge, and it is more convenient to construct the Lagrangian with a Lagrange multiplier to enforce the constraint (as in the work of da Silva [21]) such that

$$L = T - V + \int_0^1 \lambda f d\eta \tag{37}$$

where

$$f = (w')^2 + (1+u')^2 - 1 \tag{38}$$

Again, using Hamilton's principle, the PDEs coupled through the Lagrange multiplier are

$$\ddot{w} + \left[\frac{w'''}{1-(w')^2} + \frac{w'(w'')^2}{\left[1-(w')^2\right]^2} + \lambda w'\right]' = 0 \tag{39}$$

$$\ddot{u} + [\lambda(1+u')]' = 0 \tag{40}$$

subject to

$$\frac{w''}{1-(w')^2}\delta w'\Big|_0^1 = 0 \tag{41}$$

$$\left[\frac{w'''}{1-(w')^2}+\frac{w'(w'')^2}{\left[1-(w')^2\right]^2}+\lambda w'\right]\delta w\Big|_0^1 = 0 \tag{42}$$

$$\lambda(1+u')\delta u|_0^1 = 0 \tag{43}$$

$$(w')^2-1+(1+u')^2=0 \tag{44}$$

What if the inextensibility condition is applied after determining the PDEs for the fully extensible case? Here,

$$\chi_{u''}=\frac{w'w''}{1+u'} \tag{45}$$

$$\chi_{w''}=w'' \tag{46}$$

$$\chi_{u'}=-2\frac{(w'')^2}{1+u'} \tag{47}$$

$$\chi_{w'}=3w'(w'')^2-\frac{w'\left(1-3(w')^2\right)(w'')^2}{1-(w')^2} \tag{48}$$

The PDE system may now be written as

$$\ddot{w}+\left[3w'(w'')^2-\frac{w'\left(1-3(w')^2\right)(w'')^2}{1-(w')^2}+w'''\right]'=0 \tag{49}$$

$$\ddot{u}+\left[-2\frac{(w'')^2}{1+u'}+\frac{(w'')^2+w'w'''\left(1-(w')^2\right)}{(1+u')^3}\right]'=0 \tag{50}$$

which suggests that the inextensibility constraint force as a function of space may be written as follows.

$$\lambda=\frac{(w'')^2-2(w')^2(w'')^2-w'w'''\left(1-(w')^2\right)}{\left[1-(w')^2\right]^2} \tag{51}$$

This is most easily determined by comparing the constrained (via Lagrange multiplier) PDE directly governing u to that of the extensible case simplified through the inextensibility expression after applying Hamilton's principle. However, it can be shown (through significant algebra) that this expression satisfies the comparison between these two PDEs for w as well, meaning that comparing Eq. (49) to Eq. (39) yields the same result for λ.

2.4. Pure transverse deflection

Let's consider another special case of beam modeling context. Many beam dynamics problems in the literature feature the component with two fixed ends. In such a system, u is sometimes modeled as identically zero (see Refs. [18,24]). Under these circumstances, Eq. (14) may be written as

$$V=\frac{1}{2}\int_0^1\frac{(w'')^2}{\left[1+(w')^2\right]^3}+\gamma\left[\sqrt{1+(w')^2}-1\right]^2d\eta \tag{52}$$

Kinetic energy is now a function of $\dot{w}$ only, so there is no need for an additional constraint. Once again, using Hamilton's Principle,

$$\ddot{w}+\left[\left(\frac{w''}{\left[1+(w')^2\right]^3}\right)'+\frac{3w'(w'')^2}{\left[1+(w')^2\right]^4}-\gamma\left(1-\frac{1}{\sqrt{1+(w')^2}}\right)w'\right]'=0 \tag{53}$$

subject to

$$\left.\frac{w''}{\left[1+(w')^2\right]^3}\delta w'\right|_0^1 = 0 \tag{54}$$

$$\left.\left[\left(\frac{w''}{\left[1+(w')^2\right]^3}\right)' + \frac{3w'(w'')^2}{\left[1+(w')^2\right]^4} - \gamma\left(1-\frac{1}{\sqrt{1+(w')^2}}\right)w'\right]\delta w\right|_0^1 = 0 \tag{55}$$

where reducing the χ terms to match the PDE and boundary terms is a fairly straightforward exercise.

2.4.1. Validity of neglecting longitudinal displacement

Let us evaluate the validity of the $u = 0$ assumption. It can be shown that $u \neq 0$ across the length of the beam even in a system with pinned boundary conditions. To show this result, consider the definition of N_x from 4 and 11, where the variables have been nondimensionalized.

$$\tilde{N}_x = \gamma\left[\sqrt{(w')^2+(1+u')^2} - 1\right] \tag{56}$$

Now, by assuming that in a system with clamped or pinned boundary conditions at both ends, u' is small compared to 1, we can invoke the argument in Section 2.1.1 which gives rise to Eq. (17), and insert this into the previous equation.

$$\tilde{N}_x = \frac{1}{2}\gamma\left[(w')^2 + 2u'\right] \tag{57}$$

We know that $\tilde{N}_x$ is not a function of x [7]. If $\tilde{N}_x$ is not a function of x but w' is, then clearly u' must also be a function of x. Therefore, $u \neq 0$ across the beam.

To show the form of u, we can integrate Eq. (57), we get the following.

$$\tilde{N}_x = \frac{1}{2}\gamma\int_0^1 (w')^2 + 2u' d\eta$$

$$\tilde{N}_x = \frac{1}{2}\gamma\int_0^1 (w')^2 d\eta + \gamma(u(1) - u(0))$$

$$\tilde{N}_x = \frac{1}{2}\gamma\int_0^1 (w')^2 d\eta \tag{58}$$

where the boundary conditions are $u(1) = u(0) = 0$. Equating 57–58 reveals an extensible constraint equation, per se, which relates w to u explicitly:

$$\int_0^1 (w')^2 d\eta = (w')^2 + 2u' \tag{59}$$

As an example, consider the shape of u if we assume a pinned-pinned beam. Then, we can assume the solution form for w as follows.

$$w = a\sin(\pi\eta) \tag{60}$$

where a is some amplitude scaled by the beam length. Then, from Eq. (59),

$$\begin{aligned} \tfrac{\pi^2}{2}a^2 &= \pi^2 a^2\cos^2\pi\eta + 2u' \\ u' &= -\left(\tfrac{\pi a}{2}\right)^2\cos 2\pi\eta \end{aligned} \tag{61}$$

Therefore,

$$u = -\frac{\pi a^2}{8}\sin(2\pi\eta) \tag{62}$$

which shows that the beam is extended toward each end and has no extension at the midpoint.

To quantify the effect that this approximation has on the results, let us define two types of error incurred when it is assumed that $u \equiv 0$: error in absolute displacement of a point η along the beam (denoted ϵ_r) and error in its corresponding velocity (denoted ϵ_v) normalized by the position and velocity of the midpoint of the beam.

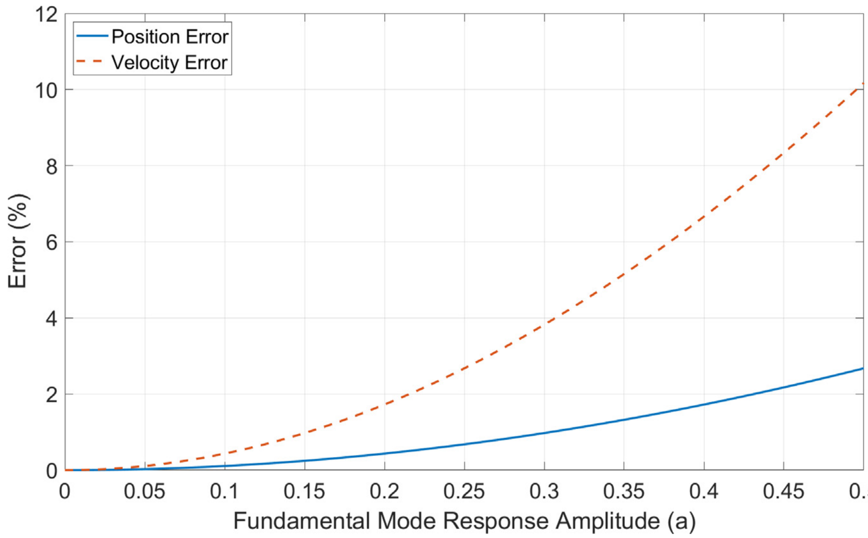

Fig. 2. Error in absolute displacement and velocity in a pinned-pinned beam as a function of nondimensional response amplitude for the fundamental mode.

$$\epsilon_r = \frac{\left|\sqrt{u^2 + w^2} - w\right|}{\left|w(\eta = \frac{1}{2})\right|} \tag{63}$$

$$\epsilon_v = \frac{\left|\sqrt{\dot{u}^2 + \dot{w}^2} - \dot{w}\right|}{\left|\dot{w}(\eta = \frac{1}{2})\right|} \tag{64}$$

Substituting in the single-mode approximations for the pinned–pinned beam response, it can be shown that these error values may be written as

$$\epsilon_r = |\sin \pi\eta| \left(\sqrt{\frac{\pi^2 a^2}{64} \left(\frac{\sin^2 2\pi\eta}{\sin^2 \pi\eta} \right)^2 + 1} - 1 \right) \tag{65}$$

$$\epsilon_v = |\sin \pi\eta| \left(\sqrt{\frac{\pi^2 a^2}{16} \left(\frac{\sin^2 2\pi\eta}{\sin^2 \pi\eta} \right)^2 + 1} - 1 \right) \tag{66}$$

Their maximum values occur, at $\eta = \frac{1}{4}$, where

$$\epsilon_r = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{\pi^2 a^2}{32} + 1} - 1 \right) \tag{67}$$

$$\epsilon_v = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{\pi^2 a^2}{8} + 1} - 1 \right) \tag{68}$$

These errors are illustrated in Fig. 2. Note that the error, particularly the error in position, is quite small even for large a. However, it is important to keep in mind that these figures represent the error incurred based on an assumed, single-mode solution for w and truncated normal strain.

3. Discussion

Equations of motion are presented without using a small-argument approximation for thin beams. These equations are presented beginning with a complete derivation of potential energy based on normal stress theory, and an expansion of the potential energy is given without conventional truncation. These are then used to examine an extensible (fixed-fixed) and an inextensible fixed-free beam. Then the characteristics of these systems are discussed by (1) providing an expression for the internal tensile force caused by the beam's extensibility, (2) quantifying the effect of ignoring longitudinal displacement in a fixed-end-beam case and (3) providing an expression for the constraint force responsible for maintaining the inextensibility condition. Additionally, a simple procedure for finding high-order polynomial nonlinearities in the potential energy expression of the component is given, using binomial expansions.

The results concerning the fixed-end beams deserve further comment, as the example shown here is for a pinned-pinned beam where the expression for the normal tension is truncated. Additional studies concerning systems with different combinations of boundary conditions and including a greater number of modes would certainly be a valuable extension of the present work. However, for the fundamental mode response case, it appears likely that neglecting u for fixed-endpoint beam vibration is a fairly good approximation. As noted in the body of the text, the maximum longitudinal displacement u occurs at the one-quarter and three-quarter length points on the pinned–pinned beam for the first mode. Even if $a = 0.5$, that is the ratio of the fundamental modal response amplitude is half the length of the beam, less than a 3% error is incurred in the point's absolute position. However, this would cause over 10% error in the absolute velocity estimate.

Additional opportunities for future work include strain energy formulations for more complicated structural elements, such as beams whose response depends appreciably on shear and torsion, as well as plates and shells.

Acknowledgments

This work is supported by the U.S. Army Research Laboratory and the Oak Ridge Associated Universities, as well as the SMART scholarship from the U.S. Department of Defense. The authors would also like to acknowledge the time and effort put into this work by the anonymous reviewers as they suggested important edits particularly in the section discussing a beam with fixed ends.

Appendix A. Energy nondimensionalization

Consider the dimensional kinetic energy

$$T = \frac{1}{2}\int_0^l \rho A\left[\dot{u}^2 + \dot{w}^2\right]dx \tag{69}$$

and the dimensional potential energy

$$V = \frac{1}{2}\int_0^l EI\frac{\left[(1+u')w'' - w'u''\right]^2}{\left[(1+u')^2 + (w')^2\right]^3} + EA\left[\sqrt{(1+u')^2 + (w')^2} - 1\right]^2 dx \tag{70}$$

Scale mass by some value $[M]$, length by some value $[L]$, and time by some value $[T]$. Dimensional variables may then be expressed as

$$\begin{aligned} &T, V = \tfrac{[M][L]^2}{[T]^2}\tilde{T}, \tilde{V} \quad && u, w = [L]\tilde{u}, \tilde{w} \\ &x = [L]\eta && t = [T]\tilde{t} \end{aligned} \tag{71}$$

Note that derivatives with respect to x and t also have a length scale. The nondimensional kinetic energy may now be written as

$$\tilde{T} = \frac{1}{2}\int_0^{\frac{l}{[L]}} \frac{\rho A l}{[M]}\left[\dot{\tilde{u}}^2 + \dot{\tilde{w}}^2\right]d\eta \tag{72}$$

where an overdot now refers to a derivative with respect to $\tilde{t}$. We choose $[L] = l$ and $[M] = \rho A l$ to simplify this expression to

$$\tilde{T} = \frac{1}{2}\int_0^1 \left[\dot{\tilde{u}}^2 + \dot{\tilde{w}}^2\right]d\eta \tag{73}$$

The nondimensional potential energy, where a prime indicates a derivative with respect to η, is

$$\tilde{V} = \frac{1}{2}\int_0^1 \frac{EI[T]^2}{\rho A l^4}\frac{\left[(1+\tilde{u}')\tilde{w}'' - \tilde{w}'\tilde{u}''\right]^2}{\left[(1+\tilde{u}')^2 + (\tilde{w}')^2\right]^3} + \frac{EA[T]^2}{\rho A l^2}\left[\sqrt{(1+\tilde{u}')^2 + (\tilde{w}')^2} - 1\right]^2 d\eta \tag{74}$$

We choose $[T]^2 = \frac{\rho A l^4}{EI}$. Now we may write

$$\tilde{V} = \frac{1}{2}\int_0^1 \frac{\left[(1+\tilde{u}')\tilde{w}'' - \tilde{w}'\tilde{u}''\right]^2}{\left[(1+\tilde{u}')^2 + (\tilde{w}')^2\right]^3} + \gamma\left[\sqrt{(1+\tilde{u}')^2 + (\tilde{w}')^2} - 1\right]^2 d\eta \tag{75}$$

where $\gamma = \frac{Al^2}{I}$.

Appendix B. Expansion of extensible beam energy

Consider the potential energy expression

$$V = \frac{1}{2}\int_0^1 \frac{[(1+u')w'' - w'u'']^2}{\left[\frac{(w')^2}{(1+u')^2}+1\right]^3 (1+u')^6} d\eta + \frac{1}{2}\int_0^1 \gamma\left[(1+u')\sqrt{\frac{(w')^2}{(1+u')^2}+1} - 1\right]^2 d\eta \tag{76}$$

The integrands may be rewritten as

$$[(1+u')w'' - w'u'']^2\left[1+\left(\frac{w'}{1+u'}\right)^2\right]^{-3}[1+u']^{-6}$$

and

$$\left[(1+u')\left(1+\left(\frac{w'}{1+u'}\right)^2\right)^{\frac{1}{2}} - 1\right]^2$$

First, we expand the binomials involving $\frac{w'}{1+u'}$ via Maclaurin series. The useful powers for the remainder of the expansion will be -3, $\frac{1}{2}$, and -10, where.

$$\begin{aligned}
&(1+a)^{-3} = 1 - 3a + 6a^2 - 10a^3 + 15a^4 + O(a^5)\\
&(1+a)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x^2 - \tfrac{1}{8}a^4 + O(a^5)\\
&(1+a)^{-10} = 1 - 10a + 55a^2 - 220a^3 + 715a^4 + O(a^5)
\end{aligned} \tag{77}$$

such that

$$[(1+u')w'' - w'u'']^2\left[1 - 3\left(\frac{w'}{1+u'}\right)^2 + 6\left(\frac{w'}{1+u'}\right)^4 + O(\epsilon_w^5)\right][1+u']^{-6}$$

and

$$\left[(1+u')\left(1+\tfrac{1}{2}\left(\frac{w'}{1+u'}\right)^2 - \tfrac{1}{8}\left(\frac{w'}{1+u'}\right)^4 + O(\epsilon_w^5)\right) - 1\right]^2$$

Collecting the $1+u'$ binomials outside of the previous expansions yields

$$[(1+u')w'' - w'u'']^2\left[(1+u')^4 - 3(w')^2(1+u')^2 + 6(w')^4 + O(\epsilon_w^5)\right][1+u']^{-10}$$

and

$$\left[(1+u')^{-3}\left((1+u')^4 + \tfrac{1}{2}(w')^2(1+u')^2 - \tfrac{1}{8}(w')^4 + O(\epsilon_w^5)\right) - 1\right]^2$$

Expanding the $1+u'$ binomials yields the expression given in the main text.

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