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Quantum majorization on semi-finite von Neumann algebras

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ABSTRACT

We extend Gour et al.'s characterization of quantum majorization via conditional min-entropy to the context of semi-finite von Neumann algebras. Our method relies on a connection between conditional min-entropy and the operator space projective tensor norm for injective von Neumann algebras. We then use this approach to generalize the tracial Hahn-Banach theorem of Helton, Klep and McCullough to vector-valued noncommutative L_1-spaces.

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1. Introduction

Majorization is a fundamental tool introduced by Hardy, Littlewood, and Pólya [13] that finds application in various fields [19]. Among the different motivations for majorization, the core idea is a notion of "disorder". For example, a probability distribution is majorized by another if it is less deviated from the uniform distribution. Recently, Gour, Jennings, Buscemi, Duan, and Marvian in [10] use the concept of "quantum majorization" as a particular partial ordering of states and processes in quantum mechanical systems.

Let H be a finite dimensional Hilbert space and $B(H)$ be the space of bounded operators acting on H. A density operator $\rho \in B(H)$ (called a state on the quantum system H in the quantum information theory literature) is positive and has trace 1. The dynamics between quantum systems is modeled by completely positive trace preserving maps (also called quantum channels) which map density operators to density operators. For two bipartite density operators ρ and σ on the tensor product Hilbert space $H_A \otimes H_B$, σ is said to be quantum majorized by ρ if there exists a linear completely positive trace preserving (CPTP) map $\Phi : B(H_B) \to B(H_B)$ such that $\sigma = \mathrm{id} \otimes \Phi(\rho)$. This concept has been studied in different contexts under various guises [23,4,3,2,16]. Intuitively, quantum majorization describes the disorder observed from the B system. This can be witnessed from the data processing inequality of conditional entropy $H(A|B)$,

$$H(A|B)_\rho \leq H(A|B)_{\mathrm{id} \otimes \Phi(\rho)} = H(A|B)_\sigma \ .$$

For a bipartite density operator $\rho \in B(H_A \otimes H_B)$, its conditional entropy is $H(A|B)_\rho := H(\rho) - H(\mathrm{tr} \otimes \mathrm{id}(\rho))$, where tr is the matrix trace and $H(\rho) = -\mathrm{tr}(\rho \log \rho)$ is the von Neumann entropy. The conditional entropy $H(A|B)_\rho$ describes the uncertainty of the bipartite density operator ρ given its information on the B system [15]. The data processing inequality says such uncertainty is monotone non-decreasing under quantum majorization. As a converse to the data processing inequality, Gour and his coauthors [10] proved the following characterization of quantum majorization using conditional min-entropy $H_{min}(A|B)$, defined as

$$H_{min}(A|B)_\rho = -\log \inf\{\mathrm{tr}(\omega) | \rho \leq 1 \otimes \omega \text{ for some positive } \omega \in B(H_B)\}. \qquad (1.1)$$

Theorem *([10]). Let H_A, H_B be finite dimensional Hilbert spaces. For two bipartite density operators ρ and σ, σ is quantum majorized by ρ if and only if for all finite dimensional H'_A and all CPTP maps $\Psi : B(H_A) \to B(H_{A'})$,*

$$H_{min}(A'|B)_{\Psi \otimes \mathrm{id}(\rho)} \leq H_{min}(A'|B)_{\Psi \otimes \mathrm{id}(\sigma)}. \qquad (1.2)$$

$H_{min}(A|B)$ is the analogue of $H(A|B)$ as the Rényi p-version at $p = \infty$ [20] and it connects to $H(A|B)$ by the quantum version of asymptotic equipartition property [26]. The "only if" direction in the above theorem follows from the data processing inequality of H_{min}, which is indeed self-evident from its definition (1.1). The other direction states that quantum majorization is actually determined by the data processing inequality of H_{min}. In [10], the above theorem has been used to characterize quantum dynamics under group symmetry and thermodynamic condition. It has further extensions from bipartite states to bipartite quantum channels [11].

In this work, we revisit Gour et al.'s theorem from a functional analytic perspective. Our starting point is the observation that the conditional min-entropy corresponds to the operator space tensor norm

$$H_{min}(A|B)_\rho = -\log \|\rho\|_{S_1(H_B)\widehat{\otimes}B(H_A)} \tag{1.3}$$

where $S_1(H_B)$ is the set of trace class operators on H_B and $S_1(H_B)\widehat{\otimes}B(H_A)$ is the operator space projective tensor product. This correspondence is based on an factorization expression for the norm of $S_1(H_B)\widehat{\otimes}B(H_A)$ that Pisier used in [21] to define noncommutative vector-valued L_p spaces. On the other hand, it is known [7,1] that the dual space of $S_1(H_B)\widehat{\otimes}B(H_A)$ is the completely bounded maps $CB(B(H_A), B(H_B))$, where quantum channels correspond to unital completely positive maps by taking adjoints. From this perspective, H_{min} is the dual of CB norm with respect to quantum channels and Gour et al.'s theorem is essentially a Hahn-Banach separation theorem. Using this approach, we prove the following characterization of quantum majorization using the projective tensor norm which extends Gour et al.'s results to the setting of tracial von Neumann algebras. We consider two semi-finite von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ equipped with normal faithful semi-finite traces $\tau_\mathcal{M}$ (resp. $\tau_\mathcal{N}$). We denote $L_1(\mathcal{M})$ (resp. $L_1(\mathcal{N})$) as the space of 1-integrable operators with respect to $\tau_\mathcal{M}$ (resp. $\tau_\mathcal{N}$). Our main theorem is

Theorem 1.1 *(cf. Theorem 3.8). Let $\mathcal{M}$ and $\mathcal{N}$ be two semi-finite von Neumann algebras. Suppose $\mathcal{M}$ is injective. Then for two density operators $\rho, \sigma \in L_1(\mathcal{M}\overline{\otimes}\mathcal{N})$, there exists a CPTP map $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M})$ such that $\Phi\otimes\mathrm{id}(\rho) = \sigma$ if and only if for any projection $e \in \mathcal{M}$ with $\tau_\mathcal{M}(e) < \infty$ and for any CPTP map $\Psi : L_1(\mathcal{N}) \to L_1(e\mathcal{M}e^{op}) \cap e\mathcal{M}e^{op}$,*

$$\|\mathrm{id}\otimes\Psi(\rho)\|_{L_1(\mathcal{M})\widehat{\otimes}e\mathcal{M}e^{op}} \geq \|\mathrm{id}\otimes\Psi(\sigma)\|_{L_1(\mathcal{M})\widehat{\otimes}e\mathcal{M}e^{op}} .$$

Here the $L_1(\mathcal{M})\widehat{\otimes}\mathcal{N}$-norm gives the analogue of H_{min} as in (1.3). We note that the assumption on injectivity is in this connection. Indeed, we show that for semi-finite von Neumann algebras, the conditional min-entropy H_{min} coincides with the projective tensor norm $L_1(\mathcal{M})\widehat{\otimes}\mathcal{N}$ if and only if $\mathcal{M}$ is injective. This can be viewed as a predual form of Haagerup's characterization of injectivity via decomposability [12]. Beyond injectivity, the information-theoretic meaning of the projective tensor norm is not clear.

The above theorem admits several variants. By taking $\mathcal{N} = l_\infty$, the commutative von Neumann algebra of bounded sequences, Theorem 1.1 concerns the quantum interpolation problem of converting an infinite family of density operators into another family of density operators using a CPTP map. On the other hand, the dual form of Theorem 1.1

provides a characterization for the factorization of CPTP maps (also known as channel factorization). A CPTP map S is quantum majorized by T if S admits a factorization $S = \Phi \circ T$ for some CPTP map Φ. Note that in finite dimensions, quantum majorization applies to CPTP maps via their Choi matrices. However, in infinite dimensions, the Choi matrix of a CPTP map is never trace class and our dual consideration is needed. Inspired by Jenčová's work [16] on statistical deficiency for CPTP maps, we also consider the approximate case when the error $\mathrm{id} \otimes \Phi(\rho) - \sigma$ is small but non-zero.

Our approach also has applications to the tracial Hahn-Banach theorem in [14]. The tracial Hahn-Banach theorem is a dual form of Effros-Wrinkler's separation theorem for matrix convex sets. We find that the duality behind the tracial Hahn-Banach theorem is the same duality as that between the operator space projective tensor product and completely bounded maps. Using an idea similar to that used in the characterization of quantum majorization, we give a tracial Hahn-Banach theorem on $L_1(\mathcal{M})\widehat{\otimes}E$ for a semifinite injective von Neumann algebra $\mathcal{M}$ and an arbitrary operator space E. If we replace $L_1(\mathcal{M})$ by an abstract operator space, our method gives some analogous results under the assumptions of 1-locally reflexivity and completely contractive approximation property. Our work in spirit connects to recent work [6] on LOCC-convertibility in semifinite von Neumann algebras.

The rest of paper is organized as follows. Section 2 reviews some basic operator space theory needed for the remainder of the paper. In Section 3, we first discuss the relation between H_{min} and the projective tensor norm and the connection to injectivity of von Neumann algebras. After that, we prove our main theorem and its variants with respect to channel factorization and the approximate case. In particular, all the results in this section apply to $B(H)$ with H being infinite dimensional. As this is arguably the case of most interest in quantum information theory, we summarize the implications for $B(H)$ in Section 3.5. Section 4 is devoted to the tracial Hahn-Banach theorem and the connection to noncommutative vector-valued L_1 space. Section 5 discusses the parallel results on the projective tensor product of abstract operator spaces.

2. Operator space preliminaries

In this section we briefly recall some operator space basics that are needed in our discussion. We refer to the books [22,7] for more information on operator space theory. We denote by $B(H)$ the bounded operator on a complex Hilbert space H and $M_n := M_n(\mathbb{C})$ the algebra of $n \times n$ complex-valued matrices. A (concrete) operator space E is a closed subspace of some $B(H)$. We denote by $M_n(E)$ the set of $n \times n$ matrices with entries from E and similarly $M_{n,m}(E)$ for $n \times m$ rectangular matrices. The space $M_n(B(H))$ is naturally isomorphic to $B(H^{(n)})$, where $H^{(n)} = \ell_2^n(H)$ is the Hilbert space direct sum of n copies of H. For all $n \geq 1$, the inclusion $M_n(E) \subset M_n(B(H)) \cong B(H^{(n)})$ induces a norm on the matrix level space $M_n(E)$ which we denote by $\|\cdot\|_{M_n(E)}$. The operator space structure of E is given by the norm sequence $\|\cdot\|_{M_n(E)}, n \geq 1$.

Given a linear map $u : E \to F$ between two operator spaces E and F, u is *completely bounded* (or CB) if its completely bounded norm (CB-norm)

$$\| u \|_{cb} := \sup_{n\geq 1} \| \mathrm{id}_n \otimes u : M_n(E) \to M_n(F) \|_{op}$$

is finite. Here id_n is the identity map on M_n. We say u is a *complete isometry* if for each n, $\mathrm{id}_n \otimes u$ is an isometry. We denote by $CB(E, F)$ the Banach space of all completely bounded maps $E \to F$ equipped with the CB-norm. Moreover, $CB(E, F)$ is again an operator space with the operator space structure given by $M_n(CB(E, F)) = CB(E, M_n(F))$. In particular, the operator space dual is defined as

$$E^* = CB(E, \mathbb{C}) \; .$$

Throughout the paper, we will use $\otimes$ for algebraic tensor product. Given two operator spaces $E \subset B(H_A)$ and $F \subset B(H_B)$, the operator space injective tensor product $E \otimes_{min} F$ is defined by the (completely) isometric embedding

$$E \otimes_{min} F \subset B(H_A \otimes_2 H_B) \tag{2.1}$$

where $H_A \otimes_2 H_B$ is the Hilbert space tensor product. Namely, $E \otimes_{min} F$ is the norm completion of $E \otimes F$ for the inclusion $E \otimes F \subset B(H_A \otimes_2 H_B)$. Via injectivity of $\otimes_{min}$, one has the (completely) isometric embedding [21, Chapter 0]

$$E^* \otimes_{min} F \subset CB(E, F) \; . \tag{2.2}$$

Another important tensor product is the projective tensor product. We denote by $\| \cdot \|_{HS}$ the Hilbert-Schmidt norm. The operator space projective tensor product $E \widehat{\otimes} F$ is defined as the completion of $E \otimes F$ with respect to the following norm,

$$\| z \|_{E\widehat{\otimes} F} = \inf \| a \|_{HS} \| x \|_{M_l(E)} \| y \|_{M_m(F)} \| b \|_{HS}$$

where the infimum runs over all factorizations of rectangular matrices a, b, and $x = (x_{i,j})_{i,j=1}^l \in M_l(E), y = (y_{p,q})_{p,q=1}^m \in M_m(F)$ such that

$$z = a(x \otimes y)b = \sum_{i,j=1}^{l} \sum_{p,q=1}^{m} a_{i,p} x_{i,j} \otimes y_{p,q} b_{j,q} \; . \tag{2.3}$$

For $z = (z_{r,s})_{r,s=1}^n \in M_n(E \otimes F)$, we consider the following factorization

$$z_{r,s} = \sum_{i,j=1}^{l} \sum_{p,q=1}^{m} a_{r,ip} x_{i,j} \otimes y_{p,q} b_{jq,s} \; , \tag{2.4}$$

where $a \in M_{n,ml}, b \in M_{ml,n}$ and $x \in M_l(E), y \in M_m(F)$. The operator space structure of $E\widehat{\otimes}F$ is defined as

$$\| z \|_{M_n(E\widehat{\otimes}F)} = \inf \| a \|_{M_{n,ml}} \| x \|_{M_l(E)} \| y \|_{M_m(F)} \| b \|_{M_{ml,n}}$$

where the infimum runs over all factorizations in (2.4). An equivalent characterization is the following duality [7,1]

$$(E\widehat{\otimes}F)^* \cong CB(E, F^*) \ . \tag{2.5}$$

For $x \in E, y \in F$ and $\Phi \in CB(E, F^*)$. The dual pairing is

$$\langle x \otimes y, \Phi \rangle = \langle \Phi(x), y \rangle_{(F^*,F)} \ .$$

Let us mention some basic examples related to our discussion. Let $\mathcal{K}(H)$ denote the space of compact operators on H and $S_1(H)$ the space of trace class operators. We have the operator space dual relations

$$S_1(H)^* = B(H) \ , \ \mathcal{K}(H)^* = S_1(H) \ , \tag{2.6}$$

where both dual pairings are given by the trace

$$\langle b, a \rangle_{(B(H),S_1(H))} = \mathrm{tr}(b^t a) \ , \langle a, c \rangle_{(S_1(H),K(H))} = \mathrm{tr}(a^t c)$$

where a^t is the transpose of a with respect to a (fixed) orthonormal basis. For two Hilbert spaces H_A and H_B, by (2.1) and (2.2) we have the isometric embedding

$$B(H_A) \otimes_{min} B(H_B) \subset B(H_A \otimes_2 H_B) \ , B(H_A) \otimes_{min} B(H_B) \subset CB(S_1(H_A), B(H_B)).$$

Indeed, one has the equality

$$B(H_A \otimes_2 H_B) \cong CB(S_1(H_A), B(H_B)). \tag{2.7}$$

Note that by (2.5) and (2.6),

$$CB(S_1(H_A), B(H_B)) = \big(S_1(H_A)\widehat{\otimes}S_1(H_B)\big)^* \ , \quad B(H_A \otimes_2 H_B) = S_1(H_A \otimes_2 H_B)^* \ .$$

For preduals, $S_1(H_A)\widehat{\otimes}S_1(H_B) \cong S_1(H_A \otimes_2 H_B)$.

Another example related to our discussion is the space $S_1(H_B)\widehat{\otimes}B(H_A)$. Let $S_2(H)$ denote the Hilbert-Schmidt operators on H. The operator space projective tensor norm on $S_1(H_B)\widehat{\otimes}B(H_A)$ admits the following expression (cf. [21]) for $x \in S_1(H_B) \otimes B(H_A)$

$$\| x \|_{S_1(H_B)\widehat{\otimes}B(H_A)} = \inf_{x=(a\otimes 1)y(1\otimes b)} \| a \|_{S_2(H_B)} \| b \|_{S_2(H_B)} \| y \|_{B(H_B)\otimes_{min} B(H_A)}$$

where the infimum is taken over all possible factorizations of $x = (a \otimes 1_A)y(b \otimes 1_A)$ with $a, b \in S_2(H_B)$ and 1_A denotes the identity operator on H_A. For positive x, it suffices to choose $a = b^*$ and, by rescaling $\| a \|_2 = 1$, we obtain

$$\begin{aligned} \| x \|_{S_1(H_B)\widehat{\otimes} B(H_A)} &= \inf\{ \| y \|_{B(H_B)\otimes_{min} B(H_A)} \mid x = (a \otimes 1)y(a^* \otimes 1) \\ &\qquad \text{for some } \| a \|_{S_2(H_B)} = 1 \} \\ &= \inf\{ \lambda \,|\, x \le \lambda \sigma \otimes I \text{ for some density operator } \sigma \in S_1(H_B) \} \ . \end{aligned}$$

Therefore, this norm on $S_1(H_B)\widehat{\otimes} B(H_A)$ corresponds to the conditional min entropy H_{min} in (1.1). That is, for a bipartite density operator ρ,

$$H_{min}(A|B)_\rho = -\log \| \rho \|_{S_1(H_B)\widehat{\otimes} B(H_A)} \ .$$

At the dual level, by (2.5) we have

$$(S_1(H_B)\widehat{\otimes} B(H_A))^* = CB(B(H_A), B(H_B)) \ . \tag{2.8}$$

Note that a CPTP map $\Phi : S_1(H_B) \to S_1(H_A)$ is completely positive trace preserving, and hence

$$\Phi \in CB(S_1(H_B), S_1(H_A)) \subset CB(B(H_A), B(H_B))$$

where $CB(S_1(H_B), S_1(H_A)) \subset CB(B(H_A), B(H_B))$ as normal CB maps by taking adjoints. Therefore, the $S_1(H_B)\widehat{\otimes} B(H_A)$ norm or equivalently H_{min}, is the dual of CB-norm with respect to quantum channels. This duality is implicitly used in Gour et al.'s arguments in [10]. In quantum information literature, the CB-norm of $CB(S_1(H_B), S_1(H_A))$ is also called the diamond norm. The diamond norm and its dual norm have been used by Jenčová in studying Le Cam's deficiency for quantum channels [16].

3. Quantum majorization on von Neumann algebras

3.1. H_{min} and injectivity of von Neumann algebras

We first discuss the connection between the conditional min entropy H_{min} and the projective tensor product in the setting of tracial von Neumann algebras. Throughout this paper, we assume that $(\mathcal{M}, \tau_{\mathcal{M}})$ and $(\mathcal{N}, \tau_{\mathcal{N}})$ are semi-finite von Neumann algebras with normal faithful semi-finite traces $\tau_{\mathcal{M}}$ (resp. $\tau_{\mathcal{N}}$). We introduce the notation

$$\mathcal{M}_0 := \cup_e e\mathcal{M}e \ ,$$

where the union runs over all projections with $\tau_{\mathcal{M}}(e) < \infty$ which forms a lattice. For $1 \le p < \infty$, the space $L_p(\mathcal{M})$ is the completion of $\mathcal{M}_0$ with respect to the L_p-norm

$$\|a\|_{L_p(\mathcal{M})} = \tau_{\mathcal{M}}(|a|^p)^{1/p} , a \in \mathcal{M}_0 .$$

We will often use the shorthand notation $\|\cdot\|_p$ for the p-norm and $\|\cdot\|_\infty$ for the operator norm in $\mathcal{M}$. Let $\mathcal{M}^{op} = \{a^{op}|a \in \mathcal{M}\}$ be the opposite algebra equipped with reversed multiplication $a^{op} \cdot b^{op} = (ba)^{op}$ and trace $\tau_{\mathcal{M}^{op}}(a^{op}) = \tau_{\mathcal{M}}(a)$. The predual of $\mathcal{M}$ can be identified with $\mathcal{M}_* = L_1(\mathcal{M}^{op})$, via the pairing $\langle a^{op}, b\rangle = \tau_{\mathcal{M}}(ab)$ for $a \in L_1(\mathcal{M})$ and $b \in \mathcal{M}$.

We will often use the normal part and the singular part of a continuous linear map between von Neumann algebras. We say a completely bounded map $\Phi : \mathcal{M} \to \mathcal{N}$ is normal if it is weak*- to weak*-topology continuous. A normal Φ admits a pre-adjoint map $\Psi : \mathcal{N}_* \to \mathcal{M}_*$ such that its adjoint $\Psi^\dagger = \Phi$. In general, a completely bounded map $\Phi : \mathcal{M} \to \mathcal{N}$ admits the decomposition $\Phi = \Phi_n + \Phi_s$ as a normal part Φ_n and a singular part Φ_s. Indeed, let e_0 be the support projection of $\mathcal{M} \subset \mathcal{M}^{**}$ in the bidual $\mathcal{M}^{**}$. Then

$$\Phi_n(x) = (\Phi^\dagger|_{\mathcal{N}_*})^\dagger(e_0 x) , \Phi_s(x) = (\Phi^\dagger|_{\mathcal{N}_*})^\dagger((1 - e_0)x)$$

where $\Phi^\dagger|_{\mathcal{N}_*} : \mathcal{N}_* \to \mathcal{M}^*$ is the restriction of the adjoint $\Phi^\dagger : \mathcal{N}^* \to \mathcal{M}^*$ on $\mathcal{N}_*$ and $(\Phi^\dagger|_{\mathcal{N}_*})^\dagger : \mathcal{M}^{**} \to \mathcal{N}$ is the adjoint of $\Phi^\dagger|_{\mathcal{N}_*}$. In particular, for a positive linear functional $\phi : \mathcal{M} \to \mathbb{C}$, we have $\phi = \phi_n + \phi_s$, where $\phi_n \in \mathcal{M}_*$ is the normal part of ϕ (that is, ϕ_n is weak*-continuous) and $\phi_s \in \mathcal{M}_*$ is singular (that is, there does not exist a non-zero weak*-continuous positive linear functional ψ on $\mathcal{M}$ such that $\psi \le \phi_s$). The dual space $\mathcal{M}^*$ is then decomposed as $\mathcal{M}^* = \mathcal{M}_* \oplus \mathcal{M}_*^\perp$ here $\mathcal{M}_*$ is the normal part and $\mathcal{M}_*^\perp = \mathcal{M}^*(1 - e_0)$ is the singular part. (See [25, Chapter 3, p. 127] for further details.)

Let $\mathcal{M} \subseteq B(H)$ (resp. $\mathcal{N} \subseteq B(K)$) be a faithful representation of $\mathcal{M}$ (resp. $\mathcal{N}$). The von Neumann algebra tensor product $\mathcal{M}\overline{\otimes}\mathcal{N}$ is the weak*-closure of $\mathcal{M} \otimes_{min} \mathcal{N}$ inside $B(H \otimes_2 K)$, and $\mathcal{M}\overline{\otimes}\mathcal{N}$ is independent of the faithful representations $\mathcal{M} \subseteq B(H)$ and $\mathcal{N} \subseteq B(K)$. The Effros-Ruan isomorphism [7] gives a complete isometry

$$\mathcal{N}\overline{\otimes}\mathcal{M} \cong CB(\mathcal{N}_*, \mathcal{M}) \cong CB(L_1(\mathcal{N}^{op}), \mathcal{M}) . \tag{3.1}$$

This isomorphism is order preserving. Indeed, a positive operator $x \in \mathcal{N}\overline{\otimes}\mathcal{M}$ corresponds to a completely positive map $T_x \in CB(L_1(\mathcal{N}^{op}), \mathcal{M})$ given by

$$T_x(\rho^{op}) = \tau_{\mathcal{N}} \otimes \mathrm{id}_{\mathcal{M}}((\rho \otimes 1)x).$$

As for the predual of (3.1), we have

$$L_1(\mathcal{M})\widehat{\otimes}L_1(\mathcal{N}) = L_1(\mathcal{M}\overline{\otimes}\mathcal{N}) = (\mathcal{M}^{op}\overline{\otimes}\mathcal{N}^{op})_* .$$

The conditional min entropy H_{min} is related to the vector-valued L_1-spaces introduced in [21]. We use the shorthand notation that for $a, b \in \mathcal{M}, y \in \mathcal{M}\overline{\otimes}\mathcal{N}$,

$$a \cdot y \cdot b := (a \otimes 1_{\mathcal{N}})y(b \otimes 1_{\mathcal{N}}).$$

We define the $L_1(\mathcal{M}, L_\infty(\mathcal{N}))$ norm for $x \in \mathcal{M}_0 \otimes \mathcal{N}$ as follows,

$$\| x \|_{L_1(\mathcal{M},L_\infty(\mathcal{N}))} = \inf\{\| a \|_{L_2(\mathcal{M})} \| y \|_{\mathcal{M}\overline{\otimes}\mathcal{N}} \| b \|_{L_2(\mathcal{M})} \mid x = a \cdot y \cdot b \ , a, b \in \mathcal{M}_0, y \in \mathcal{M}\otimes\mathcal{N}\},$$

where the infimum is over all factorizations $x = a \cdot y \cdot b$. Then $L_1(\mathcal{M}, L_\infty(\mathcal{N}))$ is defined as the completion of $\mathcal{M}_0 \otimes \mathcal{N}$ under the above norm. The triangle inequality for this norm is verified in [21, Lemma 3.5]. We will also use the shorthand notation

$$\mathcal{M}\overline{\otimes}\mathcal{N}_0 = \cup_q \mathcal{M}\overline{\otimes} q\mathcal{N}q \subset \mathcal{M}\overline{\otimes}\mathcal{N} \ ,$$

where the union runs over all projections $q \in \mathcal{N}$ with $\tau_\mathcal{N}(q) < \infty$. For $x \in \mathcal{M}\overline{\otimes}\mathcal{N}_0$, we define the $L_\infty(\mathcal{M}, L_1(\mathcal{N}))$ norm as

$$\| x \|_{L_\infty(\mathcal{M},L_1(\mathcal{N}))} = \sup\{\| a \cdot x \cdot b \|_{L_1(\mathcal{M}\overline{\otimes}\mathcal{N})} \mid \| a \|_{L_2(\mathcal{M})} = \| b \|_{L_2(\mathcal{M})} = 1 \ \} \ .$$

This norm clearly satisfies the triangle inequality. The space $L_\infty(\mathcal{M}, L_1(\mathcal{N}))$ is defined as the norm completion of $\mathcal{M}\overline{\otimes}\mathcal{N}_0$. Both spaces contain the corresponding algebraic tensor products

$$L_1(\mathcal{M}) \otimes \mathcal{N} \subset L_1(\mathcal{M}, L_\infty(\mathcal{N})) \ , \ \mathcal{M} \otimes L_1(\mathcal{N}) \subset L_\infty(\mathcal{M}, L_1(\mathcal{N})).$$

Indeed, for $a \otimes b$ with $a \in L_1(\mathcal{M})$ and $b \in \mathcal{N}$, let e_n be the spectral projection of $|a|$ for the interval $[1/n, n]$. Then $e_n a e_n \otimes b$ converges in $L_1(\mathcal{M}, L_\infty(\mathcal{N}))$ and the limit can be identified with $a \otimes b$. It is clear from the definitions that

i) a complete contraction $T : L_\infty(\mathcal{N}_1) \to L_\infty(\mathcal{N}_2)$ extends to a contraction

$$\mathrm{id}_\mathcal{M} \otimes T : L_1(\mathcal{M}, L_\infty(\mathcal{N}_1)) \to L_1(\mathcal{M}, L_\infty(\mathcal{N}_2)) \ .$$

ii) a complete contraction $S : L_1(\mathcal{N}_1) \to L_1(\mathcal{N}_2)$ extends to a contraction

$$\mathrm{id}_\mathcal{M} \otimes S : L_\infty(\mathcal{M}, L_1(\mathcal{N}_1)) \to L_\infty(\mathcal{M}, L_1(\mathcal{N}_2)) \ .$$

For the trivial case $\mathcal{N} = \mathbb{C}$, we have $L_1(\mathcal{M}, \mathbb{C}) = L_1(\mathcal{M})$ and $L_\infty(\mathcal{M}, \mathbb{C}) = L_\infty(\mathcal{M})$. In general, $L_\infty(\mathcal{M}, L_1(\mathcal{N}))$ is a subspace of $\Big(L_1(\mathcal{M}, L_\infty(\mathcal{N}))\Big)^*$. Indeed,

$$\begin{aligned} \| x \|_{L_\infty(\mathcal{M},L_1(\mathcal{N}))} &= \sup\{ \ \| a \cdot x \cdot b \|_1 \ \mid \ \| a \|_2 = \| b \|_2 = 1, a, b \in \mathcal{M}_0\} \\ &= \sup\{ \ |\tau\big(y(a \cdot x \cdot b)\big)| \ \mid \ \| a \|_2 = \| b \|_2 = 1, a, b \in \mathcal{M}_0, \| y \|_{\mathcal{M}\overline{\otimes}\mathcal{N}} = 1\} \\ &= \sup\{ \ |\tau\big((b \cdot y \cdot a)x\big)| \ \mid \ \| a \|_2 = \| b \|_2 = 1, a, b \in \mathcal{M}_0, \| y \|_{\mathcal{M}\overline{\otimes}\mathcal{N}} = 1\} \\ &= \sup\{ \ |\tau(zx)| \ \mid \ \| z \|_{L_1(\mathcal{M},L_\infty(\mathcal{N}))} = 1, z \in \mathcal{M}_0 \otimes \mathcal{N}\} \ . \end{aligned}$$

Here and in the following we will use $\tau := \tau_\mathcal{M} \otimes \tau_\mathcal{N}$ for the product trace.

Lemma 3.1. *i) For any self-adjoint* $x \in \mathcal{M}_0 \otimes \mathcal{N}$,

$$\| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} = \inf\{ \| a \|_{L_2(\mathcal{M})} \| y \|_{\mathcal{M}\overline{\otimes}\mathcal{N}} \| a^* \|_{L_2(\mathcal{M})} \mid x = a \cdot y \cdot a^*, a \in \mathcal{M}_0, y \text{ self-adjoint}\}.$$

In particular, if $x \in e\mathcal{M}e \otimes \mathcal{N}$ *for some finite projection* e *with* $\tau_{\mathcal{M}}(e) < \infty$,

$$\| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} = \inf_\sigma \| (\sigma^{-\frac{1}{2}} \otimes 1) x (\sigma^{-\frac{1}{2}} \otimes 1) \|_\infty$$

where the infimum is over all density operators σ *invertible in* $e\mathcal{M}e$.
ii) For any positive $x \in \mathcal{M}\overline{\otimes}\mathcal{N}_0$,

$$\| x \|_{L_\infty(\mathcal{M}, L_1(\mathcal{N}))} = \sup\{ \tau(a \cdot x \cdot a^*) | \ \| a \|_{L_2(\mathcal{M})} = 1 \ \}.$$

Proof. For ii), Hölder's inequality gives,

$$\begin{aligned} \| x \|_{L_\infty(\mathcal{M}, L_1(\mathcal{N}))} &= \sup_{\| a \|_2 = \| b \|_2 = 1} \| (a \otimes 1) x (b \otimes 1) \|_1 \\ &\le \sup_{\| a \|_2 = 1} \| (a \otimes 1) x^{\frac{1}{2}} \|_2 \sup_{\| b \|_2 = 1} \| x^{\frac{1}{2}} (b \otimes 1) \|_2 \\ &= \sup_{\| a \|_2 = 1} \| (a \otimes 1) x (a^* \otimes 1) \|_1^{\frac{1}{2}} \sup_{\| b \|_2 = 1} \| (b^* \otimes 1) x (b \otimes 1) \|_1^{\frac{1}{2}} \\ &= \sup_{\| a \|_2 = 1} \| (a \otimes 1) x (a^* \otimes 1) \|_1 = \sup_{\| a \|_2 = 1} \tau(a \cdot x \cdot a^*). \end{aligned}$$

For i), choose $x = (a \otimes 1) y (b \otimes 1)$ such that $a, b \in e\mathcal{M}e$ and

$$\| a \|_{L_2(\mathcal{M})} = \| b \|_{L_2(\mathcal{M})} = 1, \| y \|_{\mathcal{M}\overline{\otimes}\mathcal{N}} < \| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} + \epsilon.$$

Take $d = (aa^* + b^*b + \delta e)^{\frac{1}{2}}$. Then $d > 0$ is invertible in $e\mathcal{M}e$ and $\| d \|_2 = (2 + \delta\tau(e))^{\frac{1}{2}}$. Note that $x = x^*$ implies that

$$\begin{aligned} x &= \frac{1}{2}\Big(a \cdot y \cdot b + b^* \cdot y^* \cdot a \Big) \\ &= \frac{1}{2} d \cdot \Big(d^{-1} a \cdot y \cdot b d^{-1} + d^{-1} b^* \cdot y^* \cdot a d^{-1} \Big) \cdot d \\ &= d \cdot \tilde{y} \cdot d, \end{aligned}$$

where

$$\begin{aligned} \tilde{y} &= \frac{1}{2}\Big(d^{-1} a \cdot y \cdot b d^{-1} + d^{-1} b^* \cdot y^* \cdot a d^{-1} \Big) \\ &= \frac{1}{2} \begin{bmatrix} d^{-1} a & d^{-1} b^* \end{bmatrix} \cdot \begin{bmatrix} 0 & y \\ y^* & 0 \end{bmatrix} \cdot \begin{bmatrix} a^* d^{-1} \\ b d^{-1} \end{bmatrix}. \end{aligned}$$

Since $\left\| \begin{bmatrix} 0 & y \\ y^* & 0 \end{bmatrix} \right\|_{M_2(\mathcal{M})} = \| y \|_{\mathcal{M}}$, $\| \begin{bmatrix} d^{-1}a & d^{-1}b^* \end{bmatrix} \|_{M_{1,2}(\mathcal{M})} = \| d^{-1}(aa^* + b^*b)d^{-1} \|_{\mathcal{M}} \leq 1$ and similarly $\left\| \begin{bmatrix} a^*d^{-1} \\ bd^{-1} \end{bmatrix} \right\|_{M_{2,1}(\mathcal{M})} \leq 1$, it follows that

$$\| \tilde{y} \|_\infty \leq \frac{1}{2} \| \begin{bmatrix} d^{-1}a & d^{-1}b^* \end{bmatrix} \| \left\| \begin{bmatrix} 0 & y \\ y^* & 0 \end{bmatrix} \right\| \left\| \begin{bmatrix} a^*d^{-1} \\ bd^{-1} \end{bmatrix} \right\| \leq \frac{1}{2} \| y \|_\infty \ .$$

Thus we have $x = d \cdot \tilde{y} \cdot d$ with

$$\| d \|_2^2 \leq 2 + \delta\tau(e) \ , \| \tilde{y} \|_\infty \leq \frac{1}{2} \| y \|_\infty \ .$$

Since δ is arbitrarily small we prove the first expression in i). For the second expression, we choose

$$\sigma = \frac{1}{\| d \|_2^2} d^2 \ , \ (\sigma^{-\frac{1}{2}} \otimes 1)x(\sigma^{-\frac{1}{2}} \otimes 1) = \| d \|_2^2 \ \tilde{y} \ .$$

Then

$$\begin{aligned} \| (\sigma^{-\frac{1}{2}} \otimes 1)x(\sigma^{-\frac{1}{2}} \otimes 1) \|_\infty \leq \| d \|_2^2 \| \tilde{y} \|_\infty &\leq (1 + \frac{\delta}{2}\tau(e)) \ \| y \|_\infty \\ &\leq (1 + \frac{\delta}{2}\tau(e))(\| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} + \epsilon) \ . \end{aligned}$$

Since δ and ϵ are arbitrarily small, we prove the second expression in i). □

We define positivity and self-adjointness on $L_1(\mathcal{M}, L_\infty(\mathcal{N}))$ and $L_1(\mathcal{M}, L_\infty(\mathcal{N}))$ as follows. We say $\rho \in L_1(\mathcal{M}, L_\infty(\mathcal{N}))$ is positive (resp. self-adjoint) if there exists a positive (resp. self-adjoint) sequence $\rho_n \in \mathcal{M}_0 \otimes \mathcal{N}$ such that $\rho_n \to \rho$ in norm. For two self-adjoint operators ρ and σ, we say $\rho \leq \sigma$ if $\sigma - \rho$ is positive. The positivity and self-adjointness in $L_\infty(\mathcal{M}, L_1(\mathcal{N}))$ are defined similarly as limits of sequences in $\mathcal{M} \overline{\otimes} \mathcal{N}_0$. The next lemma shows that the $L_1(\mathcal{M}, L_\infty(\mathcal{N}))$ norm for positive elements correspond to the conditional min entropy H_{min}. Recall that $\rho \in L_1(\mathcal{M})$ is a density operator if $\rho \geq 0$ and $\tau_{\mathcal{M}}(\rho) = 1$.

Lemma 3.2. *Let $x \in L_1(\mathcal{M}, L_\infty(\mathcal{N}))$ be self-adjoint. Define*

$$\lambda(x) = \inf\{\lambda \mid x \leq \lambda\sigma \otimes 1 \ \textit{for some density operator} \ \sigma \in L_1(\mathcal{M})\} \ .$$

Then

i) $\lambda(x) \leq \| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))}$,
ii) $\lambda(x) = \| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))}$ *if x is positive.*

Proof. We first discuss the case $x \in \mathcal{M}_0 \otimes \mathcal{N}$. Suppose $x = (a \otimes 1)y(a^* \otimes 1)$ for some self-adjoint $y \in \mathcal{M} \otimes \mathcal{N}$ and $\| a \|_2 = 1$ with $a \in \mathcal{M}_0$. Then $x \leq \| y \|_\infty aa^* \otimes 1$, where $aa^* \in \mathcal{M}_0$. Then by Lemma 3.1, we have

$$\lambda(x) \leq \| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))}$$

for $x \in \mathcal{M}_0 \otimes \mathcal{N}$. Note that if $x_1 \leq \lambda_1 \sigma_1 \otimes 1$ and $x_2 \leq \lambda_2 \sigma_2 \otimes 1$, then

$$x_1 + x_2 \leq (\lambda_1 \sigma_1 + \lambda_2 \sigma_2) \otimes 1 = (\lambda_1 + \lambda_2)(\frac{\lambda_1}{\lambda_1 + \lambda_2}\sigma_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2}\sigma_2) \otimes 1 \ . \tag{3.2}$$

This implies

$$\lambda(x_1 + x_2) \leq \lambda(x_1) + \lambda(x_2) \ , \ |\lambda(x_1) - \lambda(x_2)| \leq \max\{\lambda(x_1 - x_2) \ , \lambda(x_2 - x_1)\}$$

For general x and $\epsilon > 0$, we can find a self-adjoint sequence $x_n \in \mathcal{M}_0 \otimes \mathcal{N}$ such that $x = \sum_{n=1}^{\infty} x_n$ converges absolutely and

$$\sum_n \| x_n \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} \leq \| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} + \epsilon \ .$$

By the same argument of (3.2), we have λ is also countably sub-additive. Indeed, given $\delta > 0$, we choose for each n a density operator σ_n such that $x_n \leq (\lambda_n + 2^{-n}\delta)\sigma_n \otimes 1$. Then

$$x = \sum_{n=1} x_n \leq \sum_{n=1} (2^{-n}\delta + \lambda_n)\sigma_n \otimes 1 = (\delta + \sum_{n=1} \lambda_n)(\sum_{n=1} \frac{2^{-n}\delta + \lambda_n}{\delta + \sum_{n=1} \lambda_n}\sigma_n) \otimes 1$$

which implies $\lambda(x) \leq \sum_n \lambda(x_n)$ because $\sum_{n=1} \frac{2^{-n}\delta + \lambda_n}{\delta + \sum_n \lambda_n}\sigma_n$ is a density operator and δ is arbitrary. Therefore

$$\lambda(x) \leq \sum_n \lambda(x_n) \leq \sum_n \| x_n \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} \leq \| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} + \epsilon.$$

Since ϵ is arbitrary, this proves i). To prove ii), first let $x \in e\mathcal{M}e \otimes \mathcal{N}$ be positive. If $x \leq \lambda\sigma \otimes 1$ for some density operator $\sigma \in \mathcal{M}_0$, we can choose $\tilde{\sigma} = \sigma + \delta e$ invertible in $e\mathcal{M}e$ with $\tau_{\mathcal{M}}(\tilde{\sigma}) \leq 1 + \epsilon$. Then, we have

$$0 \leq y = \tilde{\sigma}^{-\frac{1}{2}} \cdot x \cdot \tilde{\sigma}^{-\frac{1}{2}} \leq \lambda 1 \ , \ x = \tilde{\sigma}^{\frac{1}{2}} \cdot y \cdot \tilde{\sigma}^{\frac{1}{2}} \ .$$

Hence, we obtain

$$\| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} \leq \inf\{\lambda \mid x \leq \lambda\sigma \otimes 1 \ , \ \sigma \in \mathcal{M}_0 \text{ density operator}\}. \tag{3.3}$$

Then it suffices to show that $\lambda(x)$ equals the right hand side. Suppose $x \le \lambda\sigma \otimes 1$ for some density operator $\sigma \in L_1(\mathcal{M})$. Without losing generality, we can assume that σ is invertible and supported on $e\mathcal{M}e$. By definition, for any positive $y \in \mathcal{M}\overline{\otimes}\mathcal{N}_0$,

$$\lambda\tau((\sigma \otimes 1)y) \ge \tau(xy) .$$

This implies $\| (\sigma^{-\frac{1}{2}} \otimes 1)x(\sigma^{-\frac{1}{2}} \otimes 1) \| \le \lambda + \epsilon$. We modify σ to a density operator $\tilde{\sigma} \in \mathcal{M}$ such that $\tilde{\sigma} = \sigma e_{[0,k)} + k e_{[k,\infty)}$ where $e_{[0,k]}$ is the spectral projection of σ for the interval $[0, k]$. Note that for any $z \ge 0$,

$$(\min\{z,k\})^{-1} - z^{-1} = (z - \min\{z,k\})/z(\min\{z,k\}) = \begin{cases} 0, & \text{if } z \le k \\ \frac{z-k}{zk}, & \text{if } z > k. \end{cases}$$

Then by functional calculus, $\| \tilde{\sigma}^{-1} - \sigma^{-1} \|_\infty \le \frac{1}{k}$. Therefore,

$$\begin{aligned} &\| (\tilde{\sigma}^{-\frac{1}{2}} \otimes 1)x(\tilde{\sigma}^{-\frac{1}{2}} \otimes 1) \| = \| x^{\frac{1}{2}} (\tilde{\sigma}^{-1} \otimes 1)x^{\frac{1}{2}} \| \\ = &\| x^{\frac{1}{2}} (\sigma^{-1} \otimes 1)x^{\frac{1}{2}} \| + \| x^{\frac{1}{2}} (\tilde{\sigma}^{-1} \otimes 1 - \sigma^{-1} \otimes 1)x^{\frac{1}{2}} \| \le (\lambda + \epsilon) + \frac{1}{k} \| x \|_\infty . \end{aligned}$$

By choosing k large enough, we have

$$x \le (\lambda + 2\epsilon)\tilde{\sigma} \otimes 1$$

where $\| \tilde{\sigma} \|_\infty \le k$ hence belongs to $\mathcal{M}_0$. This proves ii) for positive $x \in \mathcal{M}_0 \otimes \mathcal{N}$. For a general positive element $x \in L_1(\mathcal{M}, L_\infty(\mathcal{N}))$, let x_n be a sequence of positive operators in $\mathcal{M}_0 \otimes \mathcal{N}$ such that $\| x_n - x \|_{L_1(\mathcal{M},L_\infty(\mathcal{N}))} \to 0$. Then by i), we know

$$\lambda(x) = \lim_n \lambda(x_n) = \lim_n \| x_n \|_{L_1(\mathcal{M},L_\infty(\mathcal{N}))} = \| x \|_{L_1(\mathcal{M},L_\infty(\mathcal{N}))} ,$$

which completes the proof. □

The above lemma generalizes the definition of H_{min} entropy to semi-finite von Neumann algebras. For a bipartite density operator $\rho \in L_1(\mathcal{M}\overline{\otimes}\mathcal{N})$, the H_{min} entropy of ρ conditional on $\mathcal{M}$ can be defined as

$$H_{min}(\mathcal{N}|\mathcal{M})_\rho := \begin{cases} -\log \| \rho \|_{L_1(\mathcal{M},L_\infty(\mathcal{N}))}, & \text{if } \rho \in L_1(\mathcal{M}, L_\infty(\mathcal{N})) \\ -\infty, & \text{otherwise.} \end{cases}$$

The next lemma shows that $\lambda(x)$ is attained by the duality

$$L_\infty(\mathcal{M}, L_1(\mathcal{N})) \subset \Big(L_1(\mathcal{M}, L_\infty(\mathcal{N}))\Big)^* .$$

Lemma 3.3. *Let $\rho \in L_1(\mathcal{M}, L_\infty(\mathcal{N}))$ be self-adjoint. Then*

$$\lambda(\rho) = \sup\{ \tau(x\rho) \mid x \in \mathcal{M}\overline{\otimes}\mathcal{N}_0, x \geq 0, \| x \|_{L_\infty(\mathcal{M}, L_1(\mathcal{N}))} = 1\}.$$

In particular, if $\rho \in L_1(\mathcal{M}, L_\infty(\mathcal{N}))$ is positive, then

$$\| \rho \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} = \sup\{ \tau(x\rho) \mid x \in \mathcal{M}\overline{\otimes}\mathcal{N}_0, x \geq 0, \| x \|_{L_\infty(\mathcal{M}, L_1(\mathcal{N}))} = 1\}.$$

Proof. In the proof of Lemma 3.2, we have shown $|\lambda(\rho_1) - \lambda(\rho_2)| \leq \| \rho_1 - \rho_2 \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))}$. Then by density argument, it suffices to consider $\rho \in \mathcal{M}_0 \otimes \mathcal{N}$. Let $\rho \in e\mathcal{M}e \otimes \mathcal{N}$ for some $\tau_{\mathcal{M}}(e) < \infty$. We can assume $\mathcal{M}$ is finite by restricting to $e\mathcal{M}e$. Let us first consider the case that $\mathcal{N}$ is finite. We use a standard Grothendieck-Pietsch separation argument. Let λ be a positive number such that $\lambda < \lambda(\rho)$. We know from (3.3) that for any density operator $\sigma \in \mathcal{M}_0$, $\lambda(\sigma \otimes 1) - \rho$ is not positive and hence has nontrivial negative part. Then there exists a positive $x \in L_\infty(\mathcal{M}\overline{\otimes}\mathcal{N})$ such that $\| x \|_\infty = 1$ and

$$\tau(\rho x) - \lambda \tau((\sigma \otimes 1)x) > 0.$$

Consider the weak*-compact subset

$$B = \{x \in \mathcal{M}\overline{\otimes}\mathcal{N} | \; \| x \|_\infty \leq 1, x \geq 0\}.$$

For each positive operator $\sigma \in \mathcal{M}_0$ with $\tau_{\mathcal{M}}(\sigma) \leq 1$, we define the function $f_\sigma : B \to \mathbb{R}$ as follows (we suppress the dependence on ρ since ρ is fixed)

$$f_\sigma(x) = \tau(\rho x) - \lambda \tau((\sigma \otimes 1)x) \; , x \in B.$$

These f_σ are continuous with respect to weak*-topology on B because $\mathcal{N}$ is finite and both $\sigma \otimes 1$ and ρ are in $L_1(\mathcal{M}\overline{\otimes}\mathcal{N})$. Denote $C(B, \mathbb{R})$ as the space w^*-continuous real function on B. We define two subsets

$$\mathcal{F} = \{f_\sigma \in C(B, \mathbb{R}) \mid \sigma \in \mathcal{M}_0, \sigma \geq 0, \tau_{\mathcal{M}}(\sigma) \leq 1 \}$$
$$\mathcal{F}_- = \{f \in C(B, \mathbb{R}) \mid \sup f < 0\}.$$

Both $\mathcal{F}$ and $\mathcal{F}_-$ are convex sets and $\mathcal{F}_-$ is open. Moreover, $\mathcal{F}$ and $\mathcal{F}_-$ are disjoint because for each $f_\sigma \in \mathcal{F}$, $\sup_x f_\sigma(x) > 0$. Then by the Hahn-Banach Theorem, there exists a norm-one linear function $\psi : C(B, \mathbb{R}) \to \mathbb{R}$ and a real number r such that for any $f_- \in \mathcal{F}_-$ and $f_\sigma \in \mathcal{F}$,

$$\phi(f_-) < r \leq \phi(f_\sigma) \; .$$

Because $\mathcal{F}_-$ is a cone, $r \geq 0$. Similarly, $r \leq 0$ because for any $0 < \delta < 1$, $\delta\mathcal{F} \subset \mathcal{F}$. Then $r = 0$ and ϕ is a positive linear functional because $\phi(f_-) < 0$ for any $f_- \in \mathcal{F}_-$. By the Riesz Representation Theorem, ϕ is given by a Borel probability measure μ on B. Namely,

$$\phi(f) = \int_B f(x)d\mu(x) \ .$$

Denote $x_0 = \int_B x d\mu(x)$. We have for any positive operator $\sigma \in \mathcal{M}_0$ with $\tau_{\mathcal{M}}(\sigma) \le 1$, that

$$\phi(f_\sigma) = \int_B f_\sigma(x)d\mu(x) = \int_B \tau(\rho x) - \lambda\tau((\sigma\otimes 1)x)d\mu(x) = \tau(\rho x_0) - \lambda\tau((\sigma\otimes 1)x_0) \ge 0.$$

By Lemma 3.1,

$$\tau(\rho x_0) \ge \lambda \sup\{\tau((\sigma\otimes 1)x_0)\ |\sigma \in \mathcal{M}_0,\ \tau_{\mathcal{M}}(\sigma_0) \le 1, \sigma \ge 0\} = \lambda \parallel x_0 \parallel_{L_\infty(\mathcal{M},L_1(\mathcal{N}))} \ .$$

Normalizing $\tilde{x}_0 = \parallel x_0 \parallel^{-1}_{L_\infty(\mathcal{M},L_1(\mathcal{N}))} x_0$, we have $\tau(\rho\tilde{x}_0) \ge \lambda$. This proves the case for finite $\mathcal{N}$. For semi-finite $\mathcal{N}$, we define for each projection $p \in \mathcal{N}$ with $\tau_{\mathcal{N}}(p) < \infty$,

$$\lambda_p = \inf\{\ \lambda \mid (1\otimes p)\rho(1\otimes p) \le \lambda\sigma\otimes p \text{ for some density operator } \sigma \in \mathcal{M}_0\} \ .$$

For two projections $p_1 \le p_2$, we have $\lambda_{p_1} \le \lambda_{p_2}$. Thus λ_p is monotone non-decreasing over p for the natural ordering. Since each λ_p is attainable based on the finite case, it suffices to show that $\lim_p \lambda_p \ge \lambda(\rho)$. Write $\lambda_1 = \lim_p \lambda_p$. Given $\epsilon > 0$, for each projection p we choose a density operator $\sigma_p \in \mathcal{M}$ such that

$$(1\otimes p)\rho(1\otimes p) \le (\lambda_p + \epsilon)\sigma_p \otimes p \le (\lambda_1 + \epsilon)\sigma_p \otimes p \ ,$$

where both $(1\otimes p)\rho(1\otimes p)$ and $\sigma_p \otimes p$ belongs to $\mathcal{M}\otimes p\mathcal{N}p$. We denote $\psi_p : \mathcal{M}\overline{\otimes}\mathcal{N} \to \mathbb{C}$ as the normal linear functional $\psi_p(x) = \tau((1\otimes p)\rho(1\otimes p)x)$ and $\xi_p : \mathcal{M} \to \mathbb{C}$ as the normal state $\xi_p(y) = \tau_{\mathcal{M}}(\sigma_p y)$. Let ξ be a weak*-limit point of ξ_p in $\mathcal{M}^*$. Denote $\tau_{p\mathcal{N}p}$ as the induced finite trace on $p\mathcal{N}p$. For any positive $\mathcal{M}\otimes p\mathcal{N}p$ and finite projection $q \ge p$,

$$\psi_p(x) = \psi_q(x) \le (\lambda_1 + \epsilon)\xi_q \otimes \tau_{q\mathcal{N}q}(x) = (\lambda_1 + \epsilon)\xi_q \otimes \tau_{p\mathcal{N}p}(x) \ .$$

Taking the limit over q, we have

$$\psi_p(x) \le (\lambda_1 + \epsilon)\lim_q \xi_q \otimes \tau_{p\mathcal{N}p}(x) = (\lambda_1 + \epsilon)\xi \otimes \tau_{p\mathcal{N}p}(x) \ . \tag{3.4}$$

Note that ξ is a state on $\mathcal{M}$ and it decomposes into a normal part and a singular part $\xi = \xi_n + \xi_s$. Then from (3.4) we have

$$\psi_p - (\lambda_1 + \epsilon)\xi_n \otimes \tau_{p\mathcal{N}p} \le (\lambda_1 + \epsilon)\xi_s \otimes \tau_{p\mathcal{N}p} \ .$$

Note that the right hand side $\xi_s \otimes \tau_{p\mathcal{N}p}$ is a singular positive linear functional (which is clear from [25, Corollary 3.11]). By normality, $\psi_p \leq (\lambda_1 + \epsilon)\xi_n \otimes \tau_{p\mathcal{N}p}$ as self-adjoint linear functionals for each p. Namely, for any positive $x \in \mathcal{M} \otimes p\mathcal{N}p$,

$$\tau((1 \otimes p)\rho(1 \otimes p)x) \leq (\lambda_1 + \epsilon)\tau((\sigma \otimes 1)x) \ .$$

Now we apply the same trick in Lemma 3.2 to modify σ to be an invertible density operator $\mathcal{M}$ with bounded inverse $\sigma^{-1} \in \mathcal{M}$. First, since $\mathcal{M}$ is finite, σ can be replaced by an invertible density $\sigma_1 = \frac{1}{\tau_{\mathcal{M}}(\sigma)+\delta\tau_{\mathcal{M}}(1)}(\sigma + \delta 1)$. By choosing $\delta > 0$ small enough, we have

$$\begin{aligned}\tau((1 \otimes p)\rho(1 \otimes p)x) &\leq (\lambda_1 + \epsilon)\tau((\sigma \otimes 1)x) \leq (\tau_{\mathcal{M}}(\sigma) + \delta\tau_{\mathcal{M}}(1))(\lambda_1 + \epsilon)\tau((\sigma_1 \otimes 1)x) \\ &\leq (\lambda_1 + 2\epsilon)\tau((\sigma_1 \otimes 1)x) \ .\end{aligned}$$

Then by choosing $x = (\sigma_1^{-\frac{1}{2}} \otimes p)y(\sigma_1^{-\frac{1}{2}} \otimes p)$ for positive $y \in \mathcal{M}\overline{\otimes}p\mathcal{N}p$, we have

$$\tau((\sigma_1^{-\frac{1}{2}} \otimes p)\rho((\sigma_1^{-\frac{1}{2}} \otimes p)y) \leq (\lambda_1 + 2\epsilon)\tau((\sigma_1 \otimes 1)(\sigma_1^{-\frac{1}{2}} \otimes p)y(\sigma_1^{-\frac{1}{2}} \otimes p)) = (\lambda_1 + 2\epsilon)\tau(y) \ ,$$

which implies $(\sigma_1^{-\frac{1}{2}} \otimes p)\rho(\sigma_1^{-\frac{1}{2}} \otimes p) \leq (\lambda_1 + 2\epsilon)1$. Take $\sigma_2 = g_k(\sigma_1)$ as the functional calculus of $g_k(x) = \min\{x, k\}$. We have $\| \sigma_1^{-\frac{1}{2}} - \sigma_2^{-\frac{1}{2}} \|_\infty \leq k^{-\frac{1}{2}}$ and

$$\begin{aligned}&\| (\sigma_1^{-\frac{1}{2}} \otimes p)\rho(\sigma_1^{-\frac{1}{2}} \otimes p) - (\sigma_2^{-\frac{1}{2}} \otimes p)\rho(\sigma_2^{-\frac{1}{2}} \otimes p) \| \\ \leq &\| (\sigma_1^{-\frac{1}{2}} \otimes p)\rho(\sigma_1^{-\frac{1}{2}} \otimes p) - (\sigma_1^{-\frac{1}{2}} \otimes p)\rho(\sigma_2^{-\frac{1}{2}} \otimes p) \| \\ &+ \| (\sigma_1^{-\frac{1}{2}} \otimes p)\rho(\sigma_2^{-\frac{1}{2}} \otimes p) - ((\sigma_2^{-\frac{1}{2}} \otimes p)\rho(\sigma_2^{-\frac{1}{2}} \otimes p) \| \\ \leq &k^{-\frac{1}{2}} \| \rho \|_\infty \| \sigma_1^{-\frac{1}{2}} \|_\infty + k^{-\frac{1}{2}} \| \rho \|_\infty \| \sigma_2^{-\frac{1}{2}} \|_\infty \\ \leq &2k^{-\frac{1}{2}}(\delta^{-1} - 1) \| \rho \|_\infty \ .\end{aligned}$$

By choosing k large enough (depending on ϵ, δ and $\| \rho \|_\infty$), we have

$$\begin{aligned}&(\sigma_2^{-\frac{1}{2}} \otimes p)\rho(\sigma_2^{-\frac{1}{2}} \otimes p) \\ =&(\sigma_1^{-\frac{1}{2}} \otimes p)\rho(\sigma_1^{-\frac{1}{2}} \otimes p) + \Big((\sigma_2^{-\frac{1}{2}} \otimes p)\rho(\sigma_2^{-\frac{1}{2}} \otimes p) - (\sigma_1^{-\frac{1}{2}} \otimes p)\rho(\sigma_1^{-\frac{1}{2}} \otimes p)\Big) \\ \leq&(\lambda_1 + 2\epsilon)1 + \| (\sigma_2^{-\frac{1}{2}} \otimes p)\rho(\sigma_2^{-\frac{1}{2}} \otimes p) - (\sigma_1^{-\frac{1}{2}} \otimes p)\rho(\sigma_1^{-\frac{1}{2}} \otimes p) \| \, 1 \\ \leq&(\lambda_1 + 2\epsilon)1 + 2k^{-\frac{1}{2}}(\delta^{-1} - 1) \| \rho \|_\infty \, 1 \leq (\lambda_1 + 3\epsilon)1.\end{aligned}$$

This implies that for each finite projection $p \in \mathcal{N}$,

$$(1 \otimes p)\rho(1 \otimes p) \leq (\lambda_1 + 3\epsilon)\sigma_2 \otimes p \leq (\lambda_1 + 3\epsilon)\sigma_2 \otimes 1 \ ,$$

as self-adjoint operators in $\mathcal{M} \otimes \mathcal{N}$. Then for each p and positive $x \in L_1(\mathcal{M}\overline{\otimes} p\mathcal{N}p)$,

$$\tau(\rho x) = \tau(\rho(1 \otimes p)x(1 \otimes p)) = \tau((1 \otimes p)\rho(1 \otimes p)x) \leq (\lambda_1 + 3\epsilon)\tau((\sigma_2 \otimes 1)x) \ .$$

Because $\cup_p L_1(\mathcal{M}\overline{\otimes} p\mathcal{N}p)$ is dense in $L_1(\mathcal{M}\overline{\otimes}\mathcal{N})$, we obtain that $\rho \leq (\lambda_1 + 3\epsilon)\sigma_2 \otimes 1$ as self-adjoint operators, which implies $\lambda(\rho) \leq \lambda_1$ as ϵ is arbitrary. That completes the proof. □

This next lemma is an analogue of the Choi matrix.

Lemma 3.4. *There is a contraction*

$$\begin{aligned} &L_\infty(\mathcal{M}, L_1(\mathcal{N})) \longrightarrow CB(L_1(\mathcal{M}^{op}), L_1(\mathcal{N})) \ , \\ &x \mapsto T_x \in CB(L_1(\mathcal{M}^{op}), L_1(\mathcal{N})) \ , \ \ T_x(\rho^{op}) = \tau_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{N}}((\rho \otimes 1)x). \end{aligned}$$

Moreover,

i) *for any positive x, $\| x \|_{L_\infty(\mathcal{M}, L_1(\mathcal{N}))} = \| T_x \|_{cb}$.*
ii) *T_x is completely positive if and only if x is positive.*
iii) *T_x is trace preserving if and only if $\mathrm{id} \otimes \tau_{\mathcal{N}}(x) = 1_{\mathcal{M}}$.*
iv) *for $S \in CB(L_1(\mathcal{N}), L_1(\mathcal{N}))$, $S \circ T_x = T_{\mathrm{id} \otimes S(x)}$.*
v) *for any finite rank $T : L_1(\mathcal{M}^{op}) \to L_1(\mathcal{N})$, $T = T_x$ for some $x \in \mathcal{M} \otimes L_1(\mathcal{N})$.*

Proof. By a density argument, it suffices to discuss $x \in \mathcal{M}\overline{\otimes} p\mathcal{N}p$ with $\tau_{\mathcal{N}}(p) < \infty$. Given $\rho \in L_1(\mathcal{M})$, $(\rho \otimes 1_{\mathcal{N}})x = (\rho \otimes p)x \in L_1(\mathcal{M}\overline{\otimes} p\mathcal{N}p)$ hence the map $T_x(\rho^{op}) = \tau_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{N}}((\rho \otimes 1_{\mathcal{N}})x) \in L_1(\mathcal{N})$ is well defined. For $\| \rho^{op} \|_{L_1(\mathcal{M}^{op})} = 1$, we have $\rho = ba$ for some $\| a \|_2 = \| b \|_2 = 1$. Note that $\tau_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{N}}((ba \otimes 1_{\mathcal{N}})x) = \tau_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{N}}((a \otimes 1)x(b \otimes 1))$. Then

$$\| T_x(\rho^{op}) \|_{L_1(\mathcal{N})} \leq \| a \cdot x \cdot b \|_{L_1(\mathcal{M}\overline{\otimes}\mathcal{N})} \leq \| x \|_{L_\infty(\mathcal{M}, L_1(\mathcal{N}))} \ \ .$$

Let $e_{i,j}$ be the matrix units in M_n and S_2^n be the Schatten 2-class. For the completely bounded norm, we first note that $\mathrm{id}_{M_n} \otimes T_x = T_{\phi \otimes x} : L_1(M_n(\mathcal{M})^{op}) \to L_1(M_n(\mathcal{N}))$ where $\phi = \sum_{i,j} e_{i,j} \otimes e_{i,j} \in M_n \otimes M_n$ and $\phi \otimes x \in L_\infty(M_n(\mathcal{M}), L_1(M_n(\mathcal{N})))$. Here ϕ is the Choi matrix for $\mathrm{id} : M_n \to M_n$. Given $\| a \|_{S_2^n(L_2(\mathcal{M}))} = \| b \|_{S_2^n(L_2(\mathcal{M}))} = 1$, we can write $a = \sum_k \omega_k \otimes a_k$ such that ω_k (resp. a_k) orthogonal in S_2^n (resp. $L_2(\mathcal{M})$) and $\| a_k \|_2 = 1, \sum_k \| \omega_k \|_2^2 = 1$ and similarly for $b = \sum_l \sigma_l \otimes b_l$. Then, by $\sum_k \| \omega_k \|_2^2 = \sum_l \| \sigma_l \|_2^2 = 1$,

$$\begin{aligned} \mathrm{id}_{M_n} \otimes T_x(ab) = T_{\phi \otimes x}(ab) &= \sum_{k,l} \Big(\mathrm{tr} \otimes \mathrm{id}_{M_n}((\omega_k \sigma_l \otimes 1)\phi) \Big) \otimes \Big(\tau_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{N}}((a_k b_l \otimes 1)x) \Big) \\ &= \sum_{k,l} \Big(\mathrm{tr} \otimes \mathrm{id}_{M_n}(\omega_k \cdot \phi \cdot \sigma_l) \Big) \otimes \Big(\tau_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{N}}(a_k \cdot x \cdot b_l) \Big) \end{aligned}$$

$$= \mathrm{tr} \otimes \mathrm{id}_{M_n} \otimes \tau_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{N}} \Big(\sum_{k,l} (\omega_k \cdot \phi \cdot \sigma_l) \otimes (a_k \cdot x \cdot b_l) \Big) .$$

Using bracket notation,

$$\phi = |h\rangle\langle h| \ , |h\rangle = \sum_{i=1} |i\rangle|i\rangle$$

where $\{|i\rangle\}$ is the standard basis in l_2^n. We have

$$\| \omega_k \cdot \phi \cdot \sigma_l \|_1 = \| \omega_k \otimes 1|h\rangle \|_{l_2} \| \sigma_l^* \otimes 1|h\rangle \|_{l_2} = \| \omega_k \|_2 \| \sigma_l \|_2 \ .$$

Here $\| \cdot \|_{l_2}$ is the vector norm and

$$\| \omega_k \otimes 1|h\rangle \|_{l_2}^2 = \langle h|\omega_k^*\omega_k \otimes 1|h\rangle = \mathrm{tr}(\omega_k^*\omega_k) = \| \omega_k \|_2 \ .$$

Therefore,

$$\begin{aligned} \| \sum_{k,l} (\omega_k \cdot \phi \cdot \sigma_l) \otimes (a_k \cdot x \cdot b_l) \|_1 &\leq \sum_{k,l} \| \omega_k \cdot \phi \cdot \sigma_l \|_1 \| a_k \cdot x \cdot b_l \|_1 \\ &\leq \sum_{k,l} \| \omega_k \|_2 \| \sigma_l \|_2 \| x \|_{L_\infty(\mathcal{M}, L_1(\mathcal{N}))} \leq \| x \|_{L_\infty(\mathcal{M}, L_1(\mathcal{N}))} \ . \end{aligned}$$

By $\| \mathrm{id}_{M_n} \otimes T_x(ab) \|_1 \leq \| \sum_{k,l} (\omega_k \cdot \phi \cdot \sigma_l) \otimes (a_k \cdot x \cdot b_l) \|_1$, this implies

$$\| \mathrm{id}_{M_n} \otimes T_x : L_1(M_n(\mathcal{M})^{op}) \to L_1(M_n(\mathcal{N})) \| \leq \| x \|_{L_\infty(\mathcal{M}, L_1(\mathcal{N}))} \ .$$

Then by $L_1(M_n(\mathcal{M})^{op}, \mathrm{tr} \otimes \tau_{\mathcal{M}}) \cong S_1^n(L_1(\mathcal{M}^{op}))$ and [21, Lemma 1.2], we obtain

$$\begin{aligned} \| T_x : L_1(\mathcal{M}^{op}) \to L_1(\mathcal{N}) \|_{cb} &= \sup_n \| \mathrm{id}_n \otimes T_x : S_1^n(L_1(\mathcal{M}^{op})) \to S_1^n(L_1(\mathcal{N})) \| \\ &= \sup_n \| \mathrm{id}_{M_n} \otimes T_x : L_1(M_n(\mathcal{M})^{op}) \to L_1(M_n(\mathcal{N})) \| = \| x \|_{L_\infty(\mathcal{M}, L_1(\mathcal{N}))} \ . \end{aligned}$$

Now suppose x is positive. For a density operator $\rho \in L_1(\mathcal{M}^{op})$,

$$T_x(\rho^{op}) = \tau_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{N}}((\rho \otimes 1)x) = \tau_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{N}}(\rho^{\frac{1}{2}} \cdot x \cdot \rho^{\frac{1}{2}}) \geq 0$$

Applying the same argument for $\phi \otimes x$, we know T_x is completely positive. Then taking the supremum over all density operators ρ,

$$\sup_\rho \| T_x(\rho^{op}) \|_1 = \sup_\rho \tau_{\mathcal{M}} \otimes \tau_{\mathcal{N}}(\rho^{\frac{1}{2}} \cdot x \cdot \rho^{\frac{1}{2}}) = \| x \|_{L_\infty(\mathcal{M}, L_1(\mathcal{N}))} \ .$$

Thus for positive x, we find $\| T_x \|_{cb} \leq \| x \|_{L_\infty(\mathcal{M}, L_1(\mathcal{N}))} = \| T_x \| \leq \| T_x \|_{cb}$, which proves i). For ii), we note that the “if” statement follows by the construction of T_x. To prove the "only if" statement, we conversely suppose x is not positive and we show that T_x is not completely positive. There exists a vector $h = \sum_{j=1}^{n} a_j \otimes b_j \in L_2(\mathcal{M}) \otimes_2 L_2(\mathcal{N})$ such that $a_j \in \mathcal{M}_0, b_j \in \mathcal{N}_0$, $\langle h, xh\rangle \ngeq 0$ (that is, the inner product is either not real or is negative). This means

$$\begin{aligned}\langle h, xh\rangle &= \sum_{i,j=1}^{n} \tau_{\mathcal{M}} \otimes \tau_{\mathcal{N}}((a_i^* \otimes b_i^*)x(a_j \otimes b_j)) \\ &= \tau_{\mathcal{N}}\Big(\sum_{i,j=1}^{n} b_i^* \tau_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{N}}\left((a_i^* \otimes 1)x(a_j \otimes 1)\right) b_j\Big) \ngeq 0 .\end{aligned}$$

Thus, $\Big(\tau_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{N}}\left((a_i^* \otimes 1)x(a_j \otimes 1)\right)\Big)_{i,j=1}^{n}$ is not positive in $S_1^n(L_1(\mathcal{N}))$. Note that $\omega^{op} = \sum_{i,j=1}^{n} e_{i,j} \otimes (a_i^*)^{op}(a_j)^{op} = \sum_{i,j=1}^{n} e_{i,j} \otimes (a_j a_i^*)^{op}$ is positive in $S_1^n(L_1(\mathcal{M}^{op}))$. Then T_x is not completely positive because

$$\begin{aligned} id_n \otimes T_x(\omega^{op}) &= \sum_{i,j=1}^{n} e_{i,j} \otimes \Big(\tau_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{N}}((a_j a_i^* \otimes 1)x)\Big) \\ &= \sum_{i,j} e_{i,j} \otimes \Big(\tau_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{N}}((a_i^* \otimes 1)x(a_j \otimes 1))\Big) \ngeq 0 .\end{aligned}$$

This proves ii). For any $\rho \in L_1(\mathcal{M})$, by Fubini's theorem,

$$\tau_{\mathcal{N}}(T_x(\rho^{op})) = \tau_{\mathcal{N}}\Big(\tau_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{N}}((\rho \otimes 1)x)\Big) = \tau_{\mathcal{M}}\Big(\rho\, \mathrm{id}_{\mathcal{M}} \otimes \tau_{\mathcal{N}}(x)\Big) .$$

Thus T_x is trace preserving if and only if $\mathrm{id}_{\mathcal{M}} \otimes \tau_{\mathcal{N}}(x) = 1$. This verifies iii). For iv), let $S \in CB(L_1(\mathcal{N}), L_1(\mathcal{N}))$. For $\rho \in L_1(\mathcal{M})$,

$$\begin{aligned} S \circ T_x(\rho^{op}) &= S\Big(\tau_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{N}}((\rho \otimes 1)x)\Big) \\ &= \tau_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{N}}\Big((\rho \otimes 1)\, \mathrm{id} \otimes S(x)\Big) \\ &= T_{\mathrm{id} \otimes S(x)}(\rho^{op}) .\end{aligned}$$

Finally, for v), let T be a finite rank map from $L_1(\mathcal{M}^{op})$ to $L_1(\mathcal{N})$. Then there exists finite $y_j \in \mathcal{M}$ and $z_j \in L_1(\mathcal{N})$ such that $T(\rho^{op}) = \sum_{j=1}^{n} \tau_{\mathcal{M}}(\rho y_j) z_j$. Then $T = T_x$ for $x = \sum_{j=1}^{n} y_j \otimes z_j$ which belongs to $\mathcal{M} \otimes L_1(\mathcal{N})$. That completes the proof. □

The above lemma gives a contraction

$$L_\infty(\mathcal{M}, L_1(\mathcal{N})) \to CB(L_1(\mathcal{M}^{op}), L_1(\mathcal{N})) \subset CB(\mathcal{N}^{op}, \mathcal{M}) .$$

Note that $CB(\mathcal{N}^{op}, \mathcal{M})_* = L_1(\mathcal{M}^{op}) \widehat{\otimes} L_\infty(\mathcal{N}^{op})$. The pairings for an algebraic tensor $\rho^{op} = \sum_{j=1}^{n} y_j^{op} \otimes z_j^{op} \in L_1(\mathcal{M})^{op} \otimes \mathcal{N}^{op}$ to $L_\infty(\mathcal{M}, L_1(\mathcal{N}))$ and to $CB(\mathcal{N}^{op}, \mathcal{M})$ coincide,

$$\begin{aligned}\langle x, \rho^{op}\rangle_{(L_\infty(\mathcal{M},L_1(\mathcal{N})),L_1(\mathcal{M}^{op},L_\infty(\mathcal{N}^{op}))} &= \tau_{\mathcal{M}} \otimes \tau_{\mathcal{N}}(x \sum_{j=1}^{n} y_j \otimes z_j) \\ &= \tau_{\mathcal{N}}\Big(\sum_j z_j \tau_{\mathcal{M}} \otimes \mathrm{id}((y_j \otimes 1)x)\Big) \\ &= \tau_{\mathcal{N}}\Big(\sum_j z_j T_x(y_j)\Big) = \tau_{\mathcal{N}}\Big(\sum_j T_x^{\dagger}(z_j) y_j\Big) \\ &= \langle T_x, \rho^{op}\rangle_{(CB(\mathcal{N}^{op},\mathcal{M}),L_1(\mathcal{M}^{op})\widehat{\otimes} L_\infty(\mathcal{N}^{op}))}.\end{aligned}$$

Then, for an algebraic tensor $x = \sum_{j=1}^{n} y_j \otimes z_j \in L_1(\mathcal{M}) \otimes \mathcal{N}$, we have

$$\| x \|_{L_1(\mathcal{M},L_\infty(\mathcal{N}))} \leq \| x \|_{L_1(\mathcal{M})\widehat{\otimes}\mathcal{N}} .$$

It was proved in [21, Theorem 3.4] that for hyperfinite $\mathcal{M}$ (i.e. $\mathcal{M} = \overline{(\cup_\alpha \mathcal{M}_\alpha)}^{w^*}$, where the union is of an increasing net of finite-dimensional subalgebras $\mathcal{M}_\alpha$), we have the isometric isomorphism

$$L_1(\mathcal{M}, L_\infty(\mathcal{N})) \cong L_1(\mathcal{M})\widehat{\otimes}\mathcal{N}. \tag{3.5}$$

We shall show that this isomorphism is characterized by the injectivity of $\mathcal{M}$. Recall that a von Neumann algebra $\mathcal{M}$ is *injective* if there exists an embedding $\mathcal{M} \subset B(H)$ and a completely positive projection $P : B(H) \to \mathcal{M}$ with $\| P \|= 1$. An equivalent condition is the weak* completely positive approximation property (weak*-CPAP). A von Neumann algebra $\mathcal{M}$ has weak*-CPAP if there exists a net of normal finite rank completely positive maps Φ_α such that for any $x \in \mathcal{M}$, $\Phi_\alpha(x) \to x$ in the weak* topology. In general, hyperfinite implies injective. The converse (say, when $\mathcal{M} \subset B(H)$ on a separable Hilbert space H) is a celebrated result of Connes [5]. We refer to [22] for more information about these properties.

The next theorem is a dual form of Haagerup's characterization of injectivity by decomposability [12]. It suggests that the conditional min entropy connects to the projective tensor norm if and only if $\mathcal{M}$ is injective.

Theorem 3.5. *Let $\mathcal{M}, \mathcal{N}$ be semi-finite von Neumann algebras. Suppose $\mathcal{N}$ is infinite dimensional. The following are equivalent*

i) *$\mathcal{M}$ is injective*
ii) *$L_1(\mathcal{M}, L_\infty(\mathcal{N})) \cong L_1(\mathcal{M})\widehat{\otimes}\mathcal{N}$ isomorphically*
iii) *$L_1(\mathcal{M}, L_\infty(\mathcal{N})) \cong L_1(\mathcal{M})\widehat{\otimes}\mathcal{N}$ isometrically*

In particular, $L_1(\mathcal{M}, L_\infty(\mathcal{M}^{op})) \cong L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op}$ if and only if $\mathcal{M}$ is injective.

Proof. We first prove i) ⇒ iii). Suppose $L_1(\mathcal{M}, L_\infty(\mathcal{N})) \neq L_1(\mathcal{M})\widehat{\otimes}\mathcal{N}$ isometrically. Because both spaces are norm completions of the algebraic tensor product $L_1(\mathcal{M}) \otimes \mathcal{N}$, there exists $\rho = \sum_{j=1}^n y_j \otimes z_j$ such that

$$\| \rho \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} < 1 = \| \rho \|_{L_1(\mathcal{M})\widehat{\otimes}\mathcal{N}} .$$

Then by the duality $(L_1(\mathcal{M})\widehat{\otimes}\mathcal{N})^* = CB(\mathcal{N}, \mathcal{M}^{op})$, there exists a CB map $S \in CB(\mathcal{N}, \mathcal{M}^{op})$ with $\| S \|_{cb} = 1$ such that

$$1 = \langle S, \rho \rangle = \langle \mathrm{id}, \mathrm{id}_{\mathcal{M}} \otimes S(\rho) \rangle .$$

Here we have

$$\| \mathrm{id}_{\mathcal{M}} \otimes S(\rho) \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} \leq \| S \|_{cb} \| \rho \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} < 1 .$$

If $\mathcal{M}$ is injective, then there exists a net of finite-rank, normal, unital, completely positive maps Φ_α approximating the identity map $\mathrm{id}_{\mathcal{M}}$ in the point-weak* topology. By Lemma 3.4, $\Phi_\alpha = T_{x_\alpha}$ for some $x_\alpha \in \mathcal{M}^{op} \otimes L_1(\mathcal{N}^{op})$ with

$$\| x_\alpha \|_{L_\infty(\mathcal{M}^{op}, L_1(\mathcal{N}^{op}))} = \| \Phi_\alpha \|_{cb} = 1 .$$

This leads to a contradiction:

$$\begin{aligned} 1 = \langle \mathrm{id}, \mathrm{id}_{\mathcal{M}} \otimes S(\rho) \rangle &= \lim_\alpha \langle T_{x_\alpha}, \mathrm{id}_{\mathcal{M}} \otimes S(\rho) \rangle \\ &= \lim_\alpha \langle x_\alpha, \mathrm{id}_{\mathcal{M}} \otimes S(\rho) \rangle \\ &\leq \lim_\alpha \| x_\alpha \|_{L_\infty(\mathcal{M}^{op}, L_1(\mathcal{N}^{op}))} \| \mathrm{id}_{\mathcal{M}} \otimes S(\rho) \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} \\ &\leq \| \mathrm{id}_{\mathcal{M}} \otimes S(\rho) \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} < 1. \end{aligned}$$

For ii)⇒ i), we first reduce the semi-finite $\mathcal{M}$ to the finite case. We have the decomposition $\mathcal{M} = \oplus_{i \in i}(\mathcal{M}_i \overline{\otimes} B(H_i))$ (see [25, Chapter 5, Proposition]) where $\mathcal{M}_i$ are finite von Neumann algebras and H_i are Hilbert spaces. For each $\mathcal{M}_i$, there exists a trace preserving embedding $\iota : \mathcal{M}_i \to \mathcal{M}$ and a projection $P : \mathcal{M} \to e_i \mathcal{M} e_i$ for some projection e_i such that $P \circ \iota = \mathrm{id}_{\mathcal{M}_i}$. This induces the isometric embedding

$$L_1(\mathcal{M}_i, L_\infty(\mathcal{N})) \subset L_1(\mathcal{M}, L_\infty(\mathcal{N})) \, , L_1(\mathcal{M}_i)\widehat{\otimes}\mathcal{N} \subset L_1(\mathcal{M})\widehat{\otimes}\mathcal{N} .$$

Suppose $L_1(\mathcal{M}, L_\infty(\mathcal{N})) \cong L_1(\mathcal{M})\widehat{\otimes}\mathcal{N}$ isometrically. We have for each i, $L_1(\mathcal{M}_i, L_\infty(\mathcal{N})) \cong L_1(\mathcal{M}_i)\widehat{\otimes}\mathcal{N}$ isometrically. It suffices to show that this implies $\mathcal{M}_i$ is injective.

We now assume $\mathcal{M} = \mathcal{M}_i$ finite. Let l_∞^n be the n-dimensional commutative C^*-algebra. Because $\mathcal{N}$ is infinite dimensional, for any n there exists completely positive and contractive maps (see [12, Lemma 2.7])

$$Q : l_\infty^n \to \mathcal{N} \ , \ R : \mathcal{N} \to l_\infty^n$$

such that $R \circ Q = \mathrm{id}_{l_\infty^n}$. Both $\mathrm{id}_{\mathcal{M}} \otimes R$ and $\mathrm{id}_{\mathcal{M}} \otimes Q$ extend to complete contractions

$$L_1(\mathcal{M})\widehat{\otimes} l_\infty^n \stackrel{\mathrm{id}_{\mathcal{M}}\otimes R}{\longrightarrow} L_1(\mathcal{M})\widehat{\otimes}\mathcal{N} \stackrel{\mathrm{id}_{\mathcal{M}}\otimes Q}{\longrightarrow} L_1(\mathcal{M})\widehat{\otimes} l_\infty^n \ ,$$

$$L_1(\mathcal{M}, l_\infty^n) \stackrel{\mathrm{id}_{\mathcal{M}}\otimes R}{\longrightarrow} L_1(\mathcal{M}, L_\infty(\mathcal{N})) \stackrel{\mathrm{id}_{\mathcal{M}}\otimes Q}{\longrightarrow} L_1(\mathcal{M}, l_\infty^n) \ .$$

Thus we have the isometric imbeddings

$$L_1(\mathcal{M}, l_\infty^n) \subset L_1(\mathcal{M}, L_\infty(\mathcal{N})) \text{ and } L_1(\mathcal{M})\widehat{\otimes} l_\infty^n \subset L_1(\mathcal{M})\widehat{\otimes}\mathcal{N}.$$

Suppose $L_1(\mathcal{M}, L_\infty(\mathcal{N})) \cong L_1(\mathcal{M})\widehat{\otimes}\mathcal{N}$ isomorphically. Then we have $L_1(\mathcal{M}, l_\infty^n) \cong L_1(\mathcal{M})\widehat{\otimes} l_\infty^n$ for each n, and moreover a uniform constant c such that for all n,

$$c \parallel \rho \parallel_{L_1(\mathcal{M})\widehat{\otimes} l_\infty^n} \leq \parallel \rho \parallel_{L_1(\mathcal{M}, l_\infty^n)} \leq \parallel \rho \parallel_{L_1(\mathcal{M})\widehat{\otimes} l_\infty^n} \ .$$

At the dual level, for each $T : l_\infty^n \to \mathcal{M}^{op}$,

$$\parallel T \parallel_{cb} = \parallel T_x \parallel_{cb} \leq \parallel x \parallel_{L_\infty(\mathcal{M}^{op}, l_1^n)} \leq c^{-1} \parallel T_x \parallel_{cb} \ . \tag{3.6}$$

Here $T = T_x$ as in Lemma 3.4, for $x = \sum_{j=1}^n T(e_j) \otimes e_j \in \mathcal{M}^{op} \otimes l_1^n$ with $e_j \in l_1^n$ being the dual standard basis of l_∞^n. We shall suppress the "op" notation since it is equivalent to consider $\mathcal{M}$ and $\mathcal{M}^{op}$ here. For any n unitaries u_j and a central projection q in $\mathcal{M}$, we consider $x_u = q \sum_{j=1}^n u_j \otimes e_j$. We have

$$\begin{aligned}
\parallel x_u \parallel_{L_\infty(\mathcal{M}, l_1^n)} &= \sup\{\parallel q \sum_{j=1}^n a u_j b \otimes e_j \parallel_{L_1(\mathcal{M}, l_1^\infty)} \ | \ \parallel a \parallel_{L_2(\mathcal{M})} = \parallel b \parallel_{L_2(\mathcal{M})} = 1\} \\
&= \sup\{\sum_j \parallel q a u_j b \parallel_{L_1(\mathcal{M})} \ | \ \parallel a \parallel_{L_2(\mathcal{M})} = \parallel b \parallel_{L_2(\mathcal{M})} = 1\} \\
&\geq \sum_j \tau_{\mathcal{M}}(q)^{-1} \parallel q u_j \parallel_{L_1(\mathcal{M})} \\
&= \sum_{j=1}^n 1 = n.
\end{aligned}$$

Here we have chosen $a = b = \tau_{\mathcal{M}}(q)^{-1/2} q$. Then by (3.6), we have

$$\parallel T_u \parallel_{cb} \geq c \parallel x_u \parallel_{L_\infty(\mathcal{M}, l_1^n)} = cn \ , \text{ where } T_u : l_\infty^n \to \mathcal{M} \ , \quad T_u((c_j)_j) = q \sum_{j=1}^n c_j u_j \ .$$

Then it follows from [12, Lemma 2.3 & Lemma 2.5] that $\mathcal{M}$ is injective. Since iii)$\Rightarrow$ ii) is trivial, this completes the proof. □

3.2. Quantum majorization

We now discuss quantum majorization for semi-finite von Neumann algebras. We will focus on the case where $\mathcal{M}$ is injective, because by Theorem 3.5, beyond injectivity we lose the connection between H_{min} entropy and projection tensor norm.

We say $T : L_1(\mathcal{M}) \to L_1(\mathcal{M})$ is completely positive trace preserving (resp. trace non-increasing) if its adjoint $T^\dagger : \mathcal{M}^{op} \to \mathcal{M}^{op}$ is normal completely positive and unital (resp. sub-unital). We will use the abbreviation CPTP for completely positive trace preserving, CPTNI for completely positive trace non-increasing and UCP for unital completely positive. The next proposition is a consequence of Lemma 3.3 and Theorem 3.5. All the assumptions of injectivity in later theorems is to ensure the following proposition holds.

Proposition 3.6. *Let $\mathcal{M}$ be an injective semi-finite von Neumann algebra.*

i) *For a self-adjoint $x \in L_1(\mathcal{M})\widehat{\otimes}\mathcal{N}$,*

$$\lambda(x) = \sup\{ \langle \Phi, x\rangle \mid \Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N}^{op}) \ \mathit{CPTNI} \}.$$

ii) *Define the real part of $x \in L_1(\mathcal{M})\widehat{\otimes}\mathcal{N}$ as $Re\, x = (x + x^*)/2$. Then*

$$\lambda(Re\, x) = \sup\{ Re\, \langle \Phi, x\rangle \mid \Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N}^{op}) \ \mathit{CPTNI} \}.$$

iii) *For positive ρ,*

$$\begin{aligned} \|\rho\|_{L_1(\mathcal{M})\widehat{\otimes}\mathcal{N}} &= \sup\{ \langle \Phi, \rho\rangle \mid \Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N}^{op}) \ \mathit{CPTNI} \} \\ &= \sup\{ \langle \Phi, \rho\rangle \mid \Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N}^{op}) \ \mathit{CPTP} \}. \end{aligned}$$

Proof. We first show that $\langle T, y\rangle \geq 0$ for a positive $y \in L_1(\mathcal{M})\widehat{\otimes}\mathcal{N}$ and CP $T : \mathcal{N} \to \mathcal{M}^{op}$. By a density argument, it suffices to consider $y \in \mathcal{M}_0 \otimes \mathcal{N}$. Suppose $y = (\sum_{j=1}^n a_j \otimes b_j)^*(\sum_{j=1}^n a_j \otimes b_j)$ for some $a_j \in e\mathcal{M}e$ and $b_j \in \mathcal{N}$. We have $\sum_{i,j=1}^n e_{i,j} \otimes T(b_i^* b_j) = \mathrm{id}_n \otimes T(\sum_{i,j=1}^n e_{i,j} \otimes b_i^* b_j)$ is positive in $M_n(\mathcal{M}^{op})$. Therefore,

$$\begin{aligned} \langle T, y\rangle &= \tau_{\mathcal{M}}(\sum_{i,j=1}^n (a_i^* a_j)^{op} T(b_i^* b_j)) = \sum_{i,j=1}^n \tau_{\mathcal{M}^{op}}((a_i^{op})^* T(b_i^* b_j) a_j^{op}) \\ &= \sum_{i,j=1}^n \langle a_i^{op} | T(b_i^* b_j) | a_j^{op}\rangle \geq 0, \end{aligned}$$

where $|a_j^{op}\rangle \in L_2(\mathcal{M}^{op}, \tau_{\mathcal{M}})$ is the vector of a_j^{op} in the GNS representation. Thus, $\langle T, y\rangle \geq 0$ for CP $T : \mathcal{N} \to \mathcal{M}^{op}$ and also CP $T : L_1(\mathcal{M}) \to L_1(\mathcal{N}^{op})$ as normal maps. Then for $x \leq \lambda\sigma \otimes I$ and $T : L_1(\mathcal{M}) \to L_1(\mathcal{N}^{op})$ CPTNI, we have

$$\langle T, x\rangle \leq \lambda\langle T, 1 \otimes \sigma\rangle = \lambda\tau_{\mathcal{N}}(T(\sigma)) \leq \lambda\ ,$$

which implies $\langle T, \rho \rangle \le \lambda(\rho)$. On the other hand, by Theorem 3.5 and Lemma 3.3,

$$\begin{aligned}\lambda(x) &= \sup\{ \tau(xy) \mid y \in \mathcal{M}\overline{\otimes}\mathcal{N}_0, y \ge 0, \| y \|_{L_\infty(\mathcal{M}, L_1(\mathcal{N}))} = 1 \} \\ &= \sup\{ \langle T_y, x \rangle \mid y \in \mathcal{M}\overline{\otimes}\mathcal{N}_0, y \ge 0, \| y \|_{L_\infty(\mathcal{M}, L_1(\mathcal{N}))} = 1 \} \\ &\le \sup\{ \langle T, x \rangle \mid T : L_1(\mathcal{M}) \to L_1(\mathcal{N}) \text{ CPTNI } \} \\ &\le \lambda(x) \ .\end{aligned}$$

This proves i). ii) follows from the fact that for any CP T, $\mathrm{Re}\langle T, x \rangle = \langle T, \mathrm{Re}\, x \rangle$. For iii), given a CPTNI map T, one can always find a CPTP $\tilde{T}$ such that $\tilde{T} - T$ is CP. Therefore,

$$\| \rho \|_{L_1(\mathcal{M})\widehat{\otimes}\mathcal{N}} = \lambda(\rho) = \sup_{T \text{ CPTNI}} \langle T, \rho \rangle \le \sup_{T \text{ CPTP}} \langle T, \rho \rangle \le \lambda(\rho) \ . \quad \square$$

Lemma 3.7. *Let ρ be a bipartite density operator in $L_1(\mathcal{M}\overline{\otimes}\mathcal{N})$. The set*

$$C(\rho) = \{ \Phi \otimes \mathrm{id}(\rho) \mid \Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M}) \ \mathit{CPTP} \}$$

is a closed set in $L_1(\mathcal{M}\overline{\otimes}\mathcal{N})$ with respect to the topology induced by

$$\mathcal{M}^{op} \otimes_{\min} \mathcal{N}^{op} \subset \mathcal{M}^{op}\overline{\otimes}\mathcal{N}^{op} = L_1(\mathcal{M}\overline{\otimes}\mathcal{N})^*.$$

In particular, $C(\rho)$ is a norm closed set in $L_1(\mathcal{M}\overline{\otimes}\mathcal{N})$.

Proof. Let $\sigma \in L_1(\mathcal{M}\overline{\otimes}\mathcal{N})$ and Φ_α be a net of CPTP maps such that $\Phi_\alpha \otimes \mathrm{id}(\rho) \to \sigma$ with respect to $\mathcal{M}^{op} \otimes_{\min} \mathcal{N}^{op}$. That is, for any $x \in \mathcal{M} \otimes \mathcal{N}$

$$\lim_\alpha \tau(x \Phi_\alpha \otimes \mathrm{id}(\rho)) = \tau(x\sigma) \ . \tag{3.7}$$

Taking $x = 1_\mathcal{M} \otimes 1_\mathcal{N}$, this implies $\tau(\sigma) = \lim_\alpha \tau(\Phi_\alpha \otimes \mathrm{id}(\rho)) = 1$. Note that the

$$CB(L_1(\mathcal{M}), L_1(\mathcal{M})) \subset CB(\mathcal{M}^{op}, \mathcal{M}^{op}) = (L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op})^* \ .$$

By weak*-compactness, there exists a sub-net Φ_β such that their corresponding subnet of adjoints $\Phi_\beta^\dagger : \mathcal{M}^{op} \to \mathcal{M}^{op}$ converges to some $\Phi^\dagger : \mathcal{M}^{op} \to \mathcal{M}^{op}$ in the point-weak* topology. That is, for $x \in L_1(\mathcal{M}), y^{op} \in \mathcal{M}^{op}$, we have

$$\lim_\beta \tau_\mathcal{M}\big(x \Phi_\beta^\dagger(y^{op})\big) = \tau_\mathcal{M}\big(x \Phi^\dagger(y^{op})\big) \ .$$

Then it is clear that $\Phi^\dagger$ is UCP. Note that $(\mathcal{M}^{op})^* = L_1(\mathcal{M}) \oplus L_1(\mathcal{M})^\perp$ decomposes into a normal part and a singular part. Let $\Phi : L_1(\mathcal{M}) \to (\mathcal{M}^{op})^*$ be the restriction of the double adjoint map $\Phi^{\dagger\dagger} : (\mathcal{M}^{op})^* \to (\mathcal{M}^{op})^*$. Then $\Phi_\beta \otimes \mathrm{id}(\rho) \to \Phi \otimes \mathrm{id}(\rho)$ in the sense that for any $x \in \mathcal{M} \otimes \mathcal{N}$

$$\tau(x\Phi_\beta \otimes \mathrm{id}(\rho)) = \tau(\Phi_\beta^\dagger \otimes \mathrm{id}(x)\rho) \to \Phi \otimes \mathrm{id}(\rho)(x) \ ,$$

where $\Phi \otimes \mathrm{id}(\rho) \in (\mathcal{M}^{op})^* \widehat{\otimes} L_1(\mathcal{N})$. Then by (3.7), for any $x \in \mathcal{M} \otimes \mathcal{N}$,

$$\Phi \otimes \mathrm{id}(\rho)(x) = \tau(\sigma x) := \sigma(x)$$

where the density operator σ is viewed as a normal state. Decompose the map $\Phi = \Phi_n + \Phi_s$ where $\Phi_n \in CB(L_1(\mathcal{M}), L_1(\mathcal{M}))$ is the normal part and $\Phi_s \in CB(L_1(\mathcal{M}), L_1(\mathcal{M})^\perp)$ is the singular map. Then for any $x \in \mathcal{M} \otimes \mathcal{N}$,

$$\Big(\sigma - \Phi_n \otimes \mathrm{id}(\rho)\Big)(x) = \Phi_s \otimes \mathrm{id}(\rho)(x) \tag{3.8}$$

where $\sigma - \Phi_n \otimes \mathrm{id}(\rho) \in L_1(\mathcal{M}) \widehat{\otimes} L_1(\mathcal{N})$ and $\Phi_s \otimes \mathrm{id}(\rho) \in (\mathcal{M}^{op})^* \widehat{\otimes} L_1(\mathcal{N})$. Let $\omega_1, \omega_2 : \mathcal{M} \to \mathbb{C}$ be the linear functionals defined by

$$\omega_1(y) := \Big(\sigma - \Phi_n \otimes \mathrm{id}(\rho)\Big)(y \otimes 1) \ , \ \omega_2(y) := \Phi_s \otimes \mathrm{id}(\rho)(y \otimes 1) \ , \ y \in \mathcal{M}.$$

Then ω_1 is normal and ω_2 is singular. By (3.8), $\omega_1 = \omega_2$ which implies $\omega_1 = \omega_2 = 0$. Therefore,

$$\Phi_s \otimes \mathrm{id}(\rho)(1 \otimes 1) = \omega_2(1) = 0 \ .$$

Hence $\Phi_s \otimes \mathrm{id}(\rho) = 0$. We have $\sigma = \Phi_n \otimes \mathrm{id}(\rho)$ for $\Phi_n : L_1(\mathcal{M}) \to L_1(\mathcal{M})$ CPTNI. Define $\Phi_0(x) = \tau(\Phi_n(x) - x)\omega$ for any density operator $\omega \in L_1(\mathcal{M})$. Then $\tilde{\Phi} = \Phi_n + \Phi_0$ is a CPTP map and $\tilde{\Phi} \otimes id(\rho) = \sigma$. This completes the proof. □

We say a CPTP map $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ is *entanglement-breaking* if $\Phi(\rho) = \sum_{j=1}^\infty \tau(x_j \rho)\omega_j$ for some set of x_j, $j = 1, 2, \dots$, satisfying $\sum_{j=1}^\infty x_j = 1$ and $x_j \geq 0$ (such a set $\{x_j\}$ is called a measurement in quantum mechanics) and density operators ω_j. Such a CPTP map is a quantum channel that admits a factorization through l_1^∞, which is the state space of a classical system. We note that a CP map $T : \mathcal{M} \to \mathcal{N}$ is automatically CB by Stinespring's theorem $\| T \| = \| T \|_{cb} = \| T(1) \|$ (cf. [25, Theorem 3.6]). This also holds for a CP map $\Phi : L_1(\mathcal{N}) \to L_1(\mathcal{M})$ as a pre-adjoint of normal maps, but is not necessarily true for a completely positive map $\Phi : L_1(\mathcal{N}) \to \mathcal{M}$ (for example, the identity map $\mathrm{id} : S_1(H) \to B(H)$ is CP but not CB). We now prove our main theorem with respect to quantum majorization for injective semi-finite von Neumanna algebra.

Theorem 3.8. *Let $\mathcal{M}$ and $\mathcal{N}$ be two semi-finite von Neumann algebras and let $\mathcal{M}$ be injective. Let ρ, σ be two density operators in $L_1(\mathcal{M} \overline{\otimes} \mathcal{N})$. The following are equivalent:*

- i) *there exists a CPTP map $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M})$ such that $\Phi \otimes \mathrm{id}(\rho) = \sigma$*
- ii) *for any CP and CB map $\Psi : L_1(\mathcal{N}) \to \mathcal{M}^{op}$,*

$$\| \mathrm{id} \otimes \Psi(\rho) \|_{L_1(\mathcal{M}) \widehat{\otimes} \mathcal{M}^{op}} \geq \| \mathrm{id} \otimes \Psi(\sigma) \|_{L_1(\mathcal{M}) \widehat{\otimes} \mathcal{M}^{op}}$$

iii) *for any projection* $e \in \mathcal{M}$ *with* $\tau_{\mathcal{M}}(e) < \infty$ *and for any entanglement-breaking CPTP map* $\Psi : L_1(\mathcal{N}) \to L_1(e\mathcal{M}e^{op})$ *with range* $Ran(\Psi) \subset e\mathcal{M}e^{op}$,

$$\| \mathrm{id} \otimes \Psi(\rho) \|_{L_1(\mathcal{M})\widehat{\otimes} e\mathcal{M}e^{op}} \geq \| \mathrm{id} \otimes \Psi(\sigma) \|_{L_1(\mathcal{M})\widehat{\otimes} e\mathcal{M}e^{op}}$$

Proof. The direction i)⇒ ii) and iii) follows from the factorization $\mathrm{id} \otimes \Psi(\sigma) = \Phi \otimes \mathrm{id}\Big(\mathrm{id} \otimes \Psi(\rho)\Big)$ and

$$\| \Phi \otimes \mathrm{id} : L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op} \to L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op} \| \leq \| \Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M}) \|_{cb} = 1 \ .$$

Let $C(\rho)$ be the convex set from Lemma 3.7

$$C(\rho) = \{ \ \Phi \otimes \mathrm{id}(\rho) \ | \Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M}) \text{ CPTP}\}$$

for some bipartite density operator ρ. Suppose by way of contradiction that $\sigma \notin C(\rho)$. Because $C(\rho)$ is closed with respect to the weak topology induced by $\mathcal{M}^{op} \otimes_{min} \mathcal{N}^{op}$, by the Hahn-Banach theorem there exists $x_1 \in \mathcal{M} \otimes_{min} \mathcal{N}$ such that

$$\mathrm{Re}\, \tau(\sigma x_1) > Re \ \sup_{\Phi} \tau(\Phi \otimes \mathrm{id}(\rho) x_1) \ .$$

We can replace x_1 with a finite tensor $x_2 = \sum_j a_j \otimes b_j \in \mathcal{M} \otimes \mathcal{N}$ such that $\| x_1 - x_2 \| < \epsilon$ is small enough and

$$\mathrm{Re}\ \tau(\sigma x_2) > \sup_{\Phi} \mathrm{Re}\ \tau(\mathrm{id} \otimes \Phi(\rho) x_2).$$

Take $x_3 = (x_2 + x_2^*)/2$ be the real part of x_2:

$$\begin{aligned} x_3 =& \frac{1}{2}(x_2 + x_2^*) = \frac{1}{2} \sum_j (a_j \otimes b_j + a_j^* \otimes b_j^*) \\ =& \frac{1}{4}\Big(\sum_j (a_j + a_j^*) \otimes (b_j + b_j^*) + \sum_j i(a_j - a_j^*) \otimes (-i)(b_j - b_j^*)\Big), \end{aligned} \tag{3.9}$$

which is a finite sum of tensor products of self-adjoint elements. Since σ and $\Phi \otimes \mathrm{id}(\rho)$ are positive,

$$\tau(\sigma x_3) = \mathrm{Re}\ \mathrm{tr}(\sigma x_2) > \sup_{\Phi} \mathrm{Re}\, \tau(\mathrm{id} \otimes \Phi(\rho) x_2) = \sup_{\Phi} \tau(\mathrm{id} \otimes \Phi(\rho) x_3).$$

For each j,

$$\begin{aligned} & a_j \otimes b_j + \| a_j \| \| b_j \| \ 1 \otimes 1 \\ =& \frac{1}{2}\Big((a_j + \| a_j \| \ 1) \otimes (b_j + \| b_j \| \ 1) + (\| a_j \| \ (1 - a_j)) \otimes (\| b_j \| \ (1 - b_j))\Big), \end{aligned}$$

is a sum of tensor products of positive elements. Take $K = \sum_j \| a_j \| \| b_j \|$. Then $x_4 = x_3 + K1 \otimes 1 \in \mathcal{M} \otimes \mathcal{N}$ is a sum of tensor products of positive elements. Since $\tau(\mathrm{id} \otimes \Phi(\rho)) = \tau(\sigma) = 1$, we have

$$\tau(\sigma x_4) = \tau(\sigma x_3) + K > \sup_{\Phi} \tau(\mathrm{id} \otimes \Phi(\rho) x_3) + K \geq \sup_{\Phi} \tau(\mathrm{id} \otimes \Phi(\rho) x_4) \,. \quad (3.10)$$

The opposite element $x_4^{op} \in \mathcal{M}^{op} \overline{\otimes} \mathcal{N}^{op}$ corresponds to a CP map $T \in CB(L_1(\mathcal{N}), \mathcal{M}^{op})$. Note that $\mathrm{id} \otimes T(\sigma)$ and $\mathrm{id} \otimes T(\rho) \in L_1(\mathcal{M}) \widehat{\otimes} \mathcal{M}^{op}$. Since $\mathcal{M}$ is injective, we have by Proposition 3.6

$$\begin{aligned} \tau(x_4 \sigma) &= \langle T, \sigma \rangle = \langle \mathrm{id}_{\mathcal{M}}, \mathrm{id} \otimes T(\sigma) \rangle \leq \| \mathrm{id} \otimes T(\sigma) \|_{L_1(\mathcal{M}) \widehat{\otimes} \mathcal{M}^{op}} \,, \\ \sup_{\Phi \text{ CPTP}} \tau(x_4 \Phi \otimes \mathrm{id}(\rho)) &= \sup_{\Phi} \langle T, \Phi \otimes \mathrm{id}(\rho) \rangle = \sup_{\Phi} \langle \Phi, \mathrm{id} \otimes T(\rho) \rangle \\ &= \| \mathrm{id} \otimes T(\rho) \|_{L_1(\mathcal{M}) \widehat{\otimes} \mathcal{M}^{op}} \,. \end{aligned}$$

Here the bracket is the pairing for $(L_1(\mathcal{M}) \widehat{\otimes} \mathcal{M}^{op})^* \cong CB(\mathcal{M}^{op}, \mathcal{M}^{op})$ and $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M})$ is a normal map in $CB(\mathcal{M}^{op}, \mathcal{M}^{op})$. Then the inequality (3.10) implies that

$$\| \mathrm{id} \otimes T(\sigma) \|_{L_1(\mathcal{M}) \widehat{\otimes} \mathcal{M}^{op}} > \| \mathrm{id} \otimes T(\rho) \|_{L_1(\mathcal{M}) \widehat{\otimes} \mathcal{M}^{op}}$$

which violates ii). This proves the direction ii)$\Rightarrow$ i). For the direction iii)$\Rightarrow$ i), we shall further modify T to get a CPTP map. Given $\epsilon > 0$, there exists a projection $e \in \mathcal{M}$ such that $\tau_{\mathcal{M}}(e) < \infty$ and $\| (e \otimes 1) \sigma (e \otimes 1) - \sigma \|_1 < \epsilon$. Then for small enough ϵ we have

$$\mathrm{tr}(\sigma (e \otimes 1) x_4 (e \otimes 1)) > \mathrm{tr}(\sigma x_4) - \epsilon > \sup_{\Phi} \mathrm{tr}(\mathrm{id} \otimes \Phi(\rho) x_4) \,. \quad (3.11)$$

Take $x_5 := (e \otimes 1) x_4 (e \otimes 1) = \sum_{j=1}^n c_j \otimes d_j \in e\mathcal{M}e \otimes \mathcal{N}$ as a finite sum of tensor product of positive operators. Then $x_5^{op} \in e\mathcal{M}e^{op} \otimes \mathcal{N}^{op}$ corresponds to the CP map $T_1 : L_1(\mathcal{N}) \to e\mathcal{M}e^{op}$ given by

$$T_1(\omega) = \sum_{j=1}^n \tau_{\mathcal{N}}(d_j \omega) c_j \,.$$

By (3.11), we have

$$\mathrm{tr}(\sigma x_5) > \sup_{\Phi \text{ CPTP}} \mathrm{tr}(\mathrm{id} \otimes \Phi(\rho) x_4) = \sup_{\Phi \text{ CPTP}} \langle \Phi, \mathrm{id} \otimes T(\rho) \rangle = \| \mathrm{id} \otimes T(\rho) \|_{L_1(\mathcal{M}) \widehat{\otimes} \mathcal{M}^{op}} \,.$$

Take the map $T_1(\cdot) = eT(\cdot)e$. Because the map $y \mapsto eye$ is a complete contraction from $\mathcal{M}^{op}$ to $e\mathcal{M}e^{op}$, we have

$$\| \mathrm{id} \otimes T(\rho) \|_{L_1(\mathcal{M}) \widehat{\otimes} \mathcal{M}^{op}} \geq \| (1 \otimes e) \mathrm{id} \otimes T(\rho) (1 \otimes e) \|_{L_1(\mathcal{M}) \widehat{\otimes} \mathcal{M}^{op}} = \| \mathrm{id} \otimes T_1(\rho) \|_{L_1(\mathcal{M}) \widehat{\otimes} e\mathcal{M}e^{op}} \,.$$

On the other hand,

$$\mathrm{tr}(\sigma x_5) = \langle \mathrm{id}_{\mathcal{M}}, \mathrm{id} \otimes T_1(\sigma) \rangle \leq \| \mathrm{id} \otimes T_1(\sigma) \|_{L_1(\mathcal{M}) \widehat{\otimes} e\mathcal{M}e^{op}} .$$

Thus $T_1 : L_1(\mathcal{M}) \to e\mathcal{M}e^{op}$ is a CP map and

$$\| \mathrm{id} \otimes T_1(\sigma) \|_{L_1(\mathcal{M}) \widehat{\otimes} e\mathcal{M}e^{op}} > \| \mathrm{id} \otimes T_1(\rho) \|_{L_1(\mathcal{M}) \widehat{\otimes} e\mathcal{M}e^{op}} . \tag{3.12}$$

Note that $e\mathcal{M}e^{op} \subset L_1(e\mathcal{M}e^{op})$ because $\tau_{\mathcal{M}}(e) < \infty$. Since T_1 is CP and finite rank, we have

$$\| T_1 : L_1(\mathcal{N}) \to L_1(e\mathcal{M}e^{op}) \|_{cb} = \| T_1 : L_1(\mathcal{N}) \to L_1(e\mathcal{M}e^{op}) \| < \infty .$$

Then $T_2 = \| T_1 : L_1(\mathcal{N}) \to L_1(e\mathcal{M}e^{op}) \|^{-1} T_1$ is CPTNI and satisfies the inequality (3.12).

Finally, we modify T_2 to be trace preserving. Denote by $\rho_{\mathcal{M}} = \mathrm{id} \otimes \tau_{\mathcal{N}}(\rho)$ and $\rho_{\mathcal{N}} = \tau_{\mathcal{M}} \otimes \mathrm{id}(\rho)$ the reduced density operator of ρ and similarly for σ. For the case $\rho_{\mathcal{N}} = \sigma_{\mathcal{N}}$, we define $T_3 = T_2 + T_0$ where $T_0(x) = \big(\mathrm{tr}(x) - \mathrm{tr}(T_2(x))\big) \frac{e}{\tau_{\mathcal{M}}(e)}$. Then $T_3 : L_1(\mathcal{M}) \to L_1(e\mathcal{M}e^{op})$ is CPTP. We have

$$\mathrm{id} \otimes T_3(\rho) = \mathrm{id} \otimes T_2(\rho) + \frac{\lambda_1}{\tau_{\mathcal{M}}(e)} \rho_{\mathcal{M}} \otimes e ,$$
$$\mathrm{id} \otimes T_3(\sigma) = \mathrm{id} \otimes T_2(\sigma) + \frac{\lambda_2}{\tau_{\mathcal{M}}(e)} \sigma_{\mathcal{M}} \otimes e ,$$

where $\lambda_1 = \mathrm{tr}(\rho_{\mathcal{N}}) - \mathrm{tr}(T_2(\rho_{\mathcal{N}}))$ is equal to $\lambda_2 = \mathrm{tr}(\sigma_{\mathcal{N}}) - \mathrm{tr}(T_2(\sigma_{\mathcal{N}}))$. Note that for any density operator $\omega \in L_1(\mathcal{M})$ and $\lambda > 0$

$$\mathrm{id} \otimes T_3(\rho) = \mathrm{id} \otimes T_2(\rho) + \frac{\lambda_1}{\tau_{\mathcal{M}}(e)} \rho_{\mathcal{M}} \otimes e \leq \lambda \omega \otimes e \iff \mathrm{id} \otimes T_2(\rho) \leq (\lambda \omega - \frac{\lambda_1}{\tau_{\mathcal{M}}(e)} \rho_{\mathcal{M}}) \otimes e .$$

Therefore we have

$$\begin{aligned} &\| \mathrm{id} \otimes T_3(\rho) \|_{L_1(\mathcal{M}) \widehat{\otimes} e\mathcal{M}e^{op}} \\ &= \| \mathrm{id} \otimes T_2(\rho) \|_{L_1(\mathcal{M}) \widehat{\otimes} e\mathcal{M}e^{op}} + \frac{\lambda_1}{\tau_{\mathcal{M}}(e)} \\ &< \| \mathrm{id} \otimes T_2(\sigma) \|_{L_1(\mathcal{M}) \widehat{\otimes} e\mathcal{M}e^{op}} + \frac{\lambda_1}{\tau_{\mathcal{M}}(e)} = \| \mathrm{id} \otimes T_3(\sigma) \|_{L_1(\mathcal{M}) \widehat{\otimes} e\mathcal{M}e^{op}} . \end{aligned}$$

Thus T_3 is a CPTP map that violates the condition iii). For the case $\rho_{\mathcal{N}} \neq \sigma_{\mathcal{N}}$, we denote $q_1 \in \mathcal{N}$ to be projection onto the support of $(\sigma_{\mathcal{N}} - \rho_{\mathcal{N}})_+$ and $q_2 = 1 - q_1$. Since $\mathcal{N} \neq \mathbb{C}1$ is not the trivial algebra (otherwise $\rho_{\mathcal{N}} = \sigma_{\mathcal{N}}$), we can choose two different projections $e_0, e \in \mathcal{N}$ such that $e_0 < e$ and $\tau_{\mathcal{N}}(e_0) < \tau_{\mathcal{N}}(e)$. We define the CPTP map $T_4 : L_1(\mathcal{N}) \to \mathcal{M}^{op}$ as

$$T_4(x) = \mathrm{tr}_{\mathcal{N}}(q_1 x)\frac{e_0}{\tau_{\mathcal{M}}(e_0)} + \mathrm{tr}_{\mathcal{N}}(q_2 x)\frac{e}{\tau_{\mathcal{M}}(e)} \ .$$

Denote $\sigma_{\mathcal{M},j} = \mathrm{id}\otimes \mathrm{tr}_{\mathcal{N}}\big((1\otimes q_j)\sigma\big)$ and $\rho_{\mathcal{M},j} = \mathrm{id}\otimes \mathrm{tr}_{\mathcal{N}}\big((1\otimes q_j)\sigma\big)$ with $j = 1, 2$. Note that

$$\mathrm{tr}_{\mathcal{M}}(\sigma_{\mathcal{M},1}) + \mathrm{tr}_{\mathcal{M}}(\sigma_{\mathcal{M},2}) = \mathrm{tr}_{\mathcal{M}}(\sigma_{\mathcal{M}}) = 1 \ , \mathrm{tr}_{\mathcal{M}}(\rho_{\mathcal{M},1}) + \mathrm{tr}_{\mathcal{M}}(\rho_{\mathcal{M},2}) = \mathrm{tr}_{\mathcal{M}}(\rho_{\mathcal{M}}) = 1 \ ,$$

and

$$\mathrm{tr}_{\mathcal{M}}(\sigma_{\mathcal{M},1}) - \mathrm{tr}_{\mathcal{M}}(\rho_{\mathcal{M},1}) = \tau\big((1\otimes q_1)(\sigma-\rho)\big) = \mathrm{tr}_{\mathcal{N}}\big((\sigma_{\mathcal{N}} - \rho_{\mathcal{N}})q_1\big) > 0.$$

Since $\tau_{\mathcal{N}}(e_0) < \tau_{\mathcal{N}}(e)$,

$$\begin{aligned}\|\, \mathrm{id}\otimes T_4(\sigma)\,\|_{L_1(\mathcal{M})\widehat{\otimes} e\mathcal{M}e^{op}} &= \|\, \sigma_{\mathcal{M},1}\otimes \frac{e_0}{\tau_{\mathcal{M}}(e_0)} + \sigma_{\mathcal{M},2}\otimes \frac{e}{\tau_{\mathcal{M}}(e)}\,\|_{L_1(\mathcal{M})\widehat{\otimes} e\mathcal{M}e^{op}} \\ &= \frac{\mathrm{tr}_{\mathcal{M}}(\sigma_{\mathcal{M},1})}{\tau_{\mathcal{M}}(e_0)} + \frac{\mathrm{tr}_{\mathcal{M}}(\sigma_{\mathcal{M},2})}{\tau_{\mathcal{M}}(e)} \\ &> \frac{\mathrm{tr}_{\mathcal{M}}(\rho_{\mathcal{M},1})}{\tau_{\mathcal{M}}(e_0)} + \frac{\mathrm{tr}_{\mathcal{M}}(\rho_{\mathcal{M},2})}{\tau_{\mathcal{M}}(e)} = \|\, \mathrm{id}\otimes T_4(\rho)\,\|_{L_1(\mathcal{M})\widehat{\otimes} e\mathcal{M}e^{op}} \ .\end{aligned}$$

Note that both T_3 and T_4 are entanglement-breaking. Then in both cases, we reach a contradiction to condition iii). This proves iii)⇒i). □

Remark 3.9. In the proof above, the assumption of the injectivity of $\mathcal{M}$ is only used to ensure the equivalence between $L_1(\mathcal{M}, L_\infty(\mathcal{N}))$ and $L_1(\mathcal{M})\widehat{\otimes}\mathcal{N}$. In fact, Theorem 3.8 holds for any von Neumann algebras $\mathcal{M}$ for which Proposition 3.6 iii) holds. It is possible to further extend Theorem 3.8 to general $\mathcal{M}$ by using $L_1(\mathcal{M}, L_\infty(\mathcal{N}))$ norm instead of $L_1(\mathcal{M})\widehat{\otimes}\mathcal{N}$-norm (even for non-tracial $\mathcal{M}$, see [17] for the case of $L_1(\mathcal{M}, l_\infty)$ for general $\mathcal{M}$). However, that requires further investigation of $L_1(\mathcal{M}, L_\infty(\mathcal{N}))$-space for non-injective $\mathcal{M}$, which is beyond the scope of this paper. The same remark applies to all other theorems in this section.

We shall now discuss the special case of $\mathcal{N} = l_\infty$. Let $\{\rho_i\}$ and $\{\sigma_i\}$ be two families of density operators in $L_1(\mathcal{M})$. Consider the bipartite density operator $\rho, \sigma \in L_1(\mathcal{M})\widehat{\otimes} l_1 \cong l_1(L_1(\mathcal{M}))$ given by

$$\rho = (\lambda_i \rho_i)_i \ , \ \sigma = (\lambda_i \sigma_i)_i \ ,$$

where $\lambda_i > 0, \sum_{i=1}^{\infty} \lambda_i = 1$ is a probability distribution. Then there exists a CPTP map such that $\sigma = \Phi \otimes \mathrm{id}_{l_1}(\rho)$ if and only if there exists a CPTP map Φ such that $\sigma_i = \Phi(\rho_i)$ for each i. The latter statement, called the quantum interpolation problem in [14], concerns the convertibility from one family of density operators to another using a quantum process (CPTP map). For finite families of finite dimensional density operators, it was shown in [14] that the quantum interpolation problem is solvable by semi-definite programming (SDP). The H_{min} characterization of quantum interpolation problem was used in [10] as a key lemma to prove the bipartite matrix case and has applications in the study of quantum thermal processes. A similar theorem for finite families of self-adjoint operators is obtained in [14, Theorem 7.6], which will be discussed in Section 4. The following theorem is an extension in two ways: it addresses infinite sequences and density operators on von Neumann algebras.

Theorem 3.10. *Let $\mathcal{M}$ be an injective semi-finite von Neumann algebra. Let $\{\rho_i\}_{i\in\mathbb{N}}$ and $\{\sigma_i\}_{i\in\mathbb{N}}$ be two countable families of density operators in $L_1(\mathcal{M})$. TFAE*

i) *there exists a CPTP map such that $\Phi(\rho_i) = \sigma_i$ for all $i \in \mathbb{N}$*
ii) *for any finitely supported probability distribution $(\lambda_i)_{i\in\mathbb{N}}$ and any set of density operators $\{\omega_i\} \in L_1(\mathcal{M}^{op}) \cap \mathcal{M}^{op}$*

$$\| \sum_i \lambda_i \rho_i \otimes \omega_i \|_{L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op}} \geq \| \sum_i \lambda_i \sigma_i \otimes \omega_i \|_{L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op}} \ .$$

Proof. Choose a probability distribution $(\mu_i)_{i\in\mathbb{N}}$ such that $\mu_i > 0$ for each $i \in \mathbb{N}$. Let $\rho = (\mu_i\rho_i)$ and $\sigma = (\mu_i\sigma_i)$ be density operators in $L_1(\mathcal{M})\widehat{\otimes}l_1 \cong l_1(L_1(\mathcal{M}))$. Then i)⇒ ii) again follows from the factorization $\Phi \otimes \mathrm{id}(\rho) = \sigma$ and

$$\| \Phi \otimes \mathrm{id} : L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op} \to L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op} \| \leq \| \Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M}) \|_{cb} \leq 1 \ .$$

Assume that such Φ does not exist. Then by Theorem 3.8 there exists a CPTP map $\Psi : l_1^\infty \to L_1(e\mathcal{M}^{op}e) \cap e\mathcal{M}^{op}e$ for some finite projection e such that

$$\| \mathrm{id} \otimes \Psi(\sigma) \|_{L_1(\mathcal{M})\widehat{\otimes}e\mathcal{M}^{op}e} > \| \mathrm{id} \otimes \Psi(\rho) \|_{L_1(\mathcal{M})\widehat{\otimes}e\mathcal{M}^{op}e} \ .$$

We can omit the projection e here because $L_1(e\mathcal{M}^{op}e) \subset L_1(\mathcal{M}^{op})$ and $e\mathcal{M}^{op}e \subset \mathcal{M}^{op}$ as subspaces. Note that the map Ψ constructed in Theorem 3.8 is also CB from l_1^∞ to $\mathcal{M}^{op}$. Given $\epsilon > 0$, we can choose N large enough such that $\sum_{i>N} \mu_i < \epsilon$. Write $\rho_N = (\rho_i)_{i\leq N} \oplus 0$ and $\sigma_N = (\sigma_i)_{i\leq N} \oplus 0$ as the corresponding truncated sequences. Then

$$\| \mathrm{id} \otimes \Psi(\sigma) - \mathrm{id} \otimes \Psi(\sigma_N) \|_{L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op}} \leq \| \sigma - \sigma_N \|_{L_1(\mathcal{M})\widehat{\otimes}l_1} \leq \sum_{i>N} \mu_i < \epsilon \ .$$

Thus,

$$\begin{aligned} \| \mathrm{id} \otimes \Psi(\sigma_N) \|_{L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op}} &\geq \| \mathrm{id} \otimes \Psi(\sigma) \|_{L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op}} - \epsilon \\ &> \| \mathrm{id} \otimes \Psi(\rho) \|_{L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op}} - \epsilon \\ &\geq \| \mathrm{id} \otimes \Psi(\rho_N) \|_{L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op}} \ . \end{aligned}$$

Write $\omega_i = \Psi(e_i)$ where e_i is the standard basis of l_1. We have

$$\mathrm{id}\otimes\Psi(\sigma_N)=\sum_{1\le i\le N}\mu_i\sigma_i\otimes\omega_i\ ,\ \mathrm{id}\otimes\Psi(\rho_N)=\sum_{1\le i\le N}\mu_i\rho_i\otimes\omega_i\ .$$

Renormalizing the coefficient $\lambda_i=\mu_i(\sum_{i=1}^N\mu_i)^{-1}$, we have a violation of ii). This completes the proof. □

Note that the condition ii) above only concerns finite subsets of $\{\rho_i\}$ and $\{\sigma_i\}$. This leads to the following "compactness" result. It says that to ask whether there is a CPTP map that sends an infinite family of density operators to another infinite family of density operators, it suffices to check the convertibility for every finite subfamily of the two infinite families.

Corollary 3.11. *Let $\mathcal{M}$ be an injective semi-finite von Neumann algebra. Let $\{\rho_i\}_{i\in\mathbb{N}}$ and $\{\sigma_i\}_{i\in\mathbb{N}}$ be two infinite families of density operators in $L_1(\mathcal{M})$. There exists a CPTP map Φ such that $\Phi(\rho_i)=\sigma_i$ for all $i\in\mathbb{N}$ if and only if for any finite subset $I\subset\mathbb{N}$, there exists a CPTP map $\Phi_I(\rho_i)=\sigma_i$ for all $i\in I$.*

3.3. Channel factorization

The dual picture of quantum majorization is channel factorization: given two CPTP maps T and S, determine if there exists a third CPTP Φ such that $\Phi\circ T=S$. Such a factorization relation for two CPTP maps has many implications in quantum information theory. In particular, the channel T has larger capacity than S for various communication tasks. For a finite dimensional CPTP map $\Phi: M_n\to M_m$, its Choi matrix is

$$\chi_\Phi=\sum_{i,j=1}^n e_{i,j}\otimes\Phi(e_{i,j})$$

where $e_{i,j}$ are the matrix units in M_n. As noted in [10], for two CPTP map $S,T: M_n\to M_m$, there exists a CPTP Φ such that $\Phi\circ T=S$ if and only if there exists a CPTP Φ such that $id\otimes\Phi(\chi_T)=\chi_S$. So in finite dimensions channel factorization corresponds to quantum majorization of Choi matrices. However, in the infinite dimensional case, such a correspondence fails because the Choi matrix of a CPTP map is never a density operator (since its trace is unbounded). We shall use again the duality $CB(L_1(\mathcal{M}),L_1(\mathcal{M}))\subset(L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op})^*$ to give a characterization of channel factorization on preduals of von Neumann algebras. We start with a lemma.

Lemma 3.12. *Let $T: L_1(\mathcal{N})\to L_1(\mathcal{M})$ be a CPTP map. Define the set of CPTP maps*

$$C_{post}(T)=\{\Phi\circ T\mid\Phi: L_1(\mathcal{M})\to L_1(\mathcal{M})\ CPTP\}\ ,$$
$$C_{pre}(T)=\{T\circ\Phi\mid\Phi: L_1(\mathcal{N})\to L_1(\mathcal{N})\ CPTP\}\ .$$

Then both $C_{post}(T)$ and $C_{pre}(T)$ are relatively closed in $CB(L_1(\mathcal{N}), L_1(\mathcal{M})) \subset CB(\mathcal{M}^{op}, \mathcal{N}^{op})$ for the weak-topology induced by $CB(L_1(\mathcal{N}), L_1(\mathcal{M})) \subset CB(\mathcal{M}^{op}, \mathcal{N}^{op}) = (L_1(\mathcal{N})\widehat{\otimes}\mathcal{M}^{op})^*$. Namely, both sets are closed in the point-weak topology on $CB(L_1(\mathcal{N}), L_1(\mathcal{M}))$.*

Proof. We first argue for $C_{post}(T)$. Let (Φ_α) be a net such that $\Phi_\alpha \circ T \to S$ in the weak*-topology. That is, for any $x \in L_1(\mathcal{N}), y \in \mathcal{M}$

$$\lim_\alpha \tau_{\mathcal{M}}(y\Phi_\alpha \circ T(x)) = \tau_{\mathcal{M}}(yS(x)) \, . \tag{3.13}$$

Let (Φ_β) be a sub-net such that $\Phi_\beta \to \Phi$ for some $\Phi : L_1(\mathcal{M}) \to (\mathcal{M}^{op})^*$ in the weak*-topology $CB(L_1(\mathcal{N}), (\mathcal{M}^{op})^*) \cong (L_1(\mathcal{N})\widehat{\otimes}\mathcal{M}^{op})^*$. Note that

$$CB(L_1(\mathcal{N}), (\mathcal{M}^{op})^*) \cong CB(\mathcal{M}^{op}, \mathcal{N}^{op})$$

by taking adjoint. The map $\Phi^\dagger$ is UCP because for any positive $x \in L_1(\mathcal{N})$

$$\tau_{\mathcal{M}}(x\Phi^\dagger(1)) = \lim_\beta \tau_{\mathcal{M}}(x\Phi_\beta^\dagger(1)) = \lim_\beta \tau_{\mathcal{M}}(\Phi_\beta(x)) = \tau_{\mathcal{N}}(x)$$

We have $\Phi_\beta \circ T \to \Phi \circ T$ because for any $x \in L_1(\mathcal{N}), y \in \mathcal{M}$

$$\lim_\beta \tau_{\mathcal{M}}(y\Phi_\beta \circ T(x)) = \lim_\beta \tau_{\mathcal{M}}(\Phi_\beta^\dagger(y)T(x)) = \tau_{\mathcal{M}}(\Phi^\dagger(y)T(x)) = \Phi \circ T(x)(y^{op})$$

where $\Phi \circ T(x) \in (\mathcal{M}^{op})^*$. Then by (3.13), $\Phi \circ T(x) = S(x) \in L_1(\mathcal{M})$. This implies $\Phi_n \circ T = S$ for $\Phi_n : L_1(\mathcal{M}) \to L_1(\mathcal{M})$ being the normal part of Φ. Since $\Phi_n^\dagger$ is normal CP and sub-unital, Φ_n is CPTNI. Define $\Phi_0(\rho) = \tau_{\mathcal{M}}(\Phi_n(\rho) - \rho)\sigma$ where σ is some density operator. Then $\tilde{\Phi} = \Phi_n + \Phi_0$ is CPTP. Moreover, $\Phi_0 \circ T = 0$ because both $\tilde{\Phi} \circ T$ and $\Phi_n \circ T = S$ are CPTP. Thus, we obtain $\tilde{\Phi} \circ T = \Phi_n \circ T = S$.

For $C_{pre}(T)$, let Ψ_α be a net such that $T \circ \Psi_\alpha \to S$ in the weak*-topology. Let Ψ_β be a sub-net of Ψ_α such that $\Psi_\beta \to \Psi$ for some $\Psi \in CB(L_1(\mathcal{N}), (\mathcal{M}^{op})^*)$. For any $x \in L_1(\mathcal{N})$ and $y \in \mathcal{M}$,

$$\lim_\beta \tau(yT \circ \Psi_\beta(x)) = \lim_\beta \tau(T^\dagger(y)\Psi_\beta(x)) = \Psi(x)(T^\dagger(y)) = T^{\dagger\dagger} \circ \Psi(x)(y)$$

This means $T \circ \Psi_\beta \to T^{\dagger\dagger} \circ \Psi$ in the weak*-topology of $CB(L_1(\mathcal{N}), (\mathcal{M}^{op})^*)$. Let Ψ_n be the normal part of Ψ. Since $T^{\dagger\dagger}|_{L_1(\mathcal{M})} = T$, we have

$$S = T^{\dagger\dagger} \circ \Psi_n = T^{\dagger\dagger}|_{L_1(\mathcal{M})} \circ \Psi_n = T \circ \Psi_n \, .$$

The argument to modify Φ_n to be CPTP is similar. □

We say a bipartite density operator $\rho \in L_1(\mathcal{M}\overline{\otimes}\mathcal{N})$ is *separable* if ρ can be written as $\rho = \sum_{j=1}^{\infty} \lambda_j \omega_j \otimes \sigma_j$, for some $\lambda_j \geq 0, \sum_{j=1}^{\infty} \lambda_j = 1$ and $\omega_j \in L_1(\mathcal{M}), \sigma_j \in L_1(\mathcal{N})$ are density operators.

Theorem 3.13. *Assume that $\mathcal{M}$ is an injective semi-finite von Neumann algebra. Let $T, S : L_1(\mathcal{N}) \to L_1(\mathcal{M})$ be two CPTP maps. TFAE*

i) *there exists a CPTP $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M})$ such that $\Phi \circ T = S$*
ii) *for any projection $e \in \mathcal{M}$ with $\tau(e) < \infty$ and any separable density operator $\rho \in L_1(\mathcal{N}) \otimes e\mathcal{M}e^{op}$,*

$$\| T \otimes \mathrm{id}(\rho) \|_{L_1(\mathcal{M})\widehat{\otimes} e\mathcal{M}e^{op}} \geq \| S \otimes \mathrm{id}(\rho) \|_{L_1(\mathcal{M})\widehat{\otimes} e\mathcal{M}e^{op}}$$

Proof. i)$\Rightarrow$ ii) follows from $(\Phi \circ T) \otimes \mathrm{id}(\rho) = S \otimes \mathrm{id}(\rho)$ and that the map $\Phi \otimes \mathrm{id}$ is contractive on $L_1(\mathcal{M})\widehat{\otimes} e\mathcal{M}e^{op}$. For ii) $\Rightarrow$ i), we again argue by contradiction. Suppose $S \notin C_{post}(T) = \{\Phi \circ T | \ \Phi \text{ CPTP}\}$. Then by Lemma 3.12, there exists $x_1 \in L_1(\mathcal{N})\widehat{\otimes}\mathcal{M}^{op}$ such that

$$Re\langle S, x_1\rangle > Re \sup_{\Phi \text{ CPTP}} \langle \Phi \circ T, x_1\rangle \ .$$

We can replace x_1 by a finite tensor sum $x_2 = \sum_{j=1}^{n} a_j \otimes b_j$ with $\| x_1 - x_2 \|_{L_1(\mathcal{N})\widehat{\otimes}\mathcal{M}^{op}}$ small enough. Moreover, following the same argument in (3.9), $a_j \in L_1(\mathcal{N})$ and $b_j \in \mathcal{M}^{op}$ can be self-adjoint. Note that for any $\omega \in L_1(\mathcal{N})$,

$$\langle S, \omega \otimes 1\rangle = \mathrm{tr}_{\mathcal{M}}(S(\omega)) = \mathrm{tr}_{\mathcal{N}}(\omega) = \mathrm{tr}_{\mathcal{M}}(\Phi \circ T(\omega)) = \langle \Phi \circ T, \omega \otimes 1\rangle$$

because S and $\Phi \circ T$ are trace preserving. Then we can replace x_2 by

$$x_3 = \sum_j a_j \otimes b_j + \| b_j \| \ (|a_j| \otimes 1) = \sum_j (a_j)_+ \otimes (\| b_j \| \ 1 + b_j) + (a_j)_- \otimes (\| b_j \| \ 1 - b_j)$$

which is a finite sum of positive elements. Let $e \in \mathcal{M}$ be a projection with finite trace such that

$$\left| \sum_j \tau_{\mathcal{M}}(b_j^{op} S(a_j)) - \sum_j \tau_{\mathcal{M}}(e b_j^{op} e S(a_j)) \right| < \epsilon \ .$$

Take $x_4 = (1 \otimes e)x_3(1 \otimes e)$. We have for small ϵ

$$\langle S, x_4\rangle > \langle S, x_3\rangle - \epsilon > \sup_{\Phi} \langle \Phi \circ T, x_3\rangle. \tag{3.14}$$

Since $\mathcal{M}$ is injective, we reinterpret the duality pairing and applying Proposition 3.6,

$$\langle S, x_4\rangle = \langle \mathrm{id}, S\otimes \mathrm{id}(x_4)\rangle \le \| S\otimes \mathrm{id}(x_4)\|_{L_1(\mathcal{M})\widehat{\otimes} e\mathcal{M}e^{op}} \ .$$

$$\sup_{\Phi\ CPTP} \langle \Phi\circ T, x_3\rangle = \sup_{\Phi\ CPTP} \langle \Phi, T\otimes \mathrm{id}(x_3)\rangle = \| T\otimes \mathrm{id}(x_3)\|_{L_1(\mathcal{M})\widehat{\otimes} e\mathcal{M}e^{op}}$$
$$\ge \| T\otimes \mathrm{id}(x_4)\|_{L_1(\mathcal{M})\widehat{\otimes} \mathcal{M}^{op}} \ .$$

Here the last inequality uses the fact that $\rho\mapsto e\rho e$ is a complete contraction from $\mathcal{M}$ to $e\mathcal{M}e$. Thus we have a violation of ii),

$$\| S\otimes \mathrm{id}(x_4)\|_{L_1(\mathcal{M})\widehat{\otimes} e\mathcal{M}e^{op}} > \| T\otimes \mathrm{id}(x_4)\|_{L_1(\mathcal{M})\widehat{\otimes} e\mathcal{M}e^{op}} \ .$$

Here $x_4\in L_1(\mathcal{M})\widehat{\otimes} e\mathcal{M}e^{op}$ is a finite tensor of positive element with finite trace. Replacing x_4 by its normalization, we get a separable density operator. That completes the proof. □

The above theorem gives the characterization for "post"-factorization. Similarly, we consider the "pre"-factorization, which is equivalent to the "post"-factorization of normal UCP maps.

Theorem 3.14. *Assume that $\mathcal{M}$ is injective. Let $T,S : L_1(\mathcal{M})\to L_1(\mathcal{N})$ be two CPTP maps. TFAE*

i) *there exists a CPTP $\Phi : L_1(\mathcal{M})\to L_1(\mathcal{M})$ such that $T\circ\Phi = S$,*
ii) *for any positive $x\in \mathcal{N}^{op}\otimes\mathcal{M}$,*

$$\| T^\dagger\otimes \mathrm{id}(x)\|_{\mathcal{M}^{op}\overline{\otimes}\mathcal{M}} \le \| S^\dagger\otimes \mathrm{id}(x)\|_{\mathcal{M}^{op}\overline{\otimes}\mathcal{M}}$$

Proof. By taking the adjoint, $\Phi^\dagger\circ T^\dagger = S^\dagger$ as normal UCP maps. Then i)⇒ ii) follows from

$$\| S^\dagger\otimes \mathrm{id}(x)\|_\infty = \| \Phi^\dagger\circ T^\dagger\otimes \mathrm{id}(x)\|_\infty \le \| T^\dagger\otimes \mathrm{id}(x)\|_\infty \ .$$

For ii) ⇒ i), suppose $S\notin C_{pre}(T) := \{T\circ\Phi \mid \Phi \text{ CPTP}\}$. By the same argument as for Theorem 3.13, there exists a finite tensor $x_2 = \sum_j a_j\otimes b_j \in L_1(\mathcal{M})\widehat{\otimes}\mathcal{N}^{op}$ with a_j, b_j positive such that

$$\langle S, x_2\rangle > \sup_{\Phi\ CPTP} \langle T\circ\Phi, x_2\rangle \ .$$

Then we choose a finite trace projection $e\in\mathcal{M}$ such that $ea_je\in\mathcal{M}$ are bounded and for $x_3 = (e\otimes 1)x_2(e\otimes 1) = \sum_j ea_je\otimes b_j$,

$$\langle S, x_3\rangle > \langle S, x_2\rangle - \epsilon > \sup_{\Phi\ CPTP} \langle T\circ\Phi, x_2\rangle \tag{3.15}$$

Since $\mathcal{M}$ is injective, we apply Proposition 3.6,

$$\begin{aligned}\langle S, x_3\rangle &= \langle \mathrm{id}, \mathrm{id}\otimes S^\dagger(x_3)\rangle \le \| \mathrm{id}\otimes S^\dagger(x_3)\|_{L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op}} ,\\ \langle T\circ\Phi, x_2\rangle &= \sup_{\Phi\ CPTP} \langle \Phi, \mathrm{id}\otimes T^\dagger(x_2)\rangle = \| \mathrm{id}\otimes T^\dagger(x_2)\|_{L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op}}\\ &\ge \| \mathrm{id}\otimes T^\dagger(x_3)\|_{L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op}} .\end{aligned}$$

This implies

$$\| \mathrm{id}\otimes S^\dagger(x_3)\|_{L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op}} > \| \mathrm{id}\otimes T^\dagger(x_2)\|_{L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op}} \ge \| \mathrm{id}\otimes T^\dagger(x_3)\|_{L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op}} .$$

Because $\mathrm{id}\otimes T^\dagger(x_3)$ is a positive operator in $e\mathcal{M}e\otimes\mathcal{M}^{op}$, by Lemma 3.1 we have

$$\| \mathrm{id}\otimes T^\dagger(x_3)\|_{L_1(e\mathcal{M}e)\widehat{\otimes}\mathcal{M}^{op}} = \inf_\sigma \| (\sigma^{-\frac12}\otimes 1)\, \mathrm{id}\otimes T^\dagger(x_3)(\sigma^{-\frac12}\otimes 1)\|_\infty$$

where the infimum is over all invertible density operators $\sigma\in e\mathcal{M}e$. Thus we choose an invertible density operator $\sigma\in e\mathcal{M}e$ such that

$$\begin{aligned}\| (\sigma^{-\frac12}\otimes 1)\, \mathrm{id}\otimes T^\dagger(x_3)(\sigma^{-\frac12}\otimes 1)\|_{e\mathcal{M}e\overline{\otimes}\mathcal{M}} &< \| \mathrm{id}\otimes T^\dagger(x_3)\|_{L_1(e\mathcal{M}e)\widehat{\otimes}\mathcal{M}^{op}} +\epsilon\\ &< \| \mathrm{id}\otimes S^\dagger(x_3)\|_{L_1(e\mathcal{M}e)\widehat{\otimes}\mathcal{M}^{op}}\\ &\le \| (\sigma^{-\frac12}\otimes 1)\, \mathrm{id}\otimes S^\dagger(x_3)(\sigma^{-\frac12}\otimes 1)\|_{e\mathcal{M}e\overline{\otimes}\mathcal{M}} .\end{aligned}$$

Then $x_4 = (\sigma^{-\frac12}\otimes 1)x_3(\sigma^{-\frac12}\otimes 1)$ is positive in $\mathcal{M}\otimes\mathcal{N}^{op}$, and we have

$$\| \mathrm{id}\otimes T^\dagger(x_4)\|_{\mathcal{M}\overline{\otimes}\mathcal{M}^{op}} < \| \mathrm{id}\otimes S^\dagger(x_4)\|_{\mathcal{M}\overline{\otimes}\mathcal{M}^{op}} ,$$

which is a violation to condition ii). This proves ii)⇒ i). □

3.4. Approximate case

In [16], Jenčová gives a characterization for the approximate post-channel factorization in finite dimensions that

$$\inf_{\Phi\ \mathrm{CPTP}} \| S-\Phi\circ T\|_{cb} < \delta$$

is small but nonzero. Inspired by Jenčová's work, we consider the approximate case of quantum majorization. The following lemma is an analogue of [16, Proposition 1].

Lemma 3.15. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra.*
i) For two density operators ρ,σ in $L_1(\mathcal{M})$,

$$\frac12 \| \rho-\sigma\|_1 = \sup\{\ \tau(x(\rho-\sigma)) \mid x\ge 0, \| x\|_\infty \le 1\} .$$

ii) Let $\mathcal{M}$ be injective. For two CPTP maps $T, S : L_1(\mathcal{M}) \to L_1(\mathcal{M})$,

$$\frac{1}{2} \| T - S \|_{cb} = \sup\{ \langle T - S, \rho\rangle \mid \rho \geq 0, \| \rho \|_{L_1(\mathcal{M}, L_\infty(\mathcal{M}^{op}))} \leq 1\} .$$

Proof. For i), note that

$$x = x^*, \| x \| \leq 1 \Longleftrightarrow x + 1 \geq 0, \| x + 1 \| \leq 2.$$

Since $\tau(\rho - \sigma) = 0$,

$$\begin{aligned} \| \rho - \sigma \|_1 &= \sup\{\text{Re } \tau(x(\rho - \sigma))| \; \| x \|_\infty \leq 1\} \\ &= \sup\{\tau(x(\rho - \sigma))| \; \| x \|_\infty \leq 1, x \text{ self-adjoint}\} \\ &= \sup\{\tau((x + 1)(\rho - \sigma))| \; \| x \|_\infty \leq 1, x \text{ self-adjoint}\} \\ &= \sup\{\tau(y(\rho - \sigma))| \; \| y \|_\infty \leq 2, y \geq 0\} \\ &= 2 \sup\{\tau(y(\rho - \sigma))| \; \| y \|_\infty \leq 1, y \geq 0\}. \end{aligned}$$

For ii), let x be self-adjoint and satisfy $\| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{M}^{op}))} < 1$. Apply Lemma 3.2 to x and $-x$, we have density operators $\sigma_1, \sigma_2 \in L_1(\mathcal{M})$ such that $x \geq \sigma_1 \otimes 1$ and $-x \leq x_2 \otimes 1$. Then

$$0 \leq x + \sigma_2 \otimes 1 \geq (\sigma_1 + \sigma_2) \otimes 1, \| x + \sigma_2 \otimes 1 \|_{L_1(\mathcal{M}, L_\infty(\mathcal{M}^{op}))} \leq 2 .$$

Conversely, let $y \geq 0, \| y \|_{L_1(\mathcal{M}, L_\infty(\mathcal{M}^{op}))} < 2$. Then there exists a density operator $\sigma \in L_1(\mathcal{M})$ such that $0 \leq y \leq 2\sigma \otimes 1$. Then

$$-\sigma \otimes 1 \leq y - \sigma \otimes 1 \leq \sigma \otimes 1 \; , \; \| y - \sigma \otimes 1 \|_{L_1(\mathcal{M}, L_\infty(\mathcal{M}^{op}))} \leq 1.$$

Thus we have

$$\begin{aligned} & x = x^*, \| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{M}^{op}))} \leq 1 \\ \Longleftrightarrow \; & x + 1 \otimes \sigma \geq 0, \| x + 1 \otimes \sigma \|_{L_1(\mathcal{M}, L_\infty(\mathcal{M}^{op}))} \leq 2 \text{ for some density } \sigma \in L_1(\mathcal{M}) . \end{aligned}$$

Since $\mathcal{M}$ is injective, we have $L_1(\mathcal{M}, L_\infty(\mathcal{M}^{op})) \cong L_1(\mathcal{M}) \widehat{\otimes} \mathcal{M}^{op}$. Then using the fact that $\langle T - S, \sigma \otimes 1\rangle = \tau(T(\sigma)) - \tau(S(\sigma)) = 0$, we have

$$\begin{aligned} \| T - S \|_{cb} &= \sup\{\text{Re } \langle T - S, x\rangle| \; \| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{M}^{op}))} < 1\} \\ &= \sup\{\langle T - S, x\rangle| \; \| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{M}^{op}))} < 1, x = x^*\} \\ &= \sup\{\langle T - S, x\rangle| \; \| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{M}^{op}))} < 2, x \geq 0\} \\ &= 2 \sup\{\langle T - S, x\rangle| \; \| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{M}^{op}))} < 1, x \geq 0\}. \quad \square \end{aligned}$$

Theorem 3.16. *Let $\mathcal{M}, \mathcal{N}$ be semi-finite von Neumanna algebras and $\mathcal{M}$ be injective and $\tau_{\mathcal{M}}(1) = +\infty$. Suppose ρ and σ are two density operators in $L_1(\mathcal{M}\overline{\otimes}\mathcal{N})$ such that $\tau_{\mathcal{M}} \otimes \mathrm{id}(\rho) = \tau_{\mathcal{M}} \otimes \mathrm{id}(\sigma)$. TFAE*

i) $\inf\{\| \sigma - \Phi \otimes \mathrm{id}(\rho) \|_1 \ \mid \Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M}) \text{CPTP}\} \le \delta$.
ii) *for any CPTP map $\Psi : L_1(\mathcal{N}) \to L_1(\mathcal{M}^{op})$ and $ran(\Psi) \subset \mathcal{M}^{op}$, we have*

$$\| \mathrm{id} \otimes \Psi(\sigma) \|_{L_1(\mathcal{M})\hat{\otimes}\mathcal{M}^{op}} \le \| \mathrm{id} \otimes \Psi(\rho) \|_{L_1(\mathcal{M})\hat{\otimes}\mathcal{M}^{op}} + \frac{\delta}{2} \| \Psi : L_1(\mathcal{N}) \to \mathcal{M}^{op} \|_{cb}$$

Proof. For a CPTP Ψ, we can choose $R : L_1(\mathcal{M}) \to L_1(\mathcal{M})$ CPTP such that

$$\langle R, \mathrm{id} \otimes \Psi(\sigma) \rangle \ge \| \mathrm{id} \otimes \Psi(\sigma) \|_{L_1(\mathcal{M})\hat{\otimes}\mathcal{M}^{op}} - \epsilon \ .$$

Then

$$\begin{aligned}
&\| \mathrm{id} \otimes \Psi(\sigma) \|_{L_1(\mathcal{M})\hat{\otimes}\mathcal{M}^{op}} \le \epsilon + \langle R, \mathrm{id} \otimes \Psi(\sigma) \rangle \\
\le&\epsilon + \langle R, \Phi \otimes \Psi(\rho) \rangle + \langle R, \mathrm{id} \otimes \Psi(\sigma) - \Phi \otimes \Psi(\rho) \rangle \\
\le&\epsilon + \langle R \circ \Phi, \mathrm{id} \otimes \Psi(\rho) \rangle + \langle R^\dagger \circ \Psi, \sigma - \Phi \otimes \mathrm{id}(\rho) \rangle \\
\le&\epsilon + \| \mathrm{id} \otimes \Psi(\rho) \|_{L_1(\mathcal{M})\hat{\otimes}\mathcal{M}^{op}} + \frac{1}{2} \| R^\dagger \circ \Psi : L_1(\mathcal{N}) \to \mathcal{M}^{op} \|_{cb} \| \sigma - \Phi \otimes \mathrm{id}(\rho) \|_1 \\
\le&\epsilon + \| \mathrm{id} \otimes \Psi(\rho) \|_{L_1(\mathcal{M})\hat{\otimes}\mathcal{M}^{op}} + \frac{1}{2} \| \Psi : L_1(\mathcal{N}) \to \mathcal{M}^{op} \|_{cb} \| \sigma - \Phi \otimes \mathrm{id}(\rho) \|_1
\end{aligned}$$

Here in the second last inequality we apply Lemma 3.15 i) to

$$\langle R^\dagger \circ \Psi, \sigma - \mathrm{id} \otimes \Phi(\rho) \rangle = \tau\big(x_{R^\dagger \circ \Psi}(\sigma - \Phi \otimes \mathrm{id}(\rho))\big)$$

where $x_{R^\dagger \circ \Psi} \in \mathcal{N}\overline{\otimes}\mathcal{M}$ is the operator corresponding to the map $R^\dagger \circ \Psi$ via the Effros-Ruan isomorphism

$$CB(L_1(\mathcal{N}), \mathcal{M}^{op}) \cong \mathcal{N}^{op}\overline{\otimes}\mathcal{M}^{op} \ . \tag{3.16}$$

Then i)$\Rightarrow$ ii) follows from taking the infimum over all CPTP Φ and $\epsilon \to 0$. Conversely, suppose $\inf\limits_{\Phi \ \text{CPTP}} \| \sigma - \Phi \otimes \mathrm{id}(\rho) \|_1 > \delta$. By Lemma 3.15 i), we have for $x \in \mathcal{N}\overline{\otimes}\mathcal{M}$,

$$\langle T, \sigma - \mathrm{id} \otimes \Phi(\rho) \rangle = \tau\big(x(\sigma - \Phi \otimes \mathrm{id}(\rho))\big) \le \frac{1}{2} \| T : L_1(\mathcal{N}) \to \mathcal{M}^{op} \|_{cb} \| \sigma - \Phi \otimes \mathrm{id}(\rho) \|_1 \ ,$$

where T is the map corresponding to x^{op} via the isomorphism (3.16). Because the above pairing is linear for both T and Φ, we have by Sion's minimax theorem [24],

$$
\begin{aligned}
\delta <& \inf_{\Phi \text{ CPTP}} \| \sigma - \Phi \otimes \mathrm{id}(\rho) \|_1 \\
=& 2 \inf_{\Phi \text{ CPTP}} \sup_{T\ CP, \|T\|_{cb} \le 1} \langle T, \sigma - \Phi \otimes \mathrm{id}(\rho) \rangle \\
=& 2 \sup_{T\ CP, \|T\|_{cb} \le 1} \inf_{\Phi \text{ CPTP}} \langle T, \sigma - \Phi \otimes \mathrm{id}(\rho) \rangle \\
=& 2 \sup_{T\ CP, \|T\|_{cb} \le 1} \langle T, \sigma \rangle - \sup_{\Phi \text{ CPTP}} \langle T, \Phi \otimes \mathrm{id}(\rho) \rangle
\end{aligned}
$$

Rescaling the above inequality, there exists a CP $T : L_1(\mathcal{N}) \to \mathcal{M}^{op}$ such that

$$
\langle T, \sigma \rangle - \sup_{\Phi \text{ CPTP}} \langle T, \Phi \otimes \mathrm{id}(\rho) \rangle > \frac{\delta}{2} \| T : L_1(\mathcal{N}) \to \mathcal{M}^{op} \|_{cb} \ .
$$

For a projection $e \in \mathcal{M}$, denote the map $T_e(\cdot) = eT(\cdot)e$. There exists e with $\tau_{\mathcal{M}}(e) < \infty$ such that $|\langle T, (e \otimes 1)\sigma(e \otimes 1) - \sigma \rangle|$ is small enough that

$$
\begin{aligned}
\langle T_e, \sigma \rangle =& \langle T, (e \otimes 1)\sigma(e \otimes 1) \rangle \\
&> \sup_{\Phi \text{ CPTP}} \langle T, \Phi \otimes \mathrm{id}(\rho) \rangle + \frac{\delta}{2} \| T : L_1(\mathcal{N}) \to \mathcal{M}^{op} \|_{cb} \\
&= \| \mathrm{id} \otimes T(\rho) \|_{L_1(\mathcal{M}) \widehat{\otimes} \mathcal{M}^{op}} + \frac{\delta}{2} \| T : L_1(\mathcal{N}) \to \mathcal{M}^{op} \|_{cb} \\
&\ge \| \mathrm{id} \otimes T_e(\rho) \|_{L_1(\mathcal{M}) \widehat{\otimes} e\mathcal{M}e^{op}} + \frac{\delta}{2} \| T_e : L_1(\mathcal{N}) \to e\mathcal{M}e^{op} \|_{cb} \ .
\end{aligned}
$$

Here we use Proposition 3.6 by the assumption $\mathcal{M}$ is injective

$$
\sup_{\Phi \text{ CPTP}} \langle T, \Phi \otimes \mathrm{id}(\rho) \rangle = \sup_{\Phi \text{ CPTP}} \langle \Phi, \mathrm{id} \otimes T(\rho) \rangle = \| \mathrm{id} \otimes T(\rho) \|_{L_1(\mathcal{M}) \widehat{\otimes} \mathcal{M}^{op}}
$$

and that $\| T_e : L_1(\mathcal{N}) \to e\mathcal{M}e^{op} \|_{cb} \le \| T_e : L_1(\mathcal{N}) \to \mathcal{M} \|_{cb}$. Also, we have

$$
\langle T_e, \sigma \rangle = \langle \mathrm{id}, \mathrm{id} \otimes T_e(\sigma) \rangle \le \| \mathrm{id} \otimes T_e(\sigma) \|_{L_1(\mathcal{M}) \widehat{\otimes} e\mathcal{M}e^{op}} \ .
$$

Therefore, we have a violation of ii) for $T_e : L_1(\mathcal{N}) \to e\mathcal{M}e^{op}$ is CP,

$$
\| \mathrm{id} \otimes T_e(\sigma) \|_{L_1(\mathcal{M}) \widehat{\otimes} e\mathcal{M}e^{op}} > \| \mathrm{id} \otimes T_e(\rho) \|_{L_1(\mathcal{M}) \widehat{\otimes} e\mathcal{M}e^{op}} + \frac{\delta}{2} \| T_e : L_1(\mathcal{N}) \to e\mathcal{M}e^{op} \|_{cb} \tag{3.17}
$$

By linearity, we can assume T_e is CPTNI. Denote $\rho_{\mathcal{N}} = \tau_{\mathcal{M}} \otimes \mathrm{id}(\rho)$ and $\sigma_{\mathcal{N}} = \tau_{\mathcal{M}} \otimes \mathrm{id}(\sigma)$. Because $\rho_{\mathcal{N}} = \sigma_{\mathcal{N}}$, we follow the argument in Theorem 3.8 to replace T_e by

$$
\tilde{T} = T_e + T_0 \ , \ T_0(x) = \frac{\tau_{\mathcal{M}}(x - T_e(x))}{\tau_{\mathcal{M}}(e)} e \ .
$$

Note that $\| T_0 : L_1(\mathcal{N}) \to e\mathcal{M}e^{op} \|_{cb} = \frac{1}{\tau_{\mathcal{M}}(e)}$. Then we can always choose $\tau_{\mathcal{M}}(e)$ large enough such that $\| \tilde{T} \|_{cb} - \| T_e \|_{cb}$ is small and (3.17) is satisfied for $\tilde{T}$. □

Remark 3.17. If, in addition, $\inf\{ \tau_{\mathcal{M}}(e_0) \mid e_0 \text{ nonzero projection } \} = 0$, we do not need the assumption $\rho_{\mathcal{N}} = \sigma_{\mathcal{N}}$ in Theorem 3.16. In the case of $\rho_{\mathcal{N}} \neq \sigma_{\mathcal{N}}$, by the corresponding discussion in Theorem 3.8, we have a CPTP map T_1 such that

$$\| \mathrm{id} \otimes T_1(\sigma) \|_{L_1(\mathcal{M})\hat{\otimes} e\mathcal{M}e^{op}} - \| \mathrm{id} \otimes T_1(\rho) \|_{L_1(\mathcal{M})\hat{\otimes} e\mathcal{M}e^{op}}$$
$$> (\frac{1}{\tau_{\mathcal{M}}(e_0)} - \frac{1}{\tau_{\mathcal{M}}(e)}) \tau_{\mathcal{N}}\big((\rho_{\mathcal{N}} - \sigma_{\mathcal{N}})_-\big)$$

where $e_0 \leq e$ is a sub-projection. This difference can be arbitrarily large if $\inf\limits_{e_0 \neq 0} \tau_{\mathcal{M}}(e_0) = 0$.

The following is a generalization of [16, Theorem 1].

Theorem 3.18. *Let $\mathcal{M}, \mathcal{N}$ be semi-finite von Neumanna algebras and let $\mathcal{M}$ be injective. Let $S, T : L_1(\mathcal{N}) \to L_1(\mathcal{M})$ be two CPTP maps. TFAE*

i) $\inf\{\| S - \Phi \circ T \|_{cb} \mid \Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M}) \text{CPTP}\} \leq \delta$;
ii) *for any density operator $\rho \in L_1(\mathcal{N} \overline{\otimes} \mathcal{M}^{op})$, we have*

$$\| S \otimes \mathrm{id}(\rho) \|_{L_1(\mathcal{M})\hat{\otimes}\mathcal{M}^{op}} \leq \| T \otimes \mathrm{id}(\rho) \|_{L_1(\mathcal{M})\hat{\otimes}\mathcal{M}^{op}} + \frac{\delta}{2} \| \rho \|_{L_1(\mathcal{M})\hat{\otimes}\mathcal{M}^{op}} .$$

Proof. Let $\rho \in L_1(\mathcal{N} \overline{\otimes} \mathcal{M}^{op})$ be a density operator. By Proposition 3.6, for any $\epsilon > 0$ we can choose $R : L_1(\mathcal{M}) \to L_1(\mathcal{M})$ CPTP such that

$$\langle R, S \otimes \mathrm{id}(\rho) \rangle \geq \| S \otimes \mathrm{id}(\rho) \|_{L_1(\mathcal{M})\hat{\otimes}\mathcal{M}^{op}} - \epsilon .$$

Then

$$\begin{aligned}
&\| S \otimes \mathrm{id}(\rho) \|_{L_1(\mathcal{M})\hat{\otimes}\mathcal{M}^{op}} \leq \epsilon + \langle R, S \otimes \mathrm{id}(\rho) \rangle \\
\leq & \epsilon + \langle R, \Phi \circ T \otimes \mathrm{id}(\rho) \rangle + \langle R, (S - \Phi \circ T) \otimes \mathrm{id}(\rho) \rangle \\
\leq & \epsilon + \| \Phi \circ T \otimes \mathrm{id}(\rho) \|_{L_1(\mathcal{M})\hat{\otimes}\mathcal{M}^{op}} + \langle S - \Phi \circ T, \mathrm{id} \otimes R^\dagger(\rho) \rangle \\
\leq & \epsilon + \| \Phi \circ T \otimes \mathrm{id}(\rho) \|_{L_1(\mathcal{M})\hat{\otimes}\mathcal{M}^{op}} + \frac{1}{2} \| S - \Phi \circ T \|_{cb} \| \mathrm{id} \otimes R^\dagger(\rho) \|_{L_1(\mathcal{M})\hat{\otimes}\mathcal{M}^{op}} \\
\leq & \epsilon + \| \Phi \circ T \otimes \mathrm{id}(\rho) \|_{L_1(\mathcal{M})\hat{\otimes}\mathcal{M}^{op}} + \frac{1}{2} \| S - \Phi \circ T \|_{cb} \| \rho \|_{L_1(\mathcal{M})\hat{\otimes}\mathcal{M}^{op}}
\end{aligned}$$

where in the second last inequality we used Lemma 3.15 ii) for the 1/2 factor. Then i)⇒ ii) follows from taking the infimum over all CPTP Φ and $\epsilon \to 0$. For ii) ⇒ i), suppose

$\inf_{\Phi \text{ CPTP}} \| S - \Phi \circ T \|_{cb} > \delta$. Let us use the shorthand notation $\| \cdot \|_{1,\infty} = \| \cdot \|_{L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op}}$. Using the minimax theorem [24],

$$\begin{aligned}
\delta < & \inf_{\Phi \text{ CPTP}} \| S - \Phi \circ T \|_{cb} \\
=& 2 \inf_{\Phi \text{ CPTP}} \sup_{\rho \geq 0, \|\rho\|_{1,\infty} \leq 1} \langle S - \Phi \circ T, \rho \rangle \\
=& 2 \sup_{\rho \geq 0, \|\rho\|_{1,\infty} \leq 1} \inf_{\Phi \text{CPTP}} \langle S - \Phi \circ T, \rho \rangle \\
=& 2 \sup_{\rho \geq 0, \|\rho\|_{1,\infty} \leq 1} \langle \mathrm{id}, S \otimes \mathrm{id}(\rho) \rangle - \sup_{\Phi \text{CPTP}} \langle \Phi, T \otimes \mathrm{id}(\rho) \rangle \\
\leq & 2 \sup_{\rho \geq 0, \|\rho\|_{1,\infty} \leq 1} \sup_{\Phi \text{CPTP}} \langle \Phi, S \otimes \mathrm{id}(\rho) \rangle - \sup_{\Phi \text{CPTP}} \langle \Phi, T \otimes \mathrm{id}(\rho) \rangle \\
=& 2 \sup_{\rho \geq 0, \|\rho\|_{1,\infty} \leq 1} \| S \otimes \mathrm{id}(\rho) \|_{L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op}} - \| T \otimes \mathrm{id}(\rho) \|_{L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op}},
\end{aligned}$$

where in the last equality we used Proposition 3.6 because $\mathcal{M}$ is injective. Thus there exists a positive $\rho \in L_1(\mathcal{M})\widehat{\otimes}\mathcal{M}^{op}$ violating the inequality in ii). One can then replace ρ by a bipartite density operator $\tilde{\rho}$ in $L_1(\mathcal{M}\overline{\otimes}\mathcal{M}^{op})$ as in Theorem 3.13. □

Remark 3.19. In Theorem 3.16 & 3.18, we cannot reduce condition ii) to entanglement-breaking CPTP maps and respectively separable density operator as in the case for $\delta = 0$. This is because Lemma 3.15 fails when we restrict the pairing to entanglement-breaking or separable elements.

3.5. Results in the type I setting

The results of the previous subsections subsume the case of $B(H)$ where H is infinite dimensional. However, since this is the case most relevant to quantum information theory, we briefly restate some of our results for $B(H)$ in terms of the conditional min entropy H_{min}. $H_{min}(A|B)$ is the sandwiched Rényi p-version of $H(A|B)$ at $p = \infty$ and the smooth version of $H_{min}(A|B)$ connects to $H(A|B)$ by quantum asymptotic equipartition property [26]. While the operational meaning of $H(A|B)$ is in i.i.d. asymptotic regime, $H_{min}(A|B)$ has many applications in the one shot setting ([27] and reference therein). The following theorem summarizes the results on quantum majorization, state convertibility and channel factorization.

Theorem 3.20. *Let H_A, H_B be two infinite-dimensional Hilbert spaces. The following statements hold.*

i) *For two bipartite density operators $\rho^{AB}, \sigma^{AB} \in S_1(H_A \otimes_2 H_B)$, there exists a quantum channel $\Phi : S_1(H_B) \to S_1(H_B)$ such that $\mathrm{id}_A \otimes \Phi(\rho) = \sigma$ if and only if for any entanglement-breaking channel $\Psi : S_1(H_A) \to S_1(H_A)$*

$$H_{min}(A|B)_{\Psi\otimes \mathrm{id}(\rho)} \le H_{min}(A|B)_{\Psi\otimes \mathrm{id}(\sigma)} \ .$$

ii) *For two families of density operators* $\{\rho_i\}_{i\in\mathbb{N}}$ *and* $\{\sigma_i\}_{i\in\mathbb{N}}$ *in* $B(H_B)$*, there exists a quantum channel such that* $\Phi(\rho_i) = \sigma_i$ *for all* $i \in \mathbb{N}$ *if and only if for any finitely supported probability distribution* λ_i *on* $\mathbb{N}$ *and any set of density operators* $\{\omega_i\} \in B(H_A)$

$$H_{min}(A|B)_{(\sum_i \lambda_i\omega_i\otimes\rho_i)} \le H_{min}(A|B)_{(\sum_i \lambda_i\omega_i\otimes\sigma_i)} \ .$$

iii) *For two quantum channels* $T, S : S_1(H_A) \to S_1(H_B)$*, there exists a quantum channel* Φ *such that* $\Phi \circ T = S$ *if and only if for any separable density operator* $\rho \in S_1(H_A \otimes_2 H_B)$*,*

$$H_{min}(A|B)_{\mathrm{id}\otimes T(\rho)} \le H_{min}(A|B)_{\mathrm{id}\otimes S(\rho)}.$$

The above theorem make sense even when H_{min} equals "$-\infty$". We know by Theorem 3.10 and 3.13 that it suffices to consider all finite dimensional H_A in the equivalence ii) and iii). Similarly, for the equivalence i) it suffices to consider channels $\Psi : S_1(H_A) \to S_1(H_{A'})$ into a finite dimensional H'_A. In these situations, H_{min} will always take finite values. In general, $H_{min}(A|B)$ can be "$-\infty$", where the inequalities in the above theorem are trivially satisfied.

4. Tracial convex sets in vector-valued noncommutative L_1-space

In this section, we discuss the analogue of quantum majorization in vector-valued noncommutative L_1-spaces and the connection to the tracial Hahn-Banach Theorem. Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra equipped with a normal faithful semi-finite trace τ. Let E be an operator space. The E-valued noncommutative L_1-spaces were introduced by Pisier in [21]. For $x \in \mathcal{M}_0 \otimes E$ in the algebraic tensor, we define the $L_1(\mathcal{M}, E)$ norm as follows,

$$\| x \|_{L_1(\mathcal{M},E)} = \inf\{\| a \|_{L_2(\mathcal{M})} \| b \|_{L_2(\mathcal{M})} \| y \|_{\mathcal{M}\otimes_{min} E} \ \mid x = a\cdot y\cdot b\}, \tag{4.1}$$

where the infimum runs over all factorizations $x = a\cdot y\cdot b := (a\otimes 1_E)y(b\otimes 1_E)$ with $a, b \in \mathcal{M}_0$ and $y \in \mathcal{M}\otimes E$. The space $L_1(\mathcal{M}, E)$ is defined as the norm completion of $\mathcal{M}_0 \otimes E$. The $L_1(\mathcal{M}, L_\infty(\mathcal{N}))$ space we discussed in the previous section is the special case of E being a von Neumann algebra $\mathcal{N}$. Recall that a von Neumann algebra $\mathcal{M}$ is hyperfinite if $\mathcal{M} = \overline{\cup \mathcal{M}_\alpha}$ is the w^*-closure of the union of an increasing net of finite dimensional von Nuemann algebras $\mathcal{M}_\alpha$. It was proved in [21, Theorem 3.4] that for hyperfinite $\mathcal{M}$,

$$L_1(\mathcal{M}, E) \cong L_1(\mathcal{M})\widehat{\otimes} E \tag{4.2}$$

isometrically. Namely, for hyperfinite $\mathcal{M}$, the vector-valued noncommutative L_1 space is identified with projective tensor product. Following that, we introduce the following definition of a tracial set in $L_1(\mathcal{M})\widehat{\otimes}E$.

Definition 4.1. A subset $V \subset L_1(\mathcal{M})\widehat{\otimes}E$ is called a contractively tracial set if for any CPTNI map $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M})$, $\Phi \otimes \mathrm{id}_E(V) \subset V$.

The matrix tracial sets are discussed in [14, Section 6.2] as the dual concept of matrix convex set. We refer to their definition as matrix tracial set.

Definition 4.2. A matrix contractively tracial set $(V_n)_n$ is a sequence of subsets $V_n \subset M_n(E)$ such that for any CPTNI map $\Phi : M_n \to M_m$, $\Phi \otimes \mathrm{id}(V_n) \subset V_m$.

This definition was considered in [14] for finite dimensional E. Indeed, for $\dim E = m$, each element in $V_n \subset M_n(E) \cong M_n^m$ can be identified with a finite sequence $(x_j) \in (M_n)^m$. We discuss the relations of these two definitions in the following proposition.

Proposition 4.3. *Let H be a separable Hilbert space and $(e_n)_n$ be a sequence of projections such that $\dim(e_nH) = n$ and $e_n \to 1$ weakly. Identify $M_n \cong S_1(e_nH)$ as subspace of $S_1(H)$.*

i) *Given a contractively tracial set $V \subset S_1(H)\widehat{\otimes}E$, the set*

$$V[n] = e_n \cdot V \cdot e_n$$

forms a matrix contractively tracial set such that $\overline{\cup_n V[n]} = \overline{V^{\|\cdot\|}}$.

ii) *Given a matrix contractively tracial set $(V_n) \subset M_n(E)$, the set*

$$V = \overline{(\cup_n V_n)^{\|\cdot\|}} \subset S_1(H)\widehat{\otimes}E$$

is a closed contractively tracial set such that $V[n] = \overline{V_n}$.

Proof. i) Let $e \in B(H)$ be a projection. Because the map $\rho \mapsto e\rho e$ is CPTNI on $S_1(H)$, $x \in V$ implies that $e \cdot x \cdot e \in V$. Then for any $\Phi : M_n \to M_m$ CPTNI, $\Phi \otimes \mathrm{id}(e_n \cdot x \cdot e_n) \in V[m] \subset V$. Thus $(V[n])_n$ is a matrix contractively tracial set. Moreover, for $x \in S_1(H)\widehat{\otimes}E$, $\lim_{n\to\infty} \| e_n \cdot x \cdot e_n - x \|_{S_1(H)\widehat{\otimes}E} = 0$. Then $\overline{V^{\|\cdot\|}} \subset \overline{\cup_n V[n]^{\|\cdot\|}}$ and the other inclusion follows from $V[n] \subset V$.

ii) Let $x \in V_n$. For $\Phi : S_1(H) \to S_1(H)$ CPTNI, we find that

$$e_m \cdot \Phi \otimes \mathrm{id}(x) \cdot e_m \in V_m$$

because $\rho \mapsto e_m\Phi(\rho)e_m$ can be viewed as a CPTNI map from M_n to M_m. By $\lim_{k\to\infty} e_m \cdot \Phi \otimes \mathrm{id}(x) \cdot e_m \to= \Phi \otimes \mathrm{id}(x)$, this shows that $\Phi(x) \in V$ for $x \in V_n$. Let $x_k \in V_{n(k)}$ be a sequence such that $x_k \to x$ in $S_1(H)\widehat{\otimes}E$. Then $\Phi \otimes \mathrm{id}(x_k) \to \Phi \otimes \mathrm{id}(x)$, which implies $\Phi \otimes \mathrm{id}(x) \in V$. This verifies that V is contractively tracial. In particular, the fact that $e_n \cdot x_k \cdot e_n$ converges to $e_n \cdot x \cdot e_n$ implies that $V[n] \subset \overline{V_n}$. □

The above proposition shows that Definition 4.1 and Definition 4.2 are closely related for the case $\mathcal{M} = B(H)$. In particular, they coincide for closed sets. It is easy to see that the convex hull of a contractively tracial set is again contractively tracial. In general, contractively tracial sets are not necessary convex.

The next theorem is the tracial Hahn-Banach separation theorem for convex contractively tracial sets. For matrix contractively tracial sets with $\dim E < \infty$, this was obtained in [14, Theorem 7.6]. Using the projective tensor product, we can now consider semi-finite injective $\mathcal{M}$ and a general operator space E.

Theorem 4.4. *Let $\mathcal{M}$ be an injective semi-finite von Neumann algebra. Let V be a closed convex contractively tracial set in $L_1(\mathcal{M})\widehat{\otimes}E$ and $x \in L_1(\mathcal{M})\widehat{\otimes}E$. Then $x \notin V$ if and only if there exists a CB map $T : E \to \mathcal{M}^{op}$ such that for each $y \in V$, there exists a density operator $\omega_y \in L_1(\mathcal{M})$ depending on y such that*

$$Re\ \mathrm{id} \otimes T(y) \leq \omega_y \otimes 1$$

and for any density operator ω,

$$Re\ \mathrm{id} \otimes T(x) \not\leq \omega \otimes 1.$$

Proof. The "if" direction is trivial. For the other direction, suppose $x \notin V$. Using the duality $L_1(\mathcal{M})\widehat{\otimes}E^* = CB(E, \mathcal{M}^{op})$, there exists a CB map $T : E \to \mathcal{M}^{op}$

$$\mathrm{Re}\ \langle T, x\rangle > \sup_{\rho \in V} \mathrm{Re}\ \langle T, y\rangle.$$

Reinterpreting the dual pairing,

$$\begin{aligned}\mathrm{Re}\ \langle T, x\rangle = \mathrm{Re}\ \langle \mathrm{id}_{\mathcal{M}}, \mathrm{id} \otimes T(x)\rangle &\leq \sup_{\Phi\ \mathrm{CPTNI}} \mathrm{Re}\ \langle \Phi, \mathrm{id} \otimes T(x)\rangle \\ &= \inf\{\tau(\omega) | \mathrm{Re}\ \ \mathrm{id} \otimes T(x) \leq \omega \otimes 1,\ \omega \geq 0\}\ .\end{aligned}$$

Here we used Proposition 3.6 by the assumption that $\mathcal{M}$ is injective. On the other hand, because V is contractively tracial,

$$\begin{aligned}\sup_{y \in V} \mathrm{Re}\ \langle T, y\rangle &\geq \sup_{y \in V, \Phi\ \mathrm{CPTNI}} \mathrm{Re}\ \langle T, \Phi \otimes \mathrm{id}(y)\rangle \\ &= \sup_{y \in V} \inf\{\tau(\omega) \mid \mathrm{Re}\ T \otimes \mathrm{id}(y) \leq \omega \otimes 1\ , \omega \geq 0\}.\end{aligned}$$

Take λ such that $\mathrm{Re}\ \langle T, x\rangle > \lambda > \sup_{y \in V} \mathrm{Re}\ \langle T, y\rangle$. Then for the map $\tilde{T} = \frac{1}{\lambda}T$,

$$\sup_{y\in V}\inf\{\tau(\omega)\mid \mathrm{Re}\,\tilde{T}\otimes\mathrm{id}(y)\le\omega\otimes 1\,,\omega\ge 0\}<1<\inf\{\tau(\omega)|\mathrm{Re}\,\tilde{T}\otimes\mathrm{id}(x)\le\omega\otimes 1\,,\omega\ge 0\}$$

which completes the proof. □

Using similar idea, we obtain a variant of Effros-Winkler's separation theorem [9]. Recall a CP map Φ is sub-unital if $\Phi(1)\le 1$.

Theorem 4.5. *Let E be a operator space. Let $V\subset M_n(E)$ be a closed convex set such $\Phi\otimes\mathrm{id}(V)\subset V$ for any CP sub-unital $\Phi: M_n\to M_n$. Then $x\notin V$ if and only if there exists a map $T: E\to M_n$ such that for each $y\in V$, there exists a density operator $\omega_y\in M_n$ depending on y such that*

$$Re\ \mathrm{id}\otimes T(y)\le 1\otimes\omega_y\ ,$$

and for any density operator ω,

$$Re\ \mathrm{id}\otimes T(x)\not\le 1\otimes\omega\ .$$

Proof. Suppose $x\notin V$. Because M_n is finite dimensional, we have $M_n(E)^*=S_1^n\widehat{\otimes}E^*$. Then there exists an element $T\in E^*\widehat{\otimes}S_1^n$ such that

$$\mathrm{Re}\,\langle T,x\rangle>\sup_{y\in V}\mathrm{Re}\,\langle T,y\rangle\ . \tag{4.3}$$

We identify $T\in E^*\widehat{\otimes}S_1^n$ with a map $T: E\to S_1^n$. Then the pairing on the left hand side of (4.3) can be rewritten as

$$\mathrm{Re}\,\langle T,x\rangle=\mathrm{Re}\langle\mathrm{id}_{M_n},\mathrm{id}\otimes T(x)\rangle\le\inf\{\tau(\omega)|\mathrm{Re}\ \mathrm{id}\otimes T(x)\le 1\otimes\omega\,,\omega\ge 0\}\ .$$

Here the second pairing is between $CB(M_n,M_n)=(M_n\widehat{\otimes}S_1^n)^*$. For the right hand side of (4.3),

$$\begin{aligned}\sup_{y\in V}\mathrm{Re}\,\langle T,y\rangle&=\sup_{y\in V}\sup_{\Phi\ \text{CP sub-unital}}\mathrm{Re}\,\langle T,\Phi\otimes\mathrm{id}(y)\rangle=\sup_{y\in V}\sup_{\Phi}\mathrm{Re}\langle\Phi,\mathrm{id}\otimes T(y)\rangle\\&\le\sup_{y\in V}\inf\{\tau(\omega)|\mathrm{Re}\ \mathrm{id}\otimes T(y)\le 1\otimes\omega\,,\omega\ge 0\}\ .\end{aligned}$$

Then the assertion follows from the inequality (4.3). □

Recall that a contractively matrix convex set is a sequence $(V_n)\subset M_n(E)$ such that i) for any CP sub-unital $\Phi: M_m\to M_n$, $\Phi\otimes\mathrm{id}(V_m)\subset V_n$; and ii) for any $a\in V_m, b\in V_n$, $a\oplus b\in V_{n+m}$. Effros-Winkler's theorem stated for matrix convex set admits a stronger separation: there exists a density operator ω uniform for all y such that $\mathrm{Re}\ \mathrm{id}\otimes T(y)\le 1\otimes\omega$. A similar lemma for tracial sets was given in [14, Lemma 7.4]. The above Theorem 4.5 leads to a weaker separation because we consider convex sets closed under CP sub-unital maps but not necessarily satisfies ii).

5. Norm separations on projective tensor product

In this section, we discuss the analogue of quantum majorization on projective tensor product. Recall that a operator space G is *1-locally reflexive* if for any finite dimensional operator space E, we have the complete isometry

$$CB(E, G^{**}) \cong CB(E, G)^{**} \ .$$

It is clear from the definition that $G = G^{**}$ is reflexive implies that G is 1-locally reflexive. It was proved by Effros, Junge, and Ruan [8] that the predual of von Neumann algebras are 1-locally reflexive. Another property needed in our discussion is the completely contractive approximation property (CCAP). A operator space E has the *CCAP* if there exists a net of finite rank completely contractive maps $\Phi_\alpha : E \to E$ such that for any x, $\Phi_\alpha(x) \to x$ in norm. In the setting of operator spaces, this is an analog of w^*-CPAP.

The following lemma shows that these two properties combined give the desired norm attaining property similar to Proposition 3.6. Throughout this section, we write CB for completely bounded and CC for completely contractive.

Lemma 5.1. *Let E be an operator space with the CCAP. Then $CB(E, G) \subset CB(E, G^{**})$ is w^*-dense in the sense of $CB(E, G^{**}) = (E\widehat{\otimes}G^*)^*$. If, in addition, G is 1-locally reflexive, then*

$$\| \rho \|_{E\widehat{\otimes}G^*} = \sup\{Re\, \langle \Psi, \rho\rangle \mid \Psi : E \to G \quad CC\}.$$

Proof. Let $\Phi_\alpha : E \to E$ be a net of CC maps such that $\Phi_\alpha(x) \to x$ in norm for any $x \in E$. For $\rho \in E\widehat{\otimes}G^*$ with $\| \rho \|_{E\widehat{\otimes}G^*} = 1$, we can choose a finite tensor sum $\rho_0 = \sum_{j=1}^n x_j \otimes y_j$ such that $\| \rho - \rho_0 \|_{E\widehat{\otimes}G^*} \le \epsilon$. Then for $T : E \to G^{**}$ with $\| T \|_{cb} = 1$, there exists an α such that

$$\begin{aligned} |\langle T \circ \Phi_\alpha - T, \rho\rangle| &\le |\langle T \circ \Phi_\alpha - T, \rho - \rho_0\rangle| + |\langle T \circ \Phi_\alpha - T, \rho_0\rangle| \\ &\le |\langle T \circ \Phi_\alpha - T, \rho - \rho_0\rangle| + |\langle T, \Phi_\alpha \otimes \mathrm{id}(\rho_0) - \rho_0\rangle| \le 2\epsilon + \epsilon \ . \end{aligned}$$

Let E_α be the range of Φ_α as a finite dimensional subspace of E and $T|_{E_\alpha} \in CB(E_\alpha, G^{**})$ be the restriction of T to E_α. There exists $T_\alpha \in CB(E_\alpha, G)$ such that

$$|\langle T_\alpha - T, \Phi_\alpha \otimes \mathrm{id}(\rho_0)\rangle| = |\langle (T_\alpha - T) \circ \Phi_\alpha, \rho_0\rangle| \le \epsilon \ .$$

Therefore $T_\alpha \circ \Phi_\alpha : E \to G$ is CB and

$$\begin{aligned} |\langle T_\alpha \circ \Phi_\alpha - T, \rho\rangle| \le& |\langle T \circ \Phi_\alpha - T, \rho\rangle| + |\langle (T_\alpha - T) \circ \Phi_\alpha, \rho - \rho_0\rangle| + |\langle (T_\alpha - T) \circ \Phi_\alpha, \rho_0\rangle| \\ \le& 3\epsilon + 2\epsilon + \epsilon = 6\epsilon \end{aligned}$$

which proves the w^*-density of $CB(E,G) \subset CB(E,G^{**})$. If G is 1-locally reflexive, T_α and $T_\alpha \circ \Phi_\alpha$ can be CC because we can of the isometry $CB(E_\alpha, G^*) \cong CB(E_\alpha, G)^*$. □

The following theorem is the analog of quantum majorization and channel factorization in the abstract operator space setting.

Theorem 5.2. *Let E, F, G be operator spaces. Suppose one of the following conditions holds:*

a) *G is reflexive;*
b) *G is 1-locally reflective and F has the CCAP*

Then the following two statements hold:

i) *For $\rho \in E\widehat{\otimes}F$ and $\sigma \in E\widehat{\otimes}G$, there exists a sequence of CC maps $u_n : F \to G$ such that $\mathrm{id} \otimes u_n(\rho) \to \sigma$ in the norm of $E\widehat{\otimes}G$ if and only if for any CB map $v : E \to G^*$,*

$$\| v \otimes \mathrm{id}(\rho) \|_{G^*\widehat{\otimes}F} \geq \| v \otimes \mathrm{id}(\sigma) \|_{G^*\widehat{\otimes}G} .$$

ii) *For $T \in CB(E,F)$ and $S \in CB(E,G)$, there exists a net of CC $u_\alpha : F \to G$ such that $u_\alpha \circ T \to S$ in the point-weak topology if and only if for any $x \in E \otimes G^*$,*

$$\| T \otimes \mathrm{id}(x) \|_{F\widehat{\otimes}G^*} \geq \| S \otimes \mathrm{id}(x) \|_{G\widehat{\otimes}G^*} .$$

Proof. i) The "only if" direction is easy. For the "if" part, consider the norm-closed convex set

$$C(\rho) = \overline{\{\mathrm{id} \otimes u(\rho) | u : F \to G, CC\}} \subset E\widehat{\otimes}G.$$

If $\sigma \notin C(\rho)$, there exists $v \in CB(E, G^*) = (E\widehat{\otimes}G)^*$ such that

$$\mathrm{Re}\, \langle v, \sigma \rangle > \sup_u \mathrm{Re}\, \langle v, \mathrm{id} \otimes u(\rho) \rangle .$$

Let $\iota_G : G \to G^{**}$ be the embedding. Note that

$$\mathrm{Re}\, \langle v, \sigma \rangle = \mathrm{Re}\, \langle \iota_G, v \otimes \mathrm{id}(\sigma) \rangle \leq \| v \otimes \mathrm{id}(\sigma) \|_{G^*\widehat{\otimes}G} ,$$
$$\sup_u \mathrm{Re}\, \langle v, \mathrm{id} \otimes u(\rho) \rangle = \sup_u \mathrm{Re}\, \langle u, v \otimes \mathrm{id}(\rho) \rangle = \| v \otimes \mathrm{id}(\rho) \|_{G^*\widehat{\otimes}F}$$

where the last equality follows Lemma 5.1.

ii) Suppose u_α is a net of CC maps such that $u_\alpha \circ T \to S$ in the point-weak topology. Then for any $R \in CB(G^*, G^*) = (G\widehat{\otimes}G^*)^*$ and $x \in E \otimes G^*$

$$\lim_\alpha \langle R, u_\alpha \circ T \otimes \mathrm{id}(x) \rangle = \lim_\alpha \langle u_\alpha \circ T, \mathrm{id} \otimes R(x) \rangle = \langle S, \mathrm{id} \otimes R(x) \rangle = \langle R, S \otimes \mathrm{id}(x) \rangle$$

which implies $\| T \otimes \mathrm{id}(x) \|_{F \hat{\otimes} G^*} \geq \| S \otimes \mathrm{id}(x) \|_{G \hat{\otimes} G^*}$. For the converse, consider the w^*-closure of convex set

$$C(T) = \overline{\{u \circ T | u : F \to G, CC\}}^w \subset CB(E, G^{**}) = (E \widehat{\otimes} G^*)^*.$$

If $S \notin C(T)$, there exists a $\rho \in E \widehat{\otimes} G^*$ such that

$$\mathrm{Re}\, \langle S, \rho \rangle > \sup_u \mathrm{Re}\, \langle u \circ T, \rho \rangle \ .$$

By a density argument, we can further assume $\rho \in E \otimes G^*$ in the algebraic tensor product. Note that

$$\begin{aligned} &\mathrm{Re}\, \langle S, \rho \rangle = \mathrm{Re}\, \langle \iota, S \otimes \mathrm{id}(\rho) \rangle \leq \| S \otimes \mathrm{id}(\rho) \|_{G \hat{\otimes} G^*} \ , \\ &\sup_u \mathrm{Re}\, \langle u \circ T, \rho \rangle = \sup_u \mathrm{Re}\, \langle u, T \otimes \mathrm{id}(\rho) \rangle = \| T \otimes \mathrm{id}(\rho) \|_{F \hat{\otimes} G^*} \end{aligned}$$

where again the last equality uses Lemma 5.1. □

The following proposition discusses the case when the limits in above theorem can be replaced by equality.

Proposition 5.3. *Let E, F, G be operator spaces. Let $T \in CB(E, F)$ and $\rho \in E \widehat{\otimes} F$. Suppose $G = (G_*)^*$ is a dual space. Then $\{u \circ T \mid u : F \to G,\ CC\}$ is w^*-closed in $CB(E, G)$. If, in addition, E has the CCAP or G is reflexive, $\{\mathrm{id} \otimes u(\rho) \mid u : F \to G,\ CC\}$ is norm-closed in $E \widehat{\otimes} G$.*

Proof. To prove the first statement, let $u_\alpha : F \to G$ be a net of CC maps such that $\lim_\alpha u_\alpha \circ T = S$ in the w^*-topology of $CB(E, G) = (E \widehat{\otimes} G_*)^*$. Because $CB(F, G) = (F \widehat{\otimes} G_*)^*$, we choose u as w^*-limit of (u_α) such that the subnet $u_\beta \to u$. Then $u_\beta \circ T \to u \circ T$ in the point w^*-topology hence $S = u \circ T$. For the second statement, we assume E has the CCAP or G is reflexive. Let $u_k : F \to G$ be a sequence of CC such that $\mathrm{id} \otimes u_k(\rho) \to \sigma$ in the norm of $E \hat{\otimes} G$. Choose a subsequence $u_{k_i} \to u$ in the w^*-topology for some CC u. For any $T \in CB(E, G_*)$,

$$\lim_i \langle T, \mathrm{id} \otimes u_{k_i}(\rho) \rangle = \lim_i \langle u_{k_i}, T \otimes \mathrm{id}(\rho) \rangle = \langle u, T \otimes \mathrm{id}(\rho) \rangle = \langle T, u \otimes \mathrm{id}(\rho) \rangle.$$

Thus $\mathrm{id} \otimes u_{k_i}(\rho) \to \mathrm{id} \otimes u(\rho)$ in $E \widehat{\otimes} G$ with the topology induced by $CB(E, G_*) \subset CB(E, G^*)$. Note that by Lemma 5.1, this topology is separating. Hence we have $\sigma = \lim_i \mathrm{id} \otimes u_{k_i}(\rho) = \mathrm{id} \otimes u(\rho)$. □

Theorem 5.2 also holds for Banach space tensor products. We can replace the operator space concepts with their Banach space counterparts: replace “operator spaces” by “Banach spaces”, “CB (resp. CC)” by “bounded (resp. contractive)” and “CCAP” by "metric approximation property (or 1-AP)". Moreover, all Banach spaces have 1-local reflexivity. We refer to the book [18] for definitions of the above mentioned Banach space concepts. Here we state the result analogous to Theorem 5.2. Let $\otimes_\pi$ denote the Banach space projective tensor product and $B(E, F)$ be the set of bounded maps from Banach space E to F.

Theorem 5.4. *Let E, F, G be Banach spaces. Suppose one of the following conditions holds:*

a) *G is reflexive;*
b) *F has the metric approximation property.*

Then the following two statements hold:

i) *for $\rho \in E \otimes_\pi F$ and $\sigma \in E \otimes_\pi G$, there exists a sequence of contractive maps $u_n : F \to G$ such that $\mathrm{id} \otimes u_n(\rho) \to \sigma$ in the norm of $E \otimes_\pi G$ if and only if for any bounded map $v : E \to G^*$,*

$$\| v \otimes \mathrm{id}(\rho) \|_{G^* \otimes_\pi F} \geq \| v \otimes \mathrm{id}(\sigma) \|_{G^* \otimes_\pi G} \quad .$$

ii) *for $T \in B(E, F)$ and $S \in B(E, G)$, there exists a net of contractions $u_\alpha : F \to G$ such that $u_\alpha \circ T \to S$ in the point-weak topology if and only if for any $x \in E \otimes G^*$,*

$$\| T \otimes \mathrm{id}(x) \|_{F \otimes_\pi G^*} \geq \| S \otimes \mathrm{id}(x) \|_{G \otimes_\pi G^*} \quad .$$

The proof is identical to Theorem 5.2 and the details are left to the reader.

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