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Generalized Primitive Potentials

V. Zakharov[a,b,*] and D. Zakharov[c,]**

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Abstract—Recently, we introduced a new class of bounded potentials of the one-dimensional stationary Schrödinger operator on the real axis, and a corresponding family of solutions of the KdV hierarchy. These potentials, which we call primitive, are obtained as limits of rapidly decreasing reflectionless potentials, or multisoliton solutions of KdV. In this note, we introduce generalized primitive potentials, which are obtained as limits of all rapidly decreasing potentials of the Schrödinger operator. These potentials are constructed by solving a contour problem, and are determined by a pair of positive functions on a finite interval and a functional parameter on the real axis.

Keywords: integrable systems, Schrödinger equation, primitive potentials

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This paper is concerned with the spectral properties of the one-dimensional stationary Schrödinger operator on the real axis:

$$-\psi'' + u(x)\psi = E\psi, \quad -\infty < x < \infty. \tag{1}$$

The Schrödinger operator is the auxiliary linear operator for the KdV hierarchy, the first equation of which is

$$u_t(x,t) = 6u(x,t)u_x(x,t) - u_{xxx}(x,t). \tag{2}$$

The intimate relationship between the Schrödinger operator and KdV is one of the starting points for the modern theory of integrable systems.

The KdV equation can be solved exactly using the Inverse Scattering Method (ISM) if the initial condition $u(x) = u(x,0)$ is rapidly decreasing as $x \to \pm\infty$. The ISM constructs an auxiliary function $\chi(k,x) = e^{ikx}\psi$ depending on parameter $k = \sqrt{E}$. This function satisfies a nonlocal Riemann–Hilbert problem, and has the following singularities in the k-plane: N poles on the positive imaginary axis, corresponding to KdV solitons, and a jump on the real axis determined by the reflection coefficient. The asymptotic behavior of $\chi(k,x,t)$ as $|k| \to \infty$ determines $u(x,t)$. There exists another method for constructing solutions of KdV, known as the finite-gap method. The resulting solutions $u(x,t)$ are quasiperiodic in x, and hence not rapidly decreasing at infinity. The corresponding function χ has jumps along the gaps, and can be viewed as a meromorphic function on a hyperelliptic Riemann surface that is a double cover of the k-plane branched along the gaps.

In [1–5], we constructed the closure of the set of N-soliton solutions of KdV as $N \to \infty$. The corresponding function χ has jumps on two symmetric cuts $[ia, ib]$ and $[-ib, -ia]$ on the imaginary axis. These jumps satisfy a pair of singular integral equations determined by two arbitrary positive functions R_1 and R_2 on $[a, b]$, called the *dressing functions*. The resulting solutions of KdV are called *primitive* and can be viewed as a dense soliton gas. Nabelek shows in [6] that all finite-gap potentials can be obtained in this way by a proper choice of functions R_1 and R_2.

In this note, we describe the same limiting procedure for arbitrary rapidly decreasing solutions of KdV. The resulting auxiliary function $\chi(k,x)$ has jumps on two symmetric cuts on the imaginary axis, corresponding to a soliton gas, and a jump along the real axis determined by the reflection coefficient. The corresponding potentials of the Schrödinger operator, which we call *generalized primitive potentials*, have the same doubly degenerate spectrum $[-b^2, -a^2] \cup [0, \infty)$, but are no longer reflectionless on $[0, \infty)$. The function χ is obtained by solving a Riemann–Hilbert problem involving two arbitrary positive functions R_1 and R_2 on $[a, b]$, and a functional parameter $c(k)$ on the real axis, inheriting the properties of the reflection coefficient in the rapidly decreasing case.

[a] *Skolkovo Institute of Science and Technology, Moscow, Russia*
[b] *University of Arizona, Tucson, AZ, USA*
[c] *Central Michigan University, Mount Pleasant, MI, USA*
* *e-mail: zakharov@math.arizona.edu*
** *e-mail: dvzakharov@gmail.com*

1. THE INVERSE SCATTERING TRANSFORM AS AN $\bar{\partial}$-PROBLEM

We begin by recalling the inverse scattering method (see [7–9]), and its reformulation as an $\bar{\partial}$-problem. Consider the Schrödinger operator

$$L(t) = -\frac{d^2}{dx^2} + u(x,t), \tag{3}$$

with $u(x,t)$ satisfying

$$\int_{-\infty}^{\infty} u(x,0)(1+|x|)dx < \infty \tag{4}$$

and the KdV Eq. (2). The operator $L(t)$ has a finite number of bound states with eigenvalues $-\kappa_1^2,\ldots,-\kappa_N^2$ not depending on t, and has an absolutely continuous spectrum on $[0,\infty)$. The Schrödinger equation admits two Jost solutions

$$L(t)\psi_\pm(k,x,t) = k^2\psi_\pm(k,x,t), \quad \mathrm{Im}(k) > 0, \tag{5}$$

with asymptotic behavior

$$\lim_{x\to\pm\infty} e^{\mp ikx}\psi_\pm(k,x,t) = 1. \tag{6}$$

The Jost solutions $\psi_\pm$ are analytic for $\mathrm{Im}k > 0$ and continuous for $\mathrm{Im}k \geq 0$ and have the following asymptotic behavior as $k \to \infty$ with $\mathrm{Im}k > 0$:

$$\psi_\pm(k,x,t) = e^{\pm ikx}\left(1 + Q_\pm(x,t)\frac{1}{2ik} + O\left(\frac{1}{k^2}\right)\right), \tag{7}$$

$$Q_+(x,t) = -\int_z^{\infty} u(y,t)dy, \quad Q_-(x,t) = -\int_{-\infty}^{x} u(y,t)dy. \tag{8}$$

The Jost solutions satisfy the scattering relations

$$\begin{gathered} t(k)\psi_\mp(k,x,t) \\ = \overline{\psi_\pm(k,x,t)} + r_\pm(k,t)\psi_\pm(k,x,t), \quad k \in R, \end{gathered} \tag{9}$$

where $t(k)$ and $r_\pm(k,t)$ are the transmission and reflection coefficients, respectively. These coefficients satisfy the following properties.

Proposition 1. The transmission coefficient $t(k)$ is meromorphic for $\mathrm{Im}k > 0$ and is continuous for $\mathrm{Im}k \geq 0$. It has simple poles at $i\kappa_1,\ldots,i\kappa_N$ with residues

$$\mathrm{Res}_{i\kappa_n} t(k) = i\mu_n(t)\gamma_n(t)^2, \tag{10}$$

where

$$\begin{gathered} \gamma_n(t)^{-1} = \|\psi_+(i\kappa_n,x,t)\|_2, \\ \psi_+(i\kappa_n,x,t) = \mu_n(t)\psi_-(i\kappa_n,x,t). \end{gathered} \tag{11}$$

Furthermore,

$$\begin{gathered} t(k)\overline{r_+(k,t)} + \overline{t(k)}r_-(k,t) = 0, \\ |t(k)|^2 + |r_\pm(k,t)|^2 = 1, \quad r_-(k,t) = \overline{r}_+(k,t). \end{gathered} \tag{12}$$

If we denote $r(k,t) = r_+(k,t)$, $r(k) = r(k,0)$, and $\gamma_n = \gamma_n(0)$, then

$$t(-k) = \overline{t(k)}, \quad r(-k) = \overline{r(k)}, \quad k \in \mathbb{R}, \tag{13}$$

$$\begin{gathered} |r(k)| < 1 \ \text{ for } \ k \in R, \quad k \neq 0, \\ r(0) = -1 \ \text{ if } \ |r(0)| = 1 \end{gathered} \tag{14}$$

and the function $r(k)$, if the potential $u(x)$ is smooth, decays as $O\left(\frac{1}{|k|^3}\right)$ as $|k| \to \infty$. The time evolution of the quantities $r(k,t)$ and $\gamma_n(t)$ is given by

$$r(k,t) = r(k)e^{8ik^3t}, \quad \gamma_j(t) = \gamma_n e^{4\kappa_n^3 t}. \tag{15}$$

The collection $(r(k,t), k \geq 0; \kappa_1,\ldots,\kappa_N, \gamma_1(t), \ldots, \gamma_N(t))$ is called the scattering data of the Schrödinger operator $L(t)$. We encode the scattering data as a contour problem in the following way. Consider the function

$$\chi(k,x,t) = \begin{cases} t(k)\psi_-(k,x,t)e^{ikx}, & \mathrm{Im}k > 0, \\ \psi_+(-k,x,t)e^{ikx}, & \mathrm{Im}k < 0. \end{cases} \tag{16}$$

Proposition 2. Let $(r(k);\kappa_1,\ldots,\kappa_N,\gamma_1,\ldots,\gamma_N)$ be the scattering data of the Schrödinger operator $L(0)$. Then the function $\chi(k,x,t)$ defined by (16) is the unique function satisfying the following properties:

(1) χ is meromorphic on the complex k-plane away from the real axis and has non-tangential limits

$$\chi_\pm(k,x,t) = \lim_{\varepsilon\to 0}\chi(k \pm i\varepsilon, x,t), \quad k \in \mathbb{R}; \tag{17}$$

(2) χ has a jump on the real axis satisfying

$$\begin{gathered} \chi_+(k,x,t) - \chi_-(k,x,t) \\ = r(k)e^{2ikx+8ik^3t}\chi_-(-k,x,t), \quad k \in \mathbb{R}; \end{gathered} \tag{18}$$

(3) χ has simple poles at the points $i\kappa_1,\ldots,i\kappa_N$ and no other singularities. The residues at the poles satisfy the condition

$$\begin{gathered} \mathrm{Res}_{i\kappa_n}\chi(k,x,t) = \\ = ic_n e^{-2\kappa_n x + 8\kappa_n^3 t}\chi(-i\kappa_n,x,t), \quad c_n = \gamma_n^2; \end{gathered} \tag{19}$$

(4) χ has the asymptotic behavior

$$\begin{gathered} \chi(k,x,t) = 1 + \frac{i}{2k}Q_+(x,t) + O\left(\frac{1}{k^2}\right), \\ |k| \to \infty, \quad \mathrm{Im}k \neq 0. \end{gathered} \tag{20}$$

The function χ is a solution of the equation

$$\chi'' - 2ik\chi' - u(x)\chi' = 0, \tag{21}$$

and the function $u(x,t)$ given by the formula

$$u(x,t) = \frac{d}{dx}Q_+(x,t), \tag{22}$$

is a solution of the KdV Eq. (2) satisfying condition (4).

We now reformulate, following [10], the conditions (17)–(20) that define χ as an $\bar{\partial}$-problem. Denote by

$\rho(k,x,t)$ the jump of χ on the real axis, going from the positive to the negative side, and denote $i\chi_n(x,t)$ the residue of $\chi(k,x,t)$ at $k = i\kappa_n$. Conditions (17)–(20) imply that χ has a spectral representation

$$\chi(k,x,t) = 1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho(q,x,t)}{q-k}dq + i\sum_{n=1}^{N}\frac{\chi_n(x,t)}{k-i\kappa_n}, \quad (23)$$

and that the jump $\rho(k,x,t)$ and the residues $\chi_n(x,t)$ satisfy the system

$$\begin{gathered}\rho(k,x,t) = r(k)e^{2ikx+8ik^3t}\\ \times\left[1+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho(q,x,t)}{q+k+i\varepsilon}dq - i\sum_{n=1}^{N}\frac{\chi_n(x,t)}{k+i\kappa_n}\right],\end{gathered} \quad (24)$$

$$\begin{gathered}\chi_n(x,t) = c_n e^{-2\kappa_n x+8\kappa_n^3 t}\\ \times\left[1+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho(q,x,t)}{q+i\kappa_n}dq + \sum_{m=1}^{N}\frac{\chi_m(x,t)}{\kappa_n+\kappa_m}\right].\end{gathered} \quad (25)$$

Now, consider the distribution

$$T(k) = \frac{i}{2}\delta(k_I)\theta(-k_I)r(k_R) + \pi\delta(k_R)\sum_{n=1}^{N}c_n\delta(k_I-\kappa_n), \quad (26)$$

called the *dressing function*. Here θ is the Heaviside step function, δ is the Dirac delta function, $k = k_R + ik_I$, and we use that

$$\frac{\partial}{\partial\bar{k}}\frac{1}{k} = \pi\delta(k) = \pi\delta(k_R)\delta(k_I). \quad (27)$$

The meaning of the distributions $\delta(x)\theta(\pm x)$ is that, for a function f that is possibly discontinuous at $x = 0$, we have

$$\int_{-\infty}^{\infty} f(x)\delta(x)\theta(\pm x)dx = \lim_{x\to 0^{\pm}} f(x). \quad (28)$$

A direct calculation then shows that conditions (18), (19) are equivalent to the following $\bar{\partial}$-problem (see [10]):

$$\begin{gathered}\frac{\partial\chi}{\partial\bar{k}} = T(k)e^{2ikx+8ik^3t}\chi(-k,x,t),\\ \chi\to 1 \quad\text{as}\quad k\to\infty.\end{gathered} \quad (29)$$

2. TRANSPLANTING POLES

The key observation of [1–3] is that in order to obtain finite-gap solutions of KdV from N-soliton solutions in the $N\to\infty$ limit, it is necessary to first allow χ to have poles on the negative imaginary axis. We now perform the same procedure for a generic rapidly decreasing potential.

Let $(r(k),\kappa_1,\ldots,\kappa_N,c_1,\ldots,c_N)$ be the scattering data of a potential $u(x,t)$ rapidly decreasing at infinity, and let $\chi(k,x,t)$ be the function determined by Proposition 2. Fix a subset $I\subset\{1,\ldots,N\}$, and introduce the function

$$\tilde{\chi}(k,x,t) = \chi(k,x,t)\prod_{m\in I}\frac{k-i\kappa_m}{k+i\kappa_m}, \quad (30)$$

The function $\tilde{\chi}$ has a jump on the real axis, tends to 1 as $k\to\infty$, has $N-I$ poles in the upper half-plane at $k = i\kappa_m$ for $m\notin I$, and I poles in the lower half-plane at $k = -i\kappa_m$, for $m\in I$. It follows that $\tilde{\chi}$ has the spectral representation

$$\begin{gathered}\tilde{\chi}(k,x,t) = 1+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\tilde{\rho}(q,x,t)}{q-k}dq\\ + i\sum_{m\notin I}\frac{\tilde{\chi}_m(x,t)}{k-i\kappa_m} + i\sum_{n\in I}\frac{\tilde{\chi}_m(x,t)}{k+i\kappa_m},\end{gathered} \quad (31)$$

where the jump $\tilde{\rho}$ and the residues $i\tilde{\chi}_n$ are equal to

$$\begin{gathered}\tilde{\rho}(q,x,t) = \rho(q,x,t)\prod_{m\in I}\frac{q-i\kappa_m}{q+i\kappa_m},\\ \tilde{\chi}_n(x,t) = \chi_n(x,t)\prod_{m\in I}\frac{\kappa_n-\kappa_m}{\kappa_n+\kappa_m} \quad\text{for}\quad n\notin I,\end{gathered} \quad (32)$$

$$\begin{gathered}\tilde{\chi}_m(x,t) = -2\kappa_n\chi(-i\kappa_n,x,t)\times\\ \times\prod_{m\in I\setminus\{n\}}\frac{\kappa_n+\kappa_m}{\kappa_n-\kappa_m} \quad\text{for}\quad n\in I.\end{gathered} \quad (33)$$

It follows that $\tilde{\chi}$ satisfies same $\bar{\partial}$-problem (29) as χ, but with the dressing function

$$\tilde{T}(k) = \frac{i}{2}\delta(k_I)\theta(-k_I)\tilde{r}(k_R) + \pi\delta(k_R)\sum_{n=1}^{N}\tilde{c}_n\delta(k_I-\tilde{\kappa}_n), \quad (34)$$

whose coefficients are equal to

$$\begin{gathered}\tilde{r}(k) = r(k)\prod_{m\in I}\left(\frac{k-i\kappa_m}{k+i\kappa_m}\right)^2,\\ \tilde{c}_n = c_n\prod_{m\in I}\left(\frac{\kappa_n-\kappa_m}{\kappa_n+\kappa_m}\right)^2 \quad\text{for}\quad n\notin I,\end{gathered} \quad (35)$$

$$\begin{gathered}\tilde{c}_n = -\frac{4\kappa_n^2}{c_n}\prod_{m\in I\setminus\{n\}}\left(\frac{\kappa_n+\kappa_m}{\kappa_n-\kappa_m}\right)^2 \quad\text{for}\quad n\in I,\\ \tilde{\kappa}_n = \begin{cases}\kappa_n, & n\notin I,\\ -\kappa_n, & n\in I.\end{cases}\end{gathered} \quad (36)$$

We observe that

$$\begin{gathered}\tilde{r}(-k) = \overline{\tilde{r}(k)}, \quad |\tilde{r}(k)| = |r(k)|\\ \text{for}\quad k\in R, \quad \tilde{r}(0) = r(0),\end{gathered} \quad (37)$$

hence the function $\tilde{r}$ satisfies the same properties (13), (14) as r. We also note that $\tilde{c}_n$ is positive if $n\notin I$ and negative if $n\in I$, in other words $\tilde{c}_n$ has the same sign as $\tilde{\kappa}_n$.

Finally, we observe that $\tilde{\chi}$ satisfies the same differential Eq. (21) as χ, and has the following asymptotic behavior as $|k|\to\infty$:

$$\tilde{\chi}(k,x,t) = 1 + \frac{i}{2k}\tilde{Q}_+(x,t) + O\left(\frac{1}{k^2}\right),$$
$$\tilde{Q}_+(x,t) = Q_+(x,t) - 4\sum_{m\in I}\kappa_m. \tag{38}$$

Therefore $u(x, t)$ is obtained from $\tilde{\chi}(k,x,t)$ using the same formula (22).

We have shown that if $u(x, t)$ is obtained by the ISM from the spectral data $(r(k), \kappa_1, \ldots, \kappa_N, c_1, \ldots, c_N)$, then we can change the sign of the κ_m for $m \in I$, modify the coefficients r and c_n according to (36), and the resulting function $\tilde{\chi}$ produces the same potential $u(x, t)$. The result of the above procedure is that we have transplanted some of the poles of χ to the negative imaginary axis without changing u.

Reversing this procedure, we claim the following. Let $(r(k); \kappa_1, \ldots, \kappa_N, c_1, \ldots, c_N)$ be data consisting of a function $r : \mathbb{R} \to \mathbb{R}$ satisfying properties (13), (14) and $2N$ nonzero real constants κ_n, c_n such that

$$\kappa_n \neq \pm\kappa_m \quad \text{for any} \quad n \neq m,$$
$$\kappa_n/c_n > 0 \quad \text{for all} \quad n. \tag{39}$$

Then the $\bar{\partial}$-problem (29) with the dressing function T given by (26) has a unique solution χ, and the function $u(x, t)$ given by (22) is a solution of KdV with N solitons having spectral parameters $|\kappa_1|, \ldots, |\kappa_N|$. Therefore, every solution of KdV with N solitons (and a non-trivial reflection coefficient) can be obtained using the dressing method in 2^N different ways.

3. CONSTRUCTION OF GENERALIZED PRIMITIVE POTENTIALS

We now pass to the $N \to \infty$ limit, as in [1–3]. Consider the $\bar{\partial}$-problem (29) with a dressing function T of the form (26), where r satisfies (13), (14) and the κ_n and c_n satisfy (39). Fix $0 < k_1 < k_2$, and suppose that the poles κ_n are uniformly distributed in the two intervals $[k_1, k_2]$ and $[-k_2, -k_1]$, in such a way that $\kappa_n \neq \pm\kappa_m$ for $n \neq m$. In the limit as $N \to \infty$, we obtain the following dressing function:

$$T(k) = \frac{i}{2}\delta(k_I)\theta(-k_I)r(k) + \pi\delta(k_R)$$
$$\times\left[\int_{k_1}^{k_2} R_1(p)\delta(k_I - p)dp - \int_{k_1}^{k_2} R_2(p)\delta(k_I + p)dp\right]. \tag{40}$$

We impose the same conditions on $r(k)$ as before:

$$r(-k) = \overline{r(k)}, \quad |r(k)| < 1 \quad \text{for} \quad k \neq 0,$$
$$|r(0)| \leq 1, \quad r(0) = -1 \quad \text{if} \quad |r(0)| = 1. \tag{41}$$

We require the functions R_1 and R_2 on $[k_1, k_2]$ to be positive and Hölder-continuous.

We now consider a solution χ of the $\bar{\partial}$-problem (29) with a dressing function of the form (40). Such a χ has a jump on the real axis as well as on the intervals $[ik_1, ik_2]$ and $[-ik_2, -ik_1]$ on the imaginary axis, and has the following spectral representation:

$$\chi(k,x,t) = 1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho(p,x,t)dp}{p-k}$$
$$+ i\int_{k_1}^{k_2}\frac{f(p,x,t)dp}{k-ip} + i\int_{k_1}^{k_2}\frac{g(p,x,t)dp}{k+ip}. \tag{42}$$

Denote $\chi^{\pm}(ik,x,t)$ the right and left limit values of χ on the cuts $k \in [ik_1, ik_2]$ and $k \in [-ik_1, -ik_2]$ on the imaginary axis, then

$$\chi^+(ik,x,t) - \chi^-(ik,x,t) = 2\pi i f(p,x,t),$$
$$\chi^+(ik,x,t) - \chi^-(-ik,x,t) = 2\pi i g(k,x,t). \tag{43}$$

On the real axis, χ satisfies the Riemann–Hilbert problem (18), where $\chi^{\pm}$ are the upper and lower limits of χ on the real axis. On the cuts $k \in [ik_1, ik_2]$ and $k \in [-ik_2, -ik_1]$ on the imaginary axis, the function χ satisfies the symmetric Riemann–Hilbert problem

$$f(k,x,t)$$
$$= \frac{1}{2}R_1(k)e^{-2kx+8k^3t}[\chi^+(-ik,x,t) + \chi^-(-ik,x,t)], \tag{44}$$

$$g(k,x,t)$$
$$= -\frac{1}{2}R_2(k)e^{2kx-8k^3t}[\chi^+(ik,x,t) + \chi^-(ik,x,t)]. \tag{45}$$

Together, the Riemann–Hilbert problems (18), (44), (45) are equivalent to the following system of singular integral equations on ρ, f, and g:

$$\rho(k,x,t) = r(k,x,t)e^{-2ikx-8ik^3t}\left[1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho(p,x,t)dp}{q+ik-\varepsilon}\right.$$
$$\left.- i\int_{k_1}^{k_2}\frac{f(p,x,t)dp}{k+ip} + i\int_{k_1}^{k_2}\frac{g(p,x,t)dp}{-k+ip}\right], \tag{46}$$

$$f(k,x,t) + R_1(k)e^{-2kx+8k^3t}$$
$$\times\left[\int_{k_1}^{k_2}\frac{f(p,x,t)}{k+p}dp + \int_{k_1}^{k_2}\frac{g(p,x,t)}{k-p}dp\right]$$
$$= R_1(k)e^{-2kx+8k^3t}\left[1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho(p,x,t)}{p-ik}dp\right],$$

$$= R_1(k)e^{-2kx+8k^3t}\left[1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho(p,x,t)}{p-ik}dp\right], \tag{47}$$

$$g(k,x,t)+R_2(k)e^{2kx-8k^3t} \times\left[\int_{k_1}^{k_2}\frac{f(p,x,t)}{k-p}dp+\int_{k_1}^{k_2}\frac{g(p,x,t)}{k+p}dp\right] = -R_2(k)e^{2kx-8k^3t}\left[1+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho(p,x,t)}{p+ik}dp\right]. \quad (48)$$

The system of Eqs. (46)–(48) is a complete system of equations defining generalized primitive potentials (for fixed t) and corresponding solutions of KdV. Setting $r(k)=0$ yields $\rho(k,x,t)=0$, and we obtain the system of equations describing reflectionless primitive potentials that we derived in our previous papers [1–4].

The corresponding solution $u(x, t)$ of KdV is given by the formula

$$u(x,t)=2\frac{d}{dx}\left[-\frac{1}{2\pi}\int_{-\infty}^{\infty}\rho(p,x,t)dp + \int_{k_1}^{k_2}[f(p,x,t)+g(p,x,t)]dp\right]. \quad (49)$$

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