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On symmetric primitive potentials

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The concept of a primitive potential for the Schrödinger operator on the line was introduced in Dyachenko *et al*. (2016, *Phys. D*, **333**, 148–156), Zakharov, Dyachenko *et al*. (2016, *Lett. Math. Phys.*, **106**, 731–740) and Zakharov, Zakharov *et al*. (2016, *Phys. Lett. A*, **380**, 3881–3885). Such a potential is determined by a pair of positive functions on a finite interval, called the dressing functions, which are not uniquely determined by the potential. The potential is constructed by solving a contour problem on the complex plane. In this article, we consider a reduction where the dressing functions are equal. We show that in this case, the resulting potential is symmetric, and describe how to analytically compute the potential as a power series. In addition, we establish that if the dressing functions are both equal to one, then the resulting primitive potential is the elliptic one-gap potential.

Keywords: integrable systems; Schrödinger equation; primitive potentials.

1. Introduction

One of the fundamental insights underlying the modern theory of integrable systems is the discovery of an intimate relationship between certain linear differential or difference operators, on one hand, and corresponding nonlinear equations on the other. The first of these relationships to be discovered, and arguably the most important one, is the link between the one-dimensional Schrödinger equation on the real axis

$$-\psi'' + u(x)\psi = E\psi, \quad -\infty < x < \infty, \tag{1}$$

and the Korteweg–de Vries (KdV) equation

$$u_t(x,t) = 6u(x,t)u_x(x,t) - u_{xxx}(x,t). \tag{2}$$

The study of solutions of the KdV equation has proceeded hand-in-hand with an analysis of the spectral properties of the Schrödinger operator that is applied to ψ on the left-hand side of the Schrödinger equation (1).

There are three broad methods for constructing solutions of the KdV equation, based on restricting the potentials of the Schrödinger operator. The inverse scattering method (ISM) allows us to construct potentials, and hence solutions of the KdV equation, that are rapidly vanishing as $x \to \pm\infty$. Such potentials have a finite discrete spectrum for $E < 0$ and a doubly degenerate continuous spectrum for $E > 0$, and a subset of them, corresponding to multisoliton solutions of the KdV equation, are reflectionless for positive energies. The finite-gap method, on the other hand, constructs periodic and quasi-periodic potentials of the Schrödinger operator (1) whose spectrum consists of finitely many allowed bands, one infinite, separated by forbidden gaps. These potentials are reflectionless in the allowed bands.

Both of these methods construct globally defined solutions of the KdV equation. The third method, called the dressing method [1], constructs solutions locally near a given point on the (x, t)-plane. An advantage of the method is that the constructed solutions can be quite general. However, the problem of extending such solutions to the entire (x, t)-plane is a difficult one.

Our work is motivated by a pair of related questions. First, one can ask what is the exact relationship between the ISM and the finite-gap method, and whether they can both be generalized by the dressing method. It has long been known that multisoliton solutions of the KdV equation are limits of finite-gap solutions corresponding to rational degenerations of the spectral curve. However, the converse relationship, which would consist in obtaining finite-gap solutions as limits of multisoliton solutions, has not been worked out. Additionally, one can ask which potentials of the Schrödinger operator, other than the finite-gap ones, have a band-like structure.

In the articles [2–4], the second and third authors presented a method for constructing potentials of the Schrödinger operator (1), called *primitive potentials*, that provide partial answers to these questions. Primitive potentials are constructed by directly implementing the dressing method and can be thought of as the closure of the set of multisoliton potentials. This procedure involves a reformulation of the ISM that is inherently symmetric with respect to the involution $x \to -x$, and the resulting primitive potentials are non-uniquely determined by a pair of positive, Hölder-continuous functions, called the *dressing functions*, defined on a finite interval.

In this article, we continue the study of primitive potentials. We consider primitive potentials defined by a pair of dressing functions that are equal. Such potentials are symmetric with respect to the reflection $x \to -x$. We show that the contour problem defining symmetric primitive potentials can be solved analytically, and we give an algorithm for computing the Taylor coefficients of a primitive potential. In the case, when the dressing functions are both identically equal to 1, we show that the corresponding primitive potential is the elliptic one-gap potential.

2. Primitive potentials

In this section, we recall the definition of primitive potentials, which were first introduced in the articles [2–4] as generalizations of finite-gap potentials. Primitive potentials are constructed by taking the closure of the set of N-soliton potentials as $N \to \infty$, so we begin by summarizing the ISM as a contour problem (see [5, 6]). The finite-gap method is symmetric with respect to the transformation $x \to -x$, while the ISM is not, so we give an alternative formulation of the ISM (in the reflectionless case) that takes this symmetry into account.

2.1 *The inverse scattering method*

We recall the ISM for the self-adjoint one-dimensional Schrödinger operator, following and using the notation of [6]. The Schrödinger operator

$$L(t) = -\frac{d^2}{dx^2} + u(x,t) \tag{3}$$

acts on the Sobolev space $H^2(\mathbb{R}) \subset L^2(\mathbb{R})$. We suppose that the potential $u(x,t)$ rapidly decays at infinity when $t = 0$:

$$\int_{-\infty}^{\infty} (1+|x|)(|u(x,0)| + |u_x(x,0)| + |u_{xx}(x,0)| + |u_{xxx}(x,0)|)\, dx < \infty \tag{4}$$

and satisfies the KdV equation (2). Under this assumption, the spectrum of $L(t)$ consists of an absolutely continuous part $[0,\infty)$ and a finite number of eigenvalues $-\kappa_1^2, \ldots, -\kappa_N^2$ that do not depend on t. There exist two Jost solutions $\psi_\pm(k,x,t)$ such that

$$L(t)\psi_\pm(k,x,t) = k^2\psi_\pm(k,x,t), \quad \mathrm{Im}(k) > 0, \tag{5}$$

with asymptotic behaviour

$$\lim_{x\to\pm\infty} e^{\mp ikx}\psi_\pm(k,x,t) = 1. \tag{6}$$

The Jost solutions $\psi_\pm$ are analytic for $\mathrm{Im}\, k > 0$ and continuous for $\mathrm{Im}\, k \geq 0$, and have the following asymptotic behaviour as $k \to \infty$ with $\mathrm{Im}\, k > 0$:

$$\psi_\pm(k,x,t) = e^{\pm ikx}\left(1 + Q_\pm(x,t)\frac{1}{2ik} + O\left(\frac{1}{k^2}\right)\right), \tag{7}$$

where

$$Q_+(x,t) = -\int_x^\infty u(y,t)\,dy, \quad Q_-(x,t) = -\int_{-\infty}^x u(y,t)\,dy. \tag{8}$$

The Jost solutions satisfy the scattering relations

$$T(k)\psi_\mp(k,x,t) = \overline{\psi_\pm(k,x,t)} + R_\pm(k,t)\psi_\pm(k,x,t), \quad k \in \mathbb{R}, \tag{9}$$

where $T(k)$ and $R_\pm(k,t)$ are the transmission and reflection coefficients, respectively. These coefficients satisfy the following properties:

Proposition 1 The transmission coefficient $T(k)$ is meromorphic for $\mathrm{Im}\, k > 0$ and is continuous for $\mathrm{Im}\, k \geq 0$. It has simple poles at $i\kappa_1, \ldots, i\kappa_N$ with residues

$$\mathrm{Res}_{i\kappa_n} T(k) = i\mu_n(t)\gamma_n(t)^2, \tag{10}$$

where

$$\gamma_n(t)^{-1} = ||\psi_+(i\kappa_n, x, t)||_2, \quad \psi_+(i\kappa_n, x, t) = \mu_n(t)\psi_-(i\kappa_n, x, t). \tag{11}$$

Furthermore,

$$T(k)\overline{R_+(k,t)} + \overline{T(k)}R_-(k,t) = 0, \quad |T(k)|^2 + |R_\pm(k,t)|^2 = 1. \tag{12}$$

If we denote $R(k,t) = R_+(k,t)$, $R(k) = R(k,0)$, and $\gamma_n = \gamma_n(0)$, then

$$T(-k) = \overline{T(k)}, \quad R(-k) = \overline{R(k)}, \quad k \in \mathbb{R}, \tag{13}$$

$$|R(k)| < 1 \text{ for } k \neq 0, \quad R(0) = -1 \text{ if } |R(0)| = 1, \tag{14}$$

and the function $R(k)$ is in $C^2(\mathbb{R})$ and decays as $O(1/|k|^3)$ as $|k| \to \infty$. The time evolution of the quantities $R(k,t)$ and $\gamma_n(t)$ is given by

$$R(k,t) = R(k)e^{8ik^3 t}, \quad \gamma_n(t) = \gamma_n e^{4\kappa_n^3 t}. \tag{15}$$

The collection $(R(k,t), k \geq 0; \kappa_1, \ldots, \kappa_N, \gamma_1(t), \ldots, \gamma_N(t))$ is called the scattering data of the Schrödinger operator $L(t)$. We encode the scattering data as a contour problem in the following way. Consider the function

$$\chi(k,x,t) = \begin{cases} T(k)\psi_-(k,x,t)e^{ikx}, & \operatorname{Im} k > 0, \\ \psi_+(-k,x,t)e^{ikx}, & \operatorname{Im} k < 0. \end{cases} \tag{16}$$

PROPOSITION 2 (See Theorem 2.3 in [6]) Let $(R(k); \kappa_1, \ldots, \kappa_N, \gamma_1, \ldots, \gamma_N)$ be the scattering data of the Schrödinger operator $L(0)$. Then the function $\chi(k,x,t)$ defined by (16) is the unique function satisfying the following properties:

(1) χ is meromorphic on the complex k-plane away from the real axis and has non-tangential limits

$$\chi_\pm(k,x,t) = \lim_{\varepsilon \to 0} \chi(k \pm i\varepsilon, x, t), \quad k \in \mathbb{R} \tag{17}$$

on the real axis.

(2) χ has a jump on the real axis satisfying

$$\chi_+(k,x,t) - \chi_-(k,x,t) = R(k)e^{2ikx+8ik^3 t}\chi_-(-k,x,t). \tag{18}$$

(3) χ has simple poles at the points $i\kappa_1, \ldots, i\kappa_N$ and no other singularities. The residues at the poles satisfy the condition

$$\operatorname{Res}_{i\kappa_n} \chi(k,x,t) = ic_n e^{-2\kappa_n x + 8\kappa_n^3 t}\chi(-i\kappa_n, x, t), \quad c_n = \gamma_n^2. \tag{19}$$

(4) χ has the asymptotic behaviour

$$\chi(k,x,t) = 1 + \frac{i}{2k} Q_+(x,t) + O\left(\frac{1}{k^2}\right), \quad |k| \to \infty, \quad \operatorname{Im} k \neq 0. \tag{20}$$

The function χ is a solution of the equation

$$\chi'' - 2ik\chi' - u(x)\chi' = 0, \tag{21}$$

and the function $u(x,t)$ given by the formula

$$u(x,t) = \frac{d}{dx} Q_+(x,t) \tag{22}$$

is a solution of the KdV equation (2) satisfying condition (4).

Remark 3 We note that the contour problem for χ is not symmetric with respect to the transformation $k \to -k$. The reflection coefficient $R(k)$ satisfies the symmetry condition (13), however, χ is required to have poles in the upper k-plane and be analytic in the lower k-plane. This asymmetry comes from the definition (5) of the Jost functions and is therefore ultimately of physical origin: in the ISM, we consider a quantum-mechanical particle approaching the localized potential from the *right*, in other words the method is not symmetric with respect to the transformation $x \to -x$. We will see in the next section that this asymmetry prevents us from directly relating the ISM to the finite-gap method.

It is common (see [6]) to instead consider the two-component vector $[\chi(k) \quad \chi(-k)]$. The jump condition on the real axis (18) is then replaced by a local Riemann–Hilbert problem. This Riemann–Hilbert problem includes poles on the upper and lower k-planes, but the transformation $k \to -k$ merely exchanges the components, which does not fix the asymmetry.

Remark 4 It is possible to relax the constraint $|R(k)| < 1$ for $k \neq 0$ and allow $|R(k)|$ to be equal to 1 inside two symmetric finite intervals $v < |k| < u$. In this case, the Riemann–Hilbert problem (18) is still uniquely solvable and generates a potential of the Schrödinger operator and a solution of the KdV equation. However, in this case condition (4) is not satisfied, and the potential is not rapidly decaying, at least when $x \to -\infty$. This extremely interesting case is completely unexplored.

2.2 *N-soliton solutions*

We now restrict our attention to the reflectionless case, in other words we assume that $R(k) = 0$. In this case, the function χ has no jump on the real axis and is meromorphic on the entire k-plane with simple poles at the points $i\kappa_1, \ldots, i\kappa_N$. Hence Proposition 2 reduces to the following.

Proposition 5 Let $(0; \kappa_1, \ldots, \kappa_N, \gamma_1, \ldots, \gamma_N)$ be the scattering data of the Schrödinger operator $L(0)$ with zero reflection coefficient. Then the function $\chi(k,x,t)$ defined by (16) is the unique function satisfying the following properties:

(1) χ is meromorphic on the complex k-plane with simple poles at the points $i\kappa_1, \ldots, i\kappa_N$ and no other singularities, and its residues satisfy condition (19).

(2) χ has the asymptotic behaviour (20) as $|k| \to \infty$.

The corresponding solution $u(x,t)$ of the KdV equation (2), given by formula (22), is known as the N-soliton solution. Finding this solution is a linear algebra exercise. If χ is expressed in terms of its residues

$$\chi = 1 + \sum_{n=1}^{N} \frac{\chi_n}{k - i\kappa_n}, \tag{23}$$

then plugging this into equation (19) gives a linear equation

$$\chi_n + c_n e^{-2\kappa_n x + 8\kappa_n^3 t} \sum_{m=1}^{N} \frac{\chi_m}{\kappa_n + \kappa_m} = c_n e^{-2\kappa_n x + 8\kappa_n^3 t}. \tag{24}$$

Let A be the determinant of this system:

$$A = \sum_{I \subset \{1,\ldots,N\}} \prod_{(m,n)\subset I,\, m<n} \frac{(\kappa_m - \kappa_n)^2}{(\kappa_m + \kappa_n)^2} \prod_{m \in I} q_m e^{-2\kappa_m x + 8\kappa_m^3 t}, \qquad q_m = \frac{c_m}{2\kappa_m} > 0. \tag{25}$$

Then the corresponding N-soliton solution of the KdV equation (2) is

$$u(x,t) = -2 \frac{d^2}{dx^2} \log A. \tag{26}$$

2.3 *The naïve limit $N \to \infty$*

The articles [2–4] were motivated by the following question. There exists a family of solutions of the KdV equation, called the finite-gap solutions, that are parametrized by the data of a hyperelliptic algebraic curve with real branch points and a line bundle on it. The solutions are given by the Matveev–Its formula

$$u(x,t) = -2 \frac{d^2}{dx^2} \ln \Theta(Ux + Vt + Z|B), \tag{27}$$

where $\Theta(\cdot|B)$ is the Riemann theta function of the hyperelliptic curve, and U, V and Z are certain vectors. The solution $u(x,t)$ is quasi-periodic in x and in t. It is well known that the N-soliton solutions of the KdV equation (26) can be obtained from the Matveev–Its formula by degenerating the hyperelliptic spectral curve to a rational curve with N branch points. Is it possible, conversely, to obtain the Matveev–Its formula (27) as some kind of limit of N-soliton solutions (26) when $N \to \infty$?

We may attempt to naïvely pass to the limit $N \to \infty$ in (26) in the following way. Let $[a,b]$ be an interval on the positive real axis, let R_1 be a positive Hölder-continuous function on $[a,b]$, and let μ be a non-negative measure on $[a,b]$. Consider the following integral equation

$$f(p,x,t) + \frac{R_1(p)}{\pi} e^{-2px+8p^3 t} \int_a^b \frac{f(q,x,t)}{p+q} d\mu(q) = R_1(p) e^{-2px+8p^3 t} \tag{28}$$

imposed on a function $f(p,x,t)$, where $p \in [a,b]$. Let $a = \kappa_1 < \kappa_2 < \cdots < \kappa_N = b$ be a partition of $[a,b]$ uniformly approximating μ. Replacing the above integral with the corresponding Riemann sum, and denoting $c_n = R_1(\kappa_n)(b-a)/\pi N$ and $\chi_n = f(\kappa_n)(b-a)/\pi N$, we obtain equation (24). Hence equation (28) can be seen as the limit of (24) as $N \to \infty$.

It is easy to show that (28) has a unique solution, and that the corresponding function

$$u(x,t) = -2\frac{d}{dx}\int_a^b f(p,x,t)d\mu(p) \tag{29}$$

is a bounded solution of the KdV equation, satisfying the condition $-2b < u < 0$. The solution is oscillating as $x \to -\infty$, but as $x \to +\infty$ it is clear that $f(p,x,t) \to R(k)e^{-2kx+8k^3t}$, hence $u(x,t)$ decays exponentially. In other words, $u(x,t)$ can be viewed as a superposition of an infinite number of solitons uniformly bounded away from $+\infty$. In particular, no solution obtained in this way will be an even function of x at any moment of time. It is therefore impossible to obtain the finite-gap solutions given by the Matveev–Its formula (27) in this way, since these solutions are not decreasing as $x \to +\infty$. This lack of symmetry is due to the formulation of the ISM (see Remark 3). These observations were earlier made by Krichever [7], and a rigorous study of the properties of such solutions, showing the above results, was undertaken by Girotti *et al.* [8].

2.4 *Symmetric N-soliton solutions*

In this section, we consider what happens if we try to impose by hand symmetry with respect to the spatial involution $x \mapsto -x$ at $t = 0$. We recall that an N-soliton solution of the KdV equation (26) is determined by N distinct positive parameters $\kappa_1, \ldots, \kappa_N$ and N additional positive parameters $q_1, \ldots, q_N$.

PROPOSITION 6 Let $\kappa_1, \ldots, \kappa_N$ be distinct positive numbers, and let

$$q_n = \left|\prod_{m\neq n}\frac{\kappa_n+\kappa_m}{\kappa_n-\kappa_m}\right|, \quad n = 1, \ldots, N. \tag{30}$$

Then the N-soliton solution $u(x,t)$ of the KdV equation given by (26) is symmetric at time $t = 0$:

$$u(-x,0) = u(x,0). \tag{31}$$

Proof. At time $t = 0$, the function $A(x) = A(x,t)$ is equal to

$$A(x) = 1 + q_1e^{-2\kappa_1x} + \cdots + q_Ne^{-2\kappa_Nx} + \cdots + (q_1\cdots q_N)\prod_{m<n}\frac{(\kappa_m-\kappa_n)^2}{(\kappa_m+\kappa_n)^2}e^{-2(\kappa_1+\cdots+\kappa_N)x}.$$

Denote $\Phi = \kappa_1 + \cdots + \kappa_N$. We observe that the function $\widetilde{A}(x) = e^{\Phi x}A(x)$ is symmetric: $\widetilde{A}(-x) = \widetilde{A}(x)$. Therefore, so is the corresponding solution of the KdV equation:

$$u = -2\frac{d^2}{dx^2}\log A = -\frac{d^2}{dx^2}\log\widetilde{A}. \qquad \square$$

We now observe that if we attempt to pass to the limit $N \to \infty$, for example by setting $\kappa_n = a + (b-a)n/N$, then the coefficients q_n given by (30) have small denominators and diverge. Therefore, we cannot obtain finite-gap solutions by this method.

2.5 *From the ISM to the dressing method.*

One of the main results of the articles [2–4] is a generalization of the ISM within the framework of the dressing method. This construction allows us to take the $N \to \infty$ limit of the set of N-soliton solutions and obtain finite-gap solutions. We briefly describe this generalization.

An N-soliton solution is given by Eqs. (25) and (26), where the c_n and the κ_n are the scattering data of a reflectionless potential and are therefore positive. However, formally these equations make sense under the weaker assumption that $\kappa_m + \kappa_n \neq 0$ for all m and n and that c_n/κ_n are positive. The corresponding function χ has poles on both the positive and the negative parts of the imaginary axis.

Proposition 7 Let $\kappa_1, \ldots, \kappa_N, c_1, \ldots, c_N$ be nonzero real numbers satisfying the following conditions:

(1) $\kappa_m \neq \pm\kappa_n$ for $m \neq n$.
(2) $c_n/\kappa_n > 0$ for all n.

Then there exists a unique function $\chi(k, x, t)$ satisfying the following properties:

(1) χ is meromorphic on the complex k-plane with simple poles at the points $i\kappa_1, \ldots, i\kappa_N$ and no other singularities, and its residues satisfy condition (19).
(2) χ has the asymptotic behaviour (20) as $|k| \to \infty$.

The function $u(x, t)$ given by Eqs. (25) and (26) are a solution of the KdV equation (2).

We emphasize that, for a given N, the set of solutions of the KdV equation obtained using this proposition is still the set of N-soliton solutions. Specifically, one can check that the solution given by (25) and (26) for the data $(\kappa_1, \ldots, \kappa_N, c_1, \ldots, c_N)$ are the N-soliton solution given by the scattering data $(|\kappa_1|, \ldots, |\kappa_N|, \widetilde{c}_1, \ldots, \widetilde{c}_N)$, where

$$\widetilde{c}_m = c_m \prod_{n:\,\kappa_n<0} \left(\frac{\kappa_m - \kappa_n}{\kappa_m + \kappa_n}\right)^2 \text{ if } \kappa_m > 0, \quad \widetilde{c}_m = -\frac{4\kappa_m^2}{c_m} \prod_{n:\,\kappa_n<0,\, n\neq m} \left(\frac{\kappa_m - \kappa_n}{\kappa_m + \kappa_n}\right)^2 \text{ if } \kappa_m < 0. \tag{32}$$

In other words, a N-soliton solution with a given set of positive parameters $\kappa_1, \ldots, \kappa_N$ and positive phases $c_1, \ldots, c_N$ is described by Proposition 7 in 2^N different ways, by choosing the signs of the κ_n arbitrarily and adjusting the coefficients c_n using the above formula.

We now give an informal argument why this alternative description of N-soliton potentials allows us to obtain finite-gap potentials in the $N \to \infty$ limit. In the previous two sections, we made two attempts to use formulas (25) and (26) with $\kappa_n > 0$ to produce N-soliton solutions with large N. We can either keep the q_n bounded, in which case all solitons end up on the left half-axis, or symmetrically distribute the solitons about $x = 0$, in which case the q_n (or, alternatively, the c_n) need to be large.

To obtain a symmetric distribution of N solitons using Proposition 7, we choose, as in Section 2.4, a set of parameters $\kappa_n > 0$, and set the phases q_n according to (30). We then change the signs of half of the κ_n, and change the c_n according to Eq. (32). The resulting c_n will be bounded for large N, enabling us to take the $N \to \infty$ limit.

2.6 *Primitive potentials*

In the articles [2–4], the second and third authors considered a contour problem that can be viewed as the limit of Proposition 7 as $N \to \infty$.

PROPOSITION 8 Let $0 < k_1 < k_2$, and let R_1 and R_2 be positive, Hölder-continuous functions on the interval $[k_1, k_2]$. Suppose that there exists a unique function $\chi(k, x, t)$ satisfying the following properties:

(1) χ is analytic on the complex k-plane away from the cuts $[ik_1, ik_2]$ and $[-ik_2, -ik_1]$ on the imaginary axis, and has non-tangential limits

$$\chi^{\pm}(ip, x, t) = \lim_{\varepsilon \to 0} \chi(ip \pm \varepsilon, x, t), \quad p \in (-k_2, -k_1) \cup (k_1, k_2) \tag{33}$$

on the cuts.

(2) χ has jumps on the cuts satisfying

$$\chi^{+}(ip, x, t) - \chi^{-}(ip, x, t) = iR_1(p)e^{-2px+8p^3t}[\chi^{+}(-ip, x, t) + \chi^{-}(-ip, x, t)], \tag{34}$$

$$\chi^{+}(-ip, x, t) - \chi^{-}(-ip, x, t) = -iR_2(p)e^{2px-8p^3t}[\chi^{+}(ip, x, t) + \chi^{-}(ip, x, t)], \tag{35}$$

for $p \in [k_1, k_2]$.

(3) χ has asymptotic behaviour at infinity

$$\chi(k, x, t) = 1 + \frac{i}{2k}Q(x, t) + O\left(\frac{1}{k^2}\right), \quad |k| \to \infty, \quad \operatorname{Im} k \neq 0. \tag{36}$$

(4) There exist constants $C(x, t)$ and $\alpha < 1$ such that near the points $\pm ik_1$ and $\pm ik_2$ the function χ satisfies

$$|\chi(k, x, t)| < \frac{C(x, t)}{|k \mp ik_j|^{\alpha}}, \quad k \to \pm ik_j, \quad j = 1, 2. \tag{37}$$

Then the function $u(x, t)$ given by the formula

$$u(x, t) = \frac{d}{dx}Q(x, t) \tag{38}$$

is a solution of the KdV equation (2).

We call solutions of the KdV equation obtained in this way *primitive solutions*. For fixed moments of time, we obtain *primitive potentials* of the Schrödinger operator (1).

REMARK 9 Condition (37) does not appear in the articles [2–4] and is an oversight of the authors. It is necessary, because we consider dressing functions R_1 and R_2 that do not vanish at k_1 and k_2. For such functions χ may have logarithmic or algebraic singularities at the endpoints. Condition (37) is needed to exclude trivial meromorphic solutions of the Riemann–Hilbert problem, having poles at $\pm ik_j$ and no jump on the cuts.

We also note that formulas (34) and (35) differ from the ones in [2–4] by a factor of π, this now seems to us to be a more natural normalization of the dressing functions R_1 and R_2.

REMARK 10 There is a simple observation that justifies the need to include poles in both the upper and lower half-planes when producing a finite-gap potential as a limit of N-soliton potentials as $N \to \infty$. The spectrum of a N-soliton potential determined by $\{\kappa_n, c_n\}_{n=1}^N$ is purely simple for the negative energy values $E = -\kappa_n^2$, and doubly degenerate for $E > 0$. Therefore, a limit as $N \to \infty$ of N-soliton solutions with poles in the upper half-plane will have a simple spectrum $E \in [-k_2^2, -k_1^2]$ (in the one band case) and a doubly degenerate spectrum for $E > 0$. This is precisely the structure of the spectrum of a one-sided primitive potential having $R_2 \equiv 0$, which limits to a finite-gap solution as $x \to -\infty$, but a trivial solution as $x \to \infty$.

A finite-gap potential, on the other hand, has a doubly degenerate continuous spectrum on the interior of its bands, and a simple continuous spectrum on the band ends. To produce a finite-gap potential as a limit of N-soliton potentials as $N \to \infty$, we need to include poles in both half-planes, so that in the limit we end up with two linearly independent bounded wave functions for E in the interior of a band.

A function $\chi(k, x, t)$ satisfying properties (33)–(36) can be written in the form

$$\chi(k,x,t) = 1 + \frac{i}{\pi}\int_{k_1}^{k_2} \frac{f(q,x,t)}{k - iq}dq + \frac{i}{\pi}\int_{k_1}^{k_2} \frac{g(q,x,t)}{k + iq}dq, \tag{39}$$

for some functions $f(q, x, t)$ and $g(q, x, t)$ defined for $q \in [k_1, k_2]$. Plugging this spectral representation into (34) and (35), we obtain the following system of singular integral equations on f and g for $p \in [k_1, k_2]$:

$$f(p,x,t) + \frac{R_1(p)}{\pi}e^{-2px+8p^3t}\left[\int_{k_1}^{k_2} \frac{f(q,x,t)}{p+q}dq + ⨍_{k_1}^{k_2} \frac{g(q,x,t)}{p-q}dq\right] = R_1(p)e^{-2px+8p^3t}, \tag{40}$$

$$g(p,x,t) + \frac{R_2(p)}{\pi}e^{2px-8p^3t}\left[⨍_{k_1}^{k_2} \frac{f(q,x,t)}{p-q}dq + \int_{k_1}^{k_2} \frac{g(q,x,t)}{p+q}dq\right] = -R_2(p)e^{2px-8p^3t}. \tag{41}$$

The corresponding solution of the KdV equation is equal to

$$u(x,t) = \frac{2}{\pi}\frac{d}{dx}\int_{k_1}^{k_2} \left[f(q,x,t) + g(q,x,t)\right]dq. \tag{42}$$

3. Symmetric primitive potentials

In this section, we show how to solve equations (40) and (41) analytically as Taylor series in the case when $R_1 = R_2$. Suppose that

$$R_1(p) = R_2(p) = R(p). \tag{43}$$

In this case, $g(p,x,t) = -f(p,-x,-t)$ and Eqs. (40) and (41) reduce to the single equation for all $p \in [k_1,k_2]$:

$$f(p,x,t) + \frac{R(p)}{\pi}e^{-2px+8p^3t}\left[\int_{k_1}^{k_2}\frac{f(q,x,t)}{p+q}dq - \fint_{k_1}^{k_2}\frac{f(q,-x,-t)}{p-q}dq\right] = R(p)e^{-2px+8p^3t}. \quad (44)$$

The corresponding primitive solution $u(x,t)$ of the KdV equation

$$u(x,t) = \frac{2}{\pi}\frac{d}{dx}\int_{k_1}^{k_2}\left[f(q,x,t) - f(q,-x,-t)\right]dq \quad (45)$$

satisfies the symmetry condition

$$u(-x,-t) = u(x,t). \quad (46)$$

In particular, the potential $u(x) = u(x,0)$ at $t = 0$ is symmetric:

$$u(-x) = u(x). \quad (47)$$

Figures 1 and 2 show some examples of primitive solutions computed in this manner.

REMARK 11 We emphasize that, in order for a primitive potential to be symmetric, it is sufficient but not necessary for the dressing functions R_1 and R_2 to be equal.

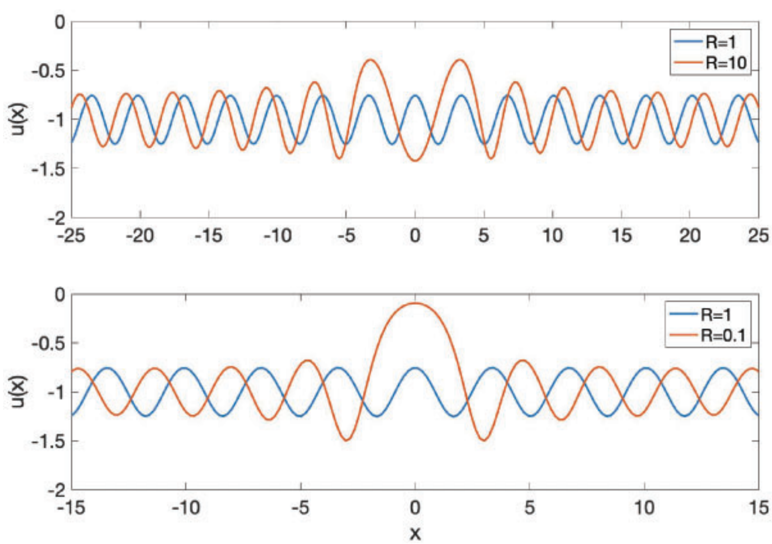

FIG. 1. Symmetric primitive potentials $u(x)$ determined by $k_1 = 0.5$, $k_2 = 1$ and various values of constant $R(p) = R$. These were computed numerically by solving (44) using Gauss–Legendre quadrature.

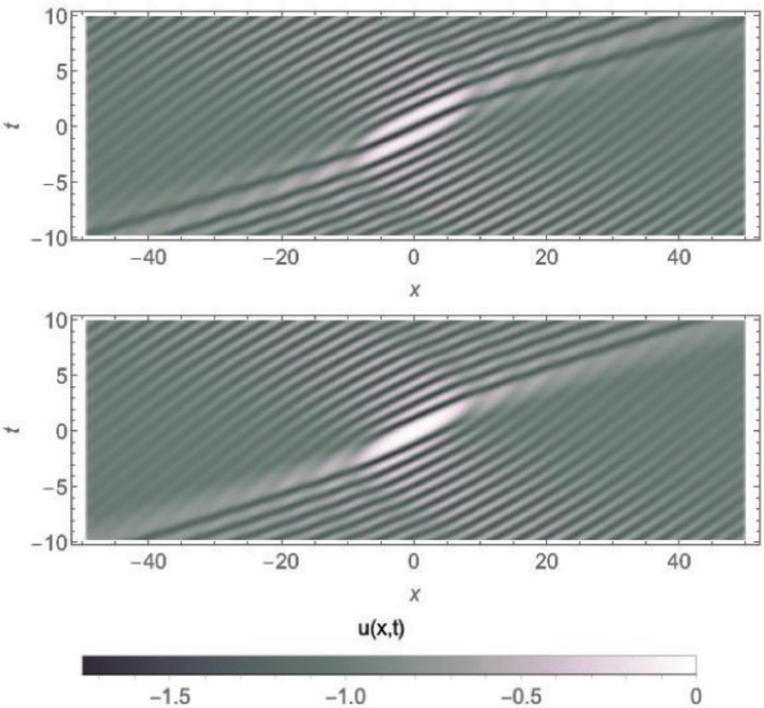

FIG. 2. Symmetric primitive potential solutions $u(x,t)$ to the KdV equation determined by $k_1 = 0.125$, $k_2 = 1$ and various values of constant $R(p) = R$. The top plot shows the primitive potential solutions determined by $R = 100$, and the bottom plot shows the primitive potential solution determined by $R = 0.01$. These were computed numerically by solving (44) using Gauss–Legendre quadrature.

We now denote $f(p,x) = f(p,x,0)$ and set $t = 0$ in Eq. (44):

$$e^{2px}f(p,x) + \frac{R(p)}{\pi}\left[\int_{k_1}^{k_2}\frac{f(q,x)}{p+q}dq - \fint_{k_1}^{k_2}\frac{f(q,-x)}{p-q}dq\right] = R(p), \quad p \in [k_1,k_2]. \tag{48}$$

We show that this equation can be solved analytically. Introduce the variable $s = p^2$ and expand $f(p,x)$ as a Taylor series in x, separating the even and odd coefficients in the following way:

$$f(p,x) = \sum_{k=0}^{\infty}\frac{1}{(2k)!}x^{2k}f_k(s) + \sum_{k=0}^{\infty}\frac{1}{(2k+1)!}x^{2k+1}\sqrt{s}h_k(s), \quad s = p^2. \tag{49}$$

Plugging this into (48) and collecting powers of x, we obtain the following system of equations on $f_k(s)$ and $h_k(s)$, where k is a non-negative integer and δ is the Kronecker delta:

$$f_k(s) + R(\sqrt{s})H[f_k](s) = R(\sqrt{s})\delta_{0k} - \sum_{i=0}^{k-1}\binom{2k}{2i}2^{2k-2i}s^{k-i}f_i(s) - \sum_{j=0}^{k-1}\binom{2k}{2j+1}2^{2k-2j-1}s^{k-j}h_j(s), \tag{50}$$

$$h_k(s) - R(\sqrt{s})H[h_k](s) = -\sum_{i=0}^{k} \binom{2k+1}{2i} 2^{2k-2i+1} s^{k-i} f_i(s) - \sum_{j=0}^{k-1} \binom{2k+1}{2j+1} 2^{2k-2j} s^{k-j} h_j(s). \tag{51}$$

Here, H is the Hilbert transform on the interval $[k_1^2, k_2^2]$:

$$H[\psi(s)] = \frac{1}{\pi} \int_{k_1^2}^{k_2^2} \frac{\psi(s')}{s'-s} ds'. \tag{52}$$

The corresponding primitive potential is given by

$$u(x) = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \int_{k_1^2}^{k_2^2} h_k(s') ds'. \tag{53}$$

Equations (50) and (51) can be solved recursively for f_k and h_k provided that we know how to invert the operators $1 \pm R(\sqrt{s})H$. This can be done explicitly using the following proposition.

Remark 12 Our method for computing the coefficients of the Taylor series in x around $x = 0$ of a symmetric primitive potential $u(x)$ is easily generalized to compute the Taylor coefficients for the series in both x and t around $(x,t) = (0,0)$ of the corresponding solution $u(x,t)$ of the KdV equation. However, including the t-dependence in the construction does not add much additional insight, and the analogue of the iterative system (50)–(51) becomes notationally complicated. We outline this generalization below and leave further details to the reader.

First, we multiply both sides of (44) by e^{2px-8p^3t}. We then differentiate both sides of the resulting integral equation by $\partial_x^k \partial_t^\ell$ to derive an integral equation for $f^{(k,\ell)}(p,0,0)$, with an inhomogenous term involving lower order derivatives of $f(p,0,0)$. Substituting $p^2 = s$ as before, we obtain a system of integral equations for $f_{k\ell}(s) = f^{(k,\ell)}(\sqrt{s},0,0)$ when $k+\ell$ is even and for $h_{k\ell}(s) = f^{(k,\ell)}(\sqrt{s},0,0)/\sqrt{s}$ when $k+\ell$ is odd. These are analogous to the system (50) and (51) for h_k and f_k. We then solve the equations for $f_{k\ell}$ and $h_{k\ell}$ inductively as follows: we proceed in order of increasing $k+\ell$, and along each diagonal $k+\ell = n$ we compute in the following order $(k,\ell) = (0,n), (n,0), (1,n-1), (n-1,1), \ldots$

Proposition 13 Let $\alpha(s)$ be a Hölder-continuous function on the interval $[k_1^2, k_2^2]$ such that $|\alpha(s)| < 1/2$ for all $s \in [k_1^2, k_2^2]$. The integral operator L_α defined by

$$L_\alpha[\psi(s)] = \psi(s) + \tan(\pi\alpha(s))H[\psi(s)] \tag{54}$$

has a unique bounded inverse as an operator on $L^p(\mathbb{R})$ for $p > 1$ and $p \neq 2$ given by

$$L_\alpha^{-1}[\varphi(s)] = \cos^2(\pi\alpha(s))\varphi(s) - \sin(\pi\alpha(s))e^{-\pi H[\alpha(s)]} H[\cos(\pi\alpha(s))e^{\pi H[\alpha(s)]}\varphi(s)]. \tag{55}$$

If α is constant, then L_α^{-1} can be written as

$$L_\alpha^{-1}[\varphi(s)] = \cos^2(\pi\alpha)\varphi(s) - \sin(\pi\alpha)\cos(\pi\alpha) \left(\frac{s-k_1^2}{k_2^2-s}\right)^\alpha H\left[\left(\frac{k_2^2-s}{s-k_1^2}\right)^\alpha \varphi(s)\right]. \tag{56}$$

Proof. We will begin by deriving the explicit representation of the inverse operator. The singular integral equation $L_\alpha[\psi(s)] = \varphi(s)$ takes the form

$$\psi(s) - \frac{\tan(\pi\alpha(s))}{\pi} \fint_{k_1^2}^{k_2^2} \frac{\psi(r)}{s-r} dr = \varphi(s). \tag{57}$$

We invert this equation to express ψ in terms of φ by reformulating it as an inhomogeneous Riemann–Hilbert problem. The function $\Psi(s)$ defined by

$$\Psi(s) = \frac{1}{\pi} \int_{k_1^2}^{k_2^2} \frac{\psi(r)}{s-r} dr$$

is holomorphic in $s \in \mathbb{C} \setminus [k_1^2, k_2^2]$. The boundary values of Ψ from the right and the left for $s \in [k_1^2, k_2^2]$ satisfy

$$\frac{i}{2}(\Psi^+(s) - \Psi^-(s)) = \psi(s), \quad \frac{1}{2}(\Psi^+(s) + \Psi^-(s)) = \frac{1}{\pi} \fint_{k_1^2}^{k_2^2} \frac{\psi(r)}{s-r} dr. \tag{58}$$

The integral equation (57) is then equivalent to the Privalov problem

$$\Psi^+(s) - e^{-2i\pi\alpha(s)}\Psi^-(s) = -2i\cos(\pi\alpha(s))e^{-i\pi\alpha(s)}\varphi(s), \tag{59}$$

where Ψ is normalized by the asymptotic behaviour $\Psi(s) \to 0$ as $s \to \infty$.

To be able to apply the Plemelj formula to solve the Privalov problem (59), we first need to remove the multiplicative factor in front of Ψ^-. We do this by looking for Ψ in the form $\Psi(s) = \Phi(s)\Xi(s)$. Here the functions $\Phi(s)$ and $\Xi(s)$ are holomorphic in $\mathbb{C} \setminus [k_1^2, k_2^2]$, and satisfy the following conditions.

The function $\Phi(s)$ satisfies the corresponding homogeneous Riemann–Hilbert problem

$$\Phi^+(s) = e^{-2i\pi\alpha(s)}\Phi^-(s)$$

and has the asymptotic behaviour $\Phi(s) \to 1$ as $s \to \infty$. Such a $\Phi(s)$ is given by

$$\Phi(s) = \exp\left(\int_{k_1^2}^{k_2^2} \frac{\alpha(r)}{s-r} dr \right).$$

The boundary values of Φ are

$$\Phi^\pm(s) = \exp(-\pi H[\alpha(s)] \mp i\pi\alpha(s)) \tag{60}$$

for $s \in [k_1^2, k_2^2]$. Note that $\Phi \to \Phi^{-1}$ under the transformation $\alpha \to -\alpha$.

The function $\Xi(s)$ satisfies the jump condition

$$\Xi^+(s) - \Xi^-(s) = \cos(\pi\alpha(s))e^{-i\pi\alpha(s)} \frac{-2i\varphi(s)}{\Phi^+(s)} = -2i\cos(\pi\alpha(s))e^{\pi H[\alpha(s)]}\varphi(s)$$

for $s \in [k_1^2, k_2^2]$ and has the asymptotic behaviour $\Xi(s) \to 0$ as $s \to \infty$. By the Plemelj formula, $\Xi(s)$ is given by

$$\Xi(s) = \frac{1}{\pi} \int_{k_1^2}^{k_2^2} \frac{\cos(\pi\alpha(r)) e^{\pi H[\alpha(r)]} \varphi(r)}{s - r} dr = H[\cos(\pi\alpha(s)) e^{\pi H[\alpha(s)]} \varphi(s)].$$

The boundary values of Ξ are

$$\Xi^{\pm}(s) = H[\cos(\pi\alpha(s)) e^{\pi H[\alpha(s)]} \varphi(s)] \mp i \cos(\pi\alpha(s)) e^{\pi H[\alpha(s)]} \varphi(s) \tag{61}$$

for $s \in [k_1^2, k_2^2]$.

We now evaluate $\psi(s)$ using (58), (60) and (61):

$$\begin{aligned} \psi(s) =& \frac{i}{2}(\Psi^+(s) - \Psi^-(s)) = \frac{i}{2}(\Phi^+(s)\Xi^+(s) - \Phi^-(s)\Xi^-(s)) \\ =& \cos^2(\pi\alpha(s))\varphi(s) - \sin(\pi\alpha(s)) e^{-\pi H[\alpha(s)]} H[\cos(\pi\alpha(s)) e^{\pi H[\alpha(s)]} \varphi(s)], \end{aligned}$$

proving the proposition. The result for constant α comes from the well-known fact that

$$\pi H[1] = \log|s - k_2^2| - \log|s - k_1^2|. \tag{62}$$

We now show that the inverse operator L_α^{-1} exists as a bounded operator on $L^p(\mathbb{R})$ for $p > 1$ and $p \neq 2$ using the bounded inverse theorem. We use results from [9] on the operator H. In particular, we use the following facts:

- The operator H is a skew-adjoint operator with adjoint $-H$ for $p > 1$ and $p \neq 2$.
- For $1 < p < 2$, H is a bounded Fredholm operator on $L^p(\mathbb{R})$ with non-trivial one-dimensional null space.
- For $p > 2$, H is a bounded Fredholm operator on $L^p(\mathbb{R})$ with trivial null space.
- For $p = 2$, H is a bounded operator with trivial null space on $L^2(\mathbb{R})$ but is not Fredholm.

Since H is skew-adjoint and bounded, so is $\tan(\pi\alpha(s))H$. Suppose $p > 1$ and $p \neq 2$. Since L_α is the sum of bounded operators on $L^p(\mathbb{R})$, it is also bounded. The operator L_α is the sum of the identity I and a skew-adjoint operator, so the spectrum of L_α is contained in $1 + i\mathbb{R}$; in particular, 0 is not in the spectrum of L_α, so L_α is injective. It follows from the explicit formula for L_α^{-1} that L_α is surjective. Since L_α is a bijective and bounded operator on a Banach space, L_α is invertible with a bounded inverse by the bounded inverse theorem. □

Using this proposition with $\alpha(s) = \tan^{-1} R(\sqrt{s})/\pi$, we can recursively solve equations (50) and (51) and obtain $u(x)$ as a power series in x. We observe that this power series converges for all values of x. Indeed, the coefficients in (50) and (51) are bounded by $N^k(2k)!/(k!^2)$, where $N = 4k_2^2$, and the Hilbert transform is a bounded operator on $L^p([k_1^2, k_2^2])$ for all $p > 1$. The inverse operators L_α^{-1} are also bounded for $p > 1$ and $p \neq 2$. Therefore the norms of f_k and h_k grow as $M^k(2k)!/(k!^2)$ for some M. The Hölder inequality for Hölder conjugates $p, q > 1$ with $p, q \neq 2$ can then be applied to bound the absolute values of the Taylor coefficients in (53) to show that the Taylor coefficients grow as $M^k(2k)!/(k!^2)$ for some M. Therefore, the power series (53) converges for all x.

4. The case of constant R

As an example, we calculate the first two coefficients of $u(x)$ as a Taylor series in the case when R is a constant positive function. Let $\alpha = \tan^{-1}(R)/\pi$, then $0 < \alpha < 1/2$. By Proposition 13, the operators

$$L_{\pm\alpha}[\psi(s)] = \psi(s) \pm \tan(\pi\alpha)H[\psi(s)]$$

are inverted by

$$L_{\pm\alpha}^{-1}[\varphi(s)] = \cos^2(\pi\alpha)\varphi(s) \mp \sin(\pi\alpha)\cos(\pi\alpha)a^{\pm 1}(s)H[a^{\mp 1}(s)\varphi(s)],$$

where the function

$$a(s) = \left(\frac{s - k_1^2}{k_2^2 - s}\right)^{\alpha} \tag{63}$$

is continuous on $[k_1^2, k_2^2)$ and has an integrable singularity at $s = k_2^2$. The equations (50) and (51) determining f_0, h_0, f_1, h_1 are

$$\begin{aligned}
L_\alpha[f_0(s)] &= \tan(\pi\alpha),\\
L_{-\alpha}[h_0(s)] &= -2f_0(s),\\
L_\alpha[f_1(s)] &= -4sh_0(s) - 4sf_0(s),\\
L_{-\alpha}[h_1(s)] &= -6f_1(s) - 12sh_0(s) - 8sf_0(s).
\end{aligned}$$

We compute

$$\begin{aligned}
L_\alpha^{-1}[1] &= \cos(\pi\alpha)a(s),\\
L_{-\alpha}^{-1}[a(s)] &= \frac{1}{2}(a(s) + a^{-1}(s)),\\
L_\alpha^{-1}[sa^{-1}(s)] &= \frac{s}{2}(a(s) + a^{-1}(s)) - \alpha(k_2^2 - k_1^2)a(s),\\
L_{-\alpha}^{-1}[sa(s)] &= \frac{s}{2}(a(s) + a^{-1}(s)) - \alpha(k_2^2 - k_1^2)a^{-1}(s).
\end{aligned}$$

We therefore obtain

$$\begin{aligned}
f_0(s) &= \tan(\pi\alpha)L_\alpha^{-1}[1] = \sin(\pi\alpha)a(s),\\
h_0(s) &= -2\sin(\pi\alpha)L_{-\alpha}^{-1}[a(s)] = -\sin(\pi\alpha)(a(s) + a^{-1}(s)),
\end{aligned}$$

$$f_1(s) = 4\sin(\pi\alpha)L_\alpha^{-1}[sa^{-1}(s)] = 2\sin(\pi\alpha)s(a(s)+a^{-1}(s)) - 4\alpha\sin(\pi\alpha)(k_2^2-k_1^2)a(s),$$
$$\begin{aligned}h_1(s) &= 24(k_2^2-k_1^2)\alpha\sin(\pi\alpha)L_{-\alpha}^{-1}[a(s)] - 8\sin(\pi\alpha)L_{-\alpha}^{-1}[sa(s)]\\ &= (k_2^2-k_1^2)\alpha\sin(\pi\alpha)(12a(s)+20a^{-1}(s)) - 4\sin(\pi\alpha)s(a(s)+a^{-1}(s)).\end{aligned}$$

The integrals

$$\int_{k_1^2}^{k_2^2} a(s)ds = \int_{k_1^2}^{k_2^2} a^{-1}(s)ds = \frac{\pi(k_2^2-k_1^2)\alpha}{\sin(\pi\alpha)},$$
$$\int_{k_1^2}^{k_2^2} sa(s)ds = \frac{\pi\alpha}{2\sin(\pi\alpha)}((k_2^4-k_1^4)+\alpha(k_2^2-k_1^2)^2),$$
$$\int_{k_1^2}^{k_2^2} sa^{-1}(s)dp = \frac{\pi\alpha}{2\sin(\pi\alpha)}((k_2^4-k_1^4)-\alpha(k_2^2-k_1^2)^2),$$

allow us to compute

$$\frac{2}{\pi}\int_{k_1^2}^{k_2^2} h_0(s)ds = -4(k_2^2-k_1^2)\alpha,$$
$$\frac{2}{\pi}\int_{k_1^2}^{k_2^2} h_1(s)ds = 8(k_2^2-k_1^2)\alpha(4(k_2^2-k_1^2)\alpha-(k_2^2+k_1^2)),$$

therefore by Equation (53) we get

$$u(x) = -4\alpha(k_2^2-k_1^2) + 4\alpha(k_2^2-k_1^2)(4\alpha(k_2^2-k_1^2)-(k_2^2+k_1^2))x^2 + O(x^4). \tag{64}$$

We know that $R=1$ (hence $\alpha = 1/4$) and $k_1 = 0$ produces the exact solution $u(x) = -k_2^2$, and indeed by the above formula we get $u_0 = -k_2^2$ and $u_1 = 0$ in this case.

Formula (64) has some interesting implications. In the limit as $R \to 0$, we observe that $u(0) \to 0$ and $u''(0) \to 0$, which is expected, since $u(x)$ becomes trivial. In the limit as $R \to \infty$, we observe that $u(0) \to -2(k_2^2-k_1^2)$ and $u''(0) \to 4(k_2^2-k_1^2)(k_2^2-3k_1^2)$. If $k_2^2 < 3k_1^2$ we see that in fact $u''(0)$ is negative for all R, while if $k_2^2 > 3k_1^2$ then $u''(0)$ is positive for sufficiently large R.

5. One-zone symmetric potential

In this section, we show that the dressing $R_1 = R_2 = 1$ on the interval $[k_1, k_2]$ produces the elliptic one-gap potential

$$u(x) = 2\wp(x + i\omega' - \omega) + e_3. \tag{65}$$

Previously, in the articles [3, 4], the second and third authors showed that this potential arises from the dressing

$$R_1(p) = \frac{1}{R_2(p)} = \sqrt{\frac{(p-k_1)(p+k_2)}{(k_2-p)(p+k_1)}}. \tag{66}$$

Our new result uses the notation and calculations of [3, 4], but relies on the results of Chapter 4.

First, we observe that if

$$R_2(p) = 1/R_1(p),$$

then equations (34) and (35) reduce to

$$\chi^+(ip,x,t) = iR_1(p)e^{-2px+8p^3t}\chi^+(-ip,x,t), \quad \chi^-(ip,x,t) = -iR_1(p)e^{-2px+8p^3t}\chi^-(-ip,x,t),$$

for $p \in [k_1,k_2]$. When $R_1(p) = 1$ and $t = 0$, the contour problem for $\chi(k,x) = \chi(k,x,0)$ is

$$\chi^+(ip,x) = ie^{-2px}\chi^+(-ip,x), \quad \chi^-(ip,x) = -ie^{-2px}\chi^-(-ip,x), \quad p \in [k_1,k_2]. \tag{67}$$

Our goal is to find the function χ satisfying (67). This can in principle be done using the inductive procedure described in Chapter 4 with $R = 1$ and $\alpha = 1/4$. However, we will need only the first Taylor coefficient. Indeed, if we set $x = 0$, then

$$f(p,0) = f_0(p) = \sin(\pi\alpha)a(s) = \frac{1}{\sqrt{2}}\left(\frac{s-k_1^2}{k_2^2-s}\right)^{1/4}.$$

Hence, we find that the function

$$\xi(k) = \chi(k,0) = 1 + \frac{i}{\pi}\int_{k_1}^{k_2}\frac{f(q,0)}{k-iq}dq - \frac{i}{\pi}\int_{k_1}^{k_2}\frac{f(q,0)}{k+iq}dq = \left(\frac{k^2+k_1^2}{k^2+k_2^2}\right)^{1/4}$$

satisfies equation (67) with $x = 0$:

$$\xi^+(ip) = i\xi^+(-ip), \quad \xi^-(ip) = -i\xi^-(-ip), \quad p \in [k_1,k_2]. \tag{68}$$

We now look for a solution of (67) in the form $\chi(k,x) = \xi(k)\chi_1(k,x)$, where $\chi_1(k,x)$ satisfies the condition

$$\chi_1^+(ip,x) = e^{-2px}\chi_1^+(-ip,x), \quad \chi_2^-(ip,x) = e^{-2px}\chi_2^-(-ip,x), \quad p \in [k_1,k_2]. \tag{69}$$

Such a function has already been found in [2, 3]. Let e_1, e_2, e_3 be defined by the equations

$$k_1^2 = e_2 - e_3, \quad k_2^2 = e_1 - e_3, \quad e_1 + e_2 + e_3 = 0.$$

Let $\wp(z) = \wp(z|\omega,\omega')$ be the Weierstrass function with half-periods ω and ω', where ω is real and ω' is purely imaginary, such that

$$e_1 = \wp(\omega), \quad e_2 = \wp(\omega + i\omega'), \quad e_3 = \wp(i\omega').$$

We introduce, as in [2, 3], the variable z via the relation

$$k^2 = e_3 - \wp(z). \tag{70}$$

This relation expresses the complex plane $\mathbb{C}$ with cuts $[ik_1, ik_2]$ and $[-ik_1, -ik_2]$ along the imaginary axis as a double cover of the period rectangle of $\wp$. The Schrödinger equation (1) with potential given by (65) is the Lamé equation

$$\varphi'' - [2\wp(x - \omega - i\omega') + \wp(z)]\varphi = 0. \tag{71}$$

The Lamé equation has a solution

$$\varphi(x,z) = \frac{\sigma(x - \omega - i\omega' + z)\sigma(\omega + i\omega')}{\sigma(x - \omega - i\omega')\sigma(\omega + i\omega' - z)} e^{-\zeta(z)x}, \tag{72}$$

which has an essential singularity $\varphi(x,z) \sim e^{-x/z}$ near the point $z = 0$ (corresponding to $k = \infty$). Therefore, the function

$$\chi_1(k,x) = \varphi(x,z)e^{-ikx} = \varphi(x,z)e^{-ix/\operatorname{sn} z} \tag{73}$$

tends to 1 as $k \to \infty$. It is easy to check that $\chi_1(k,x)$ satisfies the contour problem (69). Putting everything together, we obtain the following result.

Proposition 14 Let $k_2 > k_1 > 0$. Then the function

$$\chi(k,x) = \left(\frac{k^2 + k_1^2}{k^2 + k_2^2}\right)^{1/4} \varphi(x,z)e^{-ikx}, \quad k^2 = e_3 - \wp(z) \tag{74}$$

satisfies conditions (33)–(37) with $R_1 = R_2 = 1$ and $t = 0$. The potential $u(x)$ defined by (38) is the elliptic one-gap potential (65).

In Section 2.5, we observed that an N-soliton potential is described using the dressing method in 2^N different ways. Since primitive potentials are limits of N-soliton potentials, it is also true that a primitive potential can be described using the dressing method in multiple ways, in other words by different pairs of functions R_1 and R_2. Here, we observe an example of this behaviour: the elliptic one-gap potential can be constructed using constant dressing functions $R_1 = R_2 = 1$, or using the dressing (66).

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