

# Mixed Autonomy in Ride-Sharing Networks

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**Abstract**—We consider ride-sharing networks served by human-driven vehicles (HVs) and autonomous vehicles (AVs). We propose a model for ride-sharing in this mixed autonomy setting for a multi-location equidistant network in which a ride-sharing platform sets prices for riders, compensations for drivers of HVs, and operates AVs for a fixed price with the goal of maximizing profits. When there are more vehicles than riders at a location, we consider three vehicle-to-rider assignment possibilities: rides are assigned to HVs first; rides are assigned to AVs first; rides are assigned in proportion to the number of available HVs and AVs. Next, for each of these priority possibilities, we establish a nonconvex optimization problem characterizing the optimal profits for a network operating at a steady-state equilibrium. We then provide a convex problem which we show to have the same optimal profits, allowing for efficient computation of equilibria, and we show that all three priority possibilities result in the same maximum profits for the platform. Next, we show that, in some cases, there is a regime for which the platform will choose to mix HVs and AVs in order to maximize its profit, while in other cases, the platform will use only HVs or only AVs, depending on the relative cost of AVs. For a specific class of networks, we fully characterize these thresholds analytically and demonstrate our results on an example.

## I. INTRODUCTION

Ride-sharing platforms, also known as transportation network companies, have become commonplace due factors such as high costs of car ownership, lack of parking, and persistent traffic congestion [1]–[5]. Traditionally, rides are provided by drivers who use their personal vehicle to provide service. However, ride-sharing platforms are likely to incorporate autonomous vehicles (AVs) into their fleets in the near future [6].

Nonetheless, significant technological and regulatory hurdles remain before ride-sharing platforms could transition to 100% autonomous fleets [7], [8]. Therefore, it is likely that ride-sharing platforms will initially adopt a *mixed* framework in which AVs operate alongside conventional, human-driven vehicles (HVs) [9]–[11].

Existing research in ride-sharing has largely focused on two ends of the autonomy spectrum. On one end are futuristic *mobility-on-demand* systems consisting of only AVs [12]–[16]. On the other end, models of rider and driver behavior in conventional ride-sharing markets with only HVs and no AVs have been considered in [17]–[20].

In this paper, we study the transition from traditional ride-sharing networks to totally automated mobility-on-demand

systems. In particular, we extend the model proposed in [17], which did not consider AVs, to the mixed autonomy setting under several assumptions on the vehicle-to-rider assignment possibilities, and we analyze the resulting models. We consider a network consisting of multiple locations, and potential riders arrive at these locations with desired destinations. The ride-sharing platform sets prices for riders and compensation to drivers of HVs. In addition, the platform has the option to deploy AVs for a fixed cost. Introducing AVs leads to an important assignment choice that must be made: if both an AV and an HV are available to serve a rider, which receives preference? We consider three possible assignment rules: AVs always receive priority (AV priority); HVs always receive priority (HV priority); and priority is determined in proportion to the number of available AVs and HVs at each location (weighted priority).

We focus on the equilibrium conditions that arise in the resulting mixed autonomy deployment when the platform seeks to maximize profits. We summarize our main findings as follows: 1) In all three priority assignments, the equilibrium conditions lead to a non-convex optimization problem. Nonetheless, we develop an alternative convex problem from which an optimal solution to the original non-convex problem can be recovered. 2) We find that, surprisingly, all three priority schemes result in the same maximum profits for the platform. This is because, at an optimal equilibrium, we show that all vehicles are assigned a ride and thus the priority assignment choice is immaterial at the optimal equilibrium. 3) Lastly, we consider the ratio of AVs to HVs that will be deployed by the platform in order to maximize profits for various operating costs of AVs. We show that, in some cases, there is a regime for which the platform will choose to mix HVs and AVs vehicles in order to maximize profits, while in other cases, the platform will use only HVs or only AVs, depending on the relative cost of AVs. For a specific family of networks, we fully characterize these thresholds analytically.

The main contributions of this paper are therefore two-fold. First, we develop a new model for studying the emergence of AVs in ride-sharing networks. This model contributes substantial modifications to the foundational model developed in [17] in order to allow for the presence of AVs. Second, we conduct a detailed theoretical study of the resulting model focusing on the optimal profits obtainable by a ride-sharing platform that deploys AVs. This paper extends our preliminary work [21], which only considered AV priority assignment and a specific class of networks.

The remainder of this paper is organized as follows. Section II provides the model definitions, and Section III poses the problems of profit maximization as non-convex optimization problems. Section IV proposes an alternative convex optimization problem that provides the same optimal profits

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and from which a solution to the original problem can be recovered. In Section V, we study the relation between the AV and HV priority assignments and show that they achieve the same optimal profits. Due to its asymmetry to the AV and HV priority assignments, weighted priority assignment is introduced and studied separately in Section VI. Section VII studies a particular class of networks and fully quantifies the profit maximizing equilibrium conditions. Concluding remarks are provided in Section VIII<sup>1</sup>.

## II. PROBLEM FORMULATION

We consider an infinite horizon discrete time model of a ride-sharing network that extends the model recently proposed in [17] to accommodate a mixed autonomy setting with autonomous vehicles (AVs) and human-driven vehicles (HVs). The network operator or *platform* determines prices for rides and compensations to drivers within the network. The price of a ride may differ among locations, but does not depend on the desired destination of each rider.

In this paper, we focus on equilibrium conditions that arise when the demand pattern of riders is stationary. For example, for several hours in the early evening on weekends, there might be steady and predictable demand for rides from residential areas to entertainment districts. An alternative direction of research is to consider, for example, the transient effects of changing demand over time. While the model developed below could be utilized in such a context, we focus only on stationary demand and the resulting equilibrium conditions here.

With these considerations in mind, we study the potential benefits of adding AVs to the network to maximize the profit potential for the platform.

### A. Model Definition

We now formalize the mixed autonomous ride-sharing network described above.

**Riders.** Among a network of  $n$  equidistant locations, a mass of  $\theta_i$  potential riders arrives at location  $i \in \{1, 2, \dots, n\}$  in each period of time. Throughout, when indices are omitted from a summation expression, it is assumed the summation is over all locations 1 to  $n$ . A fraction  $\alpha_{ij} \in [0, 1]$  of riders at location  $i$  are traveling to location  $j$  so that  $\sum_j \alpha_{ij} = 1$  for all  $i$ . We assume  $\alpha_{ii} = 0$  for all  $i$  and construct the  $n$ -by- $n$  adjacency matrix  $\mathbf{A}$  as  $[\mathbf{A}]_{ij} = \alpha_{ij}$  where  $[\mathbf{A}]_{ij}$  denotes the  $ij$ -th entry of  $\mathbf{A}$ .

**Human-driven vehicles (HVs).** After each time period, a driver exits the platform with probability  $(1 - \beta)$  and serves another ride with probability  $\beta$  where  $\beta \in (0, 1)$ . Thus, a driver's expected lifetime in the network is  $(1 - \beta)^{-1}$ . Each driver has an outside option of earning  $\omega$  over the same lifetime.

**Autonomous vehicles (AVs).** The platform can choose to operate an AV in the network for a fixed cost of  $s$  each time-step. Thus,  $k = s(1 - \beta)^{-1}/\omega$  is the ratio of the cost of operating an AV for the equivalent time of a driver's expected

lifetime to the outside option earnings. Unlike HVs, it is assumed that AVs are in continual use and do not leave the platform.

**Platform.** The platform sets a price  $p_i$  for a ride from location  $i$  and correspondingly compensates a driver with  $c_i$  for providing a ride at location  $i$ . The continuous cumulative distribution of the riders' willingness to pay is denoted by  $F(\cdot)$  with support  $[0, \bar{p}]$ . That is, when confronted with a price  $p$  for a ride, a fraction  $1 - F(p)$  of riders will accept this price, and the remaining  $F(p)$  fraction will balk and leave the network without requesting a ride. Note that  $\theta_i(1 - F(p_i))$  is then the effective demand for rides at location  $i$ .

The description of the riders, HVs, and the platform is the same as that presented in [17]. In this work, we also introduce AVs as described above. As developed below, this addition substantially alters how the model behaves and is analyzed as compared to [17]. In addition, we make the following assumption throughout.

**Assumption 1.** *The network's demand pattern is stationary, i.e.,  $\mathbf{A}$  and  $\theta_i$  are fixed for all  $i$ . Moreover, the directed graph defined by adjacency matrix  $\mathbf{A}$  is strongly connected and  $\theta_i > 0$  for all  $i \in \{1, \dots, n\}$ ,  $n \geq 2$ .*

In summary, the system consists of a *platform* that sets prices, *riders* that request rides among locations, *HVs* that seek to maximize their compensation, and *AVs* managed by the platform alongside the drivers.

### B. HV and AV Priority Assignments

The number of riders willing to pay the platform's price may be less than, equal to, or greater than the total number of HVs and AVs available at that location. When it is greater than the total number of vehicles, some riders will not be served and will leave the network. When it is less than the total number of vehicles, the platform must decide how to assign riders to vehicles. Resolving this priority assignment problem is one of the main challenges presented by the model defined above as compared to the model with no AVs as proposed in [17]. When no AVs are present, it is assumed that riders are arbitrarily assigned to drivers and any remaining HVs choose to reroute to the location of highest expected earnings. In contrast, in this paper, we consider several priority assignments.

The first priority assignment, called *HV priority*, assigns riders to HVs before AVs and is appropriate if, e.g., the platform views HVs as customers that should be accommodated and given preference over AVs. We also consider an *AV priority* assignment in which AVs are assigned rides before HVs. This priority assignment is appropriate if, e.g., the platform views HVs only as a supplement when insufficient AVs are available. In Section VI, we consider a third, intermediate *weighted* priority assignment that assigns rides in proportion to the availability of vehicles, but we defer its definition and analysis until later.

We sometimes refer to the above defined model under any of the three priority assignments as a *mixed autonomy deployment*. For comparison, the *HV-only deployment* is obtained by assuming no AVs at any location. An *HV-only deployment*

<sup>1</sup>Complete proofs are contained in the extended version arXiv:1908.11711, available at <http://arxiv.org/abs/1908.11711>.

may arise by the choice of a profit-maximizing platform if the platform decides not to use any AVs; alternatively, we may consider an HV-only deployment by enforcing the constraint of no AVs at any locations, in which case it is referred to as a *forced* HV-only deployment. Similarly, the *AV-only deployment* is obtained from the mixed autonomy deployment when there are no HVs at any locations, and a *forced* AV-only deployment arises when this condition is enforced as a constraint on the system.

### C. Equilibrium Definition for HV Priority Assignment

We now turn to the equilibrium conditions of the above model that are induced by the stationary demand as characterized in Assumption 1 and by fixed prices and compensations set by the platform. An equilibrium for the system is a time-invariant distribution of the mass of riders, HVs, and AVs at each location satisfying certain equilibrium constraints, as formalized next; all variables are understood to refer to an equilibrium and therefore no time index is included.

We consider first HV priority assignment. Let  $x_i$  denote the mass of HVs at location  $i$ . Recall  $\theta_i(1 - F(p_i))$  the mass of riders willing to pay for a ride at location  $i$ . If there are fewer riders than HVs at a location, drivers can relocate to another location to provide service in the next time period. For each  $i, j \in \{1, \dots, n\}$ , let  $y_{ij}$  denote such drivers at location  $i$  who relocate to location  $j$  without providing a ride. It follows that

$$\sum_{j=1}^n y_{ij} = \max \{x_i - \theta_i(1 - F(p_i)), 0\}. \quad (1)$$

Further, let  $\delta_i$  denote the mass of new drivers who choose to enter the platform and provide service at location  $i$  at each time step. At equilibrium, it must hold that

$$x_i = \beta \left[ \sum_{j=1}^n \alpha_{ji} \min \{x_j, \theta_j(1 - F(p_j))\} + \sum_{j=1}^n y_{ji} \right] + \delta_i. \quad (2)$$

In (2), observe that  $\min \{x_j, \theta_j(1 - F(p_j))\}$  is the total demand the platform serves with HVs at location  $j$ , and therefore  $\sum_j \alpha_{ji} \min \{x_j, \theta_j(1 - F(p_j))\}$  is the mass of HVs that find themselves located at  $i$  after completing a ride.

When the demand  $\theta_i(1 - F(p_i))$  at location  $i$  exceeds the mass of available HVs  $x_i$ , the platform can choose to use AVs to meet this extra demand. Let  $z_i$  denote the mass of AVs at location  $i$ , and for each  $i, j \in \{1, \dots, n\}$ , let  $r_{ij}$  denote the AVs which do not get a ride at  $i$  and are relocated to location  $j$ . Then

$$z_i = \sum_{j=1}^n \alpha_{ji} \min \{z_j, \max \{\theta_j(1 - F(p_j)) - x_j, 0\}\} + \sum_{j=1}^n r_{ji}. \quad (3)$$

In (3), observe that  $\min \{z_j, \max \{\theta_j(1 - F(p_j)) - x_j, 0\}\}$  is the total demand that the platform serves with AVs at location  $j$ . Moreover,  $\sum_j r_{ji}$  is the mass of AVs which do not get a ride to any other location and are relocated to location  $i$ . It follows that

$$\sum_{j=1}^n r_{ij} = \max \{z_i - \max \{\theta_i(1 - F(p_i)) - x_i, 0\}, 0\}. \quad (4)$$

Note that, under HV priority assignment,  $\sum_j r_{ij}$  depends on  $x_i$ .

For each location  $i$ , define the *expected earnings*  $V_i$  to be the average total compensation earned by a driver arriving at location  $i$ . Recall that, for each ride served at location  $i$ , drivers are compensated  $c_i$  and travel to a new location according to the demand pattern  $\mathbf{A}$ . If a driver does not serve a ride due to insufficient demand, the driver earns no compensation but is free to reroute to the location with highest expected earnings. It thus follows that the expected earnings satisfy the relationship

$$V_i = \min \left\{ \frac{\theta_i(1 - F(p_i))}{x_i}, 1 \right\} \left( c_i + \sum_{k=1}^n \alpha_{ik} \beta V_k \right) + \left( 1 - \min \left\{ \frac{\theta_i(1 - F(p_i))}{x_i}, 1 \right\} \right) \beta \max_j V_j \quad (5)$$

for all locations  $i$  where we observe  $\theta_i(1 - F(p_i))/x_i$  is the fraction of drivers at location  $i$  that serve rides, provided  $\theta_i(1 - F(p_i)) \leq x_i$ .

Since drivers have an outside earnings option of  $\omega$ , they will enter the network at location  $i$  if and only if  $V_i \geq \omega$ . Moreover, the platform is able to independently adjust each compensation  $c_i$ , so a profit maximizing platform seeking to minimize  $V_i$  is able to achieve  $V_i = \omega$  for all  $i$ , so called HVs' incentive-compatibility constraints, leading to the following definition.

**Definition 1.** For some prices and compensations  $\{p_i, c_i\}_{i=1}^n$ , the collection  $\{\delta_i, x_i, y_{ij}, z_i, r_{ij}\}_{i,j=1}^n$  is an equilibrium under  $\{p_i, c_i\}_{i=1}^n$  for HV priority assignment if (1)–(4) is satisfied and  $V_i$  as defined in (5) satisfies  $V_i = \omega$  for all  $i = 1, \dots, n$  such that  $\delta_i + \sum_{j=1}^n y_{ji} > 0$ .

### D. Equilibrium Definition for AV Priority Assignment

In this subsection, we parallel the development of the previous subsection for AV priority assignment. The analogous equilibrium conditions are

$$x_i = \beta \left[ \sum_j \alpha_{ji} \min \{x_j, \max \{\theta_j(1 - F(p_j)) - z_j, 0\}\} + \sum_j y_{ji} \right] + \delta_i \quad (6)$$

$$\sum_{j=1}^n y_{ij} = \max \{x_i - \max \{\theta_i(1 - F(p_i)) - z_i, 0\}, 0\} \quad (7)$$

$$z_i = \sum_{j=1}^n \alpha_{ji} \min \{z_j, \theta_j(1 - F(p_j))\} + \sum_j r_{ji} \quad (8)$$

$$\sum_{j=1}^n r_{ij} = \max \{0, z_i - \theta_i(1 - F(p_i))\}. \quad (9)$$

In comparing (6)–(9) to (1)–(4), notice that AV priority assignment leads to  $\sum_j y_{ij}$  dependent on  $z_i$  in (7) whereas  $\sum_{j=1}^n r_{ij}$  does not depend on  $x_i$  in (9).

The expected earning  $V_i$  for a driver at location  $i$  now has the form

$$V_i = \min \left\{ \frac{M_i}{x_i}, 1 \right\} \left( c_i + \sum_{k=1}^n \alpha_{ik} \beta V_k \right) + \left( 1 - \min \left\{ \frac{M_i}{x_i}, 1 \right\} \right) \beta \max_j V_j, \quad (10)$$

$$M_i = \max \{ \theta_i (1 - F(p_i)) - z_i, 0 \}. \quad (11)$$

Again, the platform chooses compensation such that  $V_i = \omega$ .

**Definition 2.** For some prices and compensations  $\{p_i, c_i\}_{i=1}^n$ , the collection  $\{\delta_i, x_i, y_{ij}, z_i, r_{ij}\}_{i,j=1}^n$  is an equilibrium under  $\{p_i, c_i\}_{i=1}^n$  for AV priority assignment if (6)–(9) is satisfied and  $V_i$  as defined in (10)–(11) satisfies  $V_i = \omega$  for all  $i = 1, \dots, n$  such that  $\delta_i + \sum_{j=1}^n y_{ji} > 0$ .

We discuss restrictions of the present model which posits several simplifying assumptions such as equidistant locations. These simplifying assumptions allow for fundamental insights such as in Theorem 3 and in Section VII below that are not obscured or confounded by additional degrees of freedom. Moreover, such assumptions might be reasonable in certain settings. For example, about 75% of taxi rides in New York City are less than three miles<sup>2</sup>, suggesting that distance may not be a major distinguishing attribute of most rides in that market, and, as a result, the equidistance assumption would be sufficient in many situations. Equidistance is required here because we adopt a discrete time model and all parameters are on a per-ride basis. Similar to [17], normalized distance between nodes can be introduced as a new coefficient to extend the model when the equidistance assumption is relaxed. This coefficient scales ride price and driver compensation, which are then interpreted on a per-distance-unit basis.

### III. PROFIT-MAXIMIZATION FOR HV AND AV PRIORITY ASSIGNMENT

We now consider the problem of maximizing profits at equilibrium. We focus on the equilibrium under prices and compensations  $\{p_i, c_i\}_{i=1}^n$ . This analysis is reasonable when there are large populations of HVs, AVs and riders during periods of stationary rider demand. In this case, the equilibrium captures the flow constraints in (1)–(4) or (6)–(9) and the drivers' earnings constraints in (5) or (10)–(11). We first consider profit maximization with HV priority assignment and then with AV priority assignment. Under HV priority assignment, maximizing the aggregate profit across the  $n$  locations subject to the system's equilibrium constraints yields the following optimization problem:

$$\begin{aligned} \max_{\{p_i, c_i\}_{i=1}^n} & \sum_{i=1}^n [\min \{x_i + z_i, \theta_i (1 - F(p_i))\} \cdot p_i \\ & - \min \{x_i, \theta_i (1 - F(p_i))\} \cdot c_i - z_i \cdot s] \\ \text{s.t. } & \{\delta_i, x_i, y_{ij}, z_i, r_{ij}\}_{i,j=1}^n \text{ is an equilibrium under} \\ & \{p_i, c_i\}_{i=1}^n \text{ for HV priority assignment.} \end{aligned} \quad (12)$$

These equilibrium conditions capture the flow constraints of all vehicles while following HVs' incentive-compatibility constraints. However, the optimization problem (12) is difficult to analyze directly. Instead, we propose an equivalent optimization problem, followed by a lemma establishing the equivalence. To this end, consider as an alternative

$$\begin{aligned} \max_{\{p_i, \delta_i, x_i, y_{ij}, z_i, r_{ij}\}_{i,j=1}^n} & \sum_{i=1}^n p_i \theta_i (1 - F(p_i)) - \omega \sum_{i=1}^n \delta_i - s \sum_{i=1}^n z_i \\ \text{s.t. } & d_i = \theta_i (1 - F(p_i)) \\ & x_i = \beta \left[ \sum_{j=1}^n \alpha_{ji} \min \{x_j, d_j\} + \sum_{j=1}^n y_{ji} \right] + \delta_i \\ & \sum_{j=1}^n y_{ij} = \max \{x_i - d_i, 0\} \\ & z_i = \sum_{j=1}^n \alpha_{ji} \max \{d_j - x_j, 0\} + \sum_{j=1}^n r_{ji} \\ & \sum_{j=1}^n r_{ij} = z_i - \max \{d_i - x_i, 0\} \\ & p_i, \delta_i, z_i, x_i, y_{ij}, r_{ij} \geq 0 \quad \forall i, j. \end{aligned} \quad (13)$$

In a certain sense formalized in the next lemma, (13) is equivalent to (12).

**Lemma 1.** Assume HV priority assignment and consider the optimization problems (12) and (13). Under Assumption 1, an optimal solution to (13) provides an optimal solution to (12). In particular, the following hold:

- 1) If  $(1 - \beta)\omega < \bar{p}$  or  $s < \bar{p}$ , then any optimal solution  $\{p_i^*, \delta_i^*, x_i^*, y_{ij}^*, z_i^*, r_{ij}^*\}_{i,j=1}^n$  for (13) is such that  $d_i^* > 0$  for all  $i$ , i.e., some riders are served at all locations. In this case, there exist compensations  $\{c_i^*\}_{i=1}^n$  such that  $\{\delta_i^*, x_i^*, y_{ij}^*, z_i^*, r_{ij}^*\}_{i,j=1}^n$  constitutes an equilibrium under  $\{p_i^*, c_i^*\}_{i=1}^n$  for HV priority assignment. Moreover,  $\{p_i^*, c_i^*\}_{i=1}^n$  is optimal for (12).
- 2) Conversely, if  $(1 - \beta)\omega \geq \bar{p}$  and  $s \geq \bar{p}$ , then any optimal solution for (13) and any optimal equilibrium from (12) is such that  $\delta_i^* = d_i^* = x_i^* = z_i^* = 0$  for all  $i$ , i.e., no riders are served.

The proof of Lemma 1 closely follows that of [17, Lemma 1], where we adjust the claim and the proof so that it applies to the mixed autonomy setting here. In particular, we first show that the optimal value of (13) upper bounds the optimal value of (12). We then show that any feasible solution of (13) can be supported as an equilibrium in (12) for appropriate choice of compensations.

Turning now to the case of AV priority assignment, the analogous profit-maximization problem is given by (14) below and as in the case of HV priority assignment, we introduce (15) for AV priority assignment.

$$\begin{aligned} \max_{\{p_i, c_i\}_{i=1}^n} & \sum_{i=1}^n [\min \{x_i + z_i, \theta_i (1 - F(p_i))\} \cdot p_i \\ & - \min \{x_i, \max \{\theta_i (1 - F(p_i)) - z_i, 0\}\} \cdot c_i - z_i \cdot s] \\ \text{s.t. } & \{\delta_i, x_i, y_{ij}, z_i, r_{ij}\}_{i,j=1}^n \text{ is an equilibrium under} \end{aligned}$$

<sup>2</sup>As determined from almost 7 million yellow taxi trips in June 2019 available at <https://www1.nyc.gov/site/tlc/about/tlc-trip-record-data.page>

$$\{p_i, c_i\}_{i=1}^n \text{ for AV priority assignment.} \quad (14)$$

$$\begin{aligned} & \max_{\{p_i, \delta_i, x_i, y_{ij}, z_i, r_{ij}\}} \sum_{i=1}^n p_i \theta_i (1 - F(p_i)) - \omega \sum_{i=1}^n \delta_i - s \sum_{i=1}^n z_i \\ & \text{s.t. } d_i = \theta_i (1 - F(p_i)) \\ & x_i = \beta \left[ \sum_j \alpha_{ji} \max \{d_j - z_j, 0\} + \sum_j y_{ji} \right] + \delta_i \\ & \sum_{j=1}^n y_{ij} = x_i - \max \{d_i - z_i, 0\} \\ & z_i = \sum_{j=1}^n \alpha_{ji} \min \{d_j, z_j\} + \sum_{j=1}^n r_{ji} \\ & \sum_{j=1}^n r_{ij} = \max \{z_i - d_i, 0\} \\ & p_i, \delta_i, z_i, x_i, y_{ij}, r_{ij} \geq 0 \quad \forall i, j. \end{aligned} \quad (15)$$

Mirroring Lemma 1, optimization problems (14) and (15) are equivalent in a certain sense.

**Lemma 2.** Assume AV priority assignment and consider the optimization problems (14) and (15). Under Assumption 1, an optimal solution to (15) provides an optimal solution to (14). In particular, the following hold:

- 1) If  $(1 - \beta)\omega < \bar{p}$  or  $s < \bar{p}$ , then any optimal solution  $\{p_i^*, \delta_i^*, x_i^*, y_{ij}^*, z_i^*, r_{ij}^*\}_{i,j=1}^n$  for (15) is such that  $d_i^* > 0$  for all  $i$ , i.e., some riders are served at all locations. In this case, there exist compensations  $\{c_i^*\}_{i=1}^n$  such that  $\{\delta_i^*, x_i^*, y_{ij}^*, z_i^*, r_{ij}^*\}_{i,j=1}^n$  constitutes an equilibrium under  $\{p_i^*, c_i^*\}_{i=1}^n$  for AV priority assignment. Moreover,  $\{p_i^*, c_i^*\}_{i=1}^n$  is optimal for (14).
- 2) Conversely, if  $(1 - \beta)\omega \geq \bar{p}$  and  $s \geq \bar{p}$ , then any optimal solution for (15) and any optimal equilibrium from (14) is such that  $\delta_i^* = d_i^* = x_i^* = z_i^* = 0$  for all  $i$ , i.e., no riders are served.

The proof is similar to that of Lemma 1.

From Lemma 1 (resp., Lemma 2), we conclude that it is without loss of generality for us to focus on the optimization problem (13) (resp., (15)) for the rest of the paper when considering HV (resp., AV) priority assignment.

Moreover, while the objective function of (13) (resp., (15)) is not concave in general, it is concave for distributions for which the term  $p \cdot (1 - F(p))$  is concave in the fractional demand  $d = 1 - F(p)$ , which can be set by the platform by adjusting the price  $p$  (note that  $p \cdot d = d \cdot F^{-1}(1 - d)$ ). For example, the uniform distribution, exponential distribution and Pareto distribution all satisfy this concavity requirement. Throughout the rest of the paper, we focus on the case where the rider's willingness to pay is such that the revenue of the platform is concave in  $d$ .

**Assumption 2.** The cumulative distribution  $F(\cdot)$  of the riders' willingness to pay is such that  $d \cdot F^{-1}(1 - d)$  is concave in  $d$ .

Under HV (resp., AV) priority assignment, we have converted (12) (resp., (14)) to the alternative optimization problem

(13) (resp., (15)). Next, we will further convert (13) (resp., (15), henceforth written as (13)/(15)) to an alternative optimization problem that is also convex, allowing for efficient—and in some cases, closed form—solution computation.

#### IV. CONVEXIFICATION OF PROFIT MAXIMIZATION

Even when (13)/(15) possesses a concave objective function, the constraints are non-convex and cannot be simply convexified so that solving (13)/(15) remains computationally difficult, i.e., nonconvex. This section introduces alternative optimization problems of the mixed autonomy deployment for which the optimal profits will be the same as that of (13)/(15).

While the optimal profits are the same, the optimal solutions of the alternative optimization problems are not exactly the same as those calculated in the original problems (13)/(15). As a result, a main difference between the original problems and their alternatives is that, while the original problems and their optimal solutions can always be interpreted physically, the alternatives are purely mathematical problems. However, given the optimal solution of the alternative problems, we show that it is possible to compute an optimal solution for the original problems (13)/(15) with identical profit and vice versa. Moreover, by eliminating  $p_i$  using  $d_i = \theta_i(1 - F(p_i))$  in substitution, the alternative optimization problems are seen to be convex optimization problems under Assumption 2. But, for clarity, we leave  $p_i$  in the alternative optimization problems to allow for comparison to the original problems. Furthermore, the alternative optimization problems become quadratic optimization problems with linear constraints when  $F(\cdot)$  is a uniform distribution.

First, assume HV priority assignment, and consider the optimization problem given by

$$\begin{aligned} & \max_{\{p_i, \delta_i, x_i, z_i, r_{ij}\}} \sum_{i=1}^n p_i \theta_i (1 - F(p_i)) - \omega \sum_{i=1}^n \delta_i - s \sum_{i=1}^n z_i \\ & \text{s.t. } d_i = \theta_i (1 - F(p_i)) \\ & x_i = \beta \sum_{j=1}^n \alpha_{ji} x_j + \delta_i \\ & z_i = \sum_{j=1}^n \alpha_{ji} (d_j - x_j) + \sum_{j=1}^n r_{ji} \\ & \sum_{j=1}^n r_{ij} = z_i - (d_i - x_i) \\ & p_i, \delta_i, x_i, z_i, r_{ij} \geq 0 \quad \forall i, j. \end{aligned} \quad (16)$$

In the following, we regard (13) as the *original* optimization problem and (16) as the *alternative* optimization problem for HV priority assignment.

Theorem 1 below states that (13) and (16) have the same optimal profits for any  $\beta$ ,  $s$ ,  $\omega$  and adjacency matrix  $\mathbf{A}$ . Moreover, given one optimal solution for (13) or (16), it is possible to compute an optimal solution for the other.

**Theorem 1.** Assume HV priority assignment, and consider the original optimization problem (13) and alternative optimization problem (16). Let

$$\mathbf{u}^{ori*} = \{p_i^{ori*}, \delta_i^{ori*}, z_i^{ori*}, x_i^{ori*}, y_{ij}^{ori*}, r_{ij}^{ori*}\}_{i,j=1}^n \quad (17)$$

be an optimal solution for (13) and

$$\mathbf{u}^{alt*} = \{p_i^{alt*}, \delta_i^{alt*}, z_i^{alt*}, x_i^{alt*}, r_{ij}^{alt*}\}_{i,j=1}^n \quad (18)$$

be an optimal solution for (16). Then the following hold under Assumptions 1 and 2:

- The original optimization problem and the alternative problem obtain the same optimal profits for all possible choices of  $\beta$ ,  $s$ ,  $\omega$  and adjacency matrix  $\mathbf{A}$ .
- The optimal solutions satisfy  $x^{ori*} = x^{alt*}$ ,  $z^{ori*} = z^{alt*}$ ,  $p^{ori*} = p^{alt*}$  and  $\delta^{ori*} = \delta^{alt*}$ .
- If  $\theta_i(1 - F(p_i^{ori*})) \leq x_i^{ori*}$  for all  $i$  in the original optimization problem, then  $z_i^{ori*} = 0$  for all  $i$  and setting  $r_{ij}^{ori*} = y_{ij}^{ori*}$  for all  $i, j$  constitutes an optimal solution for the alternative problem.
- If  $\theta_i(1 - F(p_i^{alt*})) \leq x_i^{alt*}$  for all  $i$  in the alternative optimization problem, then  $z_i^{alt*} = 0$  for all  $i$  and setting  $y_{ij}^{alt*} = r_{ij}^{alt*}$ ,  $r_{ij}^{alt*} = 0$  constitutes an optimal solution for the original optimization problem.

*Proof Sketch.* Let  $\phi^{ori*}$  and  $\phi^{alt*}$  be the optimal profits of the two problems (13) and (16), respectively, and let  $d_i^{ori*} = \theta_i(1 - F(p_i^{ori*}))$  and  $d_i^{alt*} = \theta_i(1 - F(p_i^{alt*}))$ . To prove that the optimal profits of the two problems are equal, we first show that  $\phi^{ori*} \leq \phi^{alt*}$  and then  $\phi^{ori*} \geq \phi^{alt*}$ .

To prove the claim in the direction  $\phi^{ori*} \leq \phi^{alt*}$ , we consider three cases:  $d_i^{ori*} \geq x_i^{ori*}$  for all  $i$ ,  $d_i^{ori*} \leq x_i^{ori*}$  for all  $i$ , and the heterogeneous case when there exists some location  $i$  such that  $x_i^{ori*} > d_i^{ori*}$  and some location  $j$  such that  $x_j^{ori*} < d_j^{ori*}$ . Similarly, in the opposite direction establishing  $\phi^{ori*} \geq \phi^{alt*}$ , we consider three cases:  $d_i^{alt*} \geq x_i^{alt*}$  for all  $i$ ,  $d_i^{alt*} \leq x_i^{alt*}$  for all  $i$ , and the heterogeneous case when  $x_i^{alt*} > d_i^{alt*}$  for some location  $i$  and  $x_j^{alt*} < d_j^{alt*}$  for some location  $j$ .

For each direction, if the optimal solution of (13)/(16) falls in the first two cases, we prove the claim by showing the existence of a feasible solution for (16)/(13) with the same profit.

We next show that the heterogeneous case in the first direction, for which there exists some location  $i$  such that  $x_i^{ori*} > d_i^{ori*}$  and some location  $j$  such that  $x_j^{ori*} < d_j^{ori*}$ , is not possible. We consider an aggregated network by partitioning all locations into two or three combined nodes. Then we illustrate that this heterogeneous situation contradicts the Assumption 1 which requires the network to be strongly connected. In the other direction, we make use of the KKT conditions, which narrows the range of possible relations between  $s$  and  $\omega$ , in order to establish again the existence of a feasible solution for (16)/(13) with the same profit, completing the proof.  $\square$

Turning our attention to AV priority assignment case, consider the optimization problem

$$\begin{aligned} \max_{\{p_i, \delta_i, x_i, y_{ij}, z_i, r_{ij}\}} & \sum_{i=1}^n p_i \theta_i (1 - F(p_i)) - \omega \sum_{i=1}^n \delta_i - s \sum_{i=1}^n z_i \\ \text{s.t.} & d_i = \theta_i (1 - F(p_i)) \\ & x_i = \beta \left[ \sum_j \alpha_{ji} (d_j - z_j) + \sum_j y_{ji} \right] + \delta_i \end{aligned}$$

$$\begin{aligned} \sum_{j=1}^n y_{ij} &= x_i - (d_i - z_i) \\ z_i &= \sum_{j=1}^n \alpha_{ji} z_j \\ p_i, \delta_i, z_i, x_i, y_{ij} &\geq 0 \quad \forall i, j. \end{aligned} \quad (19)$$

Similar to above, we regard (15) as the *original* optimization problem and (19) as the *alternative* optimization problem for AV priority assignment. The next theorem mirrors Theorem 1.

**Theorem 2.** Consider the original optimization problem (15) and alternative optimization problem (19). Let

$$\mathbf{u}^{ori*} = \{p_i^{ori*}, \delta_i^{ori*}, z_i^{ori*}, x_i^{ori*}, y_{ij}^{ori*}, r_{ij}^{ori*}\}_{i,j=1}^n \quad (20)$$

be an optimal solution for (15) and

$$\mathbf{u}^{alt*} = \{p_i^{alt*}, \delta_i^{alt*}, z_i^{alt*}, x_i^{alt*}, y_{ij}^{alt*}\}_{i,j=1}^n \quad (21)$$

be an optimal solution for (19). Then the following holds under Assumptions 1 and 2:

- The original optimization problem and the alternative problem obtain the same optimal profits for all possible choices of  $\beta$ ,  $s$ ,  $\omega$  and adjacency matrix  $\mathbf{A}$ .
- The optimal solutions satisfy  $x^{ori*} = x^{alt*}$ ,  $z^{ori*} = z^{alt*}$ ,  $p^{ori*} = p^{alt*}$  and  $\delta^{ori*} = \delta^{alt*}$ .
- If  $\theta_i(1 - F(p_i^{ori*})) \leq z_i^{ori*}$  for all  $i$  in the original optimization problem, then  $x_i^{ori*} = 0$  for all  $i$  and setting  $y_{ij}^{alt*} = r_{ij}^{ori*}$  for all  $i, j$  constitutes an optimal solution for the alternative problem.
- If  $\theta_i(1 - F(p_i^{alt*})) \leq z_i^{alt*}$  for all  $i$  in the alternative optimization problem, then  $x_i^{alt*} = 0$  for all  $i$  and setting  $r_{ij}^{ori*} = y_{ij}^{alt*}$ ,  $y_{ij}^{alt*} = 0$  constitutes an optimal solution for the original optimization problem.

Corollary 1 follows from Theorems 1 and 2.

**Corollary 1.** Under Assumptions 1 and 2, the optimal profit for the mixed autonomy deployment under HV (resp., AV) priority assignment is no less than the optimal profit computed from (13)/(15) with the additional forced HV-only deployment constraint, i.e., the constraint  $z_i = 0$  for all  $i$ .

Corollary 1 emphasizes that in our model, the AVs will be introduced into the platform only if they increase the optimal profit for the platform.

## V. THE RELATION BETWEEN HV PRIORITY AND AV PRIORITY ASSIGNMENTS

Now that we have introduced the alternative optimization problems for maximizing the profits in both HV and AV priority assignments, we next compare the optimal profits for the two priority assignments. The main result of this section is Theorem 3 which shows that the two priority assignments actually lead to the same optimal profits.

Before presenting the main theorem, we first introduce some preliminary lemmas that are interesting in their own right. In the remainder of the paper, we denote an optimal solution with superscript  $*$ , e.g.,  $x_i^*$ .

The next lemma establishes that under HV priority assignment, if some location has departing AVs without passengers, then that location also does not have incoming AVs without passengers.

**Lemma 3.** *Consider the alternative optimization problem (16) for HV priority assignment under Assumptions 1 and 2. Suppose there exist some location  $i$  such that both  $x_i^* > 0$  and  $z_i^* > 0$ . Then  $d_i^* \geq x_i^*$  for all  $i$ . Moreover, for any  $i_0$ , if there exists some location  $j$  such that  $r_{i_0 j}^* > 0$ , then  $r_{j i_0}^* = 0$  for all  $j$ .*

Next, we show that if it is optimal for the platform to use both HVs and AVs at some location, then every vehicle in the network will be assigned to a ride.

**Lemma 4.** *For optimization problem (16) under Assumption 1 and 2, if there exists a location  $i_0$  such that  $x_{i_0}^* > 0$  and  $z_{i_0}^* > 0$ , then  $r_{ij}^* = 0$  for all  $i, j$ .*

Similar properties exist under AV priority assignment, as summarized in the following lemmas.

**Lemma 5.** *Consider the alternative optimization problem (19) for AV priority assignment under Assumptions 1 and 2. Suppose there exist some location  $i$  such that both  $x_i^* > 0$  and  $z_i^* > 0$ . Then  $d_i^* \geq z_i^*$  for all  $i$ . Moreover, for any  $i_0$ , if there exist some location  $j$  such that  $y_{i_0 j} > 0$ , then  $y_{j i_0} = 0$  for all  $j$ .*

**Lemma 6.** *For optimization problem (19) under Assumptions 1 and 2, if there exists a location  $i_0$  such that  $x_{i_0}^* > 0$  and  $z_{i_0}^* > 0$ , then  $y_{ij}^* = 0$  for all  $i, j$ .*

The main result of this section below uses the above lemmas to establish that a profit-maximizing platform is able to realize the same optimal profits under either the HV priority or AV priority assignments.

**Theorem 3.** *Under Assumptions 1 and 2, for any choice of  $\omega, s, \beta$  and  $\mathbf{A}$ ,  $\mathbf{u}^* = \{p_i^*, \delta_i^*, z_i^*, x_i^*, y_{ij}^*, r_{ij}^*\}_{i,j=1}^n$  is an optimal solution of the optimization problem for HV priority assignment (13) if and only if it is an optimal solution of the optimization problem for AV priority assignment (15), and therefore the optimal profits of the two optimization problems are the same.*

*Proof Sketch.* With Lemmas 4 and 6, we obtain that, when the optimal solution for HV priority assignment (13) and that for AV priority assignment (15) are under the same deployment, i.e., both are HV-only deployment, mixed autonomy deployment or AV-only deployment, they must have the same optimal solutions. Theorem 3 follows by considering the optimal profits under these same deployment conditions.  $\square$

We can then derive a threshold on the cost of AVs above which the platform does not find it optimal to deploy any AVs.

**Proposition 1.** *Under Assumptions 1 and 2, if  $k > 1$ , then, under any priority assignment, it is optimal for the platform to use an HV-only deployment, i.e., there is no benefit to introducing AVs into the ride-sharing network.*

## VI. WEIGHTED PRIORITY ASSIGNMENT

Besides assigning the rides to one type of vehicle—HVs or AVs—first, and then using the other type to satisfy any remaining demand, it is also reasonable to consider that any vehicle in the platform can be chosen randomly with equal probability. Therefore, in this section, we introduce the *weighted priority assignment* in which the platform assigns the rides at each location to HVs and AVs at that location with the same probability, i.e., in proportion to the relative fraction of HVs and AVs to the total number of vehicles.

### A. Equilibrium Definition for Weighted Priority Assignment

As described above, in weighted priority assignment, HVs and AVs are assigned to riders with equal possibility:  $\text{Prob}\{\text{rider assigned to HV}\} = \text{Prob}\{\text{rider assigned to AV}\} = \min\left\{\frac{\theta_i(1-F(p_i))}{x_i+z_i}, 1\right\}$  for all  $i$ . The resulting equilibrium constraints for the model are:

$$x_i = \beta \left[ \sum_j \alpha_{ji} \min \left\{ 1, \frac{\theta_j(1-F(p_j))}{x_j+z_j} \right\} \cdot x_j + \sum_j y_{ji} \right] + \delta_i \quad (22)$$

$$\sum_j y_{ij} = \max \left\{ 1 - \frac{\theta_i(1-F(p_i))}{x_i+z_i}, 0 \right\} \cdot x_i \quad (23)$$

$$z_i = \sum_j \alpha_{ji} \min \left\{ 1, \frac{\theta_j(1-F(p_j))}{x_j+z_j} \right\} \cdot z_j + \sum_j r_{ji} \quad (24)$$

$$\sum_j r_{ij} = \max \left\{ 0, 1 - \frac{\theta_i(1-F(p_i))}{x_i+z_i} \right\} \cdot z_i. \quad (25)$$

The expected lifetime earnings  $V_i$  for a driver at location  $i$  takes the form

$$V_i = \min \left\{ \frac{\theta_i(1-F(p_i))}{x_i+z_i}, 1 \right\} (c_i + \sum_{k=1}^n \alpha_{ik} \beta V_k) + \left( 1 - \min \left\{ \frac{\theta_i(1-F(p_i))}{x_i+z_i}, 1 \right\} \right) \beta \max_j V_j. \quad (26)$$

As before, the platform chooses compensation such that  $V_i = \omega$ .

**Definition 3.** *For some prices and compensations  $\{p_i, c_i\}_{i=1}^n$ , the collection  $\{\delta_i, x_i, y_{ij}, z_i, r_{ij}\}_{i,j=1}^n$  is an equilibrium under  $\{p_i, c_i\}_{i=1}^n$  for weighted priority assignment if (22)–(25) is satisfied and  $V_i$  as defined in (26) satisfies  $V_i = \omega$  for all  $i = 1, \dots, n$  such that  $\delta_i + \sum_{j=1}^n y_{ji} > 0$ .*

To further study weighted priority assignment, we now introduce the following assumption which ensures that the platform can make some profit by offering rides at an appropriate price.

**Assumption 3.** *The parameters  $\beta, \omega$  and  $s$  are such that  $(1-\beta)\omega < \bar{p}$  or  $s < \bar{p}$ .*

### B. Profit-Maximization Optimization Problem for Weighted Priority Assignment

We now establish the following profit-maximization problem for weighted priority assignment:

$$\begin{aligned} \max_{\{p_i, c_i\}_{i=1}^n} & \sum_{i=1}^n [\min \{x_i + z_i, \theta_i(1 - F(p_i))\} \cdot p_i \\ & - \min \left\{ x_i, \theta_i(1 - F(p_i)) \frac{x_i}{x_i + z_i} \right\} \cdot c_i - z_i \cdot s] \\ \text{s.t. } & \{\delta_i, x_i, y_{ij}, z_i, r_{ij}\}_{i,j=1}^n \text{ is an equilibrium under} \\ & \{p_i, c_i\}_{i=1}^n \text{ for weighted priority assignment.} \end{aligned} \quad (27)$$

As in Section III, we establish an equivalent optimization problem

$$\begin{aligned} \max_{\{p_i, \delta_i, x_i, y_{ij}, z_i, r_{ij}\}} & \sum_{i=1}^n p_i \theta_i(1 - F(p_i)) - \omega \sum_{i=1}^n \delta_i - s \sum_{i=1}^n z_i \\ \text{s.t. } & d_i = \theta_i(1 - F(p_i)) \\ & x_i = \beta \left[ \sum_j \alpha_{ji} d_j \frac{x_j}{x_j + z_j} + \sum_j y_{ji} \right] + \delta_i \\ & \sum_{j=1}^n y_{ij} = x_i - d_i \frac{x_i}{x_i + z_i} \\ & z_i = \sum_{j=1}^n \alpha_{ji} d_j \frac{z_j}{x_j + z_j} + \sum_{j=1}^n r_{ji} \\ & \sum_{j=1}^n r_{ij} = z_i - d_i \frac{z_i}{x_i + z_i} \\ & p_i, \delta_i, z_i, x_i, y_{ij}, r_{ij} \geq 0 \quad \forall i, j, \end{aligned} \quad (28)$$

followed by a lemma showing the equivalence.

**Lemma 7.** Assume weighted priority assignment and consider the optimization problems (27) and (28). Under Assumptions 1, 2 and 3, an optimal solution to (28) provides an optimal solution to (27). In particular, any optimal solution  $\{p_i^*, \delta_i^*, x_i^*, y_{ij}^*, z_i^*, r_{ij}^*\}$  for (28) is such that  $d_i^* > 0$  for all  $i$ , i.e., some riders are served at all locations, and there exist compensations  $\{c_i^*\}_{i=1}^n$  such that  $\{\delta_i^*, x_i^*, y_{ij}^*, z_i^*, r_{ij}^*\}_{i,j=1}^n$  constitutes an equilibrium under  $\{p_i^*, c_i^*\}_{i=1}^n$  for weighted priority assignment. Moreover,  $\{p_i^*, c_i^*\}_{i=1}^n$  is optimal for (27).

The following theorem establishes that weighted priority assignment obtains the same optimal profits as the HV and AV priority assignments, which were already shown to obtain the same optimal profits in Theorem 3.

**Theorem 4.** Under Assumptions 1, 2 and 3, for any choice of  $\omega, s, \beta$  and  $\mathbf{A}$ , a feasible solution  $\mathbf{u}$  for (13) or (15) is optimal for (13) or (15) if and only if  $\mathbf{u}$  is an optimal solution for (28).

*Proof Sketch.* By recombining the constraints in (28), we can obtain another optimization problem given by

$$\begin{aligned} \max_{\{p_i, \delta_i, x_i, y_{ij}, z_i, r_{ij}\}} & \sum_{i=1}^n p_i \theta_i(1 - F(p_i)) - \omega \sum_{i=1}^n \delta_i - s \sum_{i=1}^n z_i \\ \text{s.t. } & d_i = \theta_i(1 - F(p_i)) \end{aligned}$$

$$\begin{aligned} x_i + \beta z_i &= \beta \left[ \sum_j \alpha_{ji} d_j + \sum_j y_{ji} + \sum_j r_{ji} \right] + \delta_i \\ \sum_{j=1}^n y_{ij} + \sum_{j=1}^n r_{ij} &= x_i + z_i - d_i \\ z_i - \sum_{j=1}^n \alpha_{ji} z_j &= \sum_{j=1}^n r_{ji} - \sum_{j=1}^n \alpha_{ji} \sum_{k=1}^n r_{jk} \\ p_i, \delta_i, z_i, x_i, y_{ij}, r_{ij} &\geq 0 \quad \forall i, j. \end{aligned} \quad (29)$$

By construction, any optimal solution for (28) will be feasible for (29) and thus the optimal profit of (29) will be no less than that of (28).

By Assumptions 2 and 3, (29) is a convex optimization problem with affine constraints, and thus the KKT conditions are not only necessary, but also sufficient for optimality. By studying the KKT conditions, we conclude any optimal solution for AV priority assignment is also optimal (and feasible) for (29). Since the optimal profits for (29) are higher than or equal to that of (28), and since any optimal solution for AV priority assignment is feasible for (28), then we can conclude that any optimal solution for AV priority assignment is also optimal (and feasible) for (28).  $\square$

Theorems 3 and 4 show that, even though the three priority assignments prescribe different models for incorporating AVs into a ride-sharing platform, the resulting profits at an optimal equilibrium are the same in all three cases under Assumptions 1, 2 and 3. This is because no location will have both AVs and HVs present at an optimal equilibrium. Intuitively, on the one hand, the platform is able to set compensation for drivers and to deploy AVs as desired, so that there is considerable freedom in dictating system operation. On the other hand, locations are coupled through the rider demand pattern and cannot be managed independently by the platform, highlighting the surprising nature of this result.

### VII. CLOSED-FORM CHARACTERIZATION FOR STAR-TO-COMPLETE NETWORKS

In this section, we consider the family of *star-to-complete networks* introduced in [17]. For this large class of networks, we derive closed form expressions for the thresholds of relative cost between HVs and AVs for which the platform finds it optimal to use an HV-only deployment, AV-only deployment, or a mixed autonomy deployment.

**Definition 4.** The class of demand patterns  $(\mathbf{A}^\xi, 1)$  with  $n \geq 3$ ,  $\xi \in [0, 1]$ , and

$$\mathbf{A}^\xi = \begin{bmatrix} 0 & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ c_1 & 0 & c_2 & \cdots & c_2 \\ c_1 & c_2 & 0 & \cdots & c_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \cdots & c_2 & 0 \end{bmatrix}, \quad (30)$$

$$c_1 = \frac{\xi}{n-1} + (1-\xi), \quad c_2 = \frac{\xi}{n-1} \quad (31)$$

is the family of *star-to-complete networks*. It is a star network when  $\xi = 0$  for which we write  $\mathbf{A}^S = \mathbf{A}^0$  and a complete

network when  $\xi = 1$  for which we write  $\mathbf{A}^C = \mathbf{A}^1$ . Therefore the general adjacency matrix of a star-to-complete network can be written as  $\mathbf{A}^\xi = \xi \mathbf{A}^C + (1 - \xi) \mathbf{A}^S$ .

In addition, we make the following assumption throughout this section.

**Assumption 4.** All locations have the same mass of potential riders, which we normalize to one, i.e.,  $\theta = 1$ . Also, the riders' willingness to pay is uniformly distributed in  $[0, 1]$  so that  $F(p) = p$  for  $p \in [0, 1]$ .

Consider fixed outside option earnings  $\omega$ , and recall the parameter  $k$  determining the cost of operating AVs for the same lifetime of an HV relative to  $\omega$ . In this section, we confirm the intuition that, for large  $k$ , i.e. high relative cost of AVs, the profit maximizing strategy for the platform is an HV-only deployment, and for small  $k$ , i.e. low relative cost of AVs, the profit maximizing strategy for the platform is an AV-only deployment. We also show that in some cases, but not all, for some values of  $k$ , the platform finds it optimal to use both HVs and AVs at equilibrium, i.e., a true mixed autonomy deployment.

Recall that Proposition 1 provides a sufficient condition for when a platform will not find it optimal to use AVs. In the next Theorem, we sharpen this result for the class of star-to-complete networks and fully characterize the regions in which the profit-maximizing platform will deploy an HV-only deployment, an AV-only deployment, and a truly mixed autonomous network.

**Theorem 5.** Consider a star-to-complete network under Assumption 4. Define

$$\begin{aligned} k_1 &= \frac{1 + \beta c_1}{c_1 + 1}, \\ k_2 &= \begin{cases} 1 & \text{if } \xi \in [\frac{\beta(n-1)-1}{\beta(n-2)}, 1] \\ \frac{c_1(1+\beta) + (n-1)\beta^2 c_1^3 + 1}{(c_1+1)((n-1)\beta^2 c_1^3 + 1)} & \text{if } \xi \in [\beta_{lim}, \frac{\beta(n-1)-1}{\beta(n-2)}) \\ \frac{1+\beta c_1}{c_1+1} & \text{if } \xi \in [0, \beta_{lim}), \end{cases} \\ k_3 &= \frac{(n-1)c_1 - 1}{(1-\beta)(n-1)(1+c_1)c_1}, \\ k_4 &= \frac{(1+\beta)c_1 + (n-1)\beta c_1^3 + 1}{(c_1+1)(\beta(n-1)c_1^2 + 1)}, \end{aligned}$$

where

$$\beta_{lim} = \max \left\{ \frac{n-1}{2(1-\beta)\beta(n-2)} \left[ \beta(1-2\beta) + \sqrt{\frac{\beta^2(n-1) + 4\beta - 4}{n-1}} \right], 0 \right\}.$$

Suppose  $k_3 \geq k_1$ , equivalently,  $\beta c_1(n-1)(1-c_1+\beta c_1) \geq 1$ . When  $k \in [0, k_1]$ , it is always optimal for the platform to deploy an AV-only deployment, i.e., optimal profits are obtained with  $x_i = 0$  for all  $i$ . If  $k_1 < k_2$ , then: when  $k \in (k_1, k_2)$ , it is optimal for the platform to deploy a mixed autonomous network, i.e., optimal profits are obtained with  $x_i > 0$  and  $z_j > 0$  for some  $i, j$ ; when  $k \geq k_2$ , it is optimal for the platform to deploy an HV-only deployment, i.e., optimal profits are obtained with  $z_i = 0$  for all  $i$ . If  $k_1 \geq k_2$ , then:

when  $k > k_1$ , it is optimal for the platform to deploy an HV-only deployment.

Now suppose  $k_3 < k_1$ , equivalently,  $\beta c_1(n-1)(1-c_1+\beta c_1) \geq 1$ . When  $k \in [0, k_4]$ , it is optimal for the platform to deploy an AV-only deployment; when  $k \in (k_4, k_2)$ , it is optimal to deploy a mixed autonomy deployment; when  $k \geq k_2$ , it is optimal to deploy an HV-only deployment.

To demonstrate Theorem 5 and provide intuition for the fundamental theoretical results of this paper, we study a star-to-complete network with  $n = 3$ ,  $\xi = 0.2$ . We consider two cases:  $\beta = 0.8$  and  $\beta = 0.95$ , and we compute optimal equilibria and profits using the optimization problems formulated above. In both cases, applying Theorem 5, we can verify that  $k_3 \geq k_1$ . For the first case with  $\beta = 0.8$ , we obtain  $k_1 = 0.9053$  and  $k_2 = 0.9181$  so that  $k_1 < k_2$ . Figure 1(Top) confirms that for  $k \leq k_1$ , it is optimal for the platform to deploy only AVs, for  $k_1 < k < k_2$ , it is optimal for the platform to use both AVs and HVs, and for  $k \geq k_2$ , it is optimal for the platform to use only HVs. In contrast, when  $\beta = 0.95$  so that the expected lifetime of HVs in the network is longer, then  $k_1 = 0.9763$  and  $k_1 \geq k_2$ . Figure 1 (Bottom) confirms that for  $k \leq k_1$ , the platform finds it optimal to deploy only AVs, and for  $k > k_1$ , the platform finds it optimal to use only HVs; there is no regime in which the platform finds it optimal to use both AVs and HVs. The plots in Figure 1 are generated by solving the optimization problem (16) in MATLAB using CVX, a package for specifying and solving convex programs [22], [23]. Theorem 5 guarantees that the basic qualitative results demonstrated here apply to arbitrarily large star-to-complete networks.

It is interesting to note from the above thresholds that even if AVs are cheaper than HVs, when the price difference is small, the platform may still choose to deploy only HVs or to deploy a mix of AVs and HVs. An explanation for this observation is as follows. Recall that with probability  $1 - \beta$ , a driver leaves the network and does not seek to be matched to a new rider after finishing a ride and thus essentially provides one-way service. In contrast, AVs are assumed to remain in the network and must be recirculated to a new location. When the demand is uneven so that some destinations are more popular than others, the platform can exploit this one-way service to obtain a higher profit with HVs, even if AVs are less expensive on a per ride basis.

## VIII. CONCLUSION

We proposed three models for ride-sharing systems with mixed autonomy under different ride-assigning schemes and showed that under equilibrium conditions, the optimal profits can be computed efficiently by converting the original problems into alternative convex programs. In addition, we proved that the optimal profits of the three models are the same.

We found that the optimal profits for the ride-sharing platform with AVs in the fleet will be the same as that of the human-only network when  $k$  is large, i.e., the cost for operating an AV is relatively high compared to the outside option earnings for drivers' lifetime. In particular, in Proposition 1, we showed that if the cost of operating an AV exceeds the

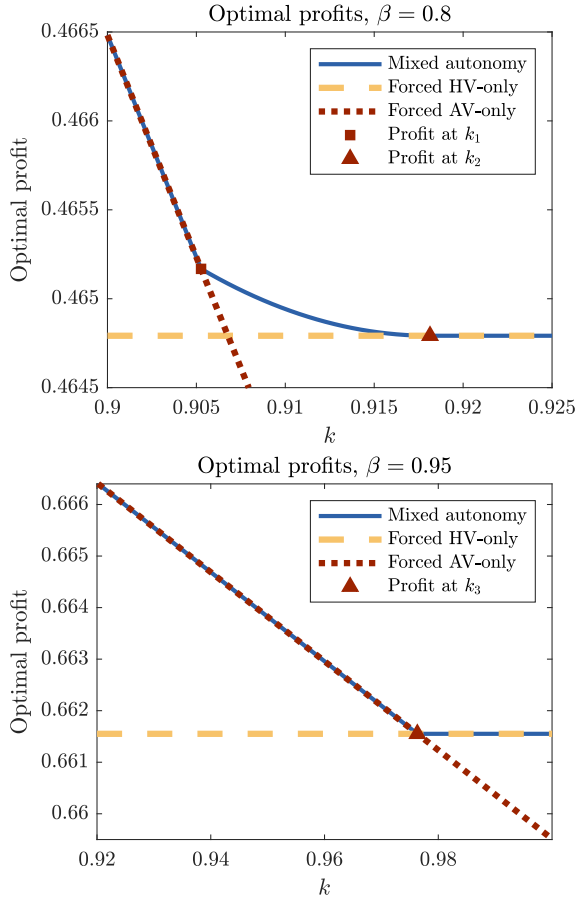


Fig. 1. Optimal profits for a star-to-complete network with  $n = 3$ ,  $\xi = 0.2$  under a mixed autonomy deployment, a forced HV-only deployment, and a forced AV-only deployment. (Top) When  $\beta = 0.8$ , it is optimal for the platform to use only AVs when  $k$ , the ratio of the cost of AVs to HVs, satisfies  $k \leq k_1 = 0.9053$ , only HVs when  $k \geq k_2 = 0.9181$ , and a mix of AVs and HVs when  $k_1 < k < k_2$ . (Bottom) When  $\beta = 0.95$ , it is optimal for the platform to use only AVs when  $k \leq k_1 = 0.9763$  and only HVs when  $k > k_1$ , and it is never optimal for the platform to use a mix of HVs and AVs.

expected compensation to a driver in the system, the platform will find it optimal to not use AVs, an intuitive result.

The case study illustrates that the platform may not necessarily find it optimal to use AVs even when the cost of operating an AV is less than the expected compensation to a driver in the system. Moreover, there are some situations when it is optimal to have both drivers and AVs in the platform. For star-to-complete networks, we quantify the conditions for which the mixed autonomy deployment allows for higher profits than a forced AV-only or forced HV-only deployment.

The model proposed and studied here includes a several simplifying assumptions that can be relaxed in future work. For example, destinations are often not equidistant and ride costs might then depend on destination. Nonetheless, these simplifying assumptions are important for illuminating fundamental properties of ride-sharing in a mixed autonomy setting.

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