DOMAINS WITHOUT DENSE STEKLOV NODAL SETS

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ABSTRACT. This article concerns the asymptotic geometric character of the nodal set of the eigenfunctions of the Steklov eigenvalue problem

$$-\Delta\phi_{\sigma_j} = 0, \quad \text{on } \Omega, \qquad \partial_\nu\phi_{\sigma_j} = \sigma_j\phi_{\sigma_j} \quad \text{on } \partial\Omega$$

in two-dimensional domains Ω. In particular, this paper presents a dense family $\mathcal{A}$ of simply-connected two-dimensional domains with analytic boundaries such that, for each $\Omega \in \mathcal{A}$, the nodal set of the eigenfunction ϕ_{σ_j} "is *not* dense at scale σ_j^{-1}". This result addresses a question put forth under "Open Problem 10" in Girouard and Polterovich, J. Spectr. Theory, 321-359 (2017). In fact, the results in the present paper establish that, for domains $\Omega \in \mathcal{A}$, the nodal sets of the eigenfunctions ϕ_{σ_j} associated with the eigenvalue σ_j have starkly different character than anticipated: they are not dense at any shrinking scale. More precisely, for each $\Omega \in \mathcal{A}$ there is a value $r_1 > 0$ such that for each j there is $x_j \in \Omega$ such that ϕ_{σ_j} does not vanish on the ball of radius r_1 around x_j.

1. INTRODUCTION

Let (M, g) be a compact Riemannian manifold with piecewise smooth boundary ∂M. The Steklov problem is given by

$$(1.1) \qquad \begin{cases} -\Delta_g\phi_\sigma = 0 & \text{in } M \\ \partial_\nu\phi_\sigma = \sigma\phi_\sigma & \text{on } \partial M. \end{cases}$$

There is a discrete sequence $0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \ldots$ of values of σ, with $\sigma_j \to \infty$ as $j \to \infty$, for which non-trivial solutions satisfying (1.1) exist [HL01]. These are the *Steklov eigenvalues* and the corresponding functions ϕ_{σ_j} are the *Steklov eigenfunctions*. This paper studies the asymptotic character of the nodal set of the eigenfunctions of the Steklov eigenvalue problem in the case M equals a bounded open set $\Omega \in \mathbb{R}^2$. In particular the results in this paper show that the nodal set of the eigenfunction ϕ_{σ_j} is *not* dense at scale σ_j^{-1} for some such sets Ω—or, more precisely, that there is a dense family $\mathcal{A}$ of simply-connected two-dimensional domains with analytic boundaries such that, for each $\Omega \in \mathcal{A}$, the eigenfunction ϕ_{σ_j} in the domain Ω remains nonzero on a j-dependent ball of j-independent radius. This result addresses a question put forth under "Open Problem 10" in [GP17].

The behavior of both the Steklov eigenvalues (see e.g. [GP17, GPPS14, LPPS17]) and eigenfunctions (see e.g. [PST19, GT19, BL15, Zhu16, Zel15, SWZ16, Sha71, HL01]) have been a topic of recent interest. When M has smooth boundary, the Steklov eigenfunctions $\phi_{\sigma_j}|_{\partial M}$ behave much like high energy Laplace eigenfunctions with eigenvalue σ_j^2. In particular, they oscillate at frequency σ_j. References [PST19, BL15, Zhu16, Zel15, SWZ16, WZ15, GRF19, Zhu15] study the nodal sets of $\phi_{\sigma_j}|_M$, giving both upper and lower bounds on its Hausdorff measure similar to those for Laplace eigenfunctions. In fact, most results regarding Steklov eigenfunctions in the interior of M extract behavior similar to that of high energy Laplace eigenfunctions.

The purpose of this article is to show that, away from the boundary of M, Steklov eigenfunctions behave *very* differently than high energy Laplace eigenfunctions. Not only do they decay rapidly

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(see [GT19, HL01]) but, at least for a dense class of analytic domains, they oscillate slowly over certain portions of the domain. Girouard–Polterovich [GP17, Open Problem 10(i)] raise the question of whether nodal sets of Steklov eigenfunctions are dense at scale σ_j^{-1} in M. One consequence of the results in the present paper is a negative answer to this question. We show that arbitrarily close to any simply-connected domain with analytic boundary $\Omega_0 \subset \mathbb{R}^2$, there is a domain Ω_1 for which the nodal sets are *not* σ_j^{-1} dense and, indeed, that there is a region within Ω_1 where the nodal set density does not increase as $\sigma_j \to \infty$. Moreover, the Steklov eigenfunctions oscillate no faster than a fixed frequency in this region. These results are summarized in the following theorem.

THEOREM **1.** *Let $\Omega_0 \subset \mathbb{R}^2$ be a bounded simply-connected domain with analytic boundary, and let $k > 0$ and $\varepsilon > 0$ be given. Then there exist a set $\Omega_1 \subset \mathbb{R}^2$ with analytic boundary given by*

$$(1.2) \qquad \partial\Omega_1 = \{x + \nu g(x) \mid x \in \partial\Omega_0\}, \qquad \|g\|_{C^k(\partial\Omega_0)} < \varepsilon$$

(where ν denotes the outward unit normal to $\partial\Omega_0$ and where g is an analytic function defined on $\partial\Omega_0$), a point $x_0 \in \Omega_1$ and numbers $0 < r_1 < r_0$, $(B(x_0, r_0) \subset \Omega_1)$ such that: for each Steklov eigenvalue σ for the domain Ω_1 there exists a point $x_\sigma \in B(x_0, r_0)$ such that $B(x_\sigma, r_1) \subset B(x_0, r_0)$ and each Steklov eigenfunction ϕ_σ of eigenvalue σ for the domain Ω_1 satisfies

$$|\phi_\sigma| > 0 \text{ on } B(x_\sigma, r_1) \subset \Omega_1.$$

Additionally, "ϕ_σ has bounded frequency on $B(x_0, r_0)$" (a precise statement follows in Theorem 2).
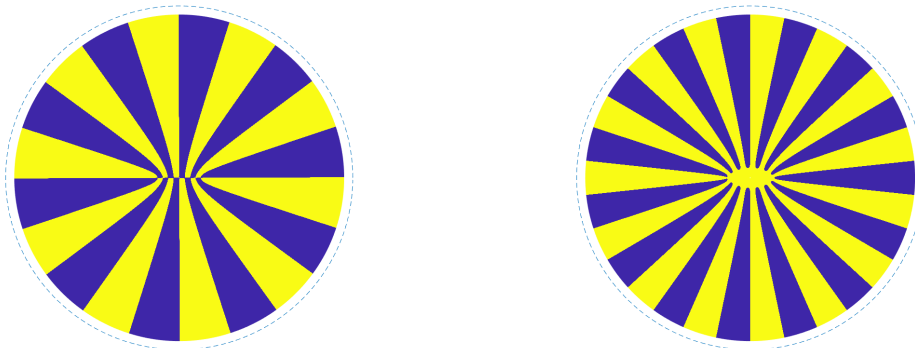

FIGURE 1. Fixed-sign sets for Steklov eigenfunctions over the elliptical domain $\Omega = x^2 + \frac{y^2}{1.01^2} = 1$. The yellow and blue regions indicate the subsets over which the eigenfunctions are positive and negative, respectively. The left and right images correspond to the eigenvalues $\sigma_{20} = 9.9502$ and $\sigma_{30} = 14.9253$, respectively. For a circle the nodal lines coincide with a set of j uniformly arranged radial lines from the center to the boundary: they are dense at scale $\sigma_j^{-1} = j^{-1}$ over the complete domain, including the origin. Under the barely-visible perturbation of the unit disc into the slightly elliptical domain Ω, regions of asymptotically fixed size on which the eigenfunction does not change sign open-up within Ω. Indeed, the nodal set corresponding to σ_{30} (right image) shows such an opening, whereas the nodal set corresponding to σ_{20} (left image) does not; cf. also Remark 1.2.

Theorem 1 is a consequence of the more precise results presented in Theorems 2 and 3 and Corollary 2.2. In particular, these results establish that, for each domain Ω in a dense class $\mathcal{A}$ of two-dimensional domains, an estimate holds for the truncation error in certain "mapped Fourier

expansions" of the eigenfunctions ϕ_σ (i.e., Fourier expansions of ϕ_σ under a change of variables). This estimate is uniformly valid over a subdomain of Ω for all eigenfunctions ϕ_σ with σ large enough. To state these results we first introduce certain conventions and notations, and we review known facts and results from complex analysis.

In what follows, and throughout the remainder of this article, $\mathbb{R}^2$ is identified with the complex plane $\mathbb{C}$, $\Omega \subset \mathbb{C}$ denotes a bounded, simply-connected open set with analytic boundary, and $D := \{z \in \mathbb{C} \mid |z| < 1\}$ denotes the open unit disc in the complex plane. Under these assumptions it follows from the Riemann mapping theorem [BK87] that there is a smooth map $f : \overline{D} \to \mathbb{C}$ such that $f|_D : D \to \Omega$ is a biholomorphism and $|\partial_z f| > 0$ on $\overline{D}$—that is to say, $f|_D : D \to \Omega$ is a biholomorphic conformal mapping of Ω up to and including $\partial\Omega$. We call such a function f a *mapping function for* Ω. Note that, denoting by ∂_r and ∂_ν the radial derivative on the boundary of D and the normal derivative on the boundary of Ω, respectively, we have $\partial_r = |\partial_z f|\partial_\nu$ and $|\partial_z f| > 0$. Thus, for $z \in \partial D$ the function

$$(1.3) \qquad u_{\sigma_j} := \phi_{\sigma_j} \circ f$$

satisfies,

$$\partial_r u_{\sigma_j}(z) = |\partial_z f(z)|\partial_\nu \phi_{\sigma_j}(f(z)) = |\partial_z f(z)|\sigma_j \phi_{\sigma_j}(f(z)),$$

and, hence, the generalized Steklov eigenvalue problem

$$(1.4) \qquad \begin{cases} -\Delta u_{\sigma_j} = 0 & \text{in } D \\ \partial_r u_{\sigma_j} = \sigma_j |\partial_z f| u_{\sigma_j} & \text{on } \partial D. \end{cases}$$

Finally we introduce notation for the relevant Fourier analysis. For $v \in C(\overline{D})$ we let

$$(1.5) \qquad \hat{v}(k) = \frac{1}{2\pi}\int_0^{2\pi} v(\cos\theta, \sin\theta)e^{-ik\theta}d\theta$$

denote the "boundary Fourier coefficients", namely, the Fourier coefficients of the restriction $v|_{\partial D}$ of v to ∂D. Where notationally useful, we write $\mathcal{F}[v] = \hat{v}$.

DEFINITION **1.1.** *We say that the Steklov problem on* Ω *satisfies the tunneling condition if there is* $m_0 > 0$ *and a mapping function* f *for* Ω, *such that for all* $K > 0$ *there is* $C_0 > 0$ *satisfying for any* m

$$|\hat{u}_\sigma(k)| \le C_0^{|k-m|}\Big(\sum_{\ell=m-m_0}^{m+m_0} |\hat{u}_\sigma(\ell)|^2\Big)^{\frac{1}{2}}, \qquad |k| \le K\sigma.$$

Lemma 4.1 shows that any tunneling Steklov problem there exist $\sigma_0 > 0$ so that for each $m \in \mathbf{Z}$ there is a constant $C > 0$ such that for $\sigma > \sigma_0$,

$$(1.6) \qquad e^{-C\sigma}\|\hat{u}_\sigma\|_{\ell^2} \le \Big(\sum_{k=m-m_0}^{m+m_0} |\hat{u}_\sigma(k)|^2\Big)^{\frac{1}{2}}.$$

This estimate and its connections with similar results in quantum mechanics motivate the "tunneling" terminology introduced in Definition 1.1. To explain this, we first review the notion of tunneling from quantum mechanics. In that setting, we consider quantum wave-functions ψ_h which are solutions to

$$(-h^2\Delta + V - E)\psi_h = 0, \qquad \|\psi_h\|_{L^2(\mathbb{R}^n)} = 1$$

where $V \in C^\infty(\mathbb{R}^n; \mathbb{R})$ satisfies $V(x) \underset{|x|\to\infty}{\longrightarrow} \infty$ and $E \in \mathbb{R}$ is the energy of ψ_h. Since ψ_h is the wave-function of a quantum particle with energy E under the influence of the potential V, the corresponding classical Hamilatonian is $p = |\xi|^2 + V - E$ and the corresponding classical particle lies on the surface $\{p = 0\}$. The classical tunneling result from quantum mechanics then states

that, despite the fact that ψ_h is classically localized to $\{p = 0\}$, for *any* U open and bounded, there is $c > 0$ such that
$$\|\psi_h\|_{L^2(U)} \geq ce^{-c/h}.$$
That is, even for $U \subset \{V(x) > E\}$, the *classically forbidden region*, there is some positive (albeit exponentially small in h) probability of finding the particle U. (We refer the reader to e.g. [Zwo12, Chapters 4, 7] for a description of the phase space perspective for studying Schrödinger operators and this type of tunneling estimate.)

Returning to the setting of Steklov eigenfunctions, recall that u_σ is an eigenfunction of the Dirichlet to Neumann map which is a pseudodifferential operator on $\partial\Omega$ with symbol $|\xi|_g$ where g is the metric on $\partial\Omega$ [Tay11, Sec. 7.11, Vol 2]. Therefore, the classical problem corresponding to the Steklov problem is the Hamiltonian flow for the Hamiltonian $|\xi|_g$ on $T^*\partial\Omega$ at energy $\sigma^{-1}|\xi|_g = 1$— which describes the motion of a free particle on $\partial\Omega$. The allowable energies for this classical problem are given by $\{\sigma^{-1}|\xi|_g = 1\}$ which, in the Fourier series representation correspond to $\sigma = |\xi|_g \sim |k|$. Thus, the classically forbidden region is $\left||\sigma^{-1}|k| - 1\right| > c > 0$. Equation (1.6) tells us that, in cases for which the Steklov problem on Ω is tunneling, Steklov eigenfunctions carry positive energy even in the classically forbidden region $\sigma^{-1}|k| \ll 1$, with an energy value that is no smaller than exponentially decaying in σ. (Using the estimates of [GT19] one can also see that Steklov eigenfunctions carry *at most* exponentially small energy in the forbidden region.)

THEOREM 2. *Assume that the Steklov problem on Ω is tunneling and let σ denote a Steklov eigenvalue for the set Ω. Let*

$$(1.7) \qquad \tilde{u}_{\sigma,\delta} := u_\sigma|_{B(0,\delta)} = \sum_{k=-\infty}^{\infty} \hat{u}_\sigma(k)r^{|k|}e^{ik\theta}, \qquad \tilde{u}_{\sigma,\delta,m} := \sum_{|k|<m} \hat{u}_\sigma(k)r^{|k|}e^{ik\theta}.$$

Then, there exist a constant $c > 0$ such that, for each integer $N > 0$, there are constants C_N, σ_0, δ_0, and $m_0 > 0$ so that for all $0 < \delta < \delta_0$, $m > m_0$, and $\sigma > \sigma_0$ the inequality

$$(1.8) \qquad \frac{\|\tilde{u}_{\sigma,\delta} - \tilde{u}_{\sigma,\delta,m}\|_{C^N(B(0,\delta))}}{\|\tilde{u}_{\sigma,\delta}\|_{L^2(B(0,\delta))}} \leq C_N(\delta^{m-N-m_0-1} + e^{-c\sigma})$$

holds.

Letting $\{\phi_{\sigma_j}\}_{j=1}^{\infty}$ denote an orthonormal basis of Steklov eigenfunctions and calling $u_{\sigma_j} = \phi_{\sigma_j} \circ f$, Theorem 2 shows in particular that

$$(1.9) \qquad u_{\sigma_j} = \sum_{|k|<m} \hat{u}_{\sigma_j}(k)r^{|k|}e^{ik\sigma} + O\left((r^{m-m_0-1} + e^{-c\sigma_j})\sqrt{\sum_{|k|<m}|\hat{u}_{\sigma_j}(k)|^2\frac{r^{2k+1}}{2k+1}}\right).$$

In other words, for r small, u_{σ_j} is well approximated by a function with finitely many Fourier modes. If there is $c > 0$ such that

$$|\hat{u}_{\sigma_j}(0)| \geq c\sqrt{\sum_{0<|k|<m}|\hat{u}_{\sigma_j}(k)|^2},$$

then we obtain

$$u_{\sigma_j} = \hat{u}_{\sigma_j}(0) + O((r + e^{-c\sigma_j})|\hat{u}_{\sigma_j}(0)|)$$

and u_{σ_j} is nearly constant on small balls centered around 0. In general, however, finitely many Fourier modes are necessary to capture the lowest-order asymptotics, as indicated in equation (1.9).

One of the main components of the proof of Theorem 1, in addition to Theorem 2, is the construction of a large class of domains Ω for which the Steklov problem is tunneling. To this end,

we introduce some additional definitions. A function $v \in C(D)$ will be said to be *boundary-band-limited* provided $\hat{v}(k) = 0$ except for a finite number of values of $k \in \mathbb{Z}$. We say that a mapping function f is *boundary band limited conformal* (BBLC) if $|\partial_z f|$ is boundary band-limited. If in addition, $|\partial_z f|\,|_{\partial D}$ is non-constant, we will write that Ω is BBLCN. Finally, we say the domain Ω is BBLC (BBLCN) if and only if a BBLC (BBLCN) mapping function, $f : D \to \Omega$ exists. We now present the main theorem of this paper.

THEOREM **3.** *Assume Ω is BBLCN. Then the Steklov problem on Ω is tunneling.*

REMARK **1.2.** It is not clear whether the elliptical and kite-shaped domains (equations (6.1) and (6.2)) considered in Figures 1, 4 and 5 satisfy the BBLCN condition or, more generally, whether they have tunneling Steklov problems (we have not as yet been able to establish that the tunneling condition holds for domains that are not BBLCN). However, domain-opening observations such as those displayed in Figure 1 and Section 6, suggest that these domains may nevertheless be tunneling. This and other domain-opening observations provide support for Conjecture 1.3 below. (Steklov eigenfunctions on a domain which satisfies the BBLCN condition, and, therefore, in view of Theorem 3, is known to be tunneling, are displayed in Figure 2.)

In view of Remark 1.2 we conjecture that every Steklov problem on an analytic domain is tunneling unless the Steklov domain Ω is a disc:

CONJECTURE **1.3.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply-connected domain with real analytic boundary that is not equal to $B(x, r)$ for any $x \in \mathbb{R}^2$, $r > 0$. Then the Steklov problem on Ω is tunneling.*

Outline of the paper. This paper is organized as follows. Section 2 shows that arbitrary analytic, bounded, simply-connected domains can be approximated arbitrarily closely by BBLCN domains. Then, Sections 3 and 4 provide proofs for Theorems 3 and 2, respectively. The numerical methods used in this paper to produce accurate Steklov eigenvalues, eigenfunctions, and associated nodal sets are presented in Section 5. Section 6, finally, illustrates the methods with numerical results for elliptical and kite-shaped domains.

REMARK **1.4.** Throughout this article we abuse notation slightly by allowing C to denote a positive constant that may change from line to line but does not depend on any of the parameters in the problem. In addition C_N is a positive constant that may change from line to line and depends only on the parameter N.

2. Approximation by tunneling domains

This section shows that any analytic domain can be approximated arbitrarily closely (in a sense made precise in Corollary 2.2) by a BBLCN domain. To do this, first let $M \geq 0$, $\alpha_i \in \mathbb{C} \setminus \overline{D}$ for $i = 1, \ldots, N$, and let $N_i \geq 1$, $i = 1, \ldots M$, and let us seek approximating BBLCN domains whose mappings $f : \overline{D} \to \mathbb{C}$ take the form

$$f(z) = \int_0^z p^2(w)dw, \qquad p(z) = \prod_{i=1}^M (z - \alpha_i)^{N_i}.$$

In words: f is the integral of the square of a polynomial with roots outside $\overline{D}$. It follows that

$$\partial_z f = \prod_{i=1}^M (z - \alpha_i)^{2N_i}, \qquad |\partial_z f| = \prod_{i=1}^M (|z - \alpha_i|^2)^{N_i}.$$

In particular,

$$|\partial_z f|(e^{i\theta}) = \prod_{i=1}^M (1 - e^{i\theta}\overline{\alpha_i} - e^{-i\theta}\alpha_i + |\alpha_i|^2)^{N_i}$$

which manifestly shows that $|\partial_z f|$ is boundary-band-limited.

We next show that an arbitrary non-vanishing analytic function on $\overline{D}$ can be approximated by the square of a polynomial.

LEMMA **2.1.** *Let $g : \overline{D} \to \mathbb{C}$ smooth with $g|_D$ analytic and $|g| > 0$ on $\overline{D}$. Then, for any $\varepsilon_0 > 0$ and $k > 0$, there are $M > 0$, α_0, $\{(\alpha_i, N_i)\}_{i=1}^M$ with $|\alpha_i| > 1$, $i = 1, \ldots, M$ such that*

$$\left\| g - \alpha_0 \prod_{i=1}^M (z - \alpha_i)^{2N_i} \right\|_{C^k(\overline{D})} < \varepsilon_0.$$

Proof. Define $h : \overline{D} \to \mathbb{C}$ by

$$h(z) = \int_0^z \frac{g'(w)}{g(w)} dw + \log(g(0))$$

Then, since D is simply-connected and $|g| > 0$ on $\overline{D}$, h is analytic in D with smooth extension to $\overline{D}$. In addition,

$$w(z) = e^{\frac{1}{2}h(z)}$$

is an analytic function on D such that $w^2(z) = g(z)$ and w extends smoothly to $\overline{D}$. Then, for all $\varepsilon > 0$, there is a polynomial p_ε such that

$$\| w(z) - p_\varepsilon(z) \|_{C^k(\overline{D})} < \varepsilon \min(\| w(z) \|_{C^k(\overline{D})}, 1)$$

In particular, since $|g| > c > 0$ on $\overline{D}$, for $0 < \varepsilon$ small enough, p_ε has no zeros in $\overline{D}$. Hence,

$$p_\varepsilon = \beta_0 \prod_{i=1}^M (z - \beta_i)^{N_i}$$

for some $|\beta_0| > 0$, $|\beta_i| > 1$, $i = 1, \ldots, M$. Multiplying by $w + p_\varepsilon$, we have

$$\begin{aligned}
\| g(z) - p_\varepsilon^2(z) \|_{C^k(\overline{D})} &= \| (w - p_\varepsilon)(w + p_\varepsilon) \|_{C^k(\overline{D})} \\
&\leq C_k \| (w - p_\varepsilon) \|_{C^k(\overline{D})} \| (w + p_\varepsilon) \|_{C^k(\overline{D})} \\
&\leq C_k \varepsilon (2 + \varepsilon) \| w \|_{C^k(\overline{D})}
\end{aligned}$$

Choosing $\varepsilon = \frac{\varepsilon_0}{C_k} \min(\frac{1}{3\| w \|_{C^k(\overline{D})}}, 1)$ proves the result with $\alpha_0 = \beta_0^2$ and $\alpha_i = \beta_i$.   □

This result can be used to approximate any analytic domain by a BBLCN domain:

COROLLARY **2.2.** *For any analytic, bounded, simply-connected domain Ω, $k > 0$, and $\varepsilon_0 > 0$ there is a BBLCN domain Ω_{ε_0} and $g_{\varepsilon_0} \in C^\infty(\partial\Omega)$ such that with ν the outward unit normal to Ω,*

(2.1) $$\partial\Omega_{\varepsilon_0} = \{ x + \nu g_{\varepsilon_0}(x) \mid x \in \partial\Omega \}, \qquad \| g_{\varepsilon_0} \|_{C^k(\partial\Omega)} < \varepsilon_0.$$

Proof. Since Ω is analytic, there is $f : \overline{D} \to \mathbb{C}$ analytic such that $f|_D : D \to \Omega$ is a biholomorphism and $|\partial_z f| > 0$ on D. Moreover, by [BK87], $\partial_z f$ has a smooth extension to $\overline{D}$. Then, applying Lemma 2.1 with $g = \partial_z f(z)$ gives

$$p_\varepsilon = \alpha_0 \prod_{i=1}^M (z - \alpha_i)^{N_i}$$

a polynomial with no roots in $\overline{D}$ such that

$$\| \partial_z f(z) - p_\varepsilon^2(z) \|_{C^{\max(k,1)}(\overline{D})} < \varepsilon.$$

Note also that adjusting p if necessary we may assume that the restriction of $|p_\varepsilon|$ to ∂D is not constant. Then, defining

$$(2.2) \qquad f_\varepsilon := \int_0^z p_\varepsilon^2(w)dw + f(0)$$

we have

$$\|f_\varepsilon - f\|_{C^{\max(k+1,2)}(\overline{D})} < \varepsilon, \qquad \partial_z f_\varepsilon = p_\varepsilon^2,$$

so that $|\partial_z f_\varepsilon|\big|_{\partial D}$ is non-constant and band limited. Moreover, since f is a biholomorphism, for $\varepsilon > 0$ small enough, f_ε is also a biholomorphism. We next show that since $\|f_\varepsilon - f\|_{C^{\max(k+1,2)}(\overline{D})} < \varepsilon$, for $\varepsilon > 0$ small enough the curve

$$\partial\Omega_\varepsilon = \{f_\varepsilon(z) \mid |z| = 1\}$$

can be expressed in the form (2.1). To do this let

$$F(t,\theta,\omega,s) = f(e^{i\theta}) - tf_\varepsilon(e^{i(\omega+\theta)}) - (1-t)f(e^{i(\omega+\theta)}) - sf'(e^{i\theta})e^{i\theta}$$

and note that $F(1,\theta,\omega,s) = 0$ if and only if

$$f_\varepsilon(e^{i(\omega+\theta)}) = f(e^{i\theta}) \pm s\nu(\theta).$$

Therefore, we aim to find $s = s(\theta)$ and $\omega = \omega(\theta)$ such that $F(1,\theta,\omega(\theta),s(\theta)) = 0$. Note that

$$\partial_s F = -f'(e^{i\theta})e^{i\theta}$$

$$\partial_\omega F = -ie^{i(\omega+\theta)}\Big(f'(e^{i(\omega+\theta)}) + t\big(f_\varepsilon'(e^{i(\omega+\theta)}) - f'(e^{i(\omega+\theta)})\big)\Big)$$

In particular,

$$\partial_\omega F = i\partial_s F + O(\varepsilon) + O(|\omega|).$$

Writing $\tilde{F} = \begin{pmatrix} \operatorname{Re} F \\ \operatorname{Im} F \end{pmatrix}$, we have

$$\partial_{s,\omega}\tilde{F} = \begin{pmatrix} -\operatorname{Re}\left(f'(e^{i\theta})e^{i\theta}\right) & \operatorname{Im}\left(f'(e^{i\theta})e^{i\theta}\right) \\ -\operatorname{Im}\left(f'(e^{i\theta})e^{i\theta}\right) & -\operatorname{Re}\left(f'(e^{i\theta})e^{i\theta}\right) \end{pmatrix} + O(|\omega| + \varepsilon)$$

Therefore, there are $C > 0$, $\delta > 0$, $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, $|\omega_0| < \delta$, $t_0 \in (-1,2)$, and $s_0 \in [-1,1]$

$$\left\|\left(\partial_{s,\omega}\tilde{F}\right)^{-1}\right\| \leq C.$$

In particular, by the implicit function theorem, there is $c > 0$ such that if $|\omega_0| < \delta$, $t_0 \in (-1,2)$, and $F(t_0,\theta_0,\omega_0,s_0) = 0$, then for $|t - t_0| < c$, $|\theta - \theta_0| < c$, $|\omega - \omega_0| < c$ and $|s - s_0| < c$, $\omega = \omega(t,\theta)$ and $s = s(t,\theta)$ are the unique solutions of $F(t,\theta,\omega,s) = 0$. In particular, since $F(0,\theta,0,0) = 0$, the solutions $s = s(t,\theta)$ and $\omega = \omega(t,\theta)$ can be continued as functions of t as long as $|\omega(t,\theta)|$ remains small.

We next note that

$$\begin{pmatrix} \partial_t\omega \\ \partial_t s \end{pmatrix} = \begin{pmatrix} \partial_\omega\tilde{F} & \partial_s\tilde{F} \end{pmatrix}^{-1}\partial_t\tilde{F} = O(\|f_\varepsilon - f\|_{L^\infty}) = O(\varepsilon),$$

and, therefore,

$$|\omega(t,\theta)| + |s(t,\theta)| \leq \int_0^t |\partial_t\omega(r,\theta)| + |\partial_t s(r,\theta)|dr \leq Ct\varepsilon.$$

Hence for ε small enough the solutions $\omega(t,\theta)$ and $s(t,\theta)$ continue to $t = 1$ and satisfy

$$|\omega(1,\theta)| + |s(1,\theta)| \leq C\varepsilon.$$

Again, using the implicit function theorem, this implies that $\omega(\theta) := \omega(1, \theta)$ and $s(\theta) := s(1, \theta)$ are 2π-periodic. Differentiating k times now yields

$$|\partial_\theta^k s| \le C_k \varepsilon,$$

finishing the proof by setting $g_{\varepsilon_0} = \pm s$ and shrinking $\varepsilon > 0$ as necessary. (Here the $\pm$ corresponds to whether $f(e^{i\theta})$ is positively $(-)$ or negatively $(+)$ oriented.) $\qquad\square$

REMARK **2.3.** Since the map f_ε in equation (2.2) may send 0 to a point z_0 close to the boundary, it is interesting to see how the Steklov eigenfunctions rearrange their nodal sets in such a way that Theorems 1 and 2 are satisfied on the image of f_ε. To demonstrate this let $|a| < 1$, consider the biholomorphic function $f(z) := \frac{z-a}{\bar{a}z-1}$, and let f_ε denote the approximant of f given by equation (2.2) with

$$(2.3) \qquad p_\varepsilon(w) = i\sqrt{1 - |a|^2} \sum_{j=0}^{N} (\bar{a}w)^j \quad \text{with } N = 20 \text{ and } a = 0.8.$$

(This polynomial was obtained as the N-th order Taylor polynomial of $\sqrt{\partial_z f}$.) In this case, according to Theorems 1 and 2, the Steklov eigenfunctions should be slowly oscillating in a σ independent neighborhood of z_0. Figure 2 displays corresponding Steklov eignfunction or various orders as well as a typical eigenfunction for the exact disc. Note the dramatic change that arises in the Steklov eigenfunctions from a barely visible boundary perturbation of the disc.

3. BBLCN DOMAINS AND TUNNELING STEKLOV PROBLEMS

This section presents a proof of Theorem 3. In preparation for that proof, let $\Omega \subset \mathbb{C}$ be a BBLCN domain, and denote by f the corresponding mapping function. Define

$$\mathcal{F}\left[\left|\partial_z f\right|\right](n) := a_n, \quad n \in \mathbb{Z} \quad \left(a_0 := \frac{1}{2\pi} \int_0^{2\pi} |\partial_z f(e^{i\theta})| \, d\theta > 0\right).$$

Since Ω is a BBLCN domain, the function $\left|\partial_z f\right|\big|_{\partial D}$ is band limited and $\left|\partial_z f\right|\big|_{\partial D}$ is not identically constant. It follows that

$$m_0 := \sup\{|n| : |a_n| \ne 0\}$$

satisfies $1 \le m_0 < \infty$.

Denoting by $\hat{u}(n)$ the boundary Fourier coefficients of an eigenfunction u, the corresponding boundary Fourier coefficients of $\partial_r u$ are given by $|n|\hat{u}(n)$. Thus, a solution to (1.4) is uniquely determined as an ℓ^2 solution to the equation

$$(3.1) \qquad |n|\hat{u}(n) = \sigma \mathcal{F}\left[u\left|\partial_z f\right|\right](n) \quad n \in \mathbb{Z}.$$

In what follows we may, and do, assume that solutions $\hat{u}$ have ℓ^2-norm equal to one.

Proof of Theorem 3. Since

$$\mathcal{F}\left[u\left|\partial_z f\right|\right](n) = \sum_m a_m \hat{u}(n - m),$$

it follows that (3.1) can be re-expressed in the form

$$(3.2) \qquad |n|\hat{u}(n) = \sum_m \sigma a_m \hat{u}(n - m).$$

From (3.2) we obtain

$$a_{-m_0}\hat{u}(n + m_0) = \frac{|n|}{\sigma}\hat{u}(n) - \sum_{m \ne -m_0} a_m \hat{u}(n - m),$$
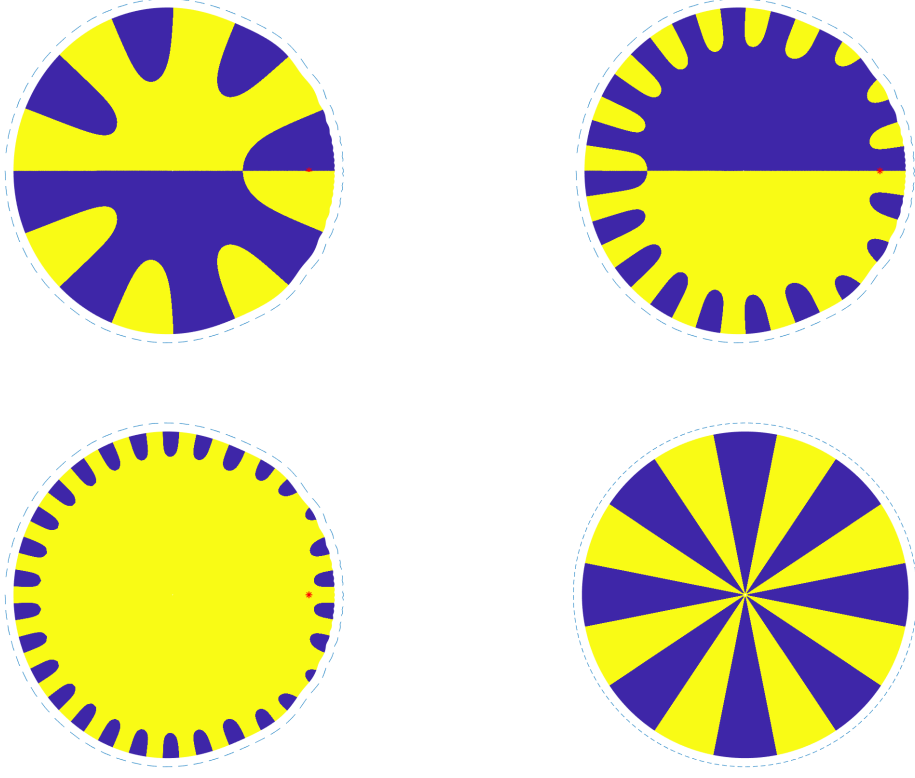
FIGURE 2. Steklov eigenfunctions on the domain Ω whose mapping function, which is given by equation (2.3), maps the center of the disk to the point $z_0 = (0.8, 0)$ (marked by red asterisks in the figures). The corresponding Steklov eigenvalues are given by $\sigma_{16} = 7.9642$ (top left), $\sigma_{40} = 19.8173$ (top right), and $\sigma_{60} = 29.8197$ (bottom left). Note that, according to Corollary 2.2 the set Ω is a BBLCN approximation to the disk. As predicted by Theorem 2, oscillations avoid a region around z_0 for high σ. The bottom-right image displays a typical eigenfunction on the exact disc. Note the dramatic change that arises in the Steklov eigenfunctions from a barely visible boundary perturbation of the disc.

and, then, for all $|n| \leq K\sigma$,

$$
\begin{aligned}
|\hat{u}(n+m_0)| &\leq |a_{-m_0}|^{-1} \Big(\frac{||n| - \sigma a_0|}{\sigma} |\hat{u}(n)| + \sum_{m \neq 0, -m_0} |a_m||\hat{u}(n-m)| \Big) \\
&\leq |a_{-m_0}|^{-1} \Big(\frac{||n| - \sigma a_0|}{\sigma} |\hat{u}(n)| + \sum_{\substack{m=-m_0+1 \\ m \neq 0}}^{m_0} |a_m||\hat{u}(n-m)| \Big) \\
&\leq |a_{-m_0}|^{-1} \max(K, \|a_m\|_{\ell^\infty}) \sum_{k=n-m_0}^{n+m_0-1} |\hat{u}(k)|.
\end{aligned}
$$

The second inequality follows from the fact that $a_n \equiv 0$ for $|n| \geq m_0$, while the third one results from the relation $a_0 > 0$ and the positivity, $\sigma > 0$, of all nontrivial eigenvalues σ, which imply that

$$||n| - \sigma a_0| \leq \max(|n|, \sigma|a_0|) \leq \sigma(\max(K, \|a_m\|_{\ell^\infty})).$$

Making an identical argument, but solving for $\hat{u}(n - m_0)$, and using that $|a_{m_0}| = |a_{-m_0}| \neq 0$, we have for all $|n| \leq K\sigma$,

(3.3)
$$|\hat{u}(n + m_0)| \leq |a_{m_0}|^{-1} \max(K, \|a_m\|_{\ell^\infty}) \sum_{k=n-m_0}^{n+m_0-1} |\hat{u}(k)|,$$

$$|\hat{u}(n - m_0)| \leq |a_{m_0}|^{-1} \max(K, \|a_m\|_{\ell^\infty}) \sum_{k=n-m_0+1}^{n+m_0} |\hat{u}(k)|.$$

We now use equation (3.3) to prove the first half of our tunneling estimate.

LEMMA **3.1.** *Let $m \in \mathbb{Z}$, $K > 0$, and*

$$A_m := \Big(\sum_{k=m-m_0}^{m+m_0} |\hat{u}(k)|^2 \Big)^{\frac{1}{2}}.$$

Then, there exists $C_0 > 0$ so that for all $\sigma > 0$ and for $-K\sigma \leq n + m \leq K\sigma$ we have

(3.4)
$$|\hat{u}(n + m)| \leq C_0^{|n|} A_m.$$

Proof. We will assume $m \geq 0$ since the other case follows similarly. The cases of $n = -m_0, \dots, m_0$ are clear if we take $C_0 \geq 1$. Suppose (3.4) holds for $-m_0 \leq n \leq \ell$ with $m_0 \leq \ell$. Then, by (3.3),

$$|\hat{u}(m + \ell + 1)| \leq |a_{m_0}|^{-1} \max(K, \|a_m\|_{\ell^\infty}) \sum_{k=\ell-2m_0+1}^{\ell} |\hat{u}(k + m)|$$

$$\leq |a_{m_0}|^{-1} \max(K, \|a_m\|_{\ell^\infty}) \sum_{k=\ell-2m_0+1}^{\ell} C_0^{|k|} A$$

Now, if $m_0 \leq \ell < 2m_0$, then

$$|\hat{u}(m + \ell + 1)| \leq |a_{m_0}|^{-1} \max(K, \|a_m\|_{\ell^\infty}) \Big(\sum_{k=0}^{\ell} C_0^k + \sum_{k=1}^{2m_0-\ell-1} C_0^k \Big) A$$

$$\leq |a_{m_0}|^{-1} \max(K, \|a_m\|_{\ell^\infty}) \Big(\frac{C_0^{\ell+1} - 1 + C_0^{2m_0-\ell+1} - C_0}{C_0 - 1} \Big) A$$

In particular, taking

$$C_0 \geq 2|a_{m_0}|^{-1} \max(K, \|a_m\|_{\ell^\infty}) + 1$$

we have

$$|\hat{u}(m + \ell + 1)| \leq C_0^{\ell+1} A.$$

Next, if $2m_0 \leq \ell$, then

$$|\hat{u}(\ell + m + 1)| \leq |a_{m_0}|^{-1} \max(K, \|a_m\|_{\ell^\infty}) A \frac{C_0^{\ell+1} - C_0^{\ell-2m_0+1}}{C_0 - 1}$$

Taking $C_0 \geq 2|a_{m_0}|^{-1} \max(K, \|a_m\|_{\ell^\infty}) + 1$ completes the proof for $-m_0 \leq n \leq K\sigma - m$.

An almost identical argument gives the $-K\sigma - m \leq n \leq 0$ case. $\qquad \square$

4. Analysis of Tunneling Steklov Problems

The proof of Theorem 2 now follows in two steps. First, we show that, for eigenfunctions of any tunneling Steklov problem, the boundary Fourier coefficients of low frequency contain a mass no smaller than exponential in σ. To finish the proof, we use the fact that the harmonic extension of $e^{in\theta}$ decays exactly as $r^{|n|}$. Examining the solution on the ball of radius $\delta > 0$ for some δ small enough, it will be shown that the low frequencies dominate the behavior of u.

LEMMA **4.1.** *Suppose that Ω has tunneling Steklov problem. Then there exist $\sigma_0 > 0$ so that for all $m > 0$ there is $C > 0$ such that for $\sigma > \sigma_0$,*

$$e^{-C\sigma}\|\hat{u}\|_{\ell^2} \leq \Big(\sum_{k=m-m_0}^{m+m_0} |\hat{u}(k)|^2\Big)^{\frac{1}{2}} =: A_m.$$

Proof. First, note that by e.g. [GT19, Corollary 1.3], for $\sigma > 3m$ there is $C > 0$ so that

$$\sum_{|k-m|\leq 2\sigma} |\hat{u}(k)|^2 \geq \|\hat{u}\|_{\ell^2}^2(1 - Ce^{-\sigma/C}).$$

By Lemma 3.1

$$\sum_{|k-m|\leq 2\sigma} |\hat{u}(k)|^2 \leq \sum_{0\leq k-m\leq 2\sigma} C_0^{2k} A_m^2 + \sum_{-2\sigma\leq k-m<0} C_0^{2|k|} A_m^2 \leq 2\frac{2C_0^{4\sigma+2}-1}{C_0^2-1}A_m^2$$

In particular,

$$\frac{C_0^2-1}{2(2C_0^{4\sigma+2}-1)}\|\hat{u}\|_{\ell^2}^2(1 - Ce^{-C\sigma}) \leq A_m^2 = \sum_{k=-m_0}^{m_0} |\hat{u}(k)|^2.$$

Taking σ_0 large enough so that $Ce^{-C\sigma} \leq \frac{1}{2}$, finishes the proof. $\square$

Proof of Theorem 2. In what follows we utilize the definitions (1.7) for a given eigenvalue $\sigma_j = \sigma$, and, for that eigenvalue we denote $\hat{u}(k) = \hat{u}_{\sigma_j}(k) = \hat{u}_\sigma(k)$. Then, applying the relation

$$(4.1) \qquad \int_{B(0,\delta)} |\sum_k b_k r^{|k|}e^{ik\theta}|^2 = \sum_k |b_k|^2\frac{2\pi\delta^{2|k|+2}}{2|k|+2},$$

which is valid for all sequences $\{b_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}$, to the right-hand equation in (1.7), for $m \geq m_0$ we obtain

$$(4.2) \qquad \|\tilde{u}_{\sigma,\delta,m}\|_{L^2}^2 = \sum_{|k|\leq m}\frac{2\pi\delta^{2|k|+2}}{2|k|+2}|\hat{u}(k)|^2 \geq 2\pi\frac{\delta^{2m_0+2}}{2m_0+2}\sum_{|k|\leq m_0}|\hat{u}(k)|^2 = 2\pi\frac{\delta^{2m_0+2}}{2m_0+2}A^2.$$

To estimate the error in approximating $u_{\sigma,\delta}$ by $\tilde{u}_{\sigma,\delta,m}$, first note that

$$\Big\|\sum_{|k|\geq 2\sigma}\hat{u}(k)r^{|k|}e^{ik\theta}\Big\|_{C^N(B(0,\delta))} \leq \sum_{|k|\geq 2\sigma}|\hat{u}(k)|\cdot\|r^{|k|}e^{ik\theta}\|_{C^N(B(0,\delta))}$$

$$\leq \Big(\sum_{|k|\geq 2\sigma}|\hat{u}(k)|^2\Big)^{\frac{1}{2}}\Big(\sum_{|k|\geq 2\sigma}\|r^{|k|}e^{ik\theta}\|_{C^N(B(0,\delta))}^2\Big)^{\frac{1}{2}}$$

$$\leq \Big(\sum_{|k|\geq 2\sigma}|\hat{u}(k)|^2\Big)^{\frac{1}{2}}\Big(\sum_{|k|\geq 2\sigma}k^{2N}\delta^{2k-2N}\Big)^{\frac{1}{2}}$$

$$\leq C_N\|\hat{u}\|_{\ell^2}\delta^{-N}\sigma^N\delta^{2\sigma}.$$

Applying Lemma 4.1 with $m = 0$, and absorbing the σ^N into the exponential factor we then obtain

$$\Big\| \sum_{|k| \geq 2\sigma} \hat{u}(k) r^{|k|} e^{ik\theta} \Big\|_{C^N(B(0,\delta))} \leq C_N \delta^{-N} \delta^{2\sigma} e^{C\sigma} A$$

where $A = A_0$ is given by

$$A := \Big(\sum_{k=-m_0}^{m_0} |\hat{u}(k)|^2 \Big)^{\frac{1}{2}},$$

We can now estimate

$$\| \sum_{|k| \geq m} \hat{u}(k) r^{|k|} e^{ik\theta} \|_{C^N(B(0,\delta))} \leq \sum_{m \leq |k| < 2\sigma} \| \hat{u}(k) r^{|k|} e^{ik\theta} \|_{C^N(B(0,\delta))} + \| \sum_{|k| \geq 2\sigma} \hat{u}(k) r^{|k|} e^{ik\theta} \|_{C^N(B(0,\delta))}$$

$$\leq \sum_{m \leq |k| < 2\sigma} |\hat{u}(k)| \cdot \| r^{|k|} e^{ik\theta} \|_{C^N(B(0,\delta))} + C_N \delta^{-N} \delta^{2\sigma} e^{C\sigma} A$$

Thus, using the definition of tunneling (Definition 1.1), we obtain

$$\| \sum_{|k| \geq m} \hat{u}(k) r^{|k|} e^{ik\theta} \|_{C^N(B(0,\delta))} \leq C_N \delta^{m-N} A \sum_{m \leq |k| < 2\sigma} C_0^{|k|} |k|^N \delta^{|k|-m} + C_N \delta^{-N} \delta^{2\sigma} e^{C\sigma} A$$

$$\leq C_N \delta^{m-N} A + C_N \delta^{-N} \delta^{2\sigma} e^{C\sigma} A$$

provided that $\delta < \frac{1}{2} C_0^{-1}$. Therefore, using (4.2),

$$\frac{\|\tilde{u}_{\sigma,\delta} - \tilde{u}_{\sigma,\delta,m}\|_{C^N(B(0,\delta))}}{\|\tilde{u}_{\sigma,\delta}\|_{L^2(B(0,\delta))}} \leq C_N \delta^{m-N-m_0-1} + C_N \delta^{2\sigma-N-m_0-1} e^{C\sigma}.$$

Thus, choosing $\delta > 0$ such that $\delta < e^{-2C}$ and taking $\sigma_0 > N + m_0 + 1$ the claim follows.   □

We can now present a proof of Theorem 1.

Proof of Theorem 1. From Corollary 2.2 we know that there exists a tunneling domain $\Omega_1 \subset \mathbb{C}$ satisfying (1.2) for the given value $\varepsilon > 0$. Let σ_0 be as in Theorem 2. Clearly, it suffices to prove the statement of the theorem for $\sigma > \sigma_0$, since for $\sigma \leq \sigma_0$ the statement follows from the fact that there are finitely many Steklov eigenvalues below σ_0 and that ϕ_σ cannot vanish in any open set. Therefore, we may and do assume $\sigma > \sigma_0$ along with the other assumptions in Theorem 2, so that, in particular, inequality (1.8) holds. In what follows we write

$$(4.3) \qquad\qquad L^2(B(0,\delta)) = L_\delta^2 \quad \text{and} \quad L^\infty(B(0,\delta)) = L_\delta^\infty$$

Fixing $m \geq m_0 + 2$, and letting $\tilde{u}_{\sigma,\delta}$ and $\tilde{u}_{\sigma,\delta,m}$ be given by (1.7) (with u_σ related to ϕ_σ via (1.3)) we note that

$$\|\tilde{u}_{\sigma,\delta,m}\|_{L_\delta^\infty} \geq \frac{\|\tilde{u}_{\sigma,\delta,m}\|_{L_\delta^2}}{\sqrt{\pi}\delta}.$$

It follows that there exists $x_0 \in B(0,\delta)$ such that

$$(4.4) \qquad\qquad |\tilde{u}_{\sigma,\delta,m}(x_0)| \geq \frac{\|\tilde{u}_{\sigma,\delta,m}\|_{L_\delta^2}}{\sqrt{\pi}\delta}.$$

Now, since $\|\tilde{u}_{\sigma,\delta,m}\|_{C^1} \leq C_{m,\delta} \|\tilde{u}_{\sigma,\delta,m}\|_{L^2}$, it follows from (4.4) that there is $r_{m,\delta} \in \mathbb{R}$, $0 < r_{m,\delta} < \delta$ (in particular, independent of σ) such that

$$(4.5) \qquad\qquad |\tilde{u}_{\sigma,\delta,m}(x)| > \frac{\|\tilde{u}_{\sigma,\delta,m}\|_{L_\delta^2}}{2\sqrt{\pi}\delta}, \qquad x \in B(x_0, r_{m,\delta}).$$

But, since $m \geq m_0 + 2$, the estimate (1.8) with $N = 0$ yields

$$(4.6) \qquad |\tilde{u}_{\sigma,\delta,m}(x)| \leq |\tilde{u}_{\sigma,\delta}(x)| + |\tilde{u}_{\sigma,\delta}(x) - \tilde{u}_{\sigma,\delta,m}(x)| \leq C_0(\delta + e^{-c\sigma})\|\tilde{u}_{\sigma,\delta}\|_{L_\delta^2} + |\tilde{u}_{\sigma,\delta}(x)|$$

and

$$(4.7) \qquad \|\tilde{u}_{\sigma,\delta}\|_{L_\delta^2} \leq \|\tilde{u}_{\sigma,\delta,m}\|_{L_\delta^2} + \|\tilde{u}_{\sigma,\delta} - \tilde{u}_{\sigma,\delta,m}\|_{L_\delta^2} \leq \|\tilde{u}_{\sigma,\delta,m}\|_{L_\delta^2} + \sqrt{\pi}\delta C_0(\delta + e^{-c\sigma})\|\tilde{u}_{\sigma,\delta}\|_{L_\delta^2}.$$

(To establish the rightmost inequality in (4.7) the relation $\|\tilde{u}_{\sigma,\delta} - \tilde{u}_{\sigma,\delta,m}\|_{L_\delta^2} \leq \sqrt{\pi}\delta\|\tilde{u}_{\sigma,\delta} - \tilde{u}_{\sigma,\delta,m}\|_{L_\delta^\infty}$ was used before the inequality (1.8) was applied.) From (4.7) we obtain

$$(4.8) \qquad \|\tilde{u}_{\sigma,\delta,m}\|_{L_\delta^2} \geq \|\tilde{u}_{\sigma,\delta}\|_{L_\delta^2} - \sqrt{\pi}\delta C_0(\delta + e^{-c\sigma})\|\tilde{u}_{\sigma,\delta}\|_{L_\delta^2}.$$

It follows from (4.5), (4.6) and (4.8) that
$$(4.9)$$
$$C_0(\delta + e^{-c\sigma})\|\tilde{u}_{\sigma,\delta}\|_{L_\delta^2} + |\tilde{u}_{\sigma,\delta}(x)| > \frac{\|\tilde{u}_{\sigma,\delta,m}\|_{L_\delta^2}}{2\sqrt{\pi}\delta} \geq \frac{\|\tilde{u}_{\sigma,\delta}\|_{L_\delta^2}}{2\sqrt{\pi}\delta} - \frac{C_0}{2}(\delta + e^{-c\sigma})\|\tilde{u}_{\sigma,\delta}\|_{L_\delta^2}, \quad x \in B(x_0, r_{m,\delta}),$$

and, therefore

$$(4.10) \qquad |\tilde{u}_{\sigma,\delta}(x)| > \|\tilde{u}_{\sigma,\delta}\|_{L_\delta^2}\left(\frac{1}{2\sqrt{\pi}\delta} - \frac{3C_0}{2}(\delta + e^{-c\sigma})\right), \qquad x \in B(x_0, r_{m,\delta}).$$

Taking δ_1 sufficiently small and $\delta \leq \delta_1$ the inequality

$$\frac{3C_0}{2}(\delta + e^{-c\sigma_0}) < \frac{1}{2\sqrt{\pi}\delta}$$

holds, and it therefore follows that for a certain constant $D > 0$ we have

$$|\tilde{u}_{\sigma,\delta}(x)| > \frac{D\|\tilde{u}_{\sigma,\delta}\|_{L_\delta^2}}{\delta} \quad \text{for} \quad x \in B(x_0, r_{m,\delta})$$

provided $\delta < \delta_1$. In particular,

$$|\phi_\sigma(x)| > 0, \qquad x \in f(B(x_0, r_{m,\delta})).$$

Since the derivative of f never vanishes, for $\delta < \delta_1$ and for a certain $E > 0$ there is a ball $\mathcal{B}$ of radius $Er_{m,\delta}$ such that ϕ_σ does not vanish on $\mathcal{B}$. The proof is now complete.
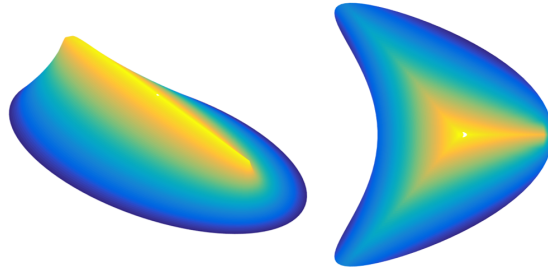
$\square$

FIGURE 3. The function λ for an ellipse (left) and a kite-shaped domain (right).

5. Numerical Formulation

5.1. **Integral representation.** Let $\Omega \subset \mathbb{R}^2$ denote a domain with, say, a C^2 boundary, and let

$$S[\phi](x) := \int_{\partial\Omega} G(x,y)\phi(y)ds(y), \quad x \in \mathbb{R}^2, \quad G(x,y) = -\frac{1}{2\pi}\log|x-y|,$$

denote the Single Layer Potential (SLP) for a given density $\phi : \partial\Omega \to \mathbb{R}$ in a certain Banach space H of functions. Both Sobolev and continuous spaces H of functions lead to well developed Fredholm theories in this context [Kre14, MM00]. It is useful to recall that, as shown e.g. in the aforementioned references, the limiting values of the potential S and its normal derivative on $\partial\Omega$ can be expressed in terms of well known "jump conditions" that involve the single and double layer boundary integral operators

$$\mathcal{S}[\phi](x) := \int_{\partial\Omega} G(x,y)\phi(y)ds(y) \quad \text{and} \quad \mathcal{T}[\phi](x) := \int_{\partial\Omega} \frac{\partial G(x,y)}{\partial\nu(x)}\phi(y)ds(y), \quad x \in \partial\Omega,$$

respectively.

In view of the jump conditions for the SLP [Kre14], use of the representation

$$(5.1) \qquad\qquad u(x) = S[\phi](x), \quad x \in \Omega,$$

for the eigenfunction u, the Steklov boundary condition in equation (1.1) gives rise to the generalized eigenvalue problem

$$(5.2) \qquad\qquad (\frac{1}{2}\mathcal{I} + \mathcal{T})[\phi] = \sigma\,\mathcal{S}[\phi] \quad \text{for} \quad x \in \partial\Omega.$$

Unfortunately, however, the single layer operator $\mathcal{S}$ on the right side of this equation is not always invertible. In order to avoid singular right-hand sides and the associated potential sensitivity to round-off errors, in what follows we utilize the Kress potential

$$(5.3) \qquad u(x) = S_0[\phi](x) = \int_{\partial\Omega} G(x,y)\left(\phi(y) - \overline{\phi}\right)ds(y) + \overline{\phi}, \quad x \in \Omega$$

(where $\overline{\phi}$ denotes the average of ϕ over $\partial\Omega$), which leads to the modified eigenvalue equation [Akh16]

$$(5.4) \qquad (\frac{1}{2}\mathcal{I} + \mathcal{T})[\phi - \overline{\phi}] = \sigma\left(\mathcal{S}[\phi - \overline{\phi}] + \overline{\phi}\right) \quad \text{for} \quad x \in \partial\Omega.$$

The right-hand operator in this equation is invertible [Kre14, Thm. 7.41], as desired. For either formulation, the evaluation of a given eigenfunction u requires evaluation of the SLP, in accordance with either (5.1) or (5.3), for the solution ϕ of the corresponding generalized eigenvalue problem (5.2) or (5.4), respectively, at all required points $x \in \Omega$.

REMARK **5.1.** Note that for a given harmonic function u in Ω, ϕ in (5.2) and that in (5.4) are *not* the same.

5.2. **Fourier expansion and exponential decay.** In terms of a given 2π-periodic parametrization $C(t)$ of $\partial\Omega$, the Steklov eigenfunction u corresponding to a given solution (ϕ, σ) of the regularized eigenvalue problem (5.4), which is given by the single layer expression (5.3), can be expressed, for a given point $x = (x_1, x_2) \in \Omega$,

$$(5.5) \qquad u(x_1, x_2) = \overline{\phi} + \frac{1}{4\pi}\int_0^{2\pi} \log\left[(x_1 - C_1(t))^2 + (x_2 - C_2(t))^2\right]\left[\phi\left(C(t)\right) - \overline{\phi}\right]\left|\dot{C}(t)\right|dt,$$

where $C(t) = (C_1(t), C_2(t))$ and where $\overline{\phi}$ denotes the average of ϕ over the curve $\partial\Omega$. Unfortunately, a direct use of this expression does not capture important elements in the eigenfunction within Ω,

such as the nodal sets, since, for analytic domains, the eigenfunctions decay exponentially fast within Ω as the frequency increases [PST19, GT19]. In regions where the actual values of the eigenfunction may be significantly below machine precision the expression (5.5) must be inaccurate: this expression can only achieve the exponentially small values via the cancellations that occur as the solution ϕ becomes more and more oscillatory. But such cancellations cannot take place numerically below the level of machine precision. In order to capture the decay explicitly within the numerical algorithm we proceed in a manner related to the construction used in [PST19].

To accurately obtain the exponentially decaying values of the Steklov eigenfunction we proceed as follows. We first consider the Fourier expansion

$$(5.6) \qquad \left[\phi\left(C(t)\right) - \overline{\phi}\right]\left|\dot{C}(t)\right| = \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} A_n e^{int}.$$

of the product $\left[\phi(C(t)) - \overline{\phi}\right]\left|\dot{C}(t)\right|$; note that, as is easily checked, the $n = 0$ term in the Fourier expansion (5.6) is indeed equal to zero. Inserting this expansion in (5.5) we obtain

$$u(x_1, x_2) = \overline{\phi} + \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} A_n B_n^0(x_1, x_2), \quad \text{where}$$

$$B_n^0(x_1, x_2) = -\frac{1}{4\pi} \int_0^{2\pi} \log\left[(x_1 - C_1(t))^2 + (x_2 - C_2(t))^2\right] e^{int} dt.$$

Then, assuming an analytic boundary, as is relevant in the context of this paper, and further assuming, for simplicity, that $C(t)$ is in fact an entire function of t (as are, for example, all parametrizations $C(t)$ given by vector Fourier series containing finitely many terms), we introduce, for $x = (x_1, x_2) \in \Omega$, the quantities

$$\lambda(x) = \sup\left\{s \geq 0 : x \neq C(t + ir) \text{ for all } r \text{ with } |r| \leq s \text{ and for all } t \in [0, 2\pi]\right\}$$

and

$$(5.7) \quad B_n(x_1, x_2, s) = -\frac{1}{4\pi} \int_0^{2\pi} \log\left[(x_1 - C_1(t + is\,\mathrm{sgn}(ns)))^2 + (x_2 - C_2(t + is\,\mathrm{sgn}(ns)))^2\right] e^{int} dt.$$

Using Cauchy's Theorem for $x = (x_1, x_2) \in \Omega$ and any $s \in \mathbb{R}$ satisfying $|s| \leq \lambda(x)$, we obtain

$$(5.8) \qquad B_n^0(x_1, x_2) = e^{-|ns|} B_n(x_1, x_2, s),$$

and, thus, letting $s = \alpha\lambda(x)$ for any $\alpha \in \mathbb{R}$ satisfying $|\alpha| \leq 1$, the eigenfunction u is given by

$$(5.9) \qquad u(x_1, x_2) = \overline{\phi} + \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} A_n e^{-|n\alpha|\lambda(x_1, x_2)} B_n(x_1, x_2, \alpha\lambda(x_1, x_2))$$

LEMMA **5.2.** *There is $C > 0$ such that for all $n > 0$,*

$$|B_n(x_1, x_2, \lambda(x_1, x_2))| \leq \frac{C}{1 + |n|}.$$

Moreover, there is $c > 0$ and a sequence $\{n_k\}_{k=1}^{\infty}$ with $|n_k| \to \infty$ such that

$$(5.10) \qquad |B_{n_k}(x_1, x_2, \lambda(x_1, x_2))| \geq \frac{c}{n_k}.$$

A proof of Lemma 5.2 is given in Appendix B. It follows from Lemma 5.2 that equation (5.8) optimally captures the exponential decay of the B_n terms as $\sigma \to \infty$. Note that this setup does not capture the exponential decay of the coefficients A_n below machine precision away from $|n| \sim \sigma$, and, therefore, the accuracy of the resulting interior eigenfunction reconstructions does not exceed that accuracy level. But the function $\lambda(x_1, x_2)$ does capture the exponential decay and the geometrical character of the eigenfunction as long as the (spatially constant) coefficients A_n for low n remain above machine precision.

FIGURE 4.   Density-plots (first and third rows) and fixed-sign sets (second and forth rows) for Steklov eigenfunctions over the elliptical domain (6.1). The eigenfunctions of orders 57 and 81 demonstrate the onset of the asymptotic character. In particular, regions of asymptotically fixed size open up. (As indicated in Section 6, the Steklov problem for the ellipse does not reduce to separated variables. Assuming Conjecture 1.3 is valid, further, the Steklov problem only reduces to separation of variables for a bounded analytic and simply-connected domain Ω in case Ω equals a disc.)

For general curves $C(t)$ no closed form expressions exist for the function $\lambda(x)$, and a numerical algorithm must be used for the evaluation of this quantity, as part of a numerical implementation of the eigenfunction expression (5.9). In our implementation the function λ was evaluated via an application of Newton's method to the nonlinear equation

$$h(z) = (x_1 - C_1(z))^2 + (x_2 - C_2(z))^2 = 0.$$

Explicit expressions can be obtained for circles and ellipses, however:

(1) For a circle of radius 1:

$$\lambda(x_1, x_2) = -\log\left(\sqrt{x_1^2 + x_2^2}\right).$$

(2) For an ellipse of semiaxes $a > b$:

(5.11) $$\lambda(x_1, x_2) = \operatorname{arcosh}\left(\frac{a}{\sqrt{a^2 - b^2}}\right) - \operatorname{Re}\left\{\operatorname{arcosh}\left(\frac{x_1 + ix_2}{\sqrt{a^2 - b^2}}\right)\right\}.$$

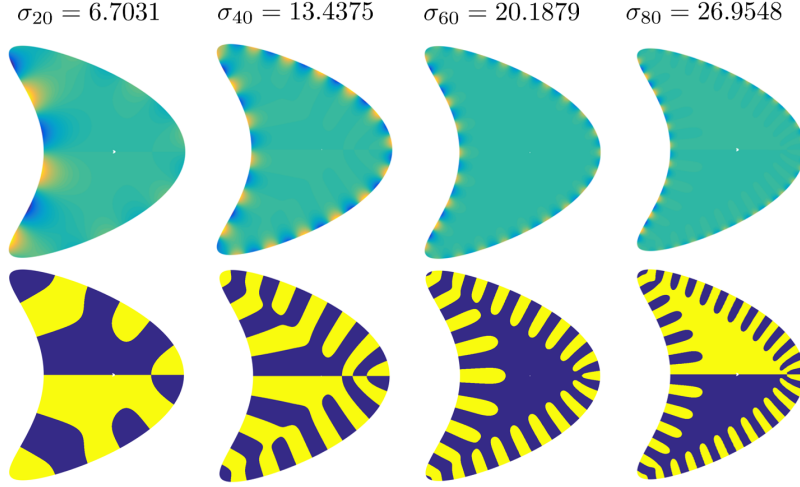The derivation of the expression (5.11) is outlined in Appendix A.

$$\sigma_{20} = 6.7031 \qquad \sigma_{40} = 13.4375 \qquad \sigma_{60} = 20.1879 \qquad \sigma_{80} = 26.9548$$

FIGURE 5. Density-plots (first row) and fixed-sign sets (second row) for Steklov eigenfunctions over the kite-shaped domain (6.2).

5.3. **Exponential decay and verification of Cauchy's theorem.** Tables 1 and 2 demonstrate the validity of equation (5.8) (since in both cases the results in the second and third columns closely agree with each other for $n \leq 50$), as well as the exponential decay of the exact coefficients B_n^0—as born by the results in the third column of these tables. The disagreement observed for $n > 50$ is caused by the lack of precision of the results in the second column beyond machine accuracy, a problem that is eliminated in the third column via an application of the relation (5.8).

n	$|B_n^0(x_1, x_2)|$	$|e^{-n0.8\lambda} B_n(x_1, x_2, 0.8\lambda)|$	Absolute B_n^0 error	Relative B_n^0 error
1	5.62e-03	5.62e-03	3.82e-16	6.79e-14
10	2.29e-06	2.29e-06	4.39e-17	1.91e-11
50	6.40e-16	6.57e-16	3.85e-17	5.86e-02
100	3.05e-17	1.30e-28	3.05e-17	2.35e+11
150	1.33e-16	5.95e-41	1.33e-16	2.23e+24
200	2.65e-16	6.58e-53	2.65e-16	4.02e+36

TABLE 1. Verification of the Cauchy-theorem-based identity (5.8) for the domain Ω bounded by the elliptical curve (6.1) with $a = 2$ and $b = 1$.

n	$|B_n^0(x_1, x_2)|$	$|e^{-n0.8\lambda} B_n(x_1, x_2, 0.8\lambda)|$	Absolute B_n^0 error	Relative B_n^0 error
1	5.83e-03	5.83e-03	4.25e-16	7.29e-14
10	5.97e-06	5.97e-06	7.18e-18	1.20e-12
50	2.33e-14	2.34e-14	3.32e-17	1.42e-03
100	1.14e-16	3.05e-25	1.14e-16	3.75e+08
150	1.27e-16	6.78e-36	1.27e-16	1.88e+19
200	2.42e-16	3.05e-45	2.42e-16	7.93e+28

TABLE 2. Same as Figure (1) but for the kite-shaped domain Ω bounded by the curve (6.2).

6. Numerical Results

Figures 4 and 5 present density plots and fixed-sign sets for Steklov eigenfunctions over domains bounded by the elliptical and kite-shaped curves parametrized by the vector functions

$$(6.1) \qquad\qquad C(t) = ((a\cos(t), b\sin(t)) \quad (0 \le t < 2\pi)$$

with $a = 2$ and $b = 1$, and

$$(6.2) \qquad\qquad C(t) = (\cos(t) + 0.65\cos(2t) - 0.65, 1.5\sin(t)) \quad (0 \le t < 2\pi),$$

respectively. These figures demonstrate, in particular, domain-opening and non-density of nodal sets as discussed in Remark 1.2.

As suggested in the caption to Figure 4, the question might be considered as to whether the Steklov problem for the ellipse can be reduced to solution of decoupled separated-variables one-dimensional problems in elliptical coordinates. As mentioned in that caption, however, a simple calculation shows that the Steklov problem does *not* separate in those variables. Note also that the observed domain-opening and non-density of nodal sets are inconsistent with separation of variables—since, if variables separate, a simple eigenfunction can be written as $u_\sigma = G(\mu)T(\tau)$, $\mu \in [0, \mu_0]$ $\tau \in [0, 2\pi)$ where μ and τ are as in Appendix A. But, since $T(\tau)$ must have $1/\sigma$ dense zeros set and $|G(\mu_0)| > 0$, we obtain that the nodal set is $1/\sigma$ dense throughout the ellipse which is indeed inconsistent with the fixed sign domains presented in the figure. An analogous argument can be used for high energy eigenfunctions, and for an arbitrary domain for which reducibility to separation of variables is assumed, even in cases involving multiple eigenvalues, once the work of Shamma [Sha71] describing the asymptotic structure of Steklov eigenfunctions is incorporated. Thus, any bounded analytic and simply-connected domain Ω for which the Steklov eigenproblem can be reduced to problems of separated variables cannot exhibit asymptotic domain-openings. It follows that any such domain Ω cannot be a tunneling domain and must thus equal a disc—at least if Conjecture 1.3 holds.

Appendix A. Function $\lambda(x)$ For An Ellipse Of Semiaxes $a > b$

Let $\gamma = \sqrt{a^2 - b^2}$ and $\mu_0 = \operatorname{arcosh}(a/\gamma)$. Using elliptical coordinates with foci $(\pm\gamma, 0)$ to represent the point $x = (x_1, x_2)$, so that $x_1 = \gamma\cosh(\mu)\cos(\tau)$ and $x_2 = \gamma\sinh(\mu)\sin(\tau)$, and letting the boundary of the ellipse be given by $C_1(t) = \gamma\cosh(\mu_0)\cos(t)$, $C_2(t) = \gamma\sinh(\mu_0)\sin(t)$,

in view of the relations $x_1 + ix_2 = \gamma \cosh(\mu + i\tau)$ and $C_1(t) + iC_2(t) = \gamma \cosh(\mu_0 + it)$ we obtain

$$(x_1 - C_1(t + is))^2 + (x_2 - C_2(t + is))^2 = \gamma^2 \left| \cosh(\mu + i\tau) - \cosh(\mu_0 + i(t + is)) \right|^2$$

(A.1)
$$= 4\gamma^2 \left| \sinh \frac{\mu + \mu_0 + i(\tau + (t + is))}{2} \right|^2 \left| \sinh \frac{\mu - \mu_0 + i(\tau - (t + is))}{2} \right|^2.$$

It follows that the left-hand side of this equation vanishes for some value of t if and only if either $s = (\mu_0 - \mu)$ or $s = (\mu_0 + \mu)$. Thus, $\lambda(x)$ equals the smallest of these two positive numbers, namely $\lambda(x) = (\mu_0 - \mu)$, which is equivalent to the desired relation (5.11).

APPENDIX B. PROOF OF LEMMA 5.2

First, let

$$h(z, x_1, x_2) := (x_1 - C_1(z))^2 + (x_2 - C_2(z))^2.$$

Then, for $|\mathrm{Im}\, z| < \lambda(x_1, x_2)$, the expression

$$\log h(z) := \int_0^z \frac{h'(s)}{h(s)} ds + \log h(0)$$

defines the principal branch of $\log h(z)$—which is, then, an analytic function in the strip $|\mathrm{Im}\, z| < \lambda$. On $\pm \mathrm{Im}\, z = \lambda$, we define

$$\log h(z) := \lim_{\varepsilon \to 0^+} h(z \mp i\varepsilon).$$

LEMMA **B.1.** *Let $h(z)$ denote an analytic function defined on an open neighborhood of the set $\{z : |\mathrm{Im}\, z| \leq \lambda\}$ which does not vanish for $|\mathrm{Im}\, z| < \lambda$, but which vanishes to order k at $z_0 = t_0 + i\lambda$. Then,*

$$\lim_{\varepsilon_1 \to 0^+} \mathrm{Im}\, \log h(z_0 + \varepsilon_1) - \lim_{\varepsilon_2 \to 0^+} \mathrm{Im}\, h(z_0 - \varepsilon_2) = k\pi.$$

Similarly, if h vanishes to order k at $z_0 = t_0 - i\lambda$,

$$\lim_{\varepsilon_1 \to 0^+} \mathrm{Im}\, \log h(z_0 + \varepsilon_1) - \lim_{\varepsilon_2 \to 0^+} \mathrm{Im}\, h(z_0 - \varepsilon_2) = -k\pi.$$

Proof. Note that for $\varepsilon > 0$ small enough $\{h(z) = 0\} \cap \{|z - z_0| < \varepsilon\} = z_0$. Therefore

$$\log h(z_0 + \varepsilon_1) - \log h(z_0 - \varepsilon_2) = \int_\Gamma \frac{h'(z)}{h(z)} dz$$

where Γ is any contour starting at $z_0 - \varepsilon_2$, ending at $z_0 + \varepsilon_1$, and lying in

$$\{\mathrm{Im}\, z \leq \lambda\} \cap B(z_0, \varepsilon).$$

In particular, let

$$\Gamma_1 = \{z_0 + \varepsilon_2 e^{it} \mid t \in [\pi, 2\pi]\}, \qquad \Gamma_2 := \{z_0 + (1 - t)\varepsilon_2 + t\varepsilon_1\}$$

and $\Gamma = \Gamma_1 \cup \Gamma_2$. Then, since

$$\frac{h'(z)}{h(z)} = \frac{k}{z - z_0}(1 + O(|z - z_0|)),$$

$$\log h(z_0 + \varepsilon_1) - \log h(z_0 - \varepsilon_2) = k\pi i + \log \varepsilon_1 - \log \varepsilon_2 + O(|\varepsilon_1 - \varepsilon_2|) + O(\varepsilon_2)$$

Letting ε_1 and ε_2 tend to zero completes the proof for the case $z_0 = t_0 + i\lambda$. The proof for $z_0 = t_0 - i\lambda$ follows by substituting z by $-z$. $\qquad \square$

LEMMA **B.2.** *Let $h(z, x_1, x_2)$ denote an analytic function on $|\operatorname{Im} z| \leq \lambda$ which vanishes to order k at $z_0 = t_0 + i\lambda$. Then for $\chi \in C_c^\infty(S^1)$ supported in a sufficiently small neighborhood of t_0, with $\chi \equiv 1$ near t_0, we have*

$$\int_{S^1} \chi(t) \log h(t + i\lambda) e^{int} dt = -\frac{2\pi k}{|n|} e^{int_0} + O(n^{-2}) \quad \text{for } n > 0.$$

Similarly if h vanishes to order k at $z_0 = t_0 - i\lambda$, we have

$$\int_{S^1} \chi(t) \log h(t - i\lambda) e^{int} dt = -\frac{2\pi k}{|n|} e^{int_0} + O(n^{-2}) \quad \text{for } n < 0.$$

Proof. We consider the first case, the second follows similarly.

Selecting $\chi(t)$ with sufficiently small support we ensure that, within the support of χ, $h(t + i\lambda)$ vanishes only at $t = t_0$. We then have

(B.1) $$\int \chi(t) \log[h(t + i\lambda)] e^{int} dt = \int \chi(t) \left(\log |h(t + i\lambda)| + i\operatorname{Im} \log[h(t + i\lambda)] \right) e^{int} dt$$

and

(B.2) $$\int \chi(t) \log |h(t + i\lambda)| e^{int} dt = \int \chi(t) \left(k \log |t - t_0| + \log |t - t_0|^{-k} |h(t + i\lambda)| \right) e^{int} dt.$$

Since $|t - t_0|^{-k} |h(t + i\lambda)|$ is smooth and bounded away from zero on the support of χ, the second term in (B.2) is $O(n^{-\infty})$.

Taking real parts in the asymptotic formula [BO99, p. 381] we obtain

(B.3) $$\int_{-1}^1 \log |t| e^{ixt} dt = -\frac{\pi}{|x|} + O(x^{-2}), \qquad x \to \infty.$$

Then, using (B.3) together with the fact that $\log 1 = 0$ we may approximate the first term on the right-hand side of (B.2) by

$$\int \chi(t) \log |h(t + i\lambda)| e^{int} dt = -\pi k e^{int_0} \frac{1}{|n|} + O(n^{-2}),$$

Let us now estimate the second term on the right-hand side of (B.1). We have

$$\int \chi(t) i \operatorname{Im} \log[h(t + i\lambda))] e^{int} dt$$

$$= \int_0^{t_0} i\chi(t) \operatorname{Im} \log[h(t + i\lambda))] e^{int} dt + \int_{t_0}^{2\pi} i\chi(t) \operatorname{Im} \log[h(t + i\lambda))] e^{int} dt$$

$$= -n^{-1} \left(\int_0^{t_0} \partial_t (\chi(t) \operatorname{Im} \log[h(t + i\lambda)]) e^{int} dt + \int_{t_0}^{2\pi} \partial_t (\chi(t) \operatorname{Im} \log[h(+ - i\lambda)]) e^{int} dt \right)$$

$$\quad - n^{-1} (e^{int_0} (\lim_{t \to t_0^+} \operatorname{Im} \log[h(t + i\lambda)]) - \lim_{t \to t_0^-} \operatorname{Im} \log[h(t + i\lambda)])$$

$$= -n^{-1} (e^{int_0} (\lim_{t \to t_0^+} \operatorname{Im} \log[h(t + i\lambda)]) - \lim_{t \to t_0^-} \operatorname{Im} \log[h(t + i\lambda)]) + O(n^{-2})$$

$$= -k\pi n^{-1} e^{int_0} + O(|n|^{-2})$$

where in the last equality Lemma B.1 was used. $\square$

We may now complete the proof of Lemma 5.2. Let $0 \leq t_1 < t_2 < \cdots < t_M < 2\pi$ denote the zeroes of $h(t + i\lambda)$ as a function of t, and let k_j $(0 \leq j \leq M)$ denote the vanishing order at $t = t_j$.

Then, by Lemma B.2, for χ_j supported close enough to t_j with $\chi_j \equiv 1$ near t_j, and $n > 0$,

$$\int \chi_j(t) \log h(t + i\lambda) e^{int} dt = -\frac{2\pi k_j e^{int_j}}{|n|} + O(n^{-2}).$$

By shrinking the support of χ_j, we may assume that $\text{supp } \chi_j \cap \chi_\ell = \emptyset$ for $\ell \neq j$. Then, since $\chi_j \equiv 1$ near t_j, $(1 - \sum_j \chi_j(t))) \log h(t + i\lambda) \in C^\infty(S^1)$ and hence

$$\int (1 - \sum_j \chi_j(t))) \log h(t + i\lambda) e^{int} dt = O(n^{-\infty}).$$

Thus in view of equation (5.7) we obtain

$$B_n(x_1, x_2, \lambda(x_1, x_2)) = \int \log h(t + i\lambda) e^{int} dt = -\frac{2\pi}{|n|} \sum_{j=1}^{M} k_j e^{int_j} + O(n^{-2})$$

Proceeding by contradiction, assume that

(B.4) $$\limsup_{n \to +\infty} n |B_n(x_1, x_2, \lambda(x_1, x_2))| = 0.$$

Then in particular,

$$\lim_{n \to +\infty} \sum_{j=1}^{M} k_j e^{int_j} = 0.$$

But we note that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \left| \sum_{j=1}^{M} k_j e^{int_j} \right|^2 = \sum_{j=1}^{M} k_j^2 + \lim_{N \to \infty} \frac{1}{N} \sum_{j \neq \ell} \sum_{n=0}^{N-1} k_j k_\ell e^{in(t_j - t_\ell)}$$

$$= \sum_{j=1}^{M} k_j^2 + \lim_{N \to \infty} \frac{1}{N} \sum_{j \neq \ell} k_j k_\ell \frac{1 - e^{iN(t_j - t_\ell)}}{1 - e^{i(t_j - t_\ell)}} = \sum_{j=1}^{M} k_j^2 > 0.$$

Recalling that

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} a_n \leq \limsup_{n \to \infty} a_n$$

we obtain

$$\lim_{n \to \infty} \sum_{j=1}^{M} k_j e^{int_j} \neq 0.$$

which contradicts (B.4).

If $h(t+i\lambda)$ does not vanish anywhere, then $h(t-i\lambda)$ vanishes at some $0 \leq t_1 < t_2 < \cdots < t_M < 2\pi$ and we may repeat the argument this time considering

$$B_n(x_1, x_2, \lambda(x_1, x_2)) = \int \log h(t - i\lambda) e^{int} dt, \qquad n < 0.$$

and taking the limit as $n \to -\infty$.

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