Maximization of Robustness of Interdependent Networks under Budget Constraints

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Abstract—We consider the problem of interlink optimization in multilayer interdependent networks under cost constraints, with the objective of maximizing the robustness of the network against component (node) failures. Diverting from the popular approaches of branching process based analysis of the failure cascades or using a supra-adjacency matrix representation of the multilayer network and employing classical metrics, in this work, we present a surrogate metric based framework for constructing interlinks to maximize the network robustness. In particular, we focus on three representative mechanisms of failure propagation, namely, connected component based cascading failure [1], load distribution in interdependent networks [2], and connectivity in demand-supply networks [3], and propose metrics to track the network robustness for each of these mechanisms. Owing to their mathematical tractability, these metrics allow us to optimize the interlink structure to enhance robustness. Furthermore, we are able to introduce the cost of construction into the interlink design problem, a practical feature largely ignored in relevant literature. We simulate the failure cascades on real world networks to compare the performance of our interlinking strategies with the state of the art heuristics and demonstrate their effectiveness.

Index Terms—Multilayer failure propagation, cost constrained optimization, network robustness, interdependent networks.

1 Introduction

Interdependent networks comprise multiple network layers, where the components of different layers are dependent on each other for their functioning [4, 5]. Over the past few years, interdependent networks have emerged as a topic of broad and current interest due to their applications in diverse areas, for instance, smart grids for power distribution [1], with interdependent network layers of power stations and communication routers; infrastructure design to control traffic congestion [6] in multi-modal transportation systems, where the different modes of traffic constitute the network layers; and recommendation systems [7], where various social networks in which the users are active, like Facebook and Twitter, are considered as the interdependent layers. The fundamental phenomenon distinguishing multilayer networks is the back-and-forth propagation of failures across the interlinks between the network layers, popularly named cascading failure. This recursive cascade of failures can increase the susceptibility of interdependent networks to node failures compared to their isolated constituents. Our objective in this work is to optimize the design of the interlinking structure between isolated network layers, maximizing the robustness of the resulting interdependent network against failure cascades.

1.1 Motivation

Research interest in interdependent networks is largely attributed to the seminal paper by Buldyrev et al. [1], where the authors study a connected component based mechanism of failure propagation in a two-layer interdependent network. Subsequently, the failure cascades corresponding to this mechanism have been studied on more general network structures [8, 9]. These works extend classical results from branching processes [10] and percolation theory [11] to multilayer interdependent networks, revealing the relationship between the network robustness and the degree distribution of the constituent nodes. To the best of our knowledge, this is the only rigorous model that can track the failure cascades in multilayer networks. This branching process (BP) based framework relies on several assumptions. In particular, the system equations tracking the robustness are exact for infinite trees and only node degree information is considered, i.e. nodes of the same degree are considered to be statistically equivalent. Although simple graph generators, like the Erdos-Renyi model, lead to locally tree-like networks, i.e. the local neighborhood of a node has a loop with vanishing probability, many topologies encountered in the real world can be dramatically different. Complex topological features, like clustering, community structure, etc., have been shown to cause non-trivial effects on the network dynamics. Modification of the classical BP framework to account for non tree-like structures is an area of active research [12, 13, 14]. However, extending these works to devise multilayer network design strategies is challenging due to the complexity of the mathematical equations tracking the robustness. It is important to note that even after the topological simplifications and degree-based characterization of nodes, the BP approach leads to self-consistent equations of robustness, which typically do not admit closed form solutions and are usually solved numerically. This lack of amenability to mathematical analysis severely restricts the applicability of the BP approach to practical network design problems involving large networks and complex topologies. In addition to the connected component based mechanism of failure propagation, other mechanisms of failure propagation have also been proposed in literature. However, studying these complex mechanisms by the BP approach is a laborious task. Some recent works [2] have made considerable progress in this area but the practicality of such results from the perspective of devising network

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design strategies is debatable, owing to the computationally expensive iterative computation of the robustness. We argue that while a rigorous modeling of the failure cascades in multilayer networks is of significant theoretical importance, it is unclear whether pursuing this avenue of research will lead to practical network design guidelines.

We can conclude from the above discussion that the computation of robustness via an exact characterization of the failure cascades is not possible for most practical applications. To circumvent this problem, researchers have used various network properties as indirect measures of robustness, for example, algebraic connectivity [15], number of spanning trees [16], and total effective resistance [17]. For the robustness of networks against component failures, most works [15, 18, 19] adopt the algebraic connectivity metric, defined as the second smallest eigenvalue of the supra-adjacency (SA) matrix representation of the multilayer network, where all links (inter- and intra-layer) in the multilayer network are stacked together: diagonal (off-diagonal) blocks indicating interactions inside (across) layers. The algebraic connectivity (λ_2) of a network is closely related to its partitioning and synchronization properties [20]. However, in the context of diverse mechanisms of failure propagation in multilayer networks, it is not clear whether λ_2 can effectively characterize the robustness. In fact, we present simulation results indicating that such metrics *cannot* capture the robustness for the mechanisms of failure propagation considered here. Additionally, the metrics based on the SA matrix are agnostic to the attack models or the propagation mechanisms. One of the most important ideas which we aim to elucidate through this paper is that the mechanism of failure propagation can significantly affect the robustness and network design guidelines do not generalize across these diverse mechanisms. This warrants the development of new metrics of robustness fine-tuned to particular mechanisms. Furthermore, these metrics should be amenable to mathematical analysis to allow us to develop network design guidelines.

To summarize, the BP approach captures the true nature of the failure cascades but is limited in its applicability due to the complexity of the self-consistent system equations, whereas the SA matrix approach leads to tractable metrics of robustness but is agnostic to the propagation mechanism. This motivates us to develop novel metrics that can characterize the robustness of multilayer networks against different mechanisms of failure propagation.

1.2 Our Contribution

The main contribution of our work is to introduce mathematically tractable metrics of robustness, which can be utilized to optimize the interlink structure of multilayer networks. As these metrics are not based on a rigorous modeling of the failure cascades, we refer to them as surrogate metrics of robustness. They were designed by studying the dynamics of the failure cascades for the different propagation mechanisms. These metrics are shown to vary monotonically with the empirical measure of robustness for both theoretical and real-world network structures. This monotonic relationship ensures that the maximization of the robustness can be achieved by the maximization of these metrics. Furthermore, the surrogate metrics enable us to consider practical design constraints with the cost of interlink construction, which has been largely overlooked in multilayer network design. The goal of this work is to optimally distribute a total budget for interlink construction, so that the robustness of the resulting interdependent network against different mechanisms of failure propagation is maximized. For three representative mechanisms, namely, i) connected component based cascading failure, ii) load distribution in interdependent networks, and iii) connectivity in demand-supply networks, we define our surrogate metrics and pose the design of interlinks as a convex optimization problem. Due to the tractable metrics, this optimization problem can be readily solved to obtain the budget allocation maximizing the proxy measures of robustness. Through extensive simulation experiments on real-world networks, we show that the surrogate metric based interlink design outperforms the state of the art heuristics.

The remainder of this paper is organized as follows. Section 2 introduces the preliminaries for this study. Section 3 contains the main contribution, including the deficiency of existing approaches, the surrogate metric based framework for optimizing interlinks, and an algorithm for learning the optimal allocation of resource distributively. We compare our interlinking strategies with the existing heuristics in Section 4. Finally, we conclude our work in Section 5 and indicate future avenues of research.

2 Preliminaries

In this section, we introduce our system model and discuss existing approaches for characterizing the robustness of multilayer networks. Finally, we describe the three representative mechanisms of failure propagation considered in this work.

2.1 System Model

Let us represent a network layer by $G=(V,E)$, where V and $E \subseteq V \times V$ are the set of nodes and intra-layer (undirected) edges, respectively. We consider a two-layer interdependent network $\mathcal{G}=(G_A, G_B, \tilde{E})$ where $G_A=(V_A, E_A)$ and $G_B=(V_B, E_B)$ are the constituent layers with $|V_A|=m$ and $|V_B|=n$. Our control variable $\tilde{E} \subseteq V_A \times V_B$ represents the weighted interdependence structure between G_A and G_B. We assume that the cost structure, explained next, and the isolated network topologies (G_A, G_B) are known and our objective is to design the inter-layer links ($\tilde{E}$) to maximize the robustness of $\mathcal{G}$.

Most practical network design problems are constrained by the availability of resource, where heterogeneous costs are associated with the construction of different links. Designers strive for the optimal allocation of resources for interlink construction under such cost constraints. Resource constraints have received limited attention in network design literature, although simplified cost structures [18], limiting the total number of interlinks, have been studied. We consider a general cost structure given by:

$$w_l = r(x_l), \tag{1}$$

where $l \in \{1, 2, \cdots, mn\}$ indexes the interlinks, and $r : \mathbb{R}^+ \to \mathbb{R}^+$ is the mapping from the allocated resource (x_l) to the interlink weight (w_l). We assume r to be increasing, concave and differentiable with respect to (w.r.t.) x_l with the additional constraint that $r(0) = 0$. It is not difficult to see that these assumptions occur frequently in practice and are also supported by economic theories, like diminishing returns and marginal utility. Our interlink optimization framework is applicable for any r satisfying these criteria. We provide results for two specific choices of r to serve as examples. Note that the interlink weights have been defined as unbounded positive real numbers. The physical interpretation of the weight can differ for various mechanisms of failure propagation and for particular cases, bounded weights might be necessary. The cost structure r is defined accordingly for such cases.

Given a total budget b, our goal is to optimally distribute it among all possible interlinks, maximizing the robustness of the constructed interdependent network. The vector $\mathbf{x} = [x_1 \cdots x_{mn}]^T \in \mathbb{R}^{mn\times 1}$ denotes the resource allocation strategy, specifying the resource allocated to all mn interlinks. The cost structure (1) maps the allocated resource $\mathbf{x}$ to the interlink structure $\tilde{E}(\mathbf{x})$. Let $\mathcal{G}(\mathbf{x})$ denote the resulting interdependent network, where $\mathcal{G}(\mathbf{x}) \triangleq (G_A, G_B, \tilde{E}(\mathbf{x}))$. Robustness can be intuitively understood as the resilience of the network against node failures, although exact definitions might vary across applications. Let $\psi^{\mathcal{M}}(\mathcal{G}(\mathbf{x}), \eta)$ denote the robustness of $\mathcal{G}(\mathbf{x})$ against cascading failures, where $\mathcal{M}$ specifies the failure propagation mechanism and η denotes the infection strength, defined as the fractional size of the nodes comprising the initial infection. The choice of the failure seeds is dependent on the attack model: the randomized attack arbitrarily chooses nodes, whereas the targeted attack preferentially chooses nodes based on some topological measure of centrality. Recent works [21] have identified the targeted attack based on betweenness as an extremely disruptive strategy. Due to this reason, we consider the betweenness-based targeted attack model for the results presented here, where an attack strength of η implies that η fraction of nodes with the highest betweenness centrality measure comprise the infection seeds. In addition to this, we also study the degree-based targeted attack and the randomized attack models. The results corresponding to these two attack models are presented in the supplementary material. Adhering to classical models [1], the initial infection is assumed to occur in layer A and the robustness is defined by the surviving fraction of nodes in layer B. Similar to works like [1, 22], we define the robustness of the multilayer network as follows.

Definition 2.1. *The robustness of an interdependent network $\mathcal{G}(\mathbf{x}) = (G_A, G_B, \tilde{E}(\mathbf{x}))$ against an attack strength η for a failure propagation mechanism $\mathcal{M}$, is defined as the fraction of nodes in layer B which survive the failure cascades triggered by the removal of η fraction of nodes from layer A. We represent this robustness by $\psi^{\mathcal{M}}(\mathcal{G}(\mathbf{x}), \eta)$.*

2.2 Existing Approaches

2.2.1 Branching Process (BP)

Extending classical works in branching processes [10] and percolation theory [11], researchers have been able to track the cascade of failures among the interdependent layers in multilayer networks. These works present equations mapping the degree distribution of the nodes of $\mathcal{G}(\mathbf{x})$ to the network robustness $\psi^{\mathcal{M}_1}(\mathcal{G}(\mathbf{x}), \eta)$ for the connected component based cascading failure mechanism, represented by $\mathcal{M}_1$. Under simplistic assumptions, such as, locally tree-like topology, infinite-sized networks, arbitrary interlinking and the randomized attack [1, 23, 24], the robustness is related to the degree distribution by inter-coupled self-consistent equations of the form:

$$\psi_A(\mathcal{G}, \eta) = \eta \cdot f_1\Big(\psi_A(\mathcal{G}, \eta), \psi_B(\mathcal{G}, \eta), \mathbf{d}_A, \mathbf{d}_B\Big) \qquad (2)$$

$$\psi_B(\mathcal{G}, \eta) = f_2\Big(\psi_A(\mathcal{G}, \eta), \psi_B(\mathcal{G}, \eta), \mathbf{d}_A, \mathbf{d}_B\Big), \qquad (3)$$

where ψ_A and ψ_B are the fractional size of surviving nodes in the two interdependent layers, $\mathbf{d}_A$ and $\mathbf{d}_B$ are the degree distribution of the two layers, and η denotes the attack strength. f_1 and f_2 are inter-coupled functions, whose details can be obtained from the references. The multiplicative factor η appears only in (2) due to the attack strategy, which only affects layer A. Although (2)-(3) can be solved for simple topologies, an iterative solution is required for most real-world applications. Thus, even under the simplified settings, the BP approach involves numerical computation of the robustness, requiring global information about the network topology and the failure spreading dynamics. When realistic conditions, like weighted interlinks and heterogeneous costs associated with interlink construction, and more advanced problems, like the optimization of the interlink structure, are considered, pursuing the BP approach does not seem promising.

2.2.2 Supra-Adjacency Matrix (SA)

The mathematical complexity of the BP approach resulted in the development of various metrics, which can approximately characterize the robustness of networks. These metrics of robustness utilize the supra-adjacency matrix [19] representation. The SA matrix for an interdependent network $\mathcal{G}(\mathbf{x}) = (G_A, G_B, \tilde{E}(\mathbf{x}))$ is defined as:

$$\mathcal{A} = \begin{bmatrix} (A_1)_{m\times m} & \tilde{E}(\mathbf{x}) \\ \tilde{E}(\mathbf{x})^T & (A_2)_{n\times n} \end{bmatrix}, \qquad (4)$$

where A_1 and A_2 are the adjacency matrices of the constituent layers G_A and G_B, respectively, and $\tilde{E}(\mathbf{x}) \in \mathbb{R}^{m\times n}$ is the interlink structure. In the context of robustness of networks against component failures, the most popular metric is the algebraic connectivity, defined as the second smallest (smallest non-zero) eigenvalue of the Laplacian L. The Laplacian matrix (L) of $\mathcal{A}$ is given by: $L = D - \mathcal{A}$, where $D \triangleq \text{diag}(\mathcal{A}\mathbf{1})$, here $\mathbf{1}$ is the all-one column vector, and $\text{diag}(\mathbf{v})$ is a diagonal matrix with entries from $\mathbf{v} = [v_1, v_2, \cdots, v_{m+n}]^T$. The surrogate measure of robustness of the network is expressed as $\lambda_2(L(\mathbf{x}))$. The relevance of λ_2 to the network robustness stems from its relationship to vertex and edge connectivity, uncovered in the classical work [20] and its later extensions. The impact of λ_2 on network dynamics is a mature area with decades of research studying it. However in the context of multilayer networks, we argue that the role of λ_2 as a metric of robustness is much less understood. This is because there is a fundamental difference between robustness problems on single and multilayer networks: cascading failure. In the former, robustness is a single step process, whereas the latter involves recursive propagation of failures among the interdependent network layers. Until any relationship between the algebraic connectivity of a multilayer network and its robustness is established, the use of λ_2 as a universal metric of robustness is debatable.

2.3 Failure Propagation in Interdependent Networks

We study three representative mechanisms in this work. Although this is not an exhaustive list, to the best of our knowledge, most failure spreading mechanisms considered in literature can be thought of as variations of these.

2.3.1 Connected Component based cascades ($\mathcal{M}_1$)

This is the most popular model for cascading failure in interdependent networks. Under this mechanism [1], node failures can occur in two ways: i) by not belonging to the largest connected component in its own layer, or ii) by failure propagated from the other layer(s) through the interlinks. At each stage of the cascades, the nodes isolated from the largest connected component become non-functional and these failures propagate to the other layer through the interlinks. We consider a generalized version of this model involving weighted interlinks, where link weight (w_l) represents the probability of propagation of failure across an interlink. Classical models consider $w_l = 1$, implying perfect propagation of failure among inter-layer neighbors. It is easy to see that $\mathcal{M}_1$ involves both local and long-range propagation of failure. The long-range propagation is due to the first condition, where only the constituents of the largest connected component are assumed to survive. If a node failure partitions the network into disconnected components, all but the largest component are assumed to fail according to $\mathcal{M}_1$. The local propagation arises from the second condition, where failure of a node can lead to failure of its inter-layer neighbors conditioned on the interlink weight.

2.3.2 Load Distribution in Interdependent Networks ($\mathcal{M}_2$)

Interdependent load distribution has recently attracted research interest due to its diverse applicability. In power distribution networks [25], the load of a station is offloaded to its neighbors upon failure. In multilayer transportation networks [26], a load distribution model is used to emulate multi-modal (e.g. airways and railways) traffic to study congestion, where upon failure of an airport, its traffic is redirected to neighboring airports and railway stations. Although the system models for these different applications have slight differences, the underlying principle is similar. Upon failure, the load of a node is fractionally offloaded to its neighbors, proportional to the link weight. A node fails when its current load, comprising the initial and the offloaded load, exceeds its capacity. Following the model in [2], we restrict $\mathcal{M}_2$ to local propagation, where the load of a node is offloaded only to its intra- and inter-layer neighbors upon failure. Note that this restriction only applies to this mechanism. $\mathcal{M}_1$ and $\mathcal{M}_3$ involve both local and long-range propagation of failures. Depending on the particular application, the load of a node in a network can be defined in many ways. It is popularly defined in terms of some topological information, such as degree [2, 27, 28] or betweenness [29, 30]. We proceed with the degree-based definition in this work. Due to the lack of works studying load distribution in multilayer networks with weighted interlinks, we extend the single layer model in [2] to multiple layers using ideas from [31], which considers a simplified model involving uniform distribution of the inter-layer offloaded load. We present the mathematical details of $\mathcal{M}_2$ in the supplementary material.

2.3.3 Connectivity in Demand-Supply Networks ($\mathcal{M}_3$)

Interdependent demand-supply networks have recently come to the attention of researchers [3]. These networks comprise a supply and a demand layer, where supply nodes feed the demand nodes to maintain their functionality. Let the supply rate of node $a \in V_A$ be denoted by s_a. Let the survival threshold of the demand node $b \in V_B$, defined as the minimum supply required for its functioning, be denoted by t_b. A demand node b survives if the following conditions hold: i) total supply available at b exceeds the threshold, i.e. $\sum_a w_{ab} s_a \geq t_b$, where w_{ab} is the weight of the interlink between a and b; and ii) b belongs to the largest connected component in the demand layer G_B. [3] studies a simplified version of this model, where w_{ab} is binary and $s_a = t_b = 1, \forall a, b$. We model the supply rate of each node (s_a) to be equal to its intra-layer degree. Note that $\mathcal{M}_3$ involves unidirectional dependence, since the supply nodes are not affected by the failed demand nodes. This is in contrast to the previous cases with back-and-forth propagation of failures. This difference in the interlink functionality leads to distinct properties, discussed in Section 4, that are not observed for $\mathcal{M}_1$ and $\mathcal{M}_2$.

It is easy to see from the above discussion that mechanisms of failure propagation can be dramatically different from each other. Even for multilayer networks with identical intra-later topology, interlink structure and initial infection seeds, distinct mechanisms can lead to significant differences in the fraction of nodes surviving in the steady state. We have illustrated this for a toy example in the supplementary material. Since the illustration requires a detailed description of the propagation mechanisms, we do not include it in the main document for brevity (see Appendix A). The main idea we want to highlight is that mechanisms of failure propagation strongly affect the robustness of multilayer networks and thus, effective metrics of robustness should be dependent on the particular mechanisms. In other words, a universal metric, capable of characterizing the robustness of networks against different mechanisms of failure propagation while easy to work with, does not exist.

3 Mechanism based interlink optimization

We start this section by formally defining the cost constrained robustness maximization problem. Next, we discuss the deficiency of the existing approaches for characterizing the robustness. Finally, we introduce our surrogate metrics and utilize them to solve the maximization problem to obtain the optimal allocation of resources for the construction of interlinks. We also present an algorithm for learning this resource allocation strategy distributively.

3.1 Problem Statement

The interlink optimization problem can be written as:

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \quad & \psi^{\mathcal{M}}(\mathcal{G}(\mathbf{x}), \eta) \\ \text{subject to} \quad & \mathbf{1}^T \mathbf{x} = b, \ \text{and} \ \mathbf{x} \succeq 0, \end{aligned} \tag{5}$$

where $\mathbf{x} \in \mathbb{R}^{mn \times 1}$ denotes a resource allocation strategy and b is the total available budget. $\psi^{\mathcal{M}}(\mathcal{G}(\mathbf{x}), \eta)$ represents the robustness of $\mathcal{G}(\mathbf{x}) = (G_A, G_B, \tilde{E}(\mathbf{x}))$ against an initial failure of strength η for a failure propagation mechanism $\mathcal{M}$. The mapping between $\mathbf{x}$ and $\tilde{E}(\mathbf{x})$ is given by the cost structure r following (1). In (5), the first constraint specifies the budget and the second constraint restricts the allocated resource to be non-negative. Note that the budget constraint has been specified as an equality. Let us consider a particular example to explain the intuition behind this. Interlink weights in $\mathcal{M}_1$ represent the probability of propagation of failure across the interconnected layers. Since inter-layer failure cascades cannot occur without interlinks, $\psi^{\mathcal{M}_1}(\mathcal{G}(\mathbf{x}), \eta)$ is maximized at $\mathbf{x} = \mathbf{0}$. It is easy to see that $\mathbf{x} = \mathbf{0}$ corresponds to a non-functional interdependent network. For instance, in smart grid for power distribution, although the absence of interlinks between the power and communication network layers eliminates the possibility of failure cascades, it compromises the fundamental property of interdependent networks: the interdependence. In such cases, if the budget for the interlink construction is constrained by an inequality ($\mathbf{1}^T \mathbf{x} \leq b$), the network design problem can produce trivial solutions. Due to this reason, we specify the budget constraint as an equality to ensure the inter-connectivity among the network layers. For specific mechanisms of failure propagation, like $\mathcal{M}_3$, where $\psi^{\mathcal{M}}(\mathcal{G}(\mathbf{x}), \eta)$ is increasing in the elements of $\mathbf{x}$, the solution to (5) also solves the case where the budget is specified as an inequality.

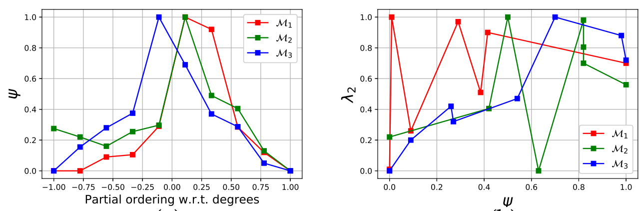
3.2 Deficiency of Existing Approaches

The two approaches for estimating the robustness are: i) the branching process (BP) approach, involving the mathematical modeling of the true nature of the failure cascades; and ii) the supra-adjacency (SA) matrix approach, involving the SA matrix representation and employing classical metrics, like algebraic connectivity, as the heuristic measures of robustness. In the following, we discuss the shortcomings of these approaches in characterizing $\psi^{\mathcal{M}}(\mathcal{G}(\mathbf{x}), \eta)$.

3.2.1 Branching Process (BP)

The development of rigorous mathematical models describing $\psi^{\mathcal{M}}(\mathcal{G}(\mathbf{x}), \eta)$ as a function of $\mathbf{x}$ for a general mechanism of failure propagation and a general cost structure is a daunting task. Although (2)-(3) allow us to compute the robustness through numerical iterations under certain conditions, it is unlikely that such equations can be utilized to solve cost constrained robustness optimization problems. Additionally, the BP approach is fundamentally limited by its reliance on the locally tree-like topology and only degree-based topological information.

Due to the difficulties in tracking the failure cascades for the distinct mechanisms, an alternative approach might be to devise various degree-based interlinking strategies and verifying their performance through simulations. Monotonic and anti-monotonic interlinking have been identified as favorable strategies under certain conditions, by both theoretical [23] and simulation based studies [32]. The monotonic (anti-monotonic) strategy interlinks the ith highest degree node in layer A to the ith highest (lowest) degree node in layer B, ties broken randomly. The simulation results are presented in Fig. 1a, where the two constituent layers and the infection seeds are fixed and the network robustness $\psi^{\mathcal{M}}(\mathcal{G}(\mathbf{x}), \eta)$ is plotted for various interlinking strategies at $\eta = 0.1$. Qualitatively similar results were observed for other values of η. We re-scale ψ for each mechanism, so that the values vary between 0 (minimum robustness) and 1 (maximum robustness) for all cases. We simplify our system model to focus on the impact of different interlinking structures. We assume that $\mathcal{G}(\mathbf{x}) = (G_A, G_B, \tilde{E}(\mathbf{x}))$ comprises layers of same size ($|V_A| = |V_B| = n$), generated by the Erdos-Renyi model. The interlinking structure is complete one-to-one, where each node $a \in V_A$ is interlinked to a unique node $b \in V_B$. We consider the simplest cost structure (linear and binary), where $w_l = x_l$ and x_l is a binary variable indicating the presence or absence of the lth interlink. These assumptions reduce the domain of $\mathbf{x}$ from the continuous space $\mathbb{R}^{mn \times 1}$ to the discrete space comprising all permutations of n indices. Note that these assumptions on the network structure are not enforced when we evaluate the performance of our interlinking strategies on real world networks in Section 4. To explore different degree-based interlinking strategies, we consider partially ordered monotonic and anti-monotonic interlinking. In Fig. 1a, an x-coordinate of +0.6 indicates that 60% of the interlinks in the network are constructed monotonically, where the positive x-coordinate denotes monotonicity, while the remaining 40% interlinks are constructed arbitrarily. The x coordinates denote the interlinking strategies while the y coordinates denote the average robustness over 500 independent instances of the interlink structure. It can be observed that no interlinking strategy is optimal for all cases, and in fact our simulation tests reveal that the maxima varies with η as well. Thus, degree-based interlinking structures cannot be used to characterize the robustness of networks against the three representative mechanisms.



(a) (b)

Figure 1: Deficiency of traditional approaches: a) Variation of ψ over different degree based interlink structures b) Variation of λ_2 with ψ.

3.2.2 Supra-Adjacency Matrix (SA)

Under this approach, the robustness of the network is measured by applying classical metrics to the SA matrix representation of the multilayer network. An important drawback of this approach is that the measure of robustness is only affected by the network topology. As a result, this approach is agnostic to the mechanisms of failure propagation as well as the attack models.

Let us evaluate the performance of algebraic connectivity (λ_2) as a metric of robustness. In Fig. 1b, we compare λ_2 to the empirical robustness $\psi^{\mathcal{M}}(\mathcal{G}(\mathbf{x}), \eta)$ for different interlinking strategies under the three mechanisms. Note that the y coordinates represent the re-scaled values of λ_2, where the absolute values of λ_2 corresponding to the three mechanisms are re-scaled to the range [0,1]. This operation was employed to improve the readability of Fig. 1 as the ranges of ψ and λ_2 for the different cases vary significantly. The interlinking structures and infections seeds are the same as that in Fig. 1a. For a metric to characterize the robustness, it should be monotone with ψ for all interlinking structures $\mathbf{x}$. Note that the simplifications of the cost structure lead to the equivalence between the resource allocation strategy and the interlink structures, i.e. $w_l = x_l$. A monotonic relationship ensures that the maximization of ψ can be achieved by maximizing λ_2. This requirement is not satisfied in Fig. 1b, implying that λ_2 is not an effective metric for measuring the robustness of networks against the mechanisms of failure propagation considered in this work. It can be observed from Fig. 1b that for all three mechanisms, the joint maximization of ψ and λ_2 is never achieved. In fact, the robustness for each mechanism is maximized at a different value of λ_2, which again is dependent on η. This highlights a key aspect of our work, that the robustness of networks against various mechanisms of failure propagation is maximized under different interlinking strategies and a universal metric of robustness, like λ_2, is not viable.

3.3 Surrogate Metrics of Robustness

Complexity of the BP approach limited its applicability to study the robustness of networks against general mechanisms of failure propagation. Employing universal metrics of robustness, like λ_2, is also not a promising direction due to the non-monotone relationship to ψ. Furthermore, Fig. 1 demonstrated that the maximization of the robustness occurs at different interlinking strategies for different mechanisms, discouraging the search for other universal metrics. This motivates us to develop new metrics of robustness specific to each mechanism. These metrics, represented by $f_0^{\mathcal{M}}(\mathbf{x})$, were designed by studying the failure cascades for the representative mechanisms. Simulation experiments reveal that the proposed metrics vary monotonically with the empirical measure of robustness for different values of attack strength (η) under all three mechanisms. This allows us to remove the dependence on η from the surrogate metrics. Using these metrics, our interlink design problem (5) becomes:

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \quad & f_0^{\mathcal{M}}(\mathbf{x}) \\ \text{subject to} \quad & \mathbf{1}^T\mathbf{x} = b, \text{ and } \mathbf{x} \succeq 0. \end{aligned} \tag{6}$$

Our objective in this work is to devise surrogate metrics $f_0^{\mathcal{M}}(\mathbf{x})$ that are monotone with the empirical value of robustness $\psi^{\mathcal{M}}(\mathcal{G}(\mathbf{x}))$ for each mechanism $\mathcal{M}$. Additionally, if the surrogate metrics are concave in the resource allocation strategy $\mathbf{x}$, then (6) becomes a convex optimization problem due to the affine constraints, allowing us to utilize the rich literature [33] studying it.

To summarize, we intend to design $f_0^{\mathcal{M}}(\mathbf{x})$ for each representative mechanism that are: i) monotone with the empirical values of robustness, to serve as a surrogate measure of robustness; ii) concave in $\mathbf{x}$, in order to pose the robustness maximization as a convex optimization problem; and iii) amenable to mathematical analysis, so that we can solve (6) to obtain interlink design strategies. In the following, we propose a family of surrogate metrics that can be utilized to study the robustness against various mechanisms of failure propagation. Later on, we choose specific metrics from this family, corresponding to the particular mechanisms studied in this work. The family of surrogate metrics can be written as:

$$f_0^{\mathcal{M}}(\mathbf{x}) = \sum_{l=1}^{mn} \mathcal{F}^{\mathcal{M}}(r(x_l), D_l^{\mathcal{M}}), \tag{7}$$

where $r(x_l)$ is the weight (w_l) of the lth interlink, $D_l^{\mathcal{M}}$ represents the contribution of interlink l towards enhancing network robustness, and $\mathcal{F}^{\mathcal{M}}$ is some function of these arguments. Note that a particular metric from this family (7) is specified by two factors: i) $D_l^{\mathcal{M}}$, capturing the importance of the lth interlink; and ii) the function $\mathcal{F}^{\mathcal{M}}$. In this work, we show that even straightforward definitions of $D_l^{\mathcal{M}}$ and $\mathcal{F}^{\mathcal{M}}$ can produce metrics which can track the robustness for the representative mechanisms and lead to better interlinking strategies than existing heuristics. In this work, we choose different models for $\mathcal{F}^{\mathcal{M}}$ and $D_l^{\mathcal{M}}$ to illustrate that there is no universal format for the surrogate metrics. Although different metrics of robustness might lead to different performance gains, we intend to elucidate through this work that even simple choices of these metrics can lead to a significant gain in performance. A comparative study on the performance of different metrics for the same mechanism of failure propagation is an important avenue of research and beyond the scope of this work.

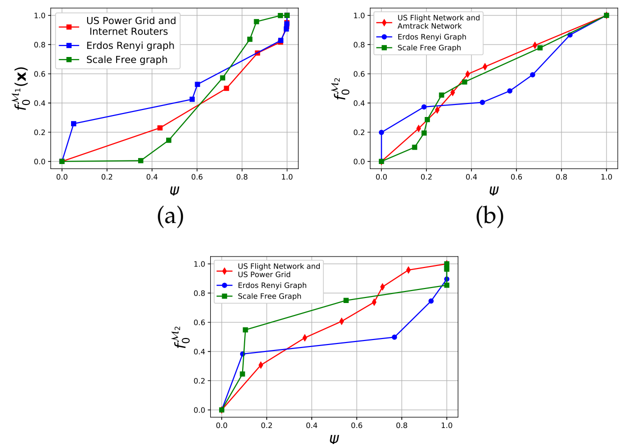
Next, we define the surrogate metrics corresponding to the three representative mechanisms and solve (6) to obtain the resource allocation strategy maximizing them. Interestingly, the surrogate metric based framework is amenable to distributed optimization and we present an algorithm that can learn the resource allocation strategy through local exchange of information among the nodes.

3.3.1 *Connected Component based cascades* ($\mathcal{M}_1$)

The basic idea behind the metric for this case is the observation that anti-monotonic ordering, where the ith highest degree node in layer A is coupled to the ith lowest degree node in layer B, outperforms the monotonic and the random ordering for the case of the degree-based targeted attacks [23, 34]. Although this observation was made under the degree-based attack model, our simulation studies reveal that similar results hold for the betweenness-based attacks as well. Intuitively, since the attacker chooses nodes of high betweenness (more important nodes), coupling them to nodes of low betweenness (less important nodes) checks the spread of failures. Using a multiplicative combination ($\mathcal{F}^{\mathcal{M}_1}(x, y) \triangleq xy$), our metric can be written as:

$$f_0^{\mathcal{M}_1}(\mathbf{x}) = \sum_{l=1}^{mn} r(x_l) D_l^{\mathcal{M}_1}. \tag{8}$$

Since we want to preferentially construct interlinks between nodes of high and low betweenness, $D_l^{\mathcal{M}_1}$ is modeled to



(a) (b) (c)

Figure 2: Plot of the surrogate metric values with the empirical network robustness for different mechanisms.

have a high value, when the betweenness difference between the endpoints of l is large. As $f_0^{\mathcal{M}_1}(\mathbf{x})$ in (8) is a weighted sum of the $D_l^{\mathcal{M}_1}$ values, maximizing it is equivalent to allocating more resource on interlinks with higher $D_l^{\mathcal{M}_1}$. Note that $D_l^{\mathcal{M}_1}$ can be defined in many ways as long as its value increases with the betweenness difference, for example:

- $D_l^{\mathcal{M}_1} = |b_{l^A} - b_{l^B}|$ (absolute),
- $D_l^{\mathcal{M}_1} = \exp(-|b_{l^A} - b_{l^B}|^{-1})$ (exponential),

where b_i is the intra-layer betweenness of i, and l^A, l^B are the two end-points of the lth interlink. Note that the proposed network design strategies hold for any arbitrary definition of $D_l^{\mathcal{M}}$, as long as it is decoupled from the interlink structure, i.e. $D_l^{\mathcal{M}}$ is not a function of $\mathbf{x}$. We consider the absolute modeling of $D_l^{\mathcal{M}_1}$ here.

The next logical step is to ascertain how well our proposed metric, given by (8), captures the robustness. The plots in Fig. 2a compare the true network robustness $\psi^{\mathcal{M}_1}(\mathcal{G}(\mathbf{x}), \eta)$ with the metric values $f_0^{\mathcal{M}_1}(\mathbf{x})$ for different interlinking structures, taking $\eta = 0.1$. We consider theoretical graph generation models along with real world networks as the intra-layer topologies. The real networks are chosen as practical candidates for interdependent networks following the representative failure propagation mechanisms. This is discussed further in Section 4. Each point reflects the average empirical robustness (x co-ordinate) and average metric value (y coordinate) over 500 independent instances of interlink structures. It can be observed that the surrogate and the empirical measures of robustness, $f_0^{\mathcal{M}_1}(\mathbf{x})$ and $\psi^{\mathcal{M}_1}(\mathcal{G}(\mathbf{x}), \eta)$, respectively, are correlated monotonically. This behavior allows us to substitute the maximization of the intractable $\psi^{\mathcal{M}_1}(\mathcal{G}(\mathbf{x}))$ with the maximization of $f_0^{\mathcal{M}_1}(\mathbf{x})$, which is a much simpler problem. The concavity and mathematical tractability of $f_0^{\mathcal{M}_1}(\mathbf{x})$ allows us to further incorporate the cost of construction of interlinks into the interlink design problem. In order to solve the robustness maximization problem (6), we need to specify our cost structure r. We remind the reader that we assume r in (1) to be increasing, concave and differentiable. For $\mathcal{M}_1$, let us choose a shifted sigmoid modeling for r, given by $r(x_l) = \frac{2}{1+e^{-\alpha_l x_l}} - 1$, where the weight of an interlink constructed by an investment of x_l resource is determined by the quality parameter $\alpha_l \in [0, 1]$, governing the ease of construction of the lth interlink, i.e. interlinks with low values of α_l are expensive to construct. Note that the shifted sigmoid representation upper bounds the interlink weight to 1. This satisfies the physical interpretation of the interlink weight as a probability, for the case of $\mathcal{M}_1$. Under this cost structure, we can utilize the KKT conditions to solve the convex optimization problem (6) to obtain the following closed form solution for the resource allocation strategy:

$$x_l^\star = \frac{1}{\alpha_l} \log\left(\frac{\sqrt{\frac{\alpha_l D_l}{2}} + \sqrt{\frac{\alpha_l D_l}{2} - \nu^\star}}{\sqrt{\frac{\alpha_l D_l}{2}} - \sqrt{\frac{\alpha_l D_l}{2} - \nu^\star}}\right), \tag{9}$$

where $x_l^\star$ is the optimal resource allocated to the lth interlink, α_l is the interlink quality parameter, $\nu^\star$ is the optimal value of the equality constraint multiplier ν, and D_l, i.e. $D_l^{\mathcal{M}_1}$, denotes the absolute modeling of the importance of the lth interlink. The mathematical details are presented in the supplementary material. It can be observed from (9) that a small $\nu^\star$ (close to 0) implies that a non-zero resource is allocated to most interlinks in the optimal condition, thereby indicating a strong interaction between the interdependent layers. $\nu^\star$ can be computed through binary search by substituting (9) into the budget constraint $\sum_l x_l^\star = b$. A distributed algorithm for obtaining this solution is presented in Algorithm 1.

3.3.2 *Load Distribution in Interdependent Networks* ($\mathcal{M}_2$)

We adopt a logarithmic modeling of $\mathcal{F}$ for this case, where $\mathcal{F}^{\mathcal{M}_2}(x, y) \triangleq \log(x + y)$. The surrogate metric of robustness for this mechanism of failure propagation can be written as:

$$f_0^{\mathcal{M}_2}(\mathbf{x}) = \sum_{l=1}^{mn} \log(D_l^{\mathcal{M}_2} + r(x_l)). \tag{10}$$

This kind of modeling is widely adopted in information theoretic studies [35], where it is desirable to allocate power optimally to different communication channels to maximize the throughput. The $D_l^{\mathcal{M}_2}$ term in (10) corresponds to the interference in each channel, while the $r(x_l)$ term corresponds to the allocated power. For interlink l that enhances the network robustness, the value of $D_l^{\mathcal{M}_2}$ should be low indicating a low-noise (high-quality) wireless channel to which more power (resource) should be allocated. On the contrary, high interference channels should receive less power. The logarithmic modeling of $\mathcal{F}$ prefers links of low $D_l^{\mathcal{M}_2}$ as opposed to the multiplicative case.

In order to define $D_l^{\mathcal{M}_2}$, let us develop a better understanding of the failure cascades. We define the *free space* of a node i as $S_i \triangleq C_i - L_i$, where C_i and L_i denote the capacity and load of the ith node, respectively. Since a node fails when its current load exceeds the capacity, the difference between the capacity and load denotes the additional space available to handle the load offloaded from neighbors. Following [36, 37], it can be conjectured that interlinking nodes of low free space to nodes of high free space is a good strategy. This is intuitive as nodes of high free space can handle the load offloaded from nodes of low free space upon failure. By coupling the nodes which are most likely to fail to the nodes which are best at handling failure, we check the spread of failures thereby enhancing robustness. We can define $D_l^{\mathcal{M}_2}$ as:

- $D_l^{\mathcal{M}_2} = |S_{l^A} - S_{l^B}|^{-1}$ (absolute)
- $D_l^{\mathcal{M}_2} = 1 - \exp(-|S_{l^A} - S_{l^B}|^{-1})$ (exponential)

These definitions ensure that links connecting nodes with high difference in free space are given preference. We use the exponential modeling for this case. Simulation tests presented in Fig. 2b reveal a monotonic relationship between $f_0^{\mathcal{M}_2}(\mathbf{x})$ and $\psi^{\mathcal{M}_2}(\mathcal{G}(\mathbf{x}), \eta)$, validating the choice of the metric. The optimal resource allocation under a shifted sigmoid r is given by:

$$x_l^\star = \frac{1}{\alpha_l} \log\left(\frac{-D_l \nu^\star + \alpha_l + \sqrt{\alpha_l^2 + \nu^{\star 2} - 2 D_l \nu^\star \alpha_l}}{(D_l - 1)\nu^\star} \right). \quad (11)$$

The details of the analysis is similar to the previous case and is omitted for brevity.

3.3.3 *Connectivity in Demand-Supply Networks ($\mathcal{M}_3$)*

Similar to the preceding case, a logarithmic modeling is used for $\mathcal{F}^{\mathcal{M}_3}$, under which the metric can be written as:

$$f_0^{\mathcal{M}_3}(\mathbf{x}) = \sum_{i=1}^{m} \log\left[\frac{1}{b_i} + \sum_{j=1}^{n} s_j r(x_{ij}) \right], \quad (12)$$

where $r(x_{ij})$ is the weight of the interlink between nodes $i \in V_A$ and $j \in V_B$, b_i is the betweenness centrality of the ith demand node, and s_j is the supply rate of j. Here, betweenness is used to measure the importance of the demand nodes in maintaining the connectivity of the demand network. As demand nodes of high significance should receive more supply, the interference term ($D_i^{\mathcal{M}_3}$) is modeled as $1/b_i$. Note that there exists a fundamental difference between (12) and the load distribution metric (10). Although both are based on a logarithmic modeling, the interlinks for these processes perform different functions. In $\mathcal{M}_2$, interlinks offload load to their neighbors independently. In $\mathcal{M}_3$, the demand node cares about the total supply it receives and the interlinks cooperate in maintaining the supply and are not independent. Due to this reason, the proposed metric is not separable w.r.t. the interlinks indexed by l but is separable w.r.t. the demand nodes indexed by i. Simulation studies (Fig. 2c) justify the choice of the metric. We use a logarithmic modeling of the cost structure, given by $r(x_l) = \log(\alpha_l x_l + 1)$, where the parameter α_l is the quality parameter. We remind the reader that the various settings of $\mathcal{F}^{\mathcal{M}}$, $D_l^{\mathcal{M}}$ and r, are chosen so as to illustrate the generality of the surrogate metric based framework. The solution for this case can be written as:

$$x_{ij}^\star = \max\left(0, \frac{1}{\nu^\star (1/b_i + w_i^\star)} - \frac{1}{\alpha_{ij}}\right), \quad (13)$$

where $w_i^\star \triangleq \sum_{j=1}^{n} s_j r(x_{ij}^\star)$ is the total supply received by the ith demand node. Note that unlike (9) and (11), (13) is self-consistent, since the right hand side of (13) is a function of $x_{ij}^\star$. This is due to the non-separability of the objective function w.r.t. the interlinks. Unlike the closed form expressions obtained in the previous cases, (13) is solved by numerical iterations.

3.4 Algorithm

We present the basic framework for obtaining the resource allocation strategy maximizing the surrogate metrics in Algorithm 1. This algorithm can be theoretically applied for any separable metric by varying the EB (evaluate budget) block corresponding to the definitions of r, $D_l^{\mathcal{M}}$ and $\mathcal{F}^{\mathcal{M}}$. An important aspect of our work is that the interlinking strategy can be obtained distributively, i.e. the constituent nodes can locally exchange information to learn the resource allocations maximizing the surrogate metrics. Distributed implementation of network design algorithms has become more of a necessity than a feature in recent years, as many practical applications involve networks of large sizes for which centralized algorithms are not feasible. We distribute the decision of finding the resource allocation strategy for each interlink to their end-points in layer A or B. Let us define the algorithm w.r.t. layer A. Algorithm 1 runs on all nodes $a \in V_A$ in order to compute the resource allocation strategy, i.e. each node a learns the resource to be allocated to all interlinks connected to it. The two main blocks of the

Algorithm 1 Optimal Resource Computation

```
1:  procedure OPT_ALLOC(α, G_A, G_B, b, Δ_th)
2:      p_1 ← 0, p_2 ← max(size(G_A), size(G_B)), Δ ← inf
3:      while Δ > Δ_th do
4:          ν ← (p_1 + p_2)/2
5:          b̃ ← Σ_l EB(a_l, ν)
6:          if b̃ > b then
7:              p_2 ← ν
8:          else
9:              p_1 ← ν
10:         Δ ← |b̃ − b|
11:     x_l ← EB(α_l, ν)
12:     return x_l, ∀l
```

distributed implementation are: i) broadcast, employed to inform every node about the budget b and Δ_{th}, and ii) distributed consensus, employed in Step 5 to compute the sum of the allocated resource. There exists extensive literature [38, 39] studying the distributed implementation of these operations. In Algorithm 1, the dual variable ν is computed at every node in Step 4 and the resource allocation strategy corresponding to ν is computed by the EB block, based on the solutions (9), (11), and (13), respectively. The sum ($\tilde{b}$) of the allocated resources is obtained by distributed consensus algorithms. This sum is compared with b to update ν in Steps 6-9. This descent process is repeated, till a threshold budget usage performance ($|\tilde{b} - b| \leq \Delta_{th}$) is achieved, to estimate $\nu^\star$. Essentially, the algorithm learns $\nu^\star$ via binary search between pre-defined upper and lower limits. The upper limit (p_2) is dependent on the objective function. We found that the maximum of the two network sizes works for all three cases. The choice has a marginal effect on the convergence time of ν. Finally in Step 11, the EB block computes the optimal resource $x_l^\star$ for each interlink using the optimal value of ν.

4 Simulation Results

We compare the performance of our interlinking strategies with the state of the art heuristics for the three mechanisms of failure propagation. We use Python and its associated libraries for our simulations. We present simulation results in two domains. Firstly, we simulate the failure cascades for different strengths of attack (η) to compare the network robustness $\psi^{\mathcal{M}}(\mathcal{G}(\mathbf{x}), \eta)$ of various interlinking strategies, under two different regimes of available budget b. Secondly, we study the variation of performance gain of these interlinking strategies with the available budget b. Instead of relying on theoretical graph generation models, we use real-world networks for the layers G_A and G_B. These real-world networks have been chosen to serve as examples of practical scenarios, where the representative mechanisms of failure propagation might occur.

4.1 Performance of different interlinking strategies

The main roadblock in the way of comparing our strategies to the existing heuristics is that since we consider a general system model with weighted interlinks and include the effects of interlink construction cost, similar models have not been studied in relevant literature, to the best of our knowledge. Most works that study the impact of interlink structure, consider unweighted links and are agnostic to the interlink construction cost. In order to provide a fair comparison, we modify the state of the art heuristics to incorporate cost. This is achieved by considering a linear combination of the heuristics with the interlink quality parameter α_l. Let us assume that for each node $a \in A$, the heuristics rank the nodes $b \in B$ in some order. The modified heuristics incorporate α_l into this ranking scheme in order to avoid spending resource on expensive interlinks with small values of α_l. Our experiments indicate that an additive modification, where the modified rank for interlink l is the sum of the original rank and α_l, works best for the cases considered here.

In the following set of results, we study the network robustness under four interlink design strategies, or more specifically, constrained resource allocation strategies: i) the random resource allocation; ii) the state of the art heuristics, based on existing works; iii) the modified state of the art heuristics, considering interlink construction cost; and iv) the surrogate metric based optimization, proposed in this work. Distinct heuristics are used for the three distinct mechanisms of failure propagation based on the relevant literature. We present the results under two different regimes distinguished by the availability of budget: stringent, where the resource availability is low and few nodes can be interlinked; and sufficient, where the resource availability is abundant and most nodes are strongly interlinked. Note that the numerical values of the budgets is not the same for all three mechanisms. These values were chosen to give the reader a visually representative idea of the comparative performance of the different resource allocation strategies at both ends of the spectrum of available budget. The simulation results for the variation of performance with available budget is presented in Section 4.2. As defined in Section 2, $\psi^{\mathcal{M}}(\mathcal{G}(\mathbf{x}), \eta)$ is measured by the fraction of nodes in V_B that survive in the steady state after the cascade of failures, initiated by an infection seed of size $\eta|V_A|$. Qualitatively similar results is obtained when the layer roles are reversed, i.e. layer B is attacked and robustness is measured w.r.t. layer A. Under the heuristic, the modified heuristic and the random allocation strategies, the total budget is divided uniformly among all nodes, and then allocated to the highest ranked inter-layer node.

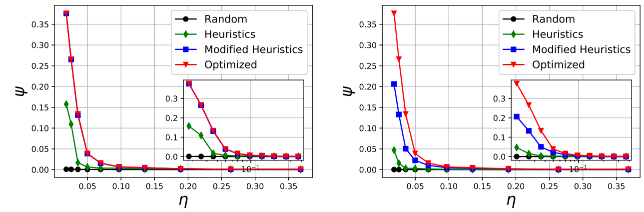


Figure 3: Connected component based cascading failure under stringent (left) and sufficient (right) budget constraints.

4.1.1 *Connected Component based cascades*

Smart power distribution grids, involving interdependent network layers of power stations and communication routers, is a classical practical example [1] of $\mathcal{M}_1$. Under the degree-based targeted attack, current art [23, 34] suggests that anti-monotonic interlinking outperforms other degree-based (random and monotonic) strategies. Supported by simulation experiments as discussed earlier, we choose the betweenness-based anti-monotonic interlinking as our heuristic design strategy, where the node with the ith highest betweenness in V_A is coupled to the node with the ith lowest betweenness in V_B. Starting from the node of the highest betweenness, for each node $a \in A$, the un-interlinked neighbors in V_B are ranked in increasing order on the basis of their betweenness and modified betweenness (additive cost correction), for the heuristic and the modified heuristic strategies, respectively. Arbitrary ranking is adopted for the random allocation.

In our simulations, we consider an Autonomous System AS-733 topology [40], representative of the network of routers comprising the Internet, as G_A and the Western States Power Grid of the United States [41] as G_B; with $|V_A| = 6474, |V_B| = 4941$. This corresponds to the problem of optimally interlinking the power stations to the communication routers to maximize the robustness of the smart grid against failure cascades. The simulation results are presented in Fig. 3. Under both stringent and sufficient budget conditions, it can be observed that the interdependent network is extremely vulnerable to node failures. For all interlinking strategies, even η as low as 0.05 leads to the failure of more than 95% of the network in the steady state. This establishes the importance of studying the robustness of multilayer networks, which can catastrophically react to node failures due to the recursive cascade of failures between the interdependent layers. Fig. 3 clearly establishes the inferiority of the random resource allocation strategy, as it almost coincides with the x axis. This indicates that random allocation can have catastrophic effects on the robustness for $\mathcal{M}_1$ and even state of the art heuristics perform much better. It is interesting to note that this phenomenon does not generalize to other mechanisms, as will be evident from Fig. 4, where performance of the random and the heuristic strategies are comparable. Since the heuristic strategies do not consider cost, we compare our interlink design algorithm w.r.t. the modified heuristics, which outperform the other two strategies in all cases. It can be observed from Fig. 3 that the performance gain of

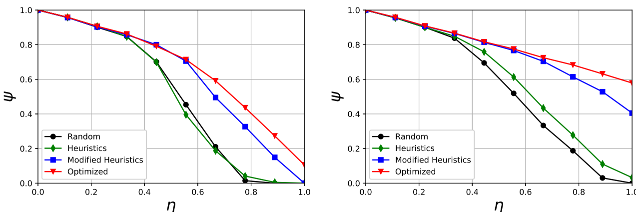


Figure 4: Load distribution in interdependent networks under stringent (left) and sufficient (right) budget constraints.

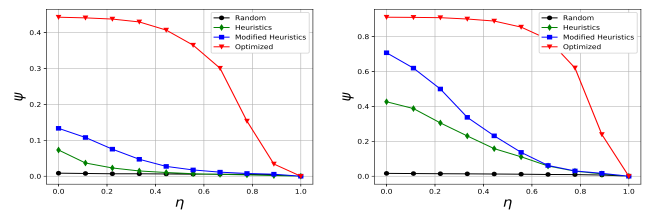


Figure 5: Connectivity in demand-supply networks under stringent (left) and sufficient (right) budget constraints.

our framework over the modified heuristics is negligible for stringent budgets but significant under sufficient budgets. The reason behind the absence of gain at stringent budgets is discussed in Section 4.2.

4.1.2 Load Distribution in Interdependent Networks

We consider a multi-modal transportation network: Amtrak railway routes [42] as G_A and the US Airport network [43] as G_B, with $V_A = 11526$ and $V_B = 1574$. The problem we are looking at is optimizing the interlinks, which may be shuttle bus services, between the airports and the railway stations to maximize the robustness of the airports against the failure of the railway stations. Adhering to the convention in related works, the load, representative of the traffic carried by the nodes, is denoted by the intra-layer degree. The capacity of a node is given by: $C_i = (1+\beta)L_i$, where L_i and C_i denote the load and capacity of node i, respectively, and $\beta = 0.5$. Simulation based works [36] from literature suggest that load based anti-monotonic interlinking is a good strategy. As a result, the heuristic strategy for $\mathcal{M}_2$ is the degree-based anti-monotonic interlinking, since the load is measured by the intra-layer degrees.

The simulation results for the two regimes are presented in Fig. 4. An interesting point to note here is that under stringent budgets, the random resource allocation can outperform the heuristics. This is counter-intuitive due to the fact that a random distribution of resources is agnostic the network topology as well as the interlink construction cost. The reason behind this observation is that the heuristic interlinking strategies do not consider cost and the recommended interlinks might end up being very expensive owing to low α_l values. This is in contrast to the previous case, where even heuristic strategies agnostic to cost performed much better than the random allocation of budget. This establishes the importance of cost constraints in real world interlink optimization problems, where for certain mechanisms ($\mathcal{M}_1$) the heuristic strategies can perform reasonably well, whereas for others ($\mathcal{M}_2$) their performance might be worse than random.

4.1.3 Connectivity in Demand-Supply networks

Demand-supply interdependencies are common in many practical instances of multilayer networks. We consider the US Power Grid network [41] as the supply layer (G_A) and the US Airport network [43] as the demand layer (G_B), with $|V_A| = 4941$ and $|V_B| = 1574$. We study the problem of interlinking the airports to the power distribution stations. Path-based interlink assignment has been identified as a good heuristic algorithm in [3] and we use it as the state of the art heuristic. In this strategy, a source and target node from the demand layer are picked randomly and all node disjoint paths between them are identified. Distinct supply nodes are connected to the individual disjoint paths. This strategy is intuitive because the failure of a supply node only affects a unique disjoint path in the demand network and the source and target demand nodes remain connected through alternative paths.

For designing a resource allocation scheme corresponding to this strategy, the supply of each node $a \in V_A$ is distributed uniformly among all constituents of each node disjoint path. Under the modified strategy, this distribution is modified so that the resource allocated to each interlink is proportional to its quality parameter α_l. The simulation results are presented in Fig. 5. Here we can observe that in contrast to the previous cases, the surrogate metric based framework has a significant performance gain even at stringent budgets. This counter-intuitive result is explained in the following.

4.2 Performance gain variation with available budget

We are also interested in studying the performance of the proposed interlinking strategies under different budget constraints b. The performance gain for a strategy is computed w.r.t. the random allocation of resource. We define the gain of an allocation strategy as the total difference between the robustness corresponding to the strategy in question and the random allocation, for the different values of η. Fig. 6 reveals that the performance gain is marginal under stringent budget conditions for $\mathcal{M}_1$ and $\mathcal{M}_2$. This is because the majority of network components are not interlinked owing to the low availability of budget. Due to this weak interlinking of the network layers, the cascade of failures between them is not pronounced, leading to similar performance for all interlinking strategies. This phenomenon is not observed for $\mathcal{M}_3$, since it does not involve any recursive propagation of failure due to the unidirectional interdependence. For this case, interlinking strategies have a pronounced effect on network performance in lower budgets. It is interesting to note that although the surrogate metric based framework outperforms the other strategies in all cases, the performance gain of the surrogate metrics varies considerably with the mechanism of failure propagation. A significant variation is also observed when other attack models, like the degree-based targeted attack and the randomized attack, are considered; the corresponding results are presented in the supplementary material. These simulation results clearly show that the surrogate metrics of robustness cognizant of the mechanism of failure propagation and the attack model outperforms the state of the art heuristics. The variation of the gain obtained by the surrogate metric based framework under different conditions reveals other interesting properties about the robustness of complex networks.

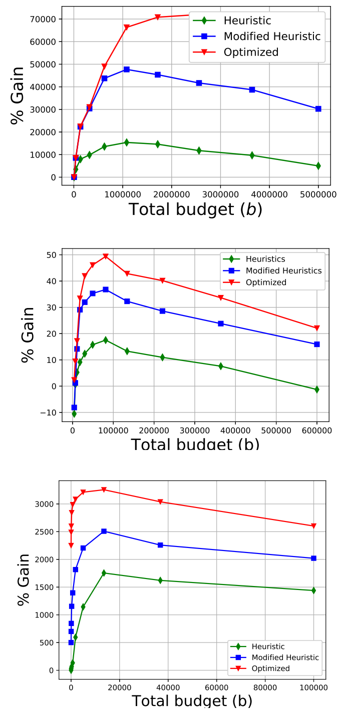


Figure 6: Variation of performance gain w.r.t budget under the betweenness-based targeted attack for: a) connected component based cascading failure (top), b) load distribution (middle), and c) connectivity in demand-supply networks (bottom).

5 Conclusion and Future Work

In this work, we model the construction of interlinks between two isolated network layers as a convex optimization problem with the objective of maximizing the robustness of the resulting interdependent network. Departing from traditional approaches of exact modeling of the failure cascades or applying classical metrics to flattened representations of multilayer networks, we propose surrogate metrics that are shown to be monotonically correlated to the empirical value of robustness for the representative mechanisms. This monotonicity justifies the substitution of the robustness $\psi^{\mathcal{M}}(\mathcal{G}(\mathbf{x}), \eta)$, which lacks mathematical models tracking its dynamics for complex failure spreading mechanisms, with the surrogate metrics $f_0^{\mathcal{M}}(\mathbf{x})$. Due to the tractability of these metrics, we are able to solve the interlink design problem and also consider the interlink construction cost. Furthermore, a framework for distributed learning of the interlinking strategy is presented, which can learn the resource allocation maximizing the surrogate metrics via local exchange of information.

In essence, this work establishes a framework for dealing with interlink optimization problems under cost constraints, which can be applied to different mechanisms of failure propagation. It is important to remember that these interlinking strategies are not guaranteed to be optimal owing to the surrogate characterization of the robustness. More sophisticated metrics, designed to faithfully capture the failure cascade properties, may outperform our surrogate metrics. This line of research is important as the performance gain from sophisticated metrics will reveal if accurate characterization of the propagation mechanisms is necessary or whether approximate characterizations, like those presented in this work, can achieve reasonably good performance.

6 Acknowledgments

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References

[1] S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, and S. Havlin. "Catastrophic cascade of failures in interdependent networks". In: *Nature* 464.7291 (2010), p. 1025.

[2] D. Lv, A. Eslami, and S. Cui. "Load-Dependent Cascading Failures in Finite-Size Erdös-Rényi Random Networks". In: *IEEE Transactions on Network Science and Engineering* 4.2 (2017), pp. 129–139.

[3] J. Zhang and E. Modiano. "Connectivity in Interdependent Networks". In: *IEEE/ACM Transactions on Networking* 26.5 (2018), pp. 2090–2103.

[4] S. Boccaletti et al. "The structure and dynamics of multilayer networks". In: *Physics Reports* 544.1 (2014), pp. 1–122.

[5] M. Kivelä et al. "Multilayer networks". In: *Journal of complex networks* 2.3 (2014), pp. 203–271.

[6] F. Tan et al. "Traffic congestion in interconnected complex networks". In: *Physical Review E* 89.6 (2014), p. 062813.

[7] T. Valles-Catala et al. "Multilayer stochastic block models reveal the multilayer structure of complex networks". In: *Physical Review X* 6.1 (2016), p. 011036.

[8] Z. Huang, C. Wang, M. Stojmenovic, and A. Nayak. "Balancing system survivability and cost of smart grid via modeling cascading failures". In: *IEEE Transactions on Emerging Topics in Computing* 1.1 (2013), pp. 45–56.

[9] G. Ranjan and Z. Zhang. "How to glue a robust smart-grid?: a finite-network theory for interdependent network robustness". In: *Proceedings on Cyber Security and Information Intelligence Research*. ACM. 2011, p. 22.

[10] L. A. Braunstein et al. "Optimal path and minimal spanning trees in random weighted networks". In: *International Journal of Bifurcation and Chaos* 17.07 (2007), pp. 2215–2255.

[11] D. S. Callaway et al. "Network robustness and fragility: Percolation on random graphs". In: *Physical review letters* 85.25 (2000), p. 5468.

[12] F. Radicchi and C. Castellano. "Beyond the locally tree-like approximation for percolation on real networks". In: *Physical Review E* 93.3 (2016), p. 030302.

[13] F. Radicchi and G. Bianconi. "Redundant interdependencies boost the robustness of multiplex networks". In: *Physical Review X* 7.1 (2017), p. 011013.

[14] D. Cellai et al. "Message passing theory for percolation models on multiplex networks with link overlap". In: *Physical Review E* 94.3 (2016), p. 032301.

[15] Y. Zheng et al. "Weighted Algebraic Connectivity Maximization for Optical Satellite Networks". In: *IEEE Access* 5 (2017), pp. 6885–6893.

[16] D. Meng, M. Fazel, and M. Mesbahi. "Online algorithms for network formation". In: (2016), pp. 135–140.

[17] H. Shakeri, N. Albin, F. D. Sahneh, P. Poggi-Corradini, and C. Scoglio. "Maximizing algebraic connectivity in interconnected networks". In: *Physical Review E* 93.3 (2016), p. 030301.

[18] A. Ghosh and S. Boyd. "Growing well-connected graphs". In: *Proceedings of the 45th IEEE Conference on Decision and Control*. IEEE. 2006, pp. 6605–6611.

[19] P. V. Mieghem. "Interconnectivity structure of a general interdependent network". In: *Physical Review E* 93.4 (2016), p. 042305.

[20] M. Fiedler. “Algebraic connectivity of graphs”. In: *Czechoslovak mathematical journal* 23.2 (1973), pp. 298–305.
[21] D. F. Rueda et al. “Reducing the impact of targeted attacks in interdependent telecommunication networks”. In: *2016 23rd International Conference on Telecommunications (ICT)*. IEEE. 2016, pp. 1–5.
[22] Z. Chen, J. Wu, Y. Xia, and X. Zhang. “Robustness of interdependent power grids and communication networks: A complex network perspective”. In: *IEEE Transactions on Circuits and Systems II* 65.1 (2018), pp. 115–119.
[23] S. Chattopadhyay, H. Dai, D. Y. Eun, and S. Hosseinalipour. “Designing Optimal Interlink Patterns to Maximize Robustness of Interdependent Networks against Cascading Failures”. In: *IEEE Transactions on Communications* 65.9 (2017), pp. 3847–3862.
[24] O. Yagan, D. Qian, J. Zhang, and D. Cochran. “Optimal allocation of interconnecting links in cyber-physical systems: Interdependence, cascading failures, and robustness”. In: *IEEE Transactions on Parallel and Distributed Systems* 23.9 (2012), pp. 1708–1720.
[25] M. Parandehgheibi and E. Modiano. “Robustness of interdependent networks: The case of communication networks and the power grid”. In: *GLOBECOM*. IEEE. 2013, pp. 2164–2169.
[26] P. Zhang et al. “The robustness of interdependent transportation networks under targeted attack”. In: *EPL (Europhysics Letters)* 103.6 (2013), p. 68005.
[27] Z. Wu et al. “Cascading failure spreading on weighted heterogeneous networks”. In: *Journal of Statistical Mechanics: Theory and Experiment* 05 (2008), p. 05013.
[28] J. Wang and L. L. Rong. “Cascade-based attack vulnerability on the US power grid”. In: *Safety Science* 47.10 (2009), pp. 1332–1336.
[29] K. I. Goh, B. Kahng, and D. Kim. “Universal behavior of load distribution in scale-free networks”. In: *Physical Review Letters* 87.27 (2001), p. 278701.
[30] A. E. Motter and Y. C. Lai. “Cascade-based attacks on complex networks”. In: *Physical Review E* 66.6 (2002), p. 065102.
[31] Y. Zhang, A. Arenas, and O. Yağan. “Cascading failures in interdependent systems under a flow redistribution model”. In: *Physical Review E* 97.2 (2018), p. 022307.
[32] J. Wang, C. Jiang, and J. Qian. “Robustness of interdependent networks with different link patterns against cascading failures”. In: *Physica A: Statistical Mechanics and its Applications* 393 (2014), pp. 535–541.
[33] S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge university press, 2004.
[34] B. Min et al. “Network robustness of multiplex networks with interlayer degree correlations”. In: *Physical Review E* 89.4 (2014), p. 042811.
[35] R. G. Gallager. *Information theory and reliable communication*. Vol. 588. Springer, 1968.
[36] X. Peng et al. “Load-induced cascading failures in interconnected networks”. In: *Nonlinear Dynamics* 82.1-2 (2015), pp. 97–105.
[37] Z. Chen et al. “Cascading failure of interdependent networks with different coupling preference under targeted attack”. In: *Chaos, Solitons & Fractals* 80 (2015), pp. 7–12.
[38] D. M. Aoyama and D. Shah. “Fast distributed algorithms for computing separable functions”. In: *IEEE Transactions on Information Theory* 54.7 (2008), pp. 2997–3007.
[39] B. Williams and T. Camp. “Comparison of broadcasting techniques for mobile ad hoc networks”. In: *Proceedings of the 3rd ACM international symposium on Mobile ad hoc networking & computing*. ACM. 2002, pp. 194–205.
[40] J. Leskovec et al. “Graph evolution: Densification and shrinking diameters”. In: *ACM Transactions on Knowledge Discovery from Data (TKDD)* 1.1 (2007), p. 2.
[41] D. J. Watts and S. H. Strogatz. “Collective dynamics of ‘small-world’networks”. In: *Nature* 393.6684 (1998), p. 440.
[42] Downloaded from the Bureau of Transportation Statistics under the US Department of Transportation https://www.bts.gov/.
[43] T. Opsahl. “Why anchorage is not (that) important: Binary ties and sample selection”. In: *http://toreopsahl.com/2011/08/12/* (2011).

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