Gram determinant solutions to nonlocal integrable discrete nonlinear Schrödinger equations via the pair reduction

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Abstract

In this paper, Gram determinant solutions of local and nonlocal integrable discrete nonlinear Schrödinger (IDNLS) equations are studied via the pair reduction. A generalized IDNLS equation is firstly introduced which possesses the single Casorati determinant solution. Two kinds of Gram determinant solutions are presented from Casorati determinant ones due to the gauge freedom. The different pair constraint conditions for wave numbers are imposed and then solutions of local and nonlocal IDNLS equations are derived in terms of Gram determinant.

Keywords: nonlocal discrete NLS equation, bilinear form, pair reduction, solution, Gram determinant

1. Introduction

The parity-time (PT) symmetry has firstly proposed in quantum mechanics since Bender and Boettcher [1] found that non-Hermitian Hamiltonians possess entirely real spectra. In nonlinear integrable system, Ablowitz and Musslimani have recently introduced a nonlocal nonlinear Schrödinger (NLS) equation with the PT-symmetric invariance. Such a nonlocal PT-symmetric NLS equation remains integrable due to the existence of a Lax pair formulation and an infinite number of conservation laws. Indeed, it can be obtained from a nonlocal reduction of the AKNS spectral problem [2, 3]. Along with this idea, many nonlocal versions of local integrable equations have been identified in both one and two space dimensions as well as in discrete case [2–11]. These nonlocal reductions include the reverse space-time symmetry, or the partially PT symmetry and the partially reverse space-time symmetry in higher dimensional case. The potential applications of nonlocal soliton equations appear in nonlinear PT symmetric media [6], or more universally in the context of "Alice-Bob events" [7, 8]. Moreover, the classical methods such as inverse scattering transform, Darboux transformation and bilinear approach have recently applied to nonlocal integrable models and gave rise to new types of solution [12–27].

Among these nonlocal system, three kinds of nonlocal integrable discrete NLS (IDNLS) equations can be reduced from the Ablowitz-Ladik (AL) spectral problem [3]. More specifically, considering the AL scattering problem

$$\upsilon_{n+1} = \begin{pmatrix} z & \psi_n \\ \phi_n & z^{-1} \end{pmatrix} \upsilon_n, \quad \upsilon_{n,t} = \begin{pmatrix} \mathrm{i}\psi_n\phi_{n-1} - \frac{\mathrm{i}}{2}(z - z^{-1})^2 & -\mathrm{i}(z\psi_n - z^{-1}\psi_{n-1}) \\ \mathrm{i}(z^{-1}\phi_n - z\phi_{n-1}) & -\mathrm{i}\phi_n\psi_{n-1} + \frac{\mathrm{i}}{2}(z - z^{-1})^2 \end{pmatrix} \upsilon_n, \tag{1}$$

where $\upsilon_n = (\upsilon_n^{(1)}, \upsilon_n^{(2)})^T$ and z is a complex spectral parameter, the discrete compatibility condition $\upsilon_{n+1,t} = (\upsilon_{m,t})_{m=n+1}$ yields the coupled system

$$\mathrm{i}\psi_{n,t} = \psi_{n+1} + \psi_{n-1} - 2\psi_n - \psi_n\phi_n(\psi_{n+1} + \psi_{n-1}), \tag{2}$$

$$-\mathrm{i}\phi_{n,t} = \phi_{n+1} + \phi_{n-1} - 2\phi_n - \phi_n\psi_n(\phi_{n+1} + \phi_{n-1}). \tag{3}$$

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It admits four different symmetry reductions [3, 12, 13, 19, 22, 24]:

(i) The standard Ablowitz-Ladik symmetry ($\phi_n = \delta\psi_n^*$) gives rise to the local IDNLS equation

$$i\psi_{n,t} = \psi_{n+1} + \psi_{n-1} - 2\psi_n - \delta\psi_n\psi_n^*(\psi_{n+1} + \psi_{n-1}), \quad \delta = \pm 1. \tag{4}$$

(ii) The discrete PT preserved symmetry ($\phi_n = \delta\psi_{-n}^*$) leads to the PT symmetric IDNLS equation

$$i\psi_{n,t} = \psi_{n+1} + \psi_{n-1} - 2\psi_n - \delta\psi_n\psi_{-n}^*(\psi_{n+1} + \psi_{n-1}), \quad \delta = \pm 1. \tag{5}$$

(iii) The reverse time symmetry ($\phi_n = \gamma\psi_n(-t)$) yields the reverse time symmetric IDNLS equation

$$i\psi_{n,t} = \psi_{n+1} + \psi_{n-1} - 2\psi_n - \gamma\psi_n\psi_n(-t)(\psi_{n+1} + \psi_{n-1}), \tag{6}$$

where γ is an arbitrary complex constant.

(iv) The reverse discrete-time symmetry ($\phi_n = \gamma\psi_{-n}(-t)$) results in the reverse discrete-time symmetric IDNLS equation

$$i\psi_{n,t} = \psi_{n+1} + \psi_{n-1} - 2\psi_n - \gamma\psi_n\psi_{-n}(-t)(\psi_{n+1} + \psi_{n-1}), \tag{7}$$

where γ is an arbitrary complex constant.

More recently, a bilinearisation-reduction approach [24] has been proposed to derive solutions of Eqs.(4)-(7). By imposing different reduction conditions, double Casoratian solutions for Eqs.(4)-(7) have been provided [24]. It is known that the local IDNLS equation (4) admits bright solition solutions in terms of the double Casorati determinant but dark soliton ones expressed as the single Casorati determinant [28]. Thus, "dark" soliton solutions for nonlocal IDNLS equations (5)-(7) need to be investigated. Apart from the single Casorati determinant form, the dark soliton solution of the local IDNLS equation (4) can be written as Gram determinant form. It motivates us to construct Gram determinant solutions for nonlocal IDNLS equations. In the previous work [27], one of authors have used the direct reduction, namely the wave number satisfying the constraint condition under the same index, to obtain Gram determinant solutions for local and nonlocal IDNLS equations (4)-(6). For instance, the derivation of the dark soliton solution for the defocusing local IDNLS Eq.(4) was realized by imposing the constraint condition $q_i = p_i^{*-1}$ [27, 28] on the before-redcution IDNLS equation. In the direct reduction, only singular soliton solutions were obtained for the nonlocal PT symmetric IDNLS Eq.(5) and only one-soliton solution was derived for the reverse time symmetric IDNLS Eq.(6) [27]. Note that in the continuous nonlocal PT-symmetric NLS equation with the nonzero boundary condition case, there existed an even number of soliton solutions related to an even number (2N) of eigenvalues [4, 26]. So it is necessary to develop a kind of pair reduction on the before-redcution IDNLS equation, in which the constraint condition is taken between a pair of wave numbers, to derive dark solutions for local and nonlocal IDNLS equations. However, for the reverse discrete-time symmetric IDNLS Eq.(7), the soliton solution can be obtained directly from the one of the before-redcution IDNLS equation without the constraint condition for wave numbers. Therefore, in the present paper, we will apply the pair reduction to construct Gram determinant solutions for the local Eq.(4), the PT symmetric Eq.(5) and the reverse time symmetric Eq.(6).

This paper is organized as follows. In Section 2, we first introduce a generalized (before-redcution) IDNLS equation whose solution is expressed in terms of the single Casorati determinant. Then we convert the Casorati determinant solution to two kinds of the Gram determinant one. In Section 3, with the help of the pair reduction, Gram determinant solutions of local and nonlocal IDNLS equations are derived by imposing different constraint conditions. The last section is a summary and discussion.

2. Gram determinant solution of the generalized IDNLS equation

2.1. Casorati determinant solution

In order to derive Gram determinant solution from the known Casorati determinant one for the generalized IDNLS equation, we first recall the derivation of Casorati determinant solution in [28, 29].

The bilinear forms for Bäcklund Transformation (BT) of Toda lattice (TL) equation

$$(aD_x - 1)\tau_{n+1}(k+1,l)\cdot\tau_n(k,l) + \tau_n(k+1,l)\tau_{n+1}(k,l) = 0, \tag{8}$$

$$(bD_y - 1)\tau_{n-1}(k,l+1)\cdot\tau_n(k,l) + \tau_n(k,l+1)\tau_{n-1}(k,l) = 0, \tag{9}$$

and the bilinear form of discrete 2-dimensional Toda lattice (D2DTL) equation

$$\tau_n(k+1,l+1)\tau_n(k,l) - \tau_n(k+1,l)\tau_n(k,l+1) = ab[\tau_n(k+1,l+1)\tau_n(k,l) - \tau_{n+1}(k+1,l)\tau_{n-1}(k,l+1)], \tag{10}$$

with the constants a and b, have the following Casorati determinant solution

$$\tau_n(k,l) = \begin{vmatrix} \varphi_1^{(n)}(k,l) & \varphi_1^{(n+1)}(k,l) & \cdots & \varphi_1^{(n+N-1)}(k,l) \\ \varphi_2^{(n)}(k,l) & \varphi_2^{(n+1)}(k,l) & \cdots & \varphi_2^{(n+N-1)}(k,l) \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_N^{(n)}(k,l) & \varphi_N^{(n+1)}(k,l) & \cdots & \varphi_N^{(n+N-1)}(k,l) \end{vmatrix}, \tag{11}$$

where $\varphi_i^{(n)}(k,l)$ are functions of continuous independent variables x, y and discrete ones k, l, satisfying the dispersion relations as follows

$$\partial_x\varphi_i^{(n)}(k,l) = \varphi_i^{(n+1)}(k,l), \quad \partial_y\varphi_i^{(n)}(k,l) = \varphi_i^{(n-1)}(k,l), \tag{12}$$

$$\Delta_k\varphi_i^{(n)}(k,l) = \varphi_i^{(n+1)}(k,l), \quad \Delta_l\varphi_i^{(n)}(k,l) = \varphi_i^{(n-1)}(k,l). \tag{13}$$

Here Δ_k and Δ_l are the backward difference operators with respect to the difference intervals a and b given by

$$\Delta_k f(k,l) = \frac{f(k,l) - f(k-1,l)}{a}, \quad \Delta_l f(k,l) = \frac{f(k,l) - f(k,l-1)}{a}, \tag{14}$$

and the Hirota's bilinear operators D_x and D_y are defined as

$$D_x^n D_y^m (a\cdot b) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m a(x,y)b(x',y')\Big|_{x=x',y=y'}.$$

The Casorati determinant solution (11) can give rise to various types of solutions for the bilinear equations (8)-(10), since the matrix elements can be taken as any functions obeying the dispersion relations (12)-(13). In the following, the function $\varphi_i^{(n)}(k,l)$ is taken as

$$\varphi_i^{(n)}(k,l) = p_i^n(1-ap_i)^{-k}\left(1 - b\frac{1}{p_i}\right)^{-l}\exp(\xi_i) + q_i^n(1-aq_i)^{-k}\left(1 - b\frac{1}{q_i}\right)^{-l}\exp(\eta_i), \tag{15}$$

with

$$\xi_i = p_i x + \frac{1}{p_i}y + \xi_{i,0}, \quad \eta_i = q_i x + \frac{1}{q_i}y + \eta_{i,0},$$

where p_i, q_i and $\xi_{i,0}$, $\eta_{i,0}$ are arbitrary constants. This kind of choice leads to soliton solutions and p_i, q_i and $\xi_{i,0}$, $\eta_{i,0}$ represent the wave numbers and phase parameters of solitons, respectively.

In order to reduce the coupled system of TL's BT (8)-(9) and D2DTL (10) to the generalized IDNLS equation, one need to impose the constraint condition for the wave numbers in (15)

$$a p_i q_i + b - p_i - q_i = 0, \tag{16}$$

which implies

$$p_i^2 \frac{1 - b\frac{1}{p_i}}{1 - a p_i} = q_i^2 \frac{1 - b\frac{1}{q_i}}{1 - a q_i}. \tag{17}$$

This condition makes $\varphi_i^{(n)}(k,l)$ in (15) satisfy

$$\varphi_i^{(n+2)}(k+1, l-1) = p_i^2 \frac{1 - b\frac{1}{p_i}}{1 - a p_i} \varphi_i^{(n)}(k,l), \tag{18}$$

and then tau functions have the relations

$$\tau_{n+2}(k+1, l-1) = \left(\prod_{i=1}^{N} p_i^2 \frac{1 - b\frac{1}{p_i}}{1 - a p_i} \right) \tau_n(k,l). \tag{19}$$

Therefore, the bilinear forms of TL's BT (8)-(9) and D2DTL (10) become

$$(aD_x - 1)\tau_{n+1}(k+1, l) \cdot \tau_n(k,l) + \tau_n(k+1,l)\tau_{n+1}(k,l) = 0, \tag{20}$$

$$(bD_y - 1)\tau_{n+1}(k+1, l) \cdot \tau_n(k,l) + \tau_{n+2}(k+1,l)\tau_{n-1}(k,l) = 0, \tag{21}$$

$$\tau_{n+1}(k+1,l)\tau_{n-1}(k-1,l) - \tau_{n+1}(k,l)\tau_{n-1}(k,l) = ab[\tau_{n+1}(k+1,l)\tau_{n-1}(k-1,l) - \tau_n(k,l)\tau_n(k,l)]. \tag{22}$$

Here the parameter l is dropped by simply taking $l = 0$. Furthermore, by using the independent variable transformation

$$x = \mathrm{i}act, \quad y = \mathrm{i}bdt, \quad \text{i.e.,} \quad -\mathrm{i}\partial_t = ac\partial_x + bd\partial_y, \tag{23}$$

where c and d are constants, and defining

$$f_n = \tau_n(0), \quad g_n = \tau_{n+1}(1), \quad h_n = \tau_{n-1}(-1), \tag{24}$$

the above bilinear equations converts to the bilinear form of the generalized IDNLS equation

$$(-\mathrm{i}D_t - c - d)g_n \cdot f_n + d g_{n+1} f_{n-1} + c g_{n-1} f_{n+1} = 0, \tag{25}$$

$$(\mathrm{i}D_t - c - d)h_n \cdot f_n + c h_{n+1} f_{n-1} + d h_{n-1} f_{n+1} = 0, \tag{26}$$

$$f_{n+1} f_{n-1} - f_n^2 = (ab - 1)(f_n^2 - g_n h_n), \tag{27}$$

which have the solution in terms of Casorati determinant as follows

$$f_n = |F| = \left| p_i^{n+j-1} e^{\xi_i} + q_i^{n+j-1} e^{\eta_i} \right|, \tag{28}$$

$$g_n = |G| = \left| \frac{p_i^{n+j}}{1 - a p_i} e^{\xi_i} + \frac{q_i^{n+j}}{1 - a q_i} e^{\eta_i} \right|, \tag{29}$$

$$h_n = |H| = \left| (1 - a p_i) p_i^{n+j-2} e^{\xi_i} + (1 - a q_i) q_i^{n+j-2} e^{\eta_i} \right|, \tag{30}$$

with

$$\xi_i = \mathrm{i}\left(ac p_i + \frac{bd}{p_i} \right) t + \xi_{i,0}, \quad \eta_i = \mathrm{i}\left(ac q_i + \frac{bd}{q_i} \right) t + \eta_{i,0} \tag{31}$$

where the wave numbers p_i and q_i need to satisfy the constraint condition (16).

Through the dependent variable transformation

$$u_n=\frac{g_n}{f_n},\ \ v_n=\frac{h_n}{f_n} \tag{32}$$

one can get the generalized IDNLS equation

$$\mathrm{i}\frac{du_n}{dt}+(c+d)u_n-[ab-(ab-1)u_nv_n](du_{n+1}+cu_{n-1})=0, \tag{33}$$

$$-\mathrm{i}\frac{dv_n}{dt}+(c+d)v_n-[ab-(ab-1)u_nv_n](cv_{n+1}+dv_{n-1})=0. \tag{34}$$

2.2. Gram determinant solution

In this section, we will transform the above Casorati determinant solution to two kinds of Gram determinant one for the generalized IDNLS equation. To this end, we change exponential functions as

$$\exp(\xi_i)=\left[\prod_{\substack{k=1\\k\neq i}}^{N}(q_k-p_i)\right]^{-1}\exp(\xi_i'),\ \ \exp(\eta_i)=\left[\prod_{\substack{k=1\\k\neq i}}^{N}(q_k-q_i)\right]^{-1}\exp(\eta_i'), \tag{35}$$

with

$$\xi_i'=\mathrm{i}\left(acp_i+\frac{bd}{p_i}\right)t+\xi_{i,0}',\ \ \eta_i'=\mathrm{i}\left(acq_i+\frac{bd}{q_i}\right)t+\eta_{i,0}',$$

and introduce the following $2N\times 2N$ diagonal matrices

$$A=\mathrm{Diag}(a_1,a_2\cdots,a_{2N}),\ \ a_i=\frac{1}{q_i-p_i}, \tag{36}$$

$$B=\mathrm{Diag}(b_1,b_2\cdots,b_{2N}),\ \ b_i=\frac{1}{p_i^ne^{\xi_i'}}, \tag{37}$$

$$C=\mathrm{Diag}(c_1,c_2,\cdots,c_{2N}),\ \ c_i=\frac{1-ap_i}{p_i}, \tag{38}$$

and a Vandermonde matrix

$$V_q=\left[(-1)^{2N-i}\sum_{\substack{1\le k_1<k_2<\cdots<k_{2N-i}\le 2N\\k_l\neq j}}\left(\prod_{l=1}^{2N-i}q_{k_l}\right)\right]_{2N\times 2N}. \tag{39}$$

For example, when $N=1$ and $N=2$, the Vandermonde matrices read

$$V_q=\begin{pmatrix}-q_2 & -q_1\\ 1 & 1\end{pmatrix}, \tag{40}$$

$$V_q=\begin{pmatrix}-q_2q_3q_4 & -q_1q_3q_4 & -q_1q_2q_4 & -q_1q_2q_3\\ q_2q_3+q_2q_4+q_3q_4 & q_1q_3+q_1q_4+q_3q_4 & q_1q_2+q_1q_4+q_2q_4 & q_1q_2+q_1q_3+q_2q_3\\ -q_2-q_3-q_4 & -q_1-q_3-q_4 & -q_1-q_2-q_4 & -q_1-q_2-q_3\\ 1 & 1 & 1 & 1\end{pmatrix}. \tag{41}$$

Due to the gauge freedom, we can find the first kind of Gram determinant solution

$$\tilde{f}_n=|\tilde{F}|=|ABFV_q|=\left|\frac{1}{p_i-q_j}+\delta_{ij}\frac{1}{p_i-q_j}\frac{q_i^ne^{\eta_i'}}{p_j^ne^{\xi_j'}}\right|_{2N\times 2N}, \tag{42}$$

$$\tilde{g}_n=|\tilde{G}|=|CABFV_q|=\left|\frac{1}{p_i-q_j}+\delta_{ij}\left[\frac{q_i(1-ap_i)}{p_j(1-aq_j)}\right]\frac{1}{p_i-q_j}\frac{q_i^ne^{\eta_i'}}{p_j^ne^{\xi_j'}}\right|_{2N\times 2N}, \tag{43}$$

$$\tilde{h}_n=|\tilde{H}|=|C^{-1}ABFV_q|=\left|\frac{1}{p_i-q_j}+\delta_{ij}\left[\frac{p_i(1-aq_i)}{q_j(1-ap_j)}\right]\frac{1}{p_i-q_j}\frac{q_i^ne^{\eta_i'}}{p_j^ne^{\xi_j'}}\right|_{2N\times 2N}, \tag{44}$$

still satisfy the bilinear IDNLS equations (25)-(27).

For the second case, we need to introduce another group of $2N \times 2N$ diagonal matrices

$$P = \mathrm{Diag}(1, 1 \cdots, 1_N; p_{N+1}, p_{N+2}, \cdots, p_{2N}), \tag{45}$$

$$Q = \mathrm{Diag}(1, 1 \cdots, 1_N; q_{N+1}, q_{N+2}, \cdots, q_{2N}). \tag{46}$$

Owing to the gauge freedom, one can derive the second kind of Gram determinant solution

$$\tilde{f}_n = |\tilde{F}| = |PABFV_qQ| = \begin{vmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{vmatrix}_{2N\times 2N}, \tag{47}$$

$$\tilde{g}_n = |\tilde{G}| = |CPABFV_qQ| = \begin{vmatrix} \mathcal{A}_{(g)} & \mathcal{B} \\ \mathcal{C} & \mathcal{D}_{(g)} \end{vmatrix}_{2N\times 2N}, \tag{48}$$

$$\tilde{h}_n = |\tilde{H}| = |C^{-1}PABFV_qQ| = \begin{vmatrix} \mathcal{A}_{(h)} & \mathcal{B} \\ \mathcal{C} & \mathcal{D}_{(h)} \end{vmatrix}_{2N\times 2N}, \tag{49}$$

with the block matrices given by

$$\mathcal{A} = \left(a_{ij}(0)\right)_{N\times N}, \quad \mathcal{A}_{(g)} = \left(a_{ij}(1)\right)_{N\times N}, \quad \mathcal{A}_{(h)} = \left(a_{ij}(-1)\right)_{N\times N}, \tag{50}$$

$$\mathcal{D} = \left(d_{ij}(0)\right)_{N\times N}, \quad \mathcal{D}_{(g)} = \left(d_{ij}(1)\right)_{N\times N}, \quad \mathcal{D}_{(h)} = \left(d_{ij}(-1)\right)_{N\times N}, \tag{51}$$

$$\mathcal{B} = \left(b_{ij}\right)_{N\times N}, \quad \mathcal{C} = \left(c_{ij}\right)_{N\times N}, \tag{52}$$

where the elements are defined by

$$a_{ij}(k) = \frac{1}{p_i - q_j} + \delta_{ij}\frac{1}{p_i - q_j}\left[\frac{q_i(1-ap_i)}{p_j(1-aq_j)}\right]^k \frac{q_i^n e^{\eta'_i}}{p_j^n e^{\xi'_j}}, \quad b_{ij} = \frac{q_{N+j}}{p_i - q_{N+j}}, \tag{53}$$

$$c_{ij} = \frac{p_{N+i}}{p_{N+i} - q_j}, \quad d_{ij}(k) = \frac{p_{N+i}q_{N+j}}{p_{N+i} - q_{N+j}} + \delta_{ij}\frac{p_{N+i}q_{N+j}}{p_{N+i} - q_{N+j}}\left[\frac{q_{N+i}(1-ap_{N+i})}{p_{N+j}(1-aq_{N+j})}\right]^k \frac{q_{N+i}^n e^{\eta'_{N+i}}}{p_{N+j}^n e^{\xi'_{N+j}}}. \tag{54}$$

This Gram determinant solution also satisfy the bilinear IDNLS equations (25)-(27).

3. Reduction to local and nonlocal IDNLS equations

3.1. *The local IDNLS equation (4)*

For the local IDNLS equation (4), we start from the second kind of Gram determinant solution (47)-(49) via the pair reduction. To be specific, by imposing the pair conditions

$$p_{N+i} = \frac{1}{q_i^*}, \quad q_{N+i} = \frac{1}{p_i^*}, \quad b = a^*, \tag{55}$$

one can find that the constraint condition (16) for p_{N+i} and q_{N+i} is complex conjugate of one for p_i and q_i. If we further restrict

$$d = c^*, \quad \xi'_{N+i,0} = -\eta'^*_{i,0}, \quad \eta'_{N+i,0} = -\xi'^*_{i,0}, \tag{56}$$

then the following relations hold

$$\xi'_{N+i} = -\eta'^*_i, \quad \eta'_{N+i} = -\xi'^*_i, \quad p_{N+i}^n e^{\xi'_{N+i}} = q_i^{*-n} e^{-\eta'^*_i}, \quad q_{N+i}^n e^{\eta'_{N+i}} = p_i^{*-n} e^{-\xi'^*_i}. \tag{57}$$

The elements in the Gram determinant solution (47)-(49) are rewritten as

$$a_{ij}(k) = \frac{1}{p_i - q_j} + \delta_{ij}\frac{1}{p_i - q_j}\left[\frac{q_i(1-ap_i)}{p_i(1-aq_i)}\right]^k \frac{q_i^n e^{\eta_i'}}{p_j^n e^{\xi_j'}},\quad b_{ij} = \frac{1}{p_i p_j^* - 1}, \tag{58}$$

$$c_{ij} = \frac{1}{1 - q_i^* q_j},\quad d_{ij}(k) = \frac{1}{p_j^* - q_i^*} + \delta_{ij}\frac{1}{p_j^* - q_i^*}\left[\frac{a - q_i^*}{a - p_i^*}\right]^k \frac{q_j^{*n} e^{\eta_j'^*}}{p_i^{*n} e^{\xi_i'^*}}. \tag{59}$$

In this case, we have $\mathcal{A}^* = \mathcal{D}^T$, $\mathcal{B}^* = \mathcal{B}^T$ and $C^* = C^T$. Notice that $a^* - p_i = q_i(1-ap_i)$ and $a^* - q_i = p_i(1-aq_i)$ which yield $\mathcal{A}^*_{(g)} = \mathcal{D}^T_{(h)}$ and $\mathcal{A}^*_{(h)} = \mathcal{D}^T_{(g)}$, thus one can get

$$\tilde{f}_n^* = |T\tilde{F}^T T| = |\tilde{F}| = \tilde{f}_n,\quad \tilde{g}_n^* = |T\tilde{H}^T T| = |\tilde{F}| = \tilde{h}_n, \tag{60}$$

where the matrix T is defined by

$$T = \begin{pmatrix} \mathbf{0} & I_{N\times N} \\ I_{N\times N} & \mathbf{0} \end{pmatrix}_{2N\times 2N}. \tag{61}$$

From the transformation (32), it immediately reaches $u_n^* = \frac{\tilde{g}_n^*}{\tilde{f}_n^*} = \frac{\tilde{h}_n}{\tilde{f}_n} = v_n$. Furthermore, through the variable transformations

$$u_n = \frac{\psi_n}{\sqrt{\alpha}}\exp\left(-\mathrm{i}n\theta + \mathrm{i}\frac{e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta}}{|a|^2}t - 2\mathrm{i}t\right),\quad \alpha = \frac{|a|^2 - 1}{\delta|a|^2},\quad c = \frac{e^{-\mathrm{i}\theta}}{|a|^2}, \tag{62}$$

with $\delta = 1(|a|^2 > 1)$ and $\delta = -1(|a|^2 < 1)$, the generalized IDNLS equation (33)-(34) reduces to the local IDNLS equation (4). Finally, we arrive at the following theorem about $2N$-soliton solution of the local IDNLS equation (4).

Theorem 3.1 The local IDNLS equation (4) has the solution

$$\psi_n = \sqrt{\frac{|a|^2 - 1}{\delta|a|^2}}\, e^{\left(\mathrm{i}n\theta - \mathrm{i}\frac{e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta}}{|a|^2}t + 2\mathrm{i}t\right)}\frac{\tilde{g}_n}{\tilde{f}_n}, \tag{63}$$

with $\delta = 1(|a|^2 > 1)$ and $\delta = -1(|a|^2 < 1)$, where tau functions are given by

$$\tilde{f}_n = \begin{vmatrix} \mathcal{A} & \mathcal{B} \\ C & \mathcal{D} \end{vmatrix}_{2N\times 2N},\quad \tilde{g}_n = \begin{vmatrix} \mathcal{A}_{(g)} & \mathcal{B} \\ C & \mathcal{D}_{(g)} \end{vmatrix}_{2N\times 2N}, \tag{64}$$

with the block matrices

$$\mathcal{A} = \left(\frac{1}{p_i - q_j} + \delta_{ij}\frac{1}{p_i - q_j}\frac{q_i^n e^{\eta_i'}}{p_j^n e^{\xi_j'}}\right)_{N\times N},\quad \mathcal{A}_{(g)} = \left(\frac{1}{p_i - q_j} + \delta_{ij}\frac{1}{p_i - q_j}\left[\frac{q_i(1-ap_i)}{p_i(1-aq_i)}\right]\frac{q_i^n e^{\eta_i'}}{p_j^n e^{\xi_j'}}\right)_{N\times N},$$

$$\mathcal{D} = \left(\frac{1}{p_j^* - q_i^*} + \delta_{ij}\frac{1}{p_j^* - q_i^*}\frac{q_j^{*n} e^{\eta_j'^*}}{p_i^{*n} e^{\xi_i'^*}}\right)_{N\times N},\quad \mathcal{D}_{(g)} = \left(\frac{1}{p_j^* - q_i^*} + \delta_{ij}\frac{1}{p_j^* - q_i^*}\left[\frac{a - q_i^*}{a - p_i^*}\right]\frac{q_j^{*n} e^{\eta_j'^*}}{p_i^{*n} e^{\xi_i'^*}}\right)_{N\times N},$$

and

$$\mathcal{B} = \left(\frac{1}{p_i p_j^* - 1}\right)_{N\times N},\quad C = \left(\frac{1}{1 - q_i^* q_j}\right)_{N\times N}.$$

Here $\xi_i' = \mathrm{i}\left(\frac{ae^{-\mathrm{i}\theta}}{|a|^2}p_i + \frac{a^* e^{\mathrm{i}\theta}}{|a|^2 p_i}\right)t + \xi_{i,0}'$ and $\eta_i' = \mathrm{i}\left(\frac{ae^{-\mathrm{i}\theta}}{|a|^2}q_i + \frac{a^* e^{\mathrm{i}\theta}}{|a|^2 q_i}\right)t + \eta_{i,0}'$, p_i, q_i $\xi_{i,0}'$,$\eta_{i,0}'$ $(i = 1, 2\cdots, N)$ and a are arbitrary complex constants, θ is an arbitrary real constant, these parameters need to satisfy the constraint conditions

$$ap_iq_i + a^* - p_i - q_i = 0. \tag{65}$$

The above solution holds for both focusing and defocusing cases in the local IDNLS equation (4). As pointed out in [28], this kind of solution corresponds to the homoclinic orbit one of the local IDNLS equation.

For example, by taking $N = 1$, tau functions for the local IDNLS equation (4) are written as:

$$f_n = \Delta_1 + \Delta_2 \left(1 + \frac{q_1^n e^{\eta_1'}}{p_1^n e^{\xi_1'}} + \frac{q_1^{*n} e^{\eta_1'^*}}{p_1^{*n} e^{\xi_1'^*}} + \frac{|q_1|^{2n} e^{\eta_1' + \eta_1'^*}}{|p_1|^{2n} e^{\xi_1' + \xi_1'^*}} \right), \tag{66}$$

$$g_n = \Delta_1 + \Delta_2 \left(1 + P_1 \frac{q_1^n e^{\eta_1'}}{p_1^n e^{\xi_1'}} + P_1^* \frac{q_1^{*n} e^{\eta_1'^*}}{p_1^{*n} e^{\xi_1'^*}} + |P_1|^2 \frac{|q_1|^{2n} e^{\eta_1' + \eta_1'^*}}{|p_1|^{2n} e^{\xi_1' + \xi_1'^*}} \right), \tag{67}$$

where

$$\Delta_1 = \frac{1}{(|p_1|^2 - 1)(|q_1|^2 - 1)}, \quad \Delta_2 = \frac{1}{|p_1 - q_1|^2}, \quad P_1 = \frac{a^* - p_1}{a^* - q_1},$$

and $\xi_1' = \mathrm{i}\left(\frac{ae^{-\mathrm{i}\theta}}{|a|^2} p_1 + \frac{a^* e^{\mathrm{i}\theta}}{|a|^2 p_1}\right) t + \xi_{1,0}'$ and $\eta_1' = \mathrm{i}\left(\frac{ae^{-\mathrm{i}\theta}}{|a|^2} q_1 + \frac{a^* e^{\mathrm{i}\theta}}{|a|^2 q_1}\right) t + \eta_{1,0}'$, p_1, q_1 $\xi_{1,0}'$,$\eta_{1,0}'$ and a are arbitrary complex constants, θ is an arbitrary real constant, they need to satisfy the constraint condition $ap_1q_1 + a^* - p_1 - q_1 = 0$.

3.2. The PT symmetric IDNLS equation (5)

For nonlocal IDNLS equation (5), we carry out the pair reduction on the first kind of Gram determinant solution (42)-(44). For this purpose, we first impose the pair conditions

$$p_{N+i} = q_i^*, \quad q_{N+i} = p_i^*, \quad a^* = a, \quad b^* = b, \tag{68}$$

which ensure that the constraint condition (16) for p_{N+i} and q_{N+i} is complex conjugate of one for p_i and q_i. If we further require

$$c^* = c, \quad d^* = d, \quad \xi_{N+i,0}' = -\eta_{i,0}'^*, \quad \eta_{N+i,0}' = -\xi_{i,0}'^*, \tag{69}$$

one can find the relations

$$\xi_{N+i}' = -\eta_i'^*, \quad \eta_{N+i}' = -\xi_i'^*, \quad p_{N+i}^n e^{\xi_{N+i}'} = q_i^{*n} e^{-\eta_i'^*}, \quad q_{N+i}^n e^{\eta_{N+i}'} = p_i^{*n} e^{-\xi_i'^*}. \tag{70}$$

The Gram determinant solution (42)-(44) can be rewritten as

$$\tilde{f}_n = |\tilde{F}| = \begin{vmatrix} \mathcal{A} & \mathcal{B} \\ C & \mathcal{D} \end{vmatrix}_{2N\times 2N}, \quad \tilde{g}_n = |\tilde{G}| = \begin{vmatrix} \mathcal{A}_{(g)} & \mathcal{B} \\ C & \mathcal{D}_{(g)} \end{vmatrix}_{2N\times 2N}, \quad \tilde{h}_n = |\tilde{H}| = \begin{vmatrix} \mathcal{A}_{(h)} & \mathcal{B} \\ C & \mathcal{D}_{(h)} \end{vmatrix}_{2N\times 2N}, \tag{71}$$

with the block matrices given by (50)-(52) whose elements are defined by

$$a_{ij}(k) = \frac{1}{p_i - q_j} + \delta_{ij} \frac{1}{p_i - q_j} \left[\frac{q_i(1 - ap_i)}{p_j(1 - aq_j)} \right]^k \frac{q_i^n e^{\eta_i'}}{p_j^n e^{\xi_j'}}, \quad b_{ij} = \frac{1}{p_i - p_j^*}, \tag{72}$$

$$c_{ij} = \frac{1}{q_i^* - q_j}, \quad d_{ij}(k) = \frac{1}{q_i^* - p_j^*} + \delta_{ij} \frac{1}{q_i^* - p_j^*} \left[\frac{p_i^*(1 - aq_i^*)}{q_i^*(1 - ap_i^*)} \right]^k \frac{p_i^{*n} e^{\eta_i'^*}}{q_j^{*n} e^{\xi_j'^*}}. \tag{73}$$

In this situation, we have $\mathcal{A}^*(-n) = -\mathcal{D}^T$, $\mathcal{A}^*(-n)_{(g)} = -\mathcal{D}^T_{(h)}$, $\mathcal{A}^*(-n)_{(h)} = -\mathcal{D}^T_{(g)}$, $\mathcal{B}^* = -\mathcal{B}^T$ and $C^* = -C^T$, thus one can get

$$\tilde{f}_{-n}^* = |-T\tilde{F}^T T| = |\tilde{F}| = \tilde{f}_n, \quad \tilde{g}_{-n}^* = |-T\tilde{H}^T T| = |\tilde{F}| = \tilde{h}_n. \tag{74}$$

From the transformation (32), it immediately reaches $u_{-n}^* = \frac{\tilde{g}_{-n}^*}{\tilde{f}_{-n}^*} = \frac{\tilde{h}_n}{\tilde{f}_n} = v_n$. Furthermore, through the variable transformations

$$u_n = \frac{\psi_n}{\sqrt{\alpha}} \exp\left(-n\theta + \mathrm{i}\frac{e^{\theta} + e^{-\theta}}{ab} t - 2\mathrm{i}t\right), \quad \alpha = \frac{ab - 1}{\delta ab}, \quad c = \frac{e^{-\theta}}{ab}, \quad d = \frac{e^{\theta}}{ab}, \tag{75}$$

with $\delta = 1(ab > 1, \text{or } ab < 0)$ and $\delta = -1(0 < ab < 1)$, we obtain the PT symmetric IDNLS equation (5). Finally, we get the following theorem about the solution of the nonlocal IDNLS equation (5).

Theorem 3.2 The nonlocal PT symmetric IDNLS equation (5) has the solution

$$\psi_n = \sqrt{\frac{ab-1}{\delta ab}} e^{\left(n\theta - \mathrm{i}\frac{e^{\theta}+e^{-\theta}}{ab}t + 2\mathrm{i}t\right)} \frac{\tilde{g}_n}{\tilde{f}_n}, \tag{76}$$

with $\delta = 1(ab > 1, \text{or } ab < 0)$ and $\delta = -1(0 < ab < 1)$, where tau functions are given by

$$\tilde{f}_n = \begin{vmatrix} \mathcal{A} & \mathcal{B} \\ C & \mathcal{D} \end{vmatrix}_{2N\times 2N}, \quad \tilde{g}_n = \begin{vmatrix} \mathcal{A}_{(g)} & \mathcal{B} \\ C & \mathcal{D}_{(g)} \end{vmatrix}_{2N\times 2N}, \tag{77}$$

with the block matrices

$$\mathcal{A} = \left(\frac{1}{p_i - q_j} + \delta_{ij}\frac{1}{p_i - q_j}\frac{q_i^n e^{\eta_i'}}{p_j^n e^{\xi_j'}}\right)_{N\times N}, \quad \mathcal{A}_{(g)} = \left(\frac{1}{p_i - q_j} + \delta_{ij}\frac{1}{p_i - q_j}\left[\frac{q_i(1-ap_i)}{p_j(1-aq_j)}\right]\frac{q_i^n e^{\eta_i'}}{p_j^n e^{\xi_j'}}\right)_{N\times N},$$

$$\mathcal{D} = \left(\frac{1}{q_i^* - p_j^*} + \delta_{ij}\frac{1}{q_i^* - p_j^*}\frac{p_i^{*n} e^{\eta_i'^*}}{q_j^{*n} e^{\xi_j'^*}}\right)_{N\times N}, \quad \mathcal{D}_{(g)} = \left(\frac{1}{q_i^* - p_j^*} + \delta_{ij}\frac{1}{q_i^* - p_j^*}\left[\frac{p_i^*(1-aq_i^*)}{q_i^*(1-ap_i^*)}\right]\frac{p_i^{*n} e^{\eta_i'^*}}{q_j^{*n} e^{\xi_j'^*}}\right)_{N\times N},$$

and

$$\mathcal{B} = \left(\frac{1}{p_i - p_j^*}\right)_{N\times N}, \quad C = \left(\frac{1}{q_i^* - q_j}\right)_{N\times N}.$$

Here $\xi_i' = \mathrm{i}\left(\frac{e^{-\theta}}{b}p_i + \frac{e^{\theta}}{ap_i}\right)t + \xi_{i,0}'$ and $\eta_i' = \mathrm{i}\left(\frac{e^{-\theta}}{b}q_i + \frac{e^{\theta}}{aq_i}\right)t + \eta_{i,0}'$, p_i, q_i $\xi_{i,0}'$ and $\eta_{i,0}'$ $(i = 1, 2\cdots, N)$ are arbitrary complex constants, a, b and θ are arbitrary real constants, these parameters need to satisfy the constraint conditions

$$ap_iq_i + b - p_i - q_i = 0. \tag{78}$$

For example, tau functions for the nonlocal IDNLS equation (5) are written as:

$$f_n = \Delta_1 + \Delta_2\left(1 + \frac{q_1^n e^{\eta_1'}}{p_1^n e^{\xi_1'}} + \frac{p_1^{*n} e^{\eta_1'^*}}{q_1^{*n} e^{\xi_1'^*}} + \frac{q_1^n p_1^{*n} e^{\eta_1' + \eta_1'^*}}{p_1^n q_1^{*n} e^{\xi_1' + \xi_1'^*}}\right), \tag{79}$$

$$g_n = \Delta_1 + \Delta_2\left(1 + P_1\frac{q_1^n e^{\eta_1'}}{p_1^n e^{\xi_1'}} + \frac{1}{P_1^*}\frac{p_1^{*n} e^{\eta_1'^*}}{q_1^{*n} e^{\xi_1'^*}} + \frac{P_1}{P_1^*}\frac{q_1^n p_1^{*n} e^{\eta_1' + \eta_1'^*}}{p_1^n q_1^{*n} e^{\xi_1' + \xi_1'^*}}\right) \tag{80}$$

where

$$\Delta_1 = \frac{1}{(p_1 - p_1^*)(q_1 - q_1^*)}, \quad \Delta_2 = -\frac{1}{|p_1 - q_1|^2}, \quad P_1 = \frac{q_1(1-ap_1)}{p_1(1-aq_1)},$$

and $\xi_1' = \mathrm{i}\left(\frac{e^{-\theta}}{b}p_1 + \frac{e^{\theta}}{ap_1}\right)t + \xi_{1,0}'$ and $\eta_1' = \mathrm{i}\left(\frac{e^{-\theta}}{b}q_1 + \frac{e^{\theta}}{aq_1}\right)t + \eta_{1,0}'$, p_1, q_1 $\xi_{1,0}'$ and $\eta_{1,0}'$ are arbitrary complex constants, a, b and θ are arbitrary real constants, they need to satisfy the constraint condition $ap_1q_1 + b - p_1 - q_1 = 0$. To investigate the asymptotic behavior of above two-soliton solution (79)-(80), we redefine the following notations:

$$\hat{\theta} = n\theta - \mathrm{i}\frac{e^{\theta}+e^{-\theta}}{ab}t + 2\mathrm{i}t, \quad \hat{\rho} = \sqrt{\frac{ab-1}{\delta ab}}, \quad \frac{q_1}{p_1} = e^{\rho + \mathrm{i}\varphi}, \quad P_1 = e^{c_0 + \mathrm{i}\theta_1}, \quad \frac{\Delta_2}{\Delta_1 + \Delta_2} = \hat{\delta}e^{c_1}, \tag{81}$$

$$\eta_1' - \xi_1' = (A + \mathrm{i}B)t + A_0 + \mathrm{i}B_0, \quad \frac{q_1^n e^{\eta_1'}}{p_1^n e^{\xi_1'}} = e^{\chi_1}, \quad \frac{p_1^{*n} e^{\eta_1'^*}}{q_1^{*n} e^{\xi_1'^*}} = e^{\chi_2}. \tag{82}$$

Without loss of generality, we assume $A > 0$ and then have the asymptotic forms as follows:

(i) Before collision $(t \to -\infty)$

Soliton 1 $(\chi_1 \approx 0, \chi_2 \approx -\infty)$:

$$\psi_n \to \hat{\rho}e^{\hat{\theta}}\frac{1 + \hat{\delta}e^{X_1^+ + c_1 + c_0}e^{\mathrm{i}(Y_1^+ + \theta_1)}}{1 + \hat{\delta}e^{X_1^+ + c_1}e^{\mathrm{i}Y_1^+}}, \quad |\psi_n|^2 \to \left|\frac{ab-1}{\delta ab}\right| e^{2n\theta + c_0}\frac{\cosh(X_1^+ + c_1 + c_0) + \hat{\delta}\cos(Y_1^+ + \theta_1)}{\cosh(X_1^+ + c_1) + \hat{\delta}\cos(Y_1^+)}. \tag{83}$$

Soliton 2 ($\chi_2 \approx 0, \chi_1 \approx -\infty$):

$$\psi_n \to \hat{\rho} e^{\hat{\theta}} \frac{1 + \hat{\delta} e^{-X_1^- + c_1 - c_0} e^{\mathrm{i}(Y_1^- + \theta_1)}}{1 + \hat{\delta} e^{-X_1^- + c_1} e^{\mathrm{i}Y_1^-}}, \quad |\psi_n|^2 \to \left|\frac{ab-1}{\delta ab}\right| e^{2n\theta - c_0} \frac{\cosh(X_1^- - c_1 + c_0) + \hat{\delta}\cos(Y_1^- + \theta_1)}{\cosh(X_1^- - c_1) + \hat{\delta}\cos(Y_1^-)}. \tag{84}$$

(ii) After collision ($t \to +\infty$)

Soliton 1 ($\chi_1 \approx 0, \chi_2 \approx +\infty$):

$$\psi_n \to \hat{\rho} e^{\hat{\theta}} \frac{p_1}{p_1^*} \frac{1 + e^{-X_1^+ - c_0} e^{-\mathrm{i}(Y_1^+ + \theta_1)}}{1 + e^{-X_1^+} e^{-\mathrm{i}Y_1^+}}, \quad |\psi_n|^2 \to \left|\frac{ab-1}{\delta ab}\right| e^{2n\theta - c_0} \frac{\cosh(X_1^+ + c_0) + \cos(Y_1^- + \theta_1)}{\cosh(X_1^+) + \hat{\delta}\cos(Y_1^+)}. \tag{85}$$

Soliton 2 ($\chi_2 \approx 0, \chi_1 \approx +\infty$):

$$\psi_n \to \hat{\rho} e^{\hat{\theta}} \frac{p_1}{p_1^*} \frac{1 + e^{X_1^- + c_0} e^{-\mathrm{i}(Y_1^- + \theta_1)}}{1 + e^{X_1^-} e^{-\mathrm{i}Y_1^-}}, \quad |\psi_n|^2 \to \left|\frac{ab-1}{\delta ab}\right| e^{2n\theta + c_0} \frac{\cosh(X_1^- + c_0) + \cos(Y_1^- + \theta_1)}{\cosh(X_1^-) + \hat{\delta}\cos(Y_1^-)}. \tag{86}$$

Here $X_1^{\pm} = n\rho \pm At \pm A_0$ and $Y_1^{\pm} = n\varphi \pm Bt \pm B_0$.

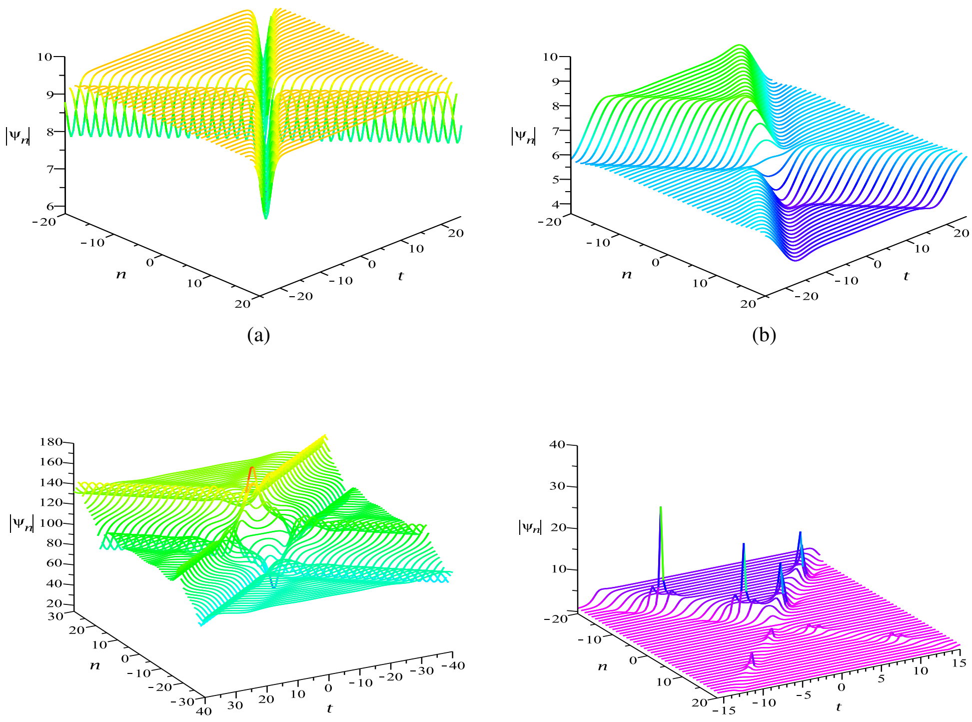

Figure 1: The solution (79)-(80) of the PT symmetric IDNLS equation (5) with the parameters (a) $a = 1$, $b = \frac{18}{5}$, $p_1 = 3 + 3\mathrm{i}$ and $q_1 = \frac{3}{5} + \frac{3}{5}\mathrm{i}$; (b) $a = 1$, $b = 3$, $p_1 = 1 + \mathrm{i}$ and $q_1 = 1 + 2\mathrm{i}$; (c) $a = 1$, $b = 11$, $p_1 = 2 + 3\mathrm{i}$ and $q_1 = 3\mathrm{i}$; (d) $a = 1$, $b = \frac{3}{2}$, $p_1 = 2 + 3\mathrm{i}$ and $q_1 = \frac{19}{20} + \frac{3}{20}\mathrm{i}$.

In the previous work [27], one of authors have applied a direct reduction $q_i = p_i^*$ to derive the solution of the nonlocal IDNLS equation (5). When $N = 1$, the solution reads

$$\psi_n = \sqrt{\frac{ab-1}{\delta ab}} e^{\left(n\theta - \mathrm{i}\frac{e^{\theta}+e^{-\theta}}{ab}t + 2\mathrm{i}t\right)} \left[\frac{p_1^*(1-ap_1)}{p_1(1-ap_1^*)}\right] \frac{1 + \frac{p_1(1-ap_1^*)}{p_1^*(1-ap_1)}\left(\frac{p_1}{p_1^*}\right)^n e^{\xi_1' + \xi_1'^*}}{1 + \left(\frac{p_1}{p_1^*}\right)^n e^{\xi_1' + \xi_1'^*}}, \tag{87}$$

where, $\delta = 1(ab < 0)$ and $\delta = -1(0 < ab < 1)$, $\xi_i' = \mathrm{i}\left(\frac{p_i}{be^{\theta}} + \frac{e^{\theta}}{ap_i}\right)t + \xi_{i,0}'$, p_1 and $\xi_{1,0}'$ are arbitrary complex parameters, a, b and θ are arbitrary real parameters and they need to satisfy $1 - ap_1 = -\frac{p_1}{p_1^*}(1 - \frac{b}{p_1})$ and $ab < 1$. For the solution (87), if we set $p_i = \exp(\alpha_i + \mathrm{i}\beta_i)$, $\left[\frac{(1-ap_1^*)}{(1-ap_1)}\right] = e^{2\mathrm{i}\gamma_1}$ and define $\xi_1 + \xi_1^* \equiv 2\vartheta_1$, then the modular square of

ψ_n is given by

$$|\psi_n|^2 = \exp(2n\theta)\left|\frac{1-ab}{ab}\right|\left[1 - \frac{2\sin(2n\beta_1+\beta_1+\gamma)\sin(\beta_1+\gamma)}{\cosh(2\vartheta_1)+\cos(2n\beta_1)}\right], \tag{88}$$

which means the singularity at $\beta_1 = k\pi$ $(k = \pm1, \pm2, \cdots)$.

However, the solution (79)-(80) may allow the nonsingular case under the suitable choice of parameters. From the above asymptotic analysis, it is found that the requirement $\varphi = B = B_0 = 0$ is an sufficient condition for the nonsingular solution. We illustrate the solution (79)-(80) in Fig.1 with the parameters $\xi'_{1,0} = \eta'_{1,0} = 0$, in which θ is taken as zero to avoid the amplitude increases/decreases exponentially. Under the special parameters satisfying $\varphi = B = B_0 = 0$, Figure 1(a) exhibits a nonsingular two-dark soliton. Other three examples in Fig.1 are singular solutions, although they look like two kink soliton, the interaction of kink and breathing soliton and two breathing-soliton, respectively.

3.3. The reverse time discrete symmetric IDNLS equation (6)

For the nonlocal IDNLS equation (6), we consider the pair reduction on the second kind of Gram determinant solution (47)-(49). More specificly, by imposing the pair conditions

$$p_{N+i} = \frac{1}{q_i}, \quad q_{N+i} = \frac{1}{p_i}, \quad b = a, \tag{89}$$

it is found that the constraint condition (16) for p_{N+i} and q_{N+i} is consistent with one for p_i and q_i. If we further require

$$d = c, \quad \xi'_{N+i,0} = -\eta'_{i,0}, \quad \eta'_{N+i,0} = -\xi'_{i,0}, \tag{90}$$

then the following relations hold

$$\xi'_{N+i} = -\eta'_i(-t), \quad \eta'_{N+i} = -\xi'_i(-t), \quad p^n_{N+i}e^{\xi'_{N+i}} = q_i^{-n}e^{-\eta'_i(-t)}, \quad q^n_{N+i}e^{\eta'_{N+i}} = p_i^{-n}e^{-\xi'_i(-t)}. \tag{91}$$

The elements in the Gram determinant solution (47)-(49) are rewritten as

$$a_{ij}(k) = \frac{1}{p_i - q_j} + \delta_{ij}\frac{1}{p_i - q_j}\left[\frac{q_i(1-ap_i)}{p_i(1-aq_i)}\right]^k \frac{q_i^n e^{\eta'_i}}{p_j^n e^{\xi'_j}}, \quad b_{ij} = \frac{1}{p_ip_j - 1}, \tag{92}$$

$$c_{ij} = \frac{1}{1 - q_iq_j}, \quad d_{ij}(k) = \frac{1}{p_j - q_i} + \delta_{ij}\frac{1}{p_j - q_i}\left[\frac{a-q_i}{a-p_i}\right]^k \frac{q_j^n e^{\eta'_j(-t)}}{p_i^n e^{\xi'_i(-t)}}. \tag{93}$$

In this case, we bave $\mathcal{A}(-t) = \mathcal{D}^T$, $\mathcal{B} = \mathcal{B}^T$ and $C = C^T$. Notice that $a - p_i = q_i(1-ap_i)$ and $a - q_i = p_i(1-aq_i)$ which yield $\mathcal{A}_{(g)}(-t) = \mathcal{D}^T_{(h)}$ and $\mathcal{A}_{(h)}(-t) = \mathcal{D}^T_{(g)}$, thus one can get

$$\tilde{f}_n(-t) = |T\tilde{F}^T T| = |\tilde{F}| = \tilde{f}_n, \quad \tilde{g}_n(-t) = |T\tilde{H}^T T| = |\tilde{F}| = \tilde{h}_n. \tag{94}$$

From the transformation (32), it immediately reaches $u_n(-t) = \frac{\tilde{g}_n(-t)}{\tilde{f}_n(-t)} = \frac{\tilde{h}_n}{\tilde{f}_n} = v_n$. Furthermore, through the variable transformations

$$u_n = \frac{\psi_n}{\sqrt{\alpha}}\exp\left[2\mathrm{i}\left(\frac{1}{a^2} - 1\right)t\right], \quad \alpha = \frac{a^2-1}{\gamma a^2}, \quad c = \frac{1}{a^2}, \tag{95}$$

with the complex constant γ, we obtain the reverse time discrete symmetric IDNLS equation (6). Finally, we get the following theorem about the solution of the nonlocal IDNLS equation (6).

Theorem 3.3 The nonlocal IDNLS equation (6) has the solution

$$\psi_n = \sqrt{\frac{a^2-1}{\gamma a^2}}\, e^{-2\mathrm{i}\left(\frac{1}{a^2}-1\right)t}\frac{\tilde{g}_n}{\tilde{f}_n}, \tag{96}$$

where tau functions are given by

$$\tilde{f}_n = \begin{vmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{vmatrix}_{2N\times 2N}, \quad \tilde{g}_n = \begin{vmatrix} \mathcal{A}_{(g)} & \mathcal{B} \\ \mathcal{C} & \mathcal{D}_{(g)} \end{vmatrix}_{2N\times 2N}, \tag{97}$$

with the block matrices

$$\mathcal{A} = \left(\frac{1}{p_i - q_j} + \delta_{ij}\frac{1}{p_i - q_j}\frac{q_i^n e^{\eta'_i}}{p_j^n e^{\xi'_j}}\right)_{N\times N}, \quad \mathcal{A}_{(g)} = \left(\frac{1}{p_i - q_j} + \delta_{ij}\frac{1}{p_i - q_j}\left[\frac{q_i(1-ap_i)}{p_i(1-aq_i)}\right]\frac{q_i^n e^{\eta'_i}}{p_j^n e^{\xi'_j}}\right)_{N\times N},$$

$$\mathcal{D} = \left(\frac{1}{p_j - q_i} + \delta_{ij}\frac{1}{p_j - q_i}\frac{q_j^n e^{\eta'_j(-t)}}{p_i^n e^{\xi'_i(-t)}}\right)_{N\times N}, \quad \mathcal{D}_{(g)} = \left(\frac{1}{p_j - q_i} + \delta_{ij}\frac{1}{p_j - q_i}\left[\frac{a-q_i}{a-p_i}\right]\frac{q_j^n e^{\eta'_j(-t)}}{p_i^n e^{\xi'_i(-t)}}\right)_{N\times N},$$

and

$$\mathcal{B} = \left(\frac{1}{p_i p_j - 1}\right)_{N\times N}, \quad \mathcal{C} = \left(\frac{1}{1 - q_i q_j}\right)_{N\times N}.$$

Here $\xi'_i = \mathrm{i}\left(p_i + \frac{1}{p_i}\right)\frac{t}{a} + \xi'_{i,0}$ and $\eta'_i = \mathrm{i}\left(q_i + \frac{1}{q_i}\right)\frac{t}{a} + \eta'_{i,0}$, p_i, q_i $\xi'_{i,0}$,$\eta'_{i,0}$ $(i = 1, 2\cdots, N)$ and a are arbitrary complex constants, these parameters need to satisfy the constraint conditions

$$ap_iq_i + a - p_i - q_i = 0. \tag{98}$$

For example, by taking $N = 1$, tau functions for the nonlocal IDNLS equation (6) are written as:

$$f_n = \Delta_1 + \Delta_2\left(1 + \frac{q_1^n e^{\eta'_1}}{p_1^n e^{\xi'_1}} + \frac{q_1^n e^{\eta'_1(-t)}}{p_1^n e^{\xi'_1(-t)}} + \frac{q_1^{2n} e^{\eta'_1+\eta'_1(-t)}}{p_1^{2n} e^{\xi'_1+\xi'_1(-t)}}\right), \tag{99}$$

$$g_n = \Delta_1 + \Delta_2\left(1 + P_1\frac{q_1^n e^{\eta'_1}}{p_1^n e^{\xi'_1}} + \frac{1}{P_1}\frac{q_1^n e^{\eta'_1(-t)}}{p_1^n e^{\xi'_1(-t)}} + \frac{q_1^{2n} e^{\eta'_1+\eta'_1(-t)}}{p_1^{2n} e^{\xi'_1+\xi'_1(-t)}}\right), \tag{100}$$

where

$$\Delta_1 = \frac{1}{(p_1^2 - 1)(q_1^2 - 1)}, \quad \Delta_2 = \frac{1}{(p_1 - q_1)^2}, \quad P_1 = \frac{a - p_1}{a - q_1},$$

and $\xi'_1 = \mathrm{i}\left(p_1 + \frac{1}{p_1}\right)\frac{t}{a} + \xi'_{1,0}$ and $\eta'_1 = \mathrm{i}\left(q_1 + \frac{1}{q_1}\right)\frac{t}{a} + \eta'_{1,0}$, p_1, q_1 $\xi'_{1,0}$,$\eta'_{1,0}$ and a are arbitrary complex constants, they need to satisfy the constraint condition $ap_1q_1 + a - p_1 - q_1 = 0$.

4. Summary and discussions

In this paper, we study Gram determinant solutions for local and nonlocal IDNLS equations via the pair reduction. Starting from the coupled system of the TL's BT and D2DTL equation, the generalized IDNLS equation with the single Casorati determinant solution is derived through the dimensional reduction. According to the gauge invariance of the bilinear form for the generalized IDNLS equation, the single Casorati determinant solution is changed as two kinds of Gram determinant solutions. Based on the different forms of the Gram determinant solution, the pair constraint conditions for wave numbers are imposed and then solutions of local and nonlocal IDNLS equations are constructed. These solutions corresponds to the even number of soliton respectively.

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