Inverse Problems

PAPER

Radiative transport model for coherent acousto-optic tomography

IOP Publishing

Inverse Problems

Inverse Problems **36** (2020) 064004 (14pp)

https://doi.org/10.1088/1361-6420/ab82ef

Radiative transport model for coherent acousto-optic tomography

Francis J Chung[1]**, Jeremy G Hoskins**[2] **and John C Schotland**[3,4]

[1] Department of Mathematics, University of Kentucky, Lexington, KY, United States of America

[2] Department of Mathematics, Yale University, New Haven, CT, United States of America

[3] Department of Mathematics and Department of Physics, University of Michigan, Ann Arbor, MI, United States of America

E-mail: fj.chung@uky.edu, jeremy.hoskins@yale.edu and schotland@umich.edu

Received 5 November 2019, revised 26 February 2020
Accepted for publication 24 March 2020
Published 25 May 2020

Abstract

We consider the problem of reconstructing the optical properties of a highly-scattering medium from coherent acousto-optic measurements. A method to solve the problem is proposed that is based on an inverse problem with internal data for a system of radiative transport equations.

Keywords: optical tomography, radiative transport, hybrid imaging, acousto-optic imaging

(Some figures may appear in colour only in the online journal)

1. Introduction

The acousto-optic effect refers to the modulation of the optical properties of a scattering medium by an acoustic wave. Acousto-optic imaging exploits this effect to combine the spectroscopic sensitivity of optical methods with the spatial resolution of ultrasound imaging. There is also a mathematical advantage. The inverse problem of acousto-optic imaging is well-posed, leading to better reconstructions than can be obtained with solely acoustic or optical imaging. Several mathematical models for acousto-optic imaging have been studied [1–8, 11]. The mathematical details of the acousto-optic inverse problem vary considerably, depending on the response of the medium to the probing optical and acoustic fields. In this paper we investigate the regime of coherent scattering, where the interaction of the medium with acoustic waves leads to a detectable frequency shift in the scattered light [13]. In this setting, we study the corresponding inverse problem within the framework of radiative transport theory. A related

[4]Author to whom any correspondence should be addressed.

question for diffuse light was examined in [10] and the inverse problem for incoherent light was investigated in [7, 11].

Consider a bounded domain $X \subset \mathbb{R}^3$ with a smooth boundary. The specific intensity at the source frequency $u(x, \theta)$ is the intensity of light at the point $x \in X$ in the direction $\theta \in S^2$. Following [13], we model light propagation by the radiative transport equation (RTE)

$$\theta \cdot \nabla u(x, \theta) + \sigma(x)u(x, \theta) = \int_{S^2} k(x, \theta, \theta')u(x, \theta')\mathrm{d}\theta' \quad \text{on } X \times S^2, \tag{1}$$

together with the boundary condition

$$u|_{\Gamma_-} = f$$

on the incoming boundary Γ_-, defined by

$$\Gamma_\pm = \{(x, \theta) \in \partial X \times S^2 : \pm\theta \cdot \nu(x) > 0\}.$$

Here σ and k denote the attenuation and scattering coefficients of the medium, respectively. For convenience, we will sometimes write the RTE as

$$\theta \cdot \nabla u = Au, \tag{2}$$

where the operator A is defined by (1).

We now recall the theory of the coherent acousto-optic effect with multiply-scattered light developed in [13]. Suppose that monochromatic light is incident on a scattering medium in which a time-harmonic acoustic plane wave also propagates. The acoustic wave modulates the dielectric permittivity of the medium, resulting in the formation of frequency-shifted optical fields at harmonics of the acoustic frequency. It can then be seen that the specific intensity of the field and its harmonics obey radiative transport equations of the form

$$\begin{aligned} \theta \cdot \nabla u_{00} &= Au_{00}, \\ \theta \cdot \nabla u_{01} &= Au_{01} + \frac{1}{2}\epsilon \cos(Q \cdot x)u_{00}, \\ \theta \cdot \nabla u_{11} &= Au_{11} + \epsilon \cos(Q \cdot x)u_{01}, \end{aligned} \tag{3}$$

with boundary conditions $u_{00}|_{\Gamma_-} = f$ and $u_{01}|_{\Gamma_-} = u_{11}|_{\Gamma_-} = 0$. Here u_{00} and u_{11} are the specific intensities of the incident and frequency-shifted light, respectively and Q is the wave vector of the acoustic wave. The quantity u_{01} is a measure of the correlation of the incident and frequency-shifted light. In addition, the parameter ϵ, which governs the strength of the acousto-optic effect, is assumed to be known.

We consider the following inverse problem: given boundary measurements of u_{00}, u_{01} and u_{11} for various f and Q, reconstruct σ and k. Note that u_{01} is singly modulated by the acoustic wave, while u_{11} is doubly modulated. As a consequence, boundary measurements of u_{01} (which is first order in the small parameter ϵ) are used to reconstruct σ and k.

It will prove useful to impose the following *a priori* conditions on the coefficients σ and k.
Regularity. The coefficients

$$\sigma \in C(X) \quad \text{and} \quad k \in C(X \times S^2 \times S^2) \quad \text{are nonnegative.} \tag{4}$$

Absorption. Scattering does not generate light; in other words there exists $c > 0$ such that

$$\inf_{x \in X}(\sigma - \rho) > c, \tag{5}$$

where

$$\rho(x) = \left\| \int_{S^2} k(x,\theta,\theta')\mathrm{d}\theta' \right\|_{L^\infty(S^2)}. \tag{6}$$

Reciprocity. Scattering is identical for incoming and outgoing light:

$$k(x,\theta,\theta') = k(x,-\theta',-\theta). \tag{7}$$

Making use of the above conditions on σ and k, and imposing an L^∞ boundary source f, the system of equations (3) has unique L^∞ solutions u_{00}, u_{01}, and u_{11} in $X \times S^2$ [11, 12] (see also proposition 15 below). Therefore for each pair σ, k satisfying the above conditions, we will define the boundary value map $\mathcal{A}^{01}_{\sigma,k} : \mathbb{R}^3 \times L^\infty(\Gamma_-) \to L^\infty(\Gamma_+)$ by

$$\mathcal{A}^{01}_{\sigma,k}(Q,f) = u_{01}|_{\Gamma_+}.$$

We are now ready to state the main result of this paper.

Theorem 1.1. *Given σ and k satisfying (4), (5) and (7), the map $(\sigma,k) \mapsto \mathcal{A}^{01}_{\sigma,k}$ is injective. Moreover, there exists $f \in L^\infty(\Gamma_-)$ such that $\mathcal{A}^{01}_{\sigma,k}(Q,f)$ suffices to recover σ. Additionally, there is a one-parameter subset of $L^\infty(\Gamma_-)$ such that if we restrict the domain of $\mathcal{A}^{01}_{\sigma,k}$ to this subset, the map from (σ,k) to the restricted map $\mathcal{A}^{01}_{\sigma,k}$ is still injective.*

We emphasize that in the above result only one boundary source is needed to recover σ, and only a one-parameter set of sources is required to reconstruct k. Three remarks should be made here. First, the proof of theorem 1.1 is constructive—we will provide an explicit method of reconstructing σ and k from $\mathcal{A}^{01}_{\sigma,k}$. Second, this construction leads to stability estimates: see proposition 2.1 and theorem 5.3. Finally, the fact that we can rely on a one-parameter set of sources is an advantage of acousto-optic tomography over inverse transport [9].

The proof of theorem 1.1 can be summarized as follows. We use the measurements of u_{01}, together with an integration by parts, to obtain an internal functional (section 2). Then we consider the forward problem for the RTE (section 3), and use the form of the solutions to analyze the internal functional. An informal description of the method of proof for theorem 1.1 is given in section 4, and in section 5 we present the full proof, along with stability estimates.

2. Internal functional

We begin by deriving the internal functional. Suppose u_{01} is as above, and $v(x,\theta)$ solves the adjoint equation

$$-\theta \cdot \nabla v = Av, \tag{8}$$

with the natural boundary condition $v|_{\Gamma_+} = g$ specified by us. (Note that solutions to the adjoint RTE (8) are precisely solutions to the regular RTE (2) under the change of variables $\theta \mapsto -\theta$.) Integrating by parts,

$$\int_X \theta \cdot \nabla u_{01} v \mathrm{d}x = -\int_X u_{01}\theta \cdot \nabla v \mathrm{d}x + \int_{\partial X} u_{01} v\, \theta \cdot n \mathrm{d}x,$$

so

$$\int_X (Au_{01} + \epsilon \cos(Q\cdot x)u_{00}) v \mathrm{d}x = \int_X u_{01} Av \mathrm{d}x + \int_{\partial X} u_{01} v\, \theta \cdot n \mathrm{d}x.$$

If we integrate in the θ variables also, then the reciprocity assumption (7) guarantees that A is self adjoint, and so

$$\int_X\int_{S^2}\epsilon\ \cos(Q\cdot x)u_{00}v\mathrm{d}\theta\mathrm{d}x=\int_{\partial X}\int_{S^2}u_{01}v\,\theta\cdot n\mathrm{d}\theta\mathrm{d}x.$$

Since $u_{01}|_{\Gamma_-}=0$, the right-hand side reduces to an integral over Γ_+, so

$$\int_X\int_{S^2}\epsilon\ \cos(Q\cdot x)u_{00}v\mathrm{d}\theta\mathrm{d}x=\int_{\Gamma_+}u_{01}g\,\theta\cdot n\,\mathrm{d}\theta\mathrm{d}x. \tag{9}$$

Here the right side of the above can be measured, so the left-hand side is also known, which means that we can recover the Fourier transform of the quantity

$$H(x)=\int_{S^2}u_{00}v\mathrm{d}\theta,$$

and we assume that in practice we can vary Q to recover H. The inverse problem is now to reconstruct σ and k from knowledge of H. Since the functional we measure depends on the boundary values we choose for u_{00} and v, we can write

$$H(x)=H_{f,g}(x)=\int_{S^2}u_{00}v\mathrm{d}\theta, \tag{10}$$

where f and g are understood to be the boundary values of u_{00} and v respectively.

The recovery of H comes with the following stability estimate.

Proposition 2.1. *If H_1 and H_2 are functionals obtained from the same initial data (f, g), but separate sets of coefficients σ_1, k_1 and σ_2, k_2, we have the stability estimate*

$$\|H_1-H_2\|_{L^\infty(X)}\lesssim\|g\|_{L^\infty(\Gamma_+)}\|\mathcal{A}^{01}_{\sigma_1,k_1}(Q,f)-\mathcal{A}^{01}_{\sigma_2,k_2}(Q,f)\|_{L^1(\mathbb{R}^3\times\Gamma_+)}. \tag{11}$$

Proof. Note that the quantity on the left side of (9) is the Fourier transform of H, and the $u_{01}|_{\Gamma_+}$ that appears on the right side can be rewritten as $\mathcal{A}^{01}_{\sigma,k}(Q,f)$. Therefore (9) says that the Fourier transform of H_1-H_2 is given by

$$\mathcal{F}(H_1-H_2)(Q)=\int_{\Gamma_+}(\mathcal{A}^{01}_{\sigma_1,k_1}(Q,f)-\mathcal{A}^{01}_{\sigma_2,k_2}(Q,f))g\,\theta\cdot n\mathrm{d}\theta\mathrm{d}x,$$

and the stability estimate follows easily. □

3. Solutions of the RTE

To make further progress, we take advantage of the collision expansion for solutions of the RTE. First we fix some terminology. For $x\in\mathbb{R}^n$, let $\hat{x}$ denote the unit vector in the direction of x, and for $x, y\in\overline{X}$, let

$$\tau(x,y)=\int_0^{|x-y|}\sigma(x-s(\widehat{x-y}))\mathrm{d}s.$$

Here $\tau(x, y)$ is the optical distance from x to y in the presence of the absorption coefficient σ, without scattering. Note that $\tau(x, y)=\tau(y, x)$. Define $\gamma_\pm: X\times S^2\to\Gamma_\pm$ by setting $\gamma_\pm(x,\theta)$ to be the (first) point in ∂X obtained by travelling from x in the $\pm\theta$ direction; we think of this as the projection of x onto ∂X in the direction $\pm\theta$. Let J be the operator that solves the non-scattering RTE

$$\theta \cdot \nabla u = -\sigma u,$$
$$u|_{\Gamma_-} = f,$$

and write J explicitly in terms of τ and γ_- as

$$Jf(x,\theta) = \mathrm{e}^{-\tau(x,\gamma_-(x,\theta))} f(\gamma_-(x,\theta),\theta). \tag{12}$$

Similarly, if we define T^{-1} to be the operator which solves the non-scattering RTE

$$\theta \cdot \nabla u + \sigma u = S,$$
$$u|_{\Gamma_-} = 0,$$

then explicitly

$$T^{-1}S(x,\theta) = \int_0^{|x-\gamma_-(x,\theta)|} \mathrm{e}^{-\tau(x,x-t\theta)} S(x-t\theta,\theta)\mathrm{d}t. \tag{13}$$

Finally, define A_2 to be the scattering operator

$$A_2 w = \int_{S^2} k(x,\theta,\theta')w(x,\theta')\mathrm{d}\theta',$$

and

$$Kw = T^{-1}A_2 w \tag{14}$$

The main result of this section is the Neumann series solution of the RTE.

Proposition 3.1. *Suppose σ and k satisfy the conditions in section 1. Then there exists $0 < C < 1$ such that*

$$\|K\|_{L^\infty(X\times S^2)\to L^\infty(X\times S^2)} < C,$$

and if u solves $\theta \cdot \nabla u = Au$ with the boundary condition $u|_{\Gamma_-} = f$, for some $f \in L^\infty(\Gamma_-)$, then u takes the form

$$u = (1 + K + K^2 + \cdots)Jf. \tag{15}$$

See [5, 9, 12] for proofs of this result. The expansion (15) is the collision expansion of u. It is useful because K is a smoothing operator, so each subsequent term of the expansion is less singular. The first term Jf corresponds to light propagation in the absence of scattering, and is called the ballistic term. The subsequent terms $K^m Jf$ correspond to light that has been scattered m times, and thus KJf is referred to as the single-scattering term, K^2Jf as the double-scattering term, and so on.

Note that analogous results also hold for the adjoint equation (8), with appropriate corresponding operators K^*, J^*, etc obtained via the change of variables $\theta \mapsto -\theta$.

The following estimates, taken from [11], will also prove useful.

Lemma 3.2. *For all $x \in X$,*

$$\|A_2(w)(x,\cdot)\|_{L^\infty(S^2)} < C_k \|w(x,\cdot)\|_{L^1(S^2)}. \tag{16}$$

Moreover

$$\|T_1^{-1}w\|_{L^\infty(X\times S^2)} < \|w\|_{L^\infty(X\times S^2)}. \tag{17}$$

4. Point-plane inversion

The main difficulty in recovering σ and k is the nonlinearity of the functional $H_{f,g}$. The basic idea for countering this difficulty is to use proposition 3.1 with carefully chosen boundary sources f and g to ensure that only the leading order terms contribute meaningfully to $H_{f,g}$. This idea is similar to that used in [11], but with the important difference that in our case, the principal term in the expansion carries no information. This point is best understood by examining what happens in the absence of scattering. In that case, the operator K vanishes and the solutions to the RTE are given solely by the ballistic term. However, now the quantity $u_{00}v$ satisfies the equation

$$\theta\cdot\nabla(u_{00}v) = 0,$$

so the solution does not vary as we move into the domain from the boundary. It follows that the leading order term in the collision expansion vanishes and we must turn to the single-scattering term. In this section we give an informal discussion of the above, first by considering each point in the domain one by one, and then foliating the domain with planes and considering each plane one at a time. In the following section, we describe how this process can be extended to consider the entire domain at once, and at the same time, make the discussion fully rigorous.

4.1. Point sources

We begin by considering one point at a time. We let $x\in X$ and consider point sources on the boundary in the direction θ_0. To do this, define for a pair $(x_0,\theta_0)\in\partial X\times S^2$ the delta distribution δ_{x_0,θ_0} so that

$$\int_{\partial X\times S^2}\delta_{x_0,\theta_0}f = f(x_0,\theta_0)$$

for any $f\in C^\infty(\partial X\times S^2)$. Now consider a solution u to the RTE with boundary data given by such a delta function. (Making this idea rigorous requires some redefinition of the notion of a solution to encompass distributions, which we do not address here. The discussion in the next section will contain a rigorous analysis in terms of approximations to delta distributions.) By proposition 3.1,

$$u = J\delta_{x_0,\theta_0} + KJ\delta_{x_0,\theta_0} + K^2J\delta_{x_0,\theta_0} + \cdots. \tag{18}$$

Here $J\delta_{x_0,\theta_0}$ is a distribution supported on the codimension four subset of the five dimensional set $X\times S^2$ given by

$$\{(x,\theta_0) : x = x_0 + c\theta_0 \quad \text{for some } c\in\mathbb{R}\}$$

The operator K integrates in one spatial dimension and two angular dimensions, so that $KJ\delta_{x_1,\theta_1}$ is supported on a codimension one subset, and all subsequent terms are less singular. Now let $x\in X$, and $\theta_1,\theta_2\in S^2$ such that $\theta_1\neq\theta_2$. We set $x_1=\gamma_-(x,\theta_1)$ and $x_2=\gamma_+(x,\theta_2)$ (see

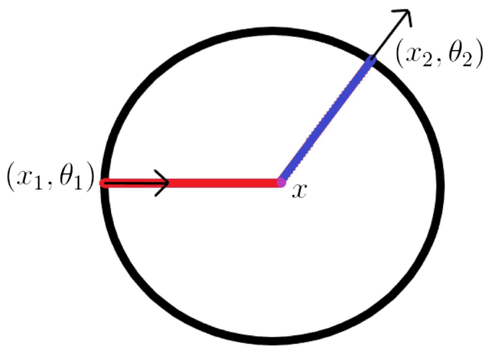

Figure 1. $H_{f,g}(x)$ represents light from the point source (x_1, θ_1), which is scattered and frequency shifted from x and observed at (x_2, θ_2).

figure 1). We define corresponding boundary sources $f = \delta_{x_1,\theta_1}$ and $g = \delta_{x_2,\theta_2}$, and consider the resulting functional

$$H_{f,g}(x) = \int_{S^2} u_{00}(x,\theta)v(x,\theta)\mathrm{d}\theta.$$

The integral identity (9) implies that the above corresponds to boundary measurements of u_{01} at x_2 in the direction of θ_2, where the acoustic wave has been focused to concentrate its support at x. Thus we expect that the leading term in the internal functional will represent light traveling along a ray from x_1 to x, scattering once at x in the direction θ_2, and exiting at x_2. Indeed, this is what we obtain when we expand u_{00} and v in terms of the collision expansion (18). Since $\theta_1 \neq \theta_2$, the leading term JfJ^*g vanishes, and so the dominant terms of $H_{f,g}$ are

$$\int_{S^2} (JfK^*J^*g + KJfJ^*g)\mathrm{d}\theta.$$

Each of these terms represents a distribution supported on a codimension one set multiplied by one supported on a codimension four set. Expanding JfK^*J^*g at x using (12) and (14) gives

$$\int_{S^2} JfK^*J^*g\mathrm{d}\theta = k(x,-\theta_1,-\theta_2)\exp(-\tau(x,x_1) - \tau(x,x_2))\delta_x(x),$$

where $\delta_x(x)$ reflects the size of the distribution at x. Similarly,

$$\int_{S^2} KJfJ^*g\mathrm{d}\theta = k(x,\theta_2,\theta_1)\exp(-\tau(x,x_1) - \tau(x,x_2))\delta_x(x).$$

By (7), these terms are identical, so to leading order and ignoring the $\delta_x(x)$ factor, we obtain

$$H_{f,g}(x) \simeq 2k(x,\theta_2,\theta_1)\exp(-\tau(x,x_1) - \tau(x,x_2)). \tag{19}$$

This formula for $H_{f,g}$ is precisely what is expected from the above discussion and figure 1.

Suppose $\tau(x,y)$ is known for all pairs (x,y), then $k(x,\theta_2,\theta_1)$ can be obtained directly from (19). If not, then we can set $x_1 = \gamma_-(x,\theta_1)$ and $x_2 = \gamma_+(x,\theta_1)$, and measure two functionals

$$H_1 = H_{\delta_{x_1,\theta_1},\delta_{x_1,\theta_1}}(x) \simeq 2k(x,\theta_1,-\theta_1)\exp(-2\tau(x,x_1))$$

$$H_2 = H_{\delta_{x_2,-\theta_1},\delta_{x_2,-\theta_1}}(x) \simeq 2k(x,\theta_1,-\theta_1)\exp(-2\tau(x,x_2)),$$

and the additional quantity

$$H_3 = \exp(-\tau(x_1,x_2)),$$

which can be obtained from the albedo map $\mathcal{A}^{00}_{\sigma,k}$ for u_{00}, applied to the point source δ_{x_1,θ_1}. By the additivity of τ, we have $\tau(x_1,x_2) = \tau(x,x_1) + \tau(x,x_2)$ and thus we get

$$\tau(x,x_1) = \frac{1}{2}(\log H_1 - \log H_2 + \log H_3).$$

Repeating this procedure gives any desired value of $\tau(x,y)$. Now, differentiating τ gives $\sigma(x)$, so we can recover both σ and k from the functional H. On the other hand, using the methods described above means that in order to obtain σ and k, we need to consider all possible point sources, which means we need four dimensions of sources. We can improve this slightly by making use of plane sources.

4.2. Plane sources

Let $\theta_0 \in S^2$ and fix a plane P parallel to θ_0 which intersects the set $\{x \in \partial X : (x,\theta_0) \in \Gamma_-\}$. Let δ_{P,θ_0} be a distribution supported on the set $P' = \{(x,\theta) \in \Gamma_- : x \in P, \theta = \theta_0\}$, so that

$$\int_{\partial X \times S^2} \delta_{P,\theta_0} f = \int_{P'} f(x,\theta_0),$$

for all $f \in C^\infty(\partial X \times S^2)$. That is, δ_{P,θ_0} is a distribution supported on a codimension three subset of the four dimensional set $\partial X \times S^2$. If we view δ_{P,θ_0} as a boundary source for the RTE and consider the collision expansion

$$u = J\delta_{P,\theta_0} + KJ\delta_{P,\theta_0} + K^2 J\delta_{P,\theta_0} + \cdots, \tag{20}$$

the leading term $J\delta_{P,\theta_0}$ is a distribution supported on a codimension 3 subset of the five-dimensional domain $X \times S^2$.

Since K integrates along one spatial dimension and two angular dimensions, $KJ\delta_{P,\theta_0}$ is supported everywhere. However, it is not actually a function since

$$KJ\delta_{P,\theta_0}(x,\theta) = T^{-1}A_2 J\delta_{P,\theta_0}(x,\theta)$$

for $x \in P$ and θ parallel to P. The spatial integral in T^{-1} is along a line fully contained in P, so it does not reduce the singularity of the distribution $A_2 J\delta_{P,\theta_0}(x,\theta)$. Therefore $KJ\delta_{P,\theta_0}(x,\theta)$ can be viewed as a function supported on $X \times S^2$ plus a distribution supported on the codimension one set $P \times S^2$. Now choosing $\theta_1 \in S^2$ and P parallel to θ_1, pick $(x_2,\theta_2) \in \Gamma_+$ so that x_2 lies in P and θ_2 is parallel to P, with $\theta_1 \neq \theta_2$. We define corresponding boundary sources $f = \delta_{P,\theta_1}$ and $g = \delta_{x_2,\theta_2}$, and consider the resulting functional $H_{f,g}(x)$ at any point x on the line through x_2 in the direction $-\theta_2$. As in the point source case, the ballistic terms vanish. By the above discussion, what remains is the term

$$KJ\delta_{P,\theta_1}(x,\theta)J^*\delta_{x_2,\theta_2},$$

which represents a codimension four distribution multiplied by a codimension one distribution.

Expanding the above using (12) and (14) gives

$$H_{f,g}(x) \simeq \int_0^{|x-\gamma_-(x,\theta_2)|} \mathrm{e}^{-\tau(\gamma_+(x,\theta_2),x-t\theta_2)-\tau(x-t\theta_2,\gamma_-(x-t\theta_2,\theta_1))} k(x-t\theta_2,\theta_2,\theta_1)\mathrm{d}t\delta_x(x) \tag{21}$$

Ignoring the $\delta_x(x)$ factor and taking the directional derivative in the direction θ_2, we get

$$\theta_2 \cdot \nabla H_{f,g}(x) \simeq k(x,\theta_2,\theta_1)\exp(-\tau(x,x_1)-\tau(x,x_2)).$$

which is just (19), and so the remainder of the reconstruction proceeds as in the point source case. Note that for each plane source δ_{P,θ_1}, we can, by varying x_2 and θ_2, recover a two dimensional collection of $k(x,\theta_1,\theta_2)$. Therefore only two dimensions of sources are needed to recover all of k and σ. In fact it's possible to do better: we can restrict ourselves to a single dimension of sources, if we use an angularly singular source such as δ_{θ_1}, and multiply by a rapidly oscillating function. This brings us to the proof of theorem 1.1.

5. Reconstruction and stability

5.1. Proof of theorem 1.1

We begin by defining the following L^∞ approximation to the delta function on S^2:

$$\delta_{\theta_1}^h(\theta) = \begin{cases} h^{-2} & \text{if } |\theta-\theta_1| < h, \\ 0 & \text{otherwise.} \end{cases}$$

Following the discussion at the end of section 4, we need a function that oscillates rapidly in the spatial directions perpendicular to θ_1. To do this, let $\theta_1 \in S^2$ and θ_3 be perpendicular to θ_1. Pick coordinates for x such that $\theta_1 = \hat{x}_1$ and $\theta_3 = \hat{x}_3$. Now consider the source

$$f_h^{\theta_1}(x,\theta) = \delta_{\theta_1}^h(\theta)\exp(\mathrm{i}x_3/h). \tag{22}$$

This complex source is not physical, but it can be recreated formally by measuring from its real and imaginary parts. Using the collision expansion, we claim the following qualitative properties for the solution of the RTE with boundary source f.

Lemma 5.1. *Suppose $f = f_h^{\theta_1}$ is as defined in (22), and u is the solution to the RTE (2) with boundary condition $u|_{\Gamma_-} = f$. Then $u = Jf + KJf + R$, where*

(*a*) *The ballistic term Jf satisfies the estimates*

$$\|Jf\|_{L^\infty(X\times S^2)} = O(h^{-2}) \quad \text{and for any fixed } x, \quad \|Jf(x,\cdot)\|_{L^1(S^2)} = O(1);$$

(*b*) *The single scattering term KJg satisfies the estimates*

$$\|KJf\|_{L^\infty(X\times S^2)} = O(1) \quad \text{and for any fixed } x, \quad \|KJf(x,\cdot)\|_{L^1(S^2)} = o(1);$$

(*c*) *and the remainder satisfies the estimate*

$$\|R\|_{L^\infty(X\times S^2)} = o(1).$$

Proof. The estimates for *Jf* follow directly from the definitions of J and f. The L^∞ norm of *KJf* follows from lemma 3.2 and the L^1 estimate for *Jf*. Now

$$Jf(x,\theta) = \mathrm{e}^{-\tau(x,\gamma_-(x,\theta))}\delta^h_{\theta_1}(\theta)\exp(\mathrm{i}\hat{x}_3\cdot\gamma_-(x,\theta)/h).$$

Therefore

$$A_2Jf(x,\theta) = \int_{S^2} k(x,\theta,\theta')\mathrm{e}^{-\tau(x,\gamma_-(x,\theta'))}\delta^h_{\theta_1}(\theta')\exp(\mathrm{i}\hat{x}_3\cdot\gamma_-(x,\theta')/h)\mathrm{d}\theta'.$$

Since $\delta^h_{\theta_1}$ is supported only for θ in a small neighbourhood of θ_1, the Lebesgue differentiation theorem guarantees that for sufficiently small h, we obtain

$$A_2Jf(x,\theta) = \mathrm{e}^{-\tau(x,\gamma_-(x,\theta_1))}k(x,\theta,\theta_1)\exp(\mathrm{i}\hat{x}_3\cdot\gamma_-(x,\theta_1)/h) + o(1).$$

Since θ_1 is perpendicular to $\hat{x}_3$, we find that

$$A_2Jf(x,\theta) = \mathrm{e}^{-\tau(x,\gamma_-(x,\theta_1))}k(x,\theta,\theta_1)\exp(\mathrm{i}x_3/h) + o(1).$$

Now we can write *KJf* as

$$T^{-1}A_2Jf(x,\theta) = \int_0^{|x-\gamma_-(x,\theta)|}\mathrm{e}^{-\tau(x,x-t\theta)}A_2Jf(x-t\theta,\theta)\mathrm{d}t.$$

If $\theta\cdot\hat{x}_3 \gg h$, then $A_2Jf(x-t\theta,\theta)$ is highly oscillatory as a function of t, and so by the Riemann–Lebesgue lemma,

$$|KJf(x,\theta)| = o(1).$$

Then it follows that

$$\|KJf(x,\cdot)\|_{L^1(S^2)} = o(1),$$

and the estimate for R follows from lemma 3.2. □

We now examine the functional $H_{f_h^{\theta_1},g_h^{\theta_2}}$ defined by $f_h^{\theta_1}$ and a boundary function $g_h^{\theta_2}$ which approximates a point source. To define $g_h^{\theta_2}$, we begin by first defining the approximation to the delta function on the boundary. For $x_0\in\partial X$, define

$$\delta^h_{x_0}(x) = \begin{cases} h^{-2} & \text{if } |x-x_0|<h,\\ 0 & \text{otherwise.}\end{cases}$$

Choose $\theta_2\in S^2$ so θ_2 is perpendicular to $\hat{x}_3$, and let

$$g_h^{\theta_2}(x,\theta) = h^2\delta^h_{\theta_2}(\theta)\delta^h_{x_0}(x). \tag{23}$$

Lemma 5.2. *Let $g = g_h^{\theta_2}$ be defined by (23), and let v solve the adjoint RTE (8) with boundary condition $v|_{\Gamma_+} = g|_{\Gamma_+}$. Then*

$$v = J^*g + K^*J^*g + R^*,$$

where

(*a*) *The ballistic term J^*g satisfies the estimates*

$$\|J^*g\|_{L^\infty(X\times S^2)} = O(h^{-2}) \quad \text{and for any fixed } x, \quad \|J^*g(x,\cdot)\|_{L^1(S^2)} = O(1);$$

(*b*) *The single scattering term K^*J^*g satisfies the estimates*

$$\|K^*J^*g\|_{L^\infty(X\times S^2)} = O(1) \quad \textit{and for any fixed } x, \quad \|K^*J^*g(x,\cdot)\|_{L^1(S^2)} = o(1);$$

(*c*) *and the remainder satisfies the estimate*

$$\|R^*\|_{L^\infty(X\times S^2)} = o(1).$$

Proof. The estimates for J^*g and the L^∞ estimate for K^*J^*g are obtained in the same manner as in lemma 5.1. To get the L^1 estimate for K^*J^*g, note that $J^*g(x,\theta)$ is only supported for x within $O(h)$ distance of the line from x_0 in direction θ_2. Therefore $A_2^*J^*g(x,\theta)$ is only supported for x within $O(h)$ distance of this line. Then for θ such that $|\theta-\theta_2| \gg h$,

$$K^*J^*g(x,\theta) = T^{*-1}A_2^*J^*g(x,\theta) = \int_0^{|x-\gamma_+(x,\theta)|} \mathrm{e}^{-\tau(x,x+t\theta)}A_2^*J^*f(x+t\theta,\theta)\mathrm{d}t$$

and the integrand is supported only in an $O(h)$ segment of the line. Therefore

$$K^*J^*g(x,\theta) = O(h) \tag{24}$$

for $|\theta-\theta_2| \gg h$, and the L^1 estimate for K^*J^*g follows.

The estimate for R^* now follows from lemma 3.2. □

Now let us consider the functional $H_{f,g}$ obtained from the sources f and g described above. Using lemmas 5.1 and 5.2 respectively, we can expand the functional as

$$\begin{aligned} H_{f,g} = &\int_{S^2} JfJ^*g\mathrm{d}\theta + \int_{S^2} KJfJ^*g\mathrm{d}\theta + \int_{S^2} RJ^*g\mathrm{d}\theta + \int_{S^2} JfK^*J^*g\mathrm{d}\theta + \int_{S^2} KJfK^*J^*g\mathrm{d}\theta \\ &+ \int_{S^2} RK^*J^*g\mathrm{d}\theta + \int_{S^2} JfR^*\mathrm{d}\theta + \int_{S^2} KJfR^*\mathrm{d}\theta + \int_{S^2} RR^*\mathrm{d}\theta. \end{aligned}$$

Assuming that $|\theta_1-\theta_2| \gg h$, the first term consists of two functions angularly supported on disjoint subsets of S^2, so it vanishes. Moreover, applying lemmas 5.1 and 5.2 shows that six of the remaining terms are $o(1)$ at best. What remains is

$$H_{f,g} = \int_{S^2} KJfJ^*g\mathrm{d}\theta + \int_{S^2} JfK^*J^*g\mathrm{d}\theta + o(1).$$

However the $\int_{S^2} JfK^*J^*g\mathrm{d}\theta$ term is not $o(1)$. Assuming that $|\theta_1-\theta_2| \gg h$, we have from (24) that $K^*J^*g(x,\theta) = O(h)$ for θ in the support of Jf. Therefore this term is $O(h)$, and as a result we are left with

$$H_{f,g} = \int_{S^2} KJfJ^*g\mathrm{d}\theta + o(1).$$

Now

$$J^*g(x,\theta) = \mathrm{e}^{-\tau(x,\gamma_+(x,\theta))}h^2\delta_{\theta_2}^h(\theta)\delta_{x_0}^h(\gamma_+(x,\theta)).$$

Therefore

$$H_{f,g}(x) = \mathrm{e}^{-\tau(x,\gamma_+(x,\theta_2))}h^2\delta_{x_0}^h(\gamma_+(x,\theta_2))KJf(x,\theta_2) + o(1).$$

For x such that $\gamma_+(x,\theta_2)$ is in the support of $\delta_{x_0}^h$, we can write

$$H_{f,g}(x) = \mathrm{e}^{-\tau(x,\gamma_+(x,\theta_2))} KJf(x,\theta_2) + o(1). \tag{25}$$

Meanwhile

$$Jf(x,\theta) = \mathrm{e}^{-\tau(x,\gamma_-(x,\theta))} \delta^h_{\theta_1}(\theta) \exp(\mathrm{i}\hat{x}_3 \cdot \gamma_-(x,\theta)/h).$$

so integrating against the scattering kernel gives

$$A_2Jf(x,\theta_2) = \mathrm{e}^{-\tau(x,\gamma_-(x,\theta_1))} \exp(\mathrm{i}\hat{x}_3 \cdot \gamma_-(x,\theta_1)/h) k(x,\theta_2,\theta_1) + o(1).$$

Since θ_1 is perpendicular to $\hat{x}_3$,

$$A_2Jf(x,\theta_2) = \mathrm{e}^{-\tau(x,\gamma_-(x,\theta_1))} \exp(\mathrm{i}x_3/h) k(x,\theta_2,\theta_1) + o(1).$$

Now $K = T^{-1}A_2$, so

$$KJf(x,\theta_2) = \int_0^{|x-\gamma_-(x,\theta_2)|} \mathrm{e}^{-\tau(x,x-t\theta_2)} A_2Jf(x-t\theta_2,\theta_2)\mathrm{d}t + o(1).$$

Substituting this into (25) gives

$$H_{f,g}(x) = \mathrm{e}^{-\tau(x,\gamma_+(x,\theta_2))} \int_0^{|x-\gamma_-(x,\theta_2)|} \mathrm{e}^{-\tau(x,x-t\theta_2)-\tau(x-t\theta_2,\gamma_-(x-t\theta_2,\theta_1))} \mathrm{e}^{\mathrm{i}\hat{x}_3\cdot(x-t\theta_2)/h} k(x-t\theta_2,\theta_2,\theta_1)\mathrm{d}t + o(1).$$

Since θ_2 is also perpendicular to $\hat{x}_3$, we can rewrite $\exp(\mathrm{i}\hat{x}_3 \cdot (x - t\theta_2)/h) = \exp(\mathrm{i}x_3/h)$. In fact, since x is known, $\exp(\mathrm{i}x_3/h)$ is also known, and we may as well assume that this is 1. Then we can write

$$H_{f,g}(x) = \mathrm{e}^{-\tau(x,\gamma_+(x,\theta_2))} \int_0^{|x-\gamma_-(x,\theta_2)|} \mathrm{e}^{-\tau(x,x-t\theta_2)} \mathrm{e}^{-\tau(x-t\theta_2,\gamma_-(x-t\theta_2,\theta_1))} k(x-t\theta_2,\theta_2,\theta_1)\mathrm{d}t + o(1).$$

Combining the remaining exponentials, we obtain

$$H_{f,g}(x) = \int_0^{|x-\gamma_-(x,\theta_2)|} \mathrm{e}^{-\tau(\gamma_+(x,\theta_2),x-t\theta_2)-\tau(x-t\theta_2,\gamma_-(x-t\theta_2,\theta_1))} k(x-t\theta_2,\theta_2,\theta_1)\mathrm{d}t + o(1).$$

Up to the $o(1)$ error, note that this is precisely equation (21), and has the same interpretation in terms of figure 2. As noted above, this result holds for x such that $\gamma_+(x,\theta_2)$ belongs to the support of $\delta^h_{x_0}$. Note that if x satisfies this condition, then so does any translation of x in the θ_2 direction. Therefore we can use the above expression to write $H_{f,g}(x - s\theta_2)$, for some parameter s, as

$$\int_0^{|x-\gamma_-(x,\theta_2)|-s} \mathrm{e}^{-\tau(\gamma_+(x,\theta_2),x-(t+s)\theta_2)-\tau(x-(t+s)\theta_2,\gamma_-(x-t\theta_2,\theta_1))} k(x-(t+s)\theta_2,\theta_2,\theta_1)\mathrm{d}t + o(1).$$

Changing variables, we get

$$\int_s^{|x-\gamma_-(x,\theta_2)|} \mathrm{e}^{-\tau(\gamma_+(x,\theta_2),x-t\theta_2)-\tau(x-t\theta_2,\gamma_-(x-t\theta_2,\theta_1))} k(x-t\theta_2,\theta_2,\theta_1)\mathrm{d}t + o(1).$$

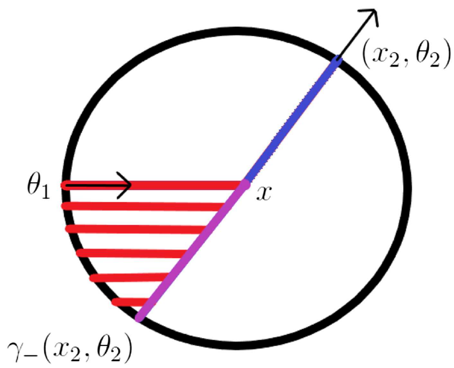

Figure 2. $H_{f,g}(x)$ represents the light from the plane source $\delta_{P,\theta_1}x_1$, scattered and frequency-shifted along the line from $\gamma_-(x, \theta_2)$ to x and then observed at (x_2, θ_2).

If we take a difference quotient with respect to s, we find that

$$\frac{H_{f,g}(x) - H_{f,g}(x - s\theta_2)}{s} = \frac{1}{s}\int_0^s \mathrm{e}^{-\tau(\gamma_+(x,\theta_2),x-t\theta_2)-\tau(x-t\theta_2,\gamma_-(x-t\theta_2,\theta_1))}k(x - t\theta_2, \theta_2, \theta_1)\mathrm{d}t + \frac{o_h(1)}{s}.$$

Here we have rewritten the $o(1)$ term as $o_h(1)$ to emphasize that this term goes to zero as $h \to 0$. If we take $0 < h \ll s \ll 1$ small, we obtain

$$\theta_2 \cdot \nabla H_{f,g}(x) = \mathrm{e}^{-\tau(\gamma_+(x,\theta_2),x)-\tau(x,\gamma_-(x,\theta_1))}k(x, \theta_2, \theta_1) + o_s(1), \tag{26}$$

where the $o_s(1)$ term goes to zero as $s \to 0$. This is the same quantity we recovered in (19) in the point source case, and the rest of the argument proceeds exactly as in section 4.1. It is useful to introduce the notation

$$\begin{aligned} F(x, \theta_1, \theta_2) &= \mathrm{e}^{-\tau(\gamma_+(x,\theta_2),x)-\tau(x,\gamma_-(x,\theta_1))}k(x, \theta_2, \theta_1) \\ &= \theta_2 \cdot \nabla H_{f_h^{\theta_1}, g_h^{\theta_2}}(x) + o(1). \end{aligned}$$

to express equation (26). Then explicitly, the discussion at the end of section 4.1 implies that

$$\tau(x, \gamma_-(x, \theta_1)) = \frac{1}{2}(\log\, F(x, \theta_1, -\theta_1) - \log\, F(x, -\theta_1, \theta_1) + \log\, \mathcal{A}^{00}_{\sigma,k}(f)(\gamma_+(x, \theta_1))), \tag{27}$$

and

$$k(x, \theta_2, \theta_1) = F(x, \theta_1, \theta_2)\mathrm{e}^{+\tau(\gamma_+(x,\theta_2),x)+\tau(x,\gamma_-(x,\theta_1))}. \tag{28}$$

Note that if θ_1 is fixed, then for a single boundary source parametrized by a choice of $\hat{x}_3$, we can, by changing v, obtain $k(x, \theta_2, \theta_1)$ for all x and all θ_2 perpendicular to $\hat{x}_3$. By rotating the choice of $\hat{x}_3$, we can then obtain $k(x, \theta_2, \theta_1)$ for all x and θ_2. Then (7) guarantees that we can recover all $k(x, \theta_1, \theta_2)$. This finishes the proof of theorem 1.1.

5.2. Stability estimates

Equations (27) and (28), combined with (26), immediately give us the following stability estimates.

Theorem 5.3. *Suppose σ_1, k_1, and σ_2, k_2 are two sets of coefficients giving rise to two functionals H_1 and H_2. Then*

$$\|\sigma_1 - \sigma_2\|_{C(X)} \leqslant \frac{1}{2}\| \log|\nabla H_1| - \log|\nabla H_2|\|_{C^1(X)}$$

and

$$\|k_1 - k_2\|_{C(X\times S^2\times S^2)} \leqslant \sup_{x,y\in X} \exp(2\tau(x,y))\|H_1 - H_2\|_{C^1(X)}.$$

Acknowledgments

This work was supported in part by the NSF Grant DMS-1912821 and the AFOSR Grant FA9550-19-1-0320.

ORCID iDs

John C Schotland https://orcid.org/0000-0003-0545-1962

References

[1] Ammari H, Bossy E, Garnier J, Nguyen L H and Seppecher L 2014 A reconstruction algorithm for ultrasound-modulated diffuse optical tomography *Proc. Am. Math. Soc.* **142** 3221
[2] Ammari H, Bossy E, Garnier J and Seppecher L 2012 Acousto-electromagnetic tomography *SIAM J. App. Math.* **72** 1592–617
[3] Ammari H, Garnier J, Nguyen L H and Seppecher L 2013 Reconstruction of a piecewise smooth absorption coefficient by an acousto-optic process *Commun. PDE* **38** 1737–62
[4] Ammari H, Nguyen L H and Seppecher L 2014 Reconstruction and stability in acousto-optic imaging for absorption maps with bounded variation *J. Funct. Anal.* **267** 4361–98
[5] Bal G, Chung F J and Schotland J C 2016 Ultrasound modulated bioluminescence tomography and controllability of the radiative transport equation *SIAM J. Math. Anal.* **48** 1332–47
[6] Bal G and Moskow S 2014 Local inversions in ultrasound-modulated optical tomography *Inverse Problems* **30** 025005
[7] Bal G and Schotland J 2010 Inverse scattering and acousto-optic imaging *Phys. Rev. Lett.* **104** 043902
[8] Bal G and Schotland J 2014 Ultrasound modulated bioluminescence tomography *Phys. Rev.* E **89** 031101
[9] Choulli M and Stefanov P 1999 An inverse boundary value problem for the stationary transport equation *Osaka J. Math.* **36** 87–104
[10] Chung F J, Hoskins J and Schotland J 2020 Coherent acousto-optic tomography with diffuse light *Optics Letters* **45** 1623–6
[11] Chung F J and Schotland J C 2017 Inverse transport and acousto-optic imaging *SIAM J. Math. Anal.* **49** 4704–21
[12] Dautray R and Lions J-L 1993 *Mathematical Analysis and Numerical Methods for Science and Technology* (Berlin: Springer) vol 6
[13] Hoskins J G and Schotland J C 2017 Acousto-optic effect in random media *Phys. Rev.* E **95** 033002