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Phase Separation in the Advective Cahn–Hilliard Equation

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Abstract

The Cahn–Hilliard equation is a classic model of phase separation in binary mixtures that exhibits spontaneous coarsening of the phases. We study the Cahn–Hilliard equation with an imposed advection term in order to model the stirring and eventual mixing of the phases. The main result is that if the imposed advection is sufficiently mixing, then no phase separation occurs, and the solution instead converges exponentially to a homogeneous mixed state. The mixing effectiveness of the imposed drift is quantified in terms of the dissipation time of the associated advection–hyperdiffusion equation, and we produce examples of velocity fields with a small dissipation time. We also study the relationship between this quantity and the dissipation time of the standard advection–diffusion equation.

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1 Introduction

Spinodal decomposition refers to the phase separation of a binary mixture, such as alloys that are quenched below their critical temperature. A well-studied model is the Cahn–Hilliard equation (Cahn and Hilliard 1958; Cahn 1961), where the evolution of the normalized concentration difference c between the two phases is governed by the equation

$$\partial_t c + \gamma D \, \Delta^2 c = D\Delta(c^3 - c). \tag{1.1}$$

Here, $D > 0$ is a mobility parameter, and $\sqrt{\gamma}$ is the Cahn number, which is related to the surface tension at the interface between phases. The coefficient γD is a hyper-diffusion that regularizes the equation at small length scales by overcoming the destabilizing $-D\Delta c$ term. The concentration c is normalized such that the regions $\{c = 1\}$ and $\{c = -1\}$ represent domains that are pure in each phase. For simplicity, we will only consider (1.1) on the d-dimensional torus $\mathbb{T}^d$.

When γ is small, solutions to (1.1) spontaneously form domains with $c = \pm 1$ separated by thin transition regions (see Fig. 1). This has been well studied by many authors [see, for instance, (Elliott and Songmu 1986; Elliott 1989; Pego 1989)], and the underlying mechanism can be understood as follows. The free energy of this system, $\mathcal{E}$, can be decomposed into the sum of the chemical free energy, $\mathcal{E}_{\text{chem}}$, and the interfacial free energy, $\mathcal{E}_{\text{int}}$, where

$$\mathcal{E}_{\text{chem}} \stackrel{\text{def}}{=} \tfrac{1}{4} \int_{\mathbb{T}^d} (c^2 - 1)^2 \, \mathrm{d}x \quad \text{and} \quad \mathcal{E}_{\text{int}} \stackrel{\text{def}}{=} \tfrac{1}{2}\gamma \int_{\mathbb{T}^d} |\nabla c|^2 \, \mathrm{d}x.$$

Using (1.1), one can directly check that $\mathcal{E}$ decreases with time, and hence, solutions should approach minimizers of $\mathcal{E}$ after a long time. Minimizing the chemical free energy $\mathcal{E}_{\text{chem}}$ favors forming domains where $c = \pm 1$. Minimizing the interfacial free energy $\mathcal{E}_{\text{int}}$ favors interfaces of thickness $\sqrt{\gamma}$ separating the domains. As a result, the typical behavior of equation (1.1) is to spontaneously phase-separate as in Fig. 1.

In this paper, we study the effect of stirring on spontaneous phase separation. When subjected to an incompressible stirring velocity field $u(t, x)$, equation (1.1) is modified to

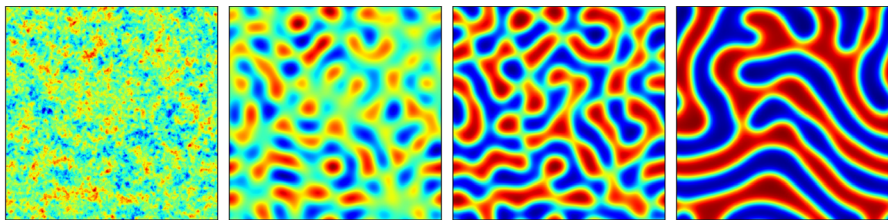$$\partial_t c + u \cdot \nabla c + \gamma \Delta^2 c = \Delta(c^3 - c). \tag{1.2}$$

Fig. 1 Solution of the Cahn–Hilliard equation (1.1) on the 2D torus with $D = 0.001$, $\gamma = .01/(2\pi)^2$ at times $t = 0, 2, 5$ and 20. The phases $c = 1$ and $c = -1$ are red and blue, respectively, and $c = 0$ is green
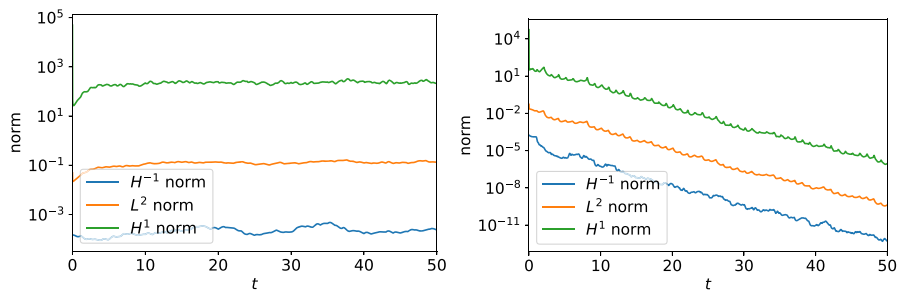
Fig. 2 Decay of H^1, L^2, and H^{-1} norms for the random shear flow for $A = 0.5$ (left) and $A = 2$ (right). On the left, the norms settle to equilibrium values; on the right, they decay exponentially
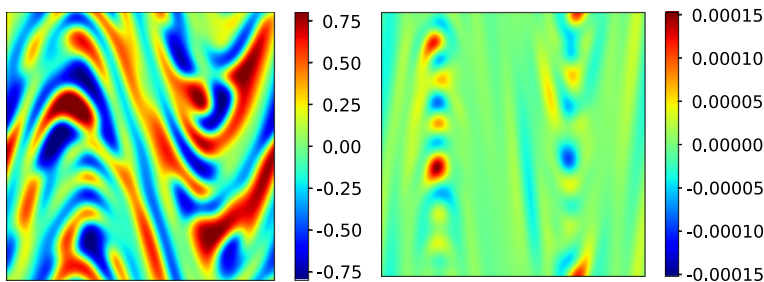

Fig. 3 Final concentration c for the two cases in Figure 2

For simplicity, we have set the mobility parameter D to be 1. The advective Cahn–Hilliard equation (1.2) has been studied by many authors (Chan et al. 1988; Läuger et al. 1995; Náraigh and Thiffeault 2007a, b, 2008; Liu et al. 2013) for both passive and active advection. Under a strong shear flow, for instance, it is known that solutions to (1.2) equilibrate along the flow direction and spontaneously phase separates in the direction perpendicular to the flow (Berthier 2001; Bray 2003; Shou and Chakrabarti 2000; Hashimoto et al. 1995).

Our main result is to show that if the stirring velocity field is sufficiently mixing, then no phase separation occurs. More precisely, we show that if the *dissipation time* of u is small enough, then c converges exponentially to the total concentration $\bar{c} = \int_{\mathbb{T}^d} c_0 \, dx$, where c_0 denotes the initial data. This is illustrated by the numerical simulations in Fig. 2, where the velocity field u was chosen to be alternating horizontal and vertical shear flows with randomized phases (see Pierrehumbert 1994; Náraigh and Thiffeault 2007a, 2008). When the shear amplitude, A, is small, the norms of the solution settle to some nonzero value after a large time. As the amplitude is increased, the flow mixes faster, and we see the solution decays exponentially to $\bar{c} = 0$ (Fig. 3).

1.1 Decay of the Advective Cahn–Hilliard equation

To state our main result, we need to first introduce the notion of *dissipation time*. Let u be a divergence-free vector field and consider the equation

$$\partial_t \theta + u \cdot \nabla \theta + \gamma(-\Delta)^\alpha \theta = 0, \tag{1.3}$$

with $\alpha > 0$, periodic boundary conditions, and mean-zero initial data. For $\alpha = 1$, this is the *advection–diffusion equation*; for $\alpha = 2$, it is the *advection–hyperdiffusion equation*. Incompressibility of u and the Poincaré inequality immediately imply that $\|\theta_t\|_{L^2}$ is decreasing as a function of t, and

$$\|\theta(s+t)\|_{L^2} \leq e^{-(2\pi)^{2\alpha}\gamma t} \|\theta(s)\|_{L^2}. \tag{1.4}$$

Thus, we are guaranteed

$$\|\theta(s+t)\|_{L^2} \leq \tfrac{1}{2}\|\theta(s)\|_{L^2}, \quad \text{for every} \quad t \geq \frac{\ln 2}{(2\pi)^{2\alpha}\gamma}, \tag{1.5}$$

and every $s \geq 0$. However, u generates gradients through filamentation, which causes solutions to dissipate $\|\theta(t)\|_{L^2}$ faster. This may result in the lower bound in (1.5) being attained at much smaller times, and the smallest time t at which this happens is known as the *dissipation time* [see, for instance, (Fannjiang and Wołowski 2003; Feng and Iyer 2019)].

Definition 1.1 (Dissipation time). Let $\mathcal{S}^{u,\alpha}_{s,t}$ be the solution operator to (1.3) on $\mathbb{T}^d \times (0, \infty)$. That is, for any $f \in L^2(\mathbb{T}^d)$, the function $\theta(t) = \mathcal{S}^{u,\alpha}_{s,t} f$ solves (1.3) with initial data $\theta(s) = f$, and periodic boundary conditions. The *dissipation time* of u is

$$\tau^*_\alpha(u, \gamma) \stackrel{\text{def}}{=} \inf \left\{ t \geq 0 \,\middle|\, \|\mathcal{S}^{u,\alpha}_{s,s+t}\|_{\dot{L}^2 \to \dot{L}^2} \leq \tfrac{1}{2} \text{ for all } s \geq 0 \right\}. \tag{1.6}$$

Here, $\dot{L}^2$ is the space of all mean-zero, square integrable functions on the torus $\mathbb{T}^d$.

While this definition makes sense for any $\alpha > 0$, we are mainly interested in the case when α is either 1 or 2. Note that (1.5) implies $\tau^*_\alpha(u, \gamma) \leq O(1/\gamma)$ as $\gamma \to 0$. If, however, u is mixing, then this can be dramatically improved [see, for instance, (Constantin et al. 2008; Zlatoš 2010; Coti Zelati et al. 2018; Wei 2018; Feng and Iyer 2019; Feng 2019)]. In fact, (Feng and Iyer 2019) bound $\tau^*_1(u, \gamma)$ explicitly in terms of the mixing rate of u. Moreover, when u is exponentially mixing, Coti Zelati et al. (2018), Wei (2018), Feng (2019) show that $\tau^*_1(u, \gamma) \leq O(|\ln \gamma|^2)$ as $\gamma \to 0$.

With this notion, we can now state our main result.

Theorem 1.2 *Let $d \in \{2, 3\}$, $u \in L^\infty((0, \infty); W^{1,\infty}(\mathbb{T}^d))$, and c be the solution of (1.2) with initial data $c_0 \in H^2(\mathbb{T}^d)$.*

(1) *When $d = 2$, for any $\beta > 1$, $\mu > 0$, there exists a time*

$$T_0 = T_0(\|c_0 - \bar{c}\|_{L^2}, \bar{c}, \beta, \gamma, \mu)$$

such that if $\tau_2^*(u, \gamma) < T_0$, then for every $t \geq 0$, we have

$$\|c(t) - \bar{c}\|_{L^2} \leq \beta e^{-\mu t} \|c_0 - \bar{c}\|_{L^2}. \tag{1.7}$$

(2) *When $d = 3$, for any $\beta > 1$, $\mu > 0$, there exists a time*

$$T_1 = T_1(\|c_0 - \bar{c}\|_{L^2}, \bar{c}, \beta, \gamma, \mu)$$

such that if

$$(1 + \|\nabla u\|_{L^\infty})^{1/2} \tau_2^*(u, \gamma) < T_1, \tag{1.8}$$

then (1.7) still holds for every $t \geq 0$.

Remark 1.3 The times T_0 and T_1 can be computed explicitly, as can be seen from the proof of the theorem and equations (2.16) and (2.27).

We emphasize that T_0 and T_1 only depend on the mean concentration $\bar{c}$, the variance of the initial data $\|c_0 - \bar{c}\|_{L^2}^2$, the Cahn number $\sqrt{\gamma}$, and the exponential decay constants β, μ. Once T_0 and T_1 are determined from these parameters, in order to apply Theorem 1.2, we need to produce velocity fields u that satisfy $\tau_2^*(u, \gamma) < T_0$ when $d = 2$, and the condition (1.8) when $d = 3$. We do this in Sect. 1.2, by using sufficiently mixing flows with a large amplitude. In general, however, smallness of the dissipation time (such as the conditions required in Theorem 1.2) is weaker than mixing, and there may be simpler examples of velocity fields that satisfy the requirements of Theorem 1.2.

Several authors have used mixing properties of the advection term to quench reactions, prevent blow-up, and stem the growth of nonlinear PDEs [see, for instance, (Fannjiang et al. 2006; Hou and Lei 2009; Berestycki et al. 2010; Kiselev and Xu 2016; Bedrossian and He 2017; Iyer et al. 2019)]. Our results are similar in spirit to those in Iyer et al. (2019), where the authors used related ideas to prove decay of solutions to a large class of nonlinear parabolic equations. These results were formulated for second-order PDEs where the diffusive term is the Laplacian, but they can easily be generalized to apply when the diffusive term is the bi-Laplacian as we have in (1.2). Unfortunately, the assumptions required for the results in Iyer et al. (2019) to apply are not satisfied by the nonlinear term, even when $d = 2$, and thus, we cannot use them here.

Our 3D result is qualitatively different (and weaker) from the 2D case and from the results in Iyer et al. (2019). Indeed, Theorem 1.2 in 2D and all the results in Iyer et al. (2019) only rely on smallness of the dissipation times τ_1^* or τ_2^*. In 3D, however, Theorem 1.2 now requires smallness of $(1 + \|\nabla u\|)_{L^\infty}^{1/2} \tau_2^*$. The reason for this is that we are able to estimate the nonlinear term $\|\Delta(c^3 - c)\|_{L^2}$ by $\|c - \bar{c}\|_{L^2} \|\Delta c\|_{L^2}^2$ in 2D and by $\|\nabla c\|_{L^2} \|\Delta c\|_{L^2}^2$ in 3D. The growth of $\|c - \bar{c}\|_{L^2}$ can easily be controlled independent of the advecting flow, and so the 2D result can be formulated only in

terms of the dissipation time $\tau_2^*(u, \gamma)$. The quantity $\|\nabla c\|_{L^2}$, however, is expected to depend intrinsically on (and grow with) the advecting flow, and the result in 3D involves both $\tau_2^*(u, \gamma)$ and the size of the flow (condition (1.8)).

In the next section, we produce velocity fields where this is arbitrarily small. We remark, however, that while we can find velocity fields for which $(1 + \|\nabla u\|_{L^\infty})^{1/2}\tau_2^*$ is arbitrarily small, it appears impossible to produce velocity fields for which $(1 + \|\nabla u\|_{L^\infty})\tau_2^*$ is arbitrarily small. To see this, the proof in Poon (1996) [see also equation (9) in Miles and Doering (2018)] can be easily adapted to obtain the lower bound

$$\tau_2^*(u, \gamma) \geq \frac{1}{C\|u\|_{C^2}} \ln\left(1 + \frac{C\|u\|_{C^2}}{\gamma}\right)$$

for some explicit dimensional constant C. (Here, by $\|u\|_{C^2}$ we mean the spatial C^2 norm $\sup_t \|u\|_{C^2(\mathbb{T}^d)}$.) When $\tau_2^*(u, \gamma)$ is small, we expect $\|u\|_{C^2}$ to be large, and in this case the above shows $(1 + \|u\|_{C^2})\tau_2^*(u, \gamma)$ grows at least logarithmically with $\|u\|_{C^2}$.

1.2 Incompressible Velocity Fields with Small Dissipation Time

In order to apply Theorem 1.2, we need to produce incompressible velocity fields u for which $\tau_2^*(u, \gamma)$ is arbitrarily small when $d = 2$, and for which $(1 + \|\nabla u\|_{L^\infty}^{1/2})\tau_2^*(u, \gamma)$ is arbitrarily small when $d = 3$. We do this here by rescaling *mixing flows*. This has been studied previously by Constantin et al. (2008), Kiselev et al. (2008), Zlatoš (2010), Coti Zelati et al. (2018), Feng and Iyer (2019), and Feng (2019) when the diffusive term is the standard Laplacian. With minor modification, the proofs can be adapted to our context, where the diffusive term is the bi-Laplacian.

Proposition 1.4 *Let* $v \in L^\infty([0, \infty); C^2(\mathbb{T}^d))$, *and define* $u_A(x, t) = Av(x, At)$. *If* v *is* weakly mixing *with rate function h, then*

$$\tau_2^*(u_A, \gamma) \xrightarrow{A \to \infty} 0.$$

If further v *is* strongly mixing *with rate function h, and*

$$t\, h(t) \xrightarrow{t \to \infty} 0, \tag{1.9}$$

then

$$(1 + \|\nabla u_A\|_{L^\infty})^{1/2}\tau_2^*(u_A, \gamma) \xrightarrow{A \to \infty} 0.$$

For ease of presentation, we defer the definition of weak and strong mixing used above to Sect. 3 (see Definition 3.1). To the best of our knowledge, the existence of smooth, *time-independent* or even *time-periodic*, mixing flows on the torus is open. Various interesting and explicit examples of mixing flows were constructed in Yao and Zlatoš (2017), Alberti et al. (2019), Elgindi and Zlatoš (2019). Unfortunately, none of these examples are spatially regular enough to be used in Proposition 1.4.

Fortunately, there are many known examples of (spatially) smooth, *time-dependent*, flows on the torus that are exponentially mixing, and any such flow *will* satisfy the

conditions required by Proposition 1.4. The simplest example we are aware of is to use alternating horizontal/vertical sinusoidal shear flows with randomized phases. These were introduced by Pierrehumbert (1994) and used to produce our Figure 2. One can show that these flows, and a variety of other examples, are exponentially mixing using techniques in Bedrossian et al. (2019).

We also remark that the mixing requirement in Proposition 1.4 is morally much stronger than what is needed in order to apply Theorem 1.2. Indeed, for Theorem 1.2 one only needs flows whose dissipation time τ_2^* is sufficiently small. Proposition 1.4 ensures smallness of $\tau_2^*(u_A, \gamma)$ by using the property that the flow v sends a fraction of the total energy to high frequencies, which then gets rapidly damped by the diffusion. The mixing assumptions on v, however, ensure a much stronger property, namely that the flow v eventually sends all the energy to high frequencies [see Drivas et al. (2019) for a longer discussion]. Thus, the mixing hypothesis in Proposition 1.4 is most likely much stronger than what may be needed to apply Theorem 1.2. In theory, it should also be easier to find flows directly satisfying the requirements of Theorem 1.2, without using Proposition 1.4.

When the diffusion operator is the standard Laplacian, this was done in Iyer et al. (2019). Here, the authors showed that for any $\tau_0 > 0$, there exists a sufficiently strong and fine cellular flow, u, for which $\tau_1^*(u, \gamma) < \tau_0$. This provides a simple, explicit, smooth, *time independent* family of velocity fields with arbitrarily small dissipation time (when the diffusion operator is the standard Laplacian), and in Iyer et al. (2019) the authors used it to prevent blow-up in the Keller–Segel and other second-order, nonlinear, parabolic PDEs.

We expect that for any $\tau_0 > 0$, one can also construct sufficiently strong and fine cellular flows for which $\tau_2^*(u, \gamma) < \tau_0$. (We recall here $\tau_2^*(u, \gamma)$ is the dissipation time when the diffusion operator is the bi-Laplacian.) Unfortunately, the proof in Iyer et al. (2019) does not generalize, and thus, we are presently unable to produce cellular flows for which $\tau_2^*(u, \gamma)$ is small enough, or for which (1.8) holds.

1.3 Relationships Between the Various Dissipation Times

Since for any $\alpha, \gamma > 0$, the quantity $\tau_\alpha^*(u, \gamma)$ is a measure of the rate at which u mixes, it is natural to study its behavior as α and γ vary. When $\alpha = 1$, the behavior of $\tau_\alpha^*(u, \gamma)$ as $\gamma \to 0$ was recently studied in Coti Zelati et al. (2018), Feng and Iyer (2019), Feng (2019) and quantified in terms of the mixing rate. We will instead study the behavior of $\tau_\alpha^*(u, \gamma)$ when γ is fixed and α varies. Moreover, since $\tau_1^*(u, \gamma)$ and $\tau_2^*(u, \gamma)$ are particularly interesting from a physical point of view, we focus our attention on the relationship between these two quantities. Our first result is an upper bound for $\tau_2^*(u, \gamma)$ in terms of $\tau_1^*(u, \gamma)$.

Lemma 1.5 *There exists an explicit dimensional constant C such that for every divergence-free $u \in L^\infty([0, \infty); C^2(\mathbb{T}^d))$, and every $\gamma > 0$, we have*

$$\tau_2^*(u, \gamma) \le C\tau_1^*(u, \gamma)(1 + \|u\|_{C^2} \tau_1^*(u, \gamma)). \tag{1.10}$$

Since velocity fields with small $\tau_1^*(u, \gamma)$ are known, one use of Lemma 1.5 is to produce velocity fields for which $\tau_2^*(u, \gamma)$ and $(1 + \|\nabla u\|_{L^\infty})^{1/2} \tau_2^*(u, \gamma)$ are small. For instance, if u is mixing at a sufficiently fast rate, then the results of Wei (2018), Coti Zelati et al. (2018), Feng and Iyer (2019), Feng (2019) along with Lemma 1.5 can be used to produce velocity fields for which $\tau_2^*(u, \gamma)$ and $(1 + \|\nabla u\|_{L^\infty})^{1/2} \tau_2^*(u, \gamma)$ are arbitrarily small. Lemma 1.5, however, cannot be used to produce cellular flows for which $\tau_2^*(u, \gamma)$ is arbitrarily small. Indeed, with the τ_1^* bound in Iyer et al. (2019), or even the best expected heuristic for cellular flows, the right-hand side of (1.10) diverges.

1.4 Plan of the Paper

In Sect. 2, we prove our main result (Theorem 1.2). In Sect. 3 we recall the definition of weak and strong mixing and prove Proposition 1.4. In Sect. 4, we prove Lemma 1.5 bounding τ_2^* in terms of τ_1^*. Finally, for completeness, we conclude with an appendix estimating the dissipation time τ_2^* in terms of the mixing rate of the advecting velocity field.

2 Decay of the Advective Cahn–Hilliard Equation

This section is devoted to the proof of Theorem 1.2. We begin by recalling the well-known existence of global strong solutions to equation (1.2). Elliott and Songmu Elliott and Songmu (1986) proved well-posedness in the absence of advection. Since the advection is a first-order linear term, their proof can easily be adapted to our setting. We state the result here for convenience.

Proposition 2.1 *Let $\gamma > 0$, $u \in L^\infty([0, \infty); W^{1,\infty}(\mathbb{T}^d))$ be divergence-free and $c_0 \in H^2(\mathbb{T}^d)$. There exists a unique strong solution to (1.2) in the space*

$$c(t, x) \in L^2_{\text{loc}}([0, \infty); H^4(\mathbb{T}^d)) \cap L^\infty_{\text{loc}}([0, \infty); H^2(\mathbb{T}^d)) \cap H^1_{\text{loc}}([0, \infty); L^2(\mathbb{T}^d)).$$

For the remainder of this section, let $\beta > 1$, $\gamma > 0$, and $\mu > 0$ be as in the statement of Theorem 1.2. Without loss of generality, we may further assume $\beta \in (1, 2]$. We also fix a divergence-free velocity field $u \in L^\infty([0, \infty); W^{1,\infty}(\mathbb{T}^d))$, $c_0 \in H^2(\mathbb{T}^d)$ and let c be the unique strong solution to equation (1.2) with initial data c_0. The existence of such a solution is guaranteed by Proposition 2.1.

The main idea behind the proof of Theorem 1.2 is to split the analysis into two cases. First, when the time average of $\|\Delta c\|_{L^2}$ is large, standard energy estimates will show that the variance of c decreases exponentially. Second, when the time average of $\|\Delta c\|_{L^2}^2$ is small, we will use the advection term to show that the variance of c still decreases exponentially, at a comparable rate.

We begin with a lemma handling the first case.

Lemma 2.2 *For any $t_0 \geq 0$ and $\beta > 1$, we have*

$$\sup_{0 \leq \tau \leq \gamma \ln \beta} \|c(t_0 + \tau) - \bar{c}\|_{L^2}^2 \leq \beta \|c(t_0) - \bar{c}\|_{L^2}^2. \tag{2.1}$$

Moreover, if for some $\tau \in (0, \gamma \ln \beta)$ and $\mu > 0$ we have

$$\frac{1}{\tau} \int_{t_0}^{t_0+\tau} \|\Delta c\|_{L^2}^2 \, ds \geq \frac{\beta + 2\gamma\mu}{\gamma^2} \|c(t_0) - \bar{c}\|_{L^2}^2, \tag{2.2}$$

then

$$\|c(t_0 + \tau) - \bar{c}\|_{L^2} \leq e^{-\mu\tau} \|c(t_0) - \bar{c}\|_{L^2}. \tag{2.3}$$

For clarity of presentation, we momentarily postpone the proof of Lemma 2.2. We will now treat the two- and three-dimensional cases separately.

2.1 The Two-Dimensional Case

Suppose the time average of $\|\Delta c\|_{L^2}^2$ is small. In this case, we will show that if $\tau_2^*(u, \gamma)$ is small enough, then the variance of c still decreases by a constant fraction after time $\tau_2^*(u, \gamma)$.

Lemma 2.3 *For any $t_0 \geq 0$, there exists a time*

$$T_0' = T_0'(\|c(t_0) - \bar{c}\|_{L^2}, \bar{c}, \beta, \gamma, \mu) \in (0, \gamma \ln \beta]$$

such that if

$$\tau_2^*(u, \gamma) \leq T_0'(\|c(t_0) - \bar{c}\|_{L^2}, \beta, \gamma, \mu, \bar{c}), \tag{2.4a}$$

$$\frac{1}{\tau_2^*(u, \gamma)} \int_{t_0}^{t_0+\tau_2^*(u,\gamma)} \|\Delta c\|_{L^2}^2 \, ds \leq \frac{\beta + 2\gamma\mu}{\gamma^2} \|c(t_0) - \bar{c}\|_{L^2}^2, \tag{2.4b}$$

then (2.3) still holds at time $\tau = \tau_2^*(u, \gamma)$. Moreover, the time T_0' can be chosen to be decreasing as a function of $\|c(t_0) - \bar{c}\|_{L^2}$.

Remark The time T_0' can be computed explicitly in terms of $\|c(t_0) - \bar{c}\|_{L^2}$, β, γ, μ, and $\bar{c}$, as can be seen from (2.16).

Momentarily postponing the proof of Lemma 2.3, we prove Theorem 1.2 in 2D.
Proof of Theorem 1.2 when $d = 2$. Define

$$T_0 = \min\left\{T_0', \frac{\ln \beta}{2\mu}\right\},$$

where T_0' is the time given by Lemma 2.3 with $t_0 = 0$. For conciseness, let $\tau_2^* = \tau_2^*(u, \gamma)$, and suppose $\tau_2^* < T_0'$. If

$$\frac{1}{\tau_2^*} \int_0^{\tau_2^*} \|\Delta c\|_{L^2}^2 \, ds \geq \frac{\beta + 2\gamma\mu}{\gamma^2} \|c(t_0) - \bar{c}\|_{L^2}^2, \tag{2.5}$$

and since $T_0' < \gamma \ln \beta$ by choice, Lemma 2.2 applies and we must have

$$\|c(\tau_2^*) - \bar{c}\|_{L^2} \le e^{-\mu\tau_2^*}\|c_0 - \bar{c}\|_{L^2}. \tag{2.6}$$

If on the other hand (2.5) does not hold, then Lemma 2.3 applies and (2.6) still holds.

Since T_0' is a decreasing function of $\|c - \bar{c}\|_{L^2}$, we may restart the above argument at time τ_2^*. Proceeding inductively, we find

$$\|c(n\tau_2^*) - \bar{c}\|_{L^2} \le e^{-\mu n\tau_2^*}\|c_0 - \bar{c}\|_{L^2},$$

for all $n \in \mathbb{N}$.

Now for any time $t \ge 0$, let $n \in \mathbb{N}$ be such that $t \in (n\tau_2^*, (n+1)\tau_2^*)$. Since $t - n\tau_2^* \le \tau_2^* \le \gamma \ln \beta$, Lemma 2.2 applies and (2.1) yields

$$\|c(t) - \bar{c}\|_{L^2} \le \sqrt{\beta}\|c(n\tau_2^*) - \bar{c}\|_{L^2} \le \sqrt{\beta}e^{-\mu n\tau_2^*}\|c_0 - \bar{c}\|_{L^2}$$
$$\le \sqrt{\beta}e^{-\mu t + \mu\tau_2^*}\|c_0 - \bar{c}\|_{L^2} \le \beta e^{-\mu t}\|c_0 - \bar{c}\|_{L^2}.$$

The last inequality follows from $\tau_2^* \le \ln \beta/(2\mu)$. This completes the proof. $\qquad\square$

2.2 The Three-Dimensional Case

In this case, in order to prove the analog of Lemma 2.3, we need a stronger assumption on $\tau_2^*(u, \gamma)$.

Lemma 2.4 *For any $t_0 \ge 0$, there exists a time $T_1' = T_1'(\|c(t_0) - \bar{c}\|_{L^2}, \bar{c}, \beta, \gamma, \mu)$ such that if*

$$(1 + \|\nabla u\|_{L^\infty})^{1/2}\tau_2^*(u, \gamma) \le T_1', \tag{2.7}$$

$$\frac{1}{2\tau_2^*(u, \gamma)}\int_{t_0}^{t_0+2\tau_2^*(u,\gamma)} \|\Delta c\|_{L^2}^2 \, ds \le \frac{\beta + 2\gamma\mu}{\gamma^2}\|c(t_0) - \bar{c}\|_{L^2}^2, \tag{2.8}$$

then

$$\|c(t_0 + 2\tau_2^*(u, \gamma)) - \bar{c}\|_{L^2} \le e^{-2\mu\tau_2^*(u,\gamma)}\|c(t_0) - \bar{c}\|_{L^2}. \tag{2.9}$$

Moreover, the time T_1' can be chosen to be decreasing as a function of $\|c(t_0) - \bar{c}\|_{L^2}$.

Remark The time T_1' can be computed explicitly in terms of $\|c(t_0) - \bar{c}\|_{L^2}$, β, γ, μ, and $\bar{c}$, as can be seen from (2.27).

Momentarily postponing the proof of Lemma 2.4, we prove Theorem 1.2 in 3D.
Proof of Theorem 1.2 when $d = 3$. Let T_1' be the time given by Lemma 2.4 with $t_0 = 0$, and define

$$T_1 = \min\left\{T_1', \frac{\ln \beta}{4\mu}\right\}.$$

The remainder of the proof is now identical to the proof when $d = 2$ (page 8) with Lemma 2.3 replaced with Lemma 2.4. $\qquad\square$

2.3 Variance Decay in 2D (Lemmas 2.2 and 2.3)

It now remains to prove the lemmas. The variance decay when $\|\Delta c\|_{L^2}$ is large follows directly from the energy inequality in both 2D and 3D. We prove this first.

Proof of Lemma 2.2 For simplicity and without loss of generality, we assume $t_0 = 0$. Multiplying equation (1.2) by $c - \bar{c}$ and integrating over $\mathbb{T}^d$, we obtain

$$
\begin{aligned}
\partial_t \|c - \bar{c}\|_{L^2}^2 &= 2\langle \Delta(c^3 - c - \gamma \Delta c), c - \bar{c}\rangle \\
&\leq -6\|c\nabla c\|_{L^2}^2 + 2\|c - \bar{c}\|_{L^2}\|\Delta c\|_{L^2} - 2\gamma\|\Delta c\|_{L^2}^2.
\end{aligned}
\tag{2.10}
$$

Here, the notation $\langle f, g\rangle = \int_{\mathbb{T}^d} fg\, dx$ denotes the standard L^2 inner-product on $\mathbb{T}^d$. Drop the first term in (2.10) and apply Young's inequality to find

$$
\partial_t \|c - \bar{c}\|_{L^2}^2 \leq -\gamma\|\Delta c\|_{L^2}^2 + \frac{1}{\gamma}\|c - \bar{c}\|_{L^2}^2,
\tag{2.11}
$$

and hence,

$$
\|c(t) - \bar{c}\|_{L^2}^2 \leq \|c_0 - \bar{c}\|_{L^2}^2\, e^{t/\gamma} \quad \text{for all } t \geq 0.
\tag{2.12}
$$

In particular, if $t \in (0, \gamma \ln \beta)$, we see that (2.1) holds with $t_0 = 0$.

For (2.3), note that integration of (2.11) from 0 to τ with (2.1) and (2.2) gives

$$
\|c(\tau) - \bar{c}\|_{L^2}^2 \leq \|c_0 - \bar{c}\|_{L^2}^2 \left(1 + \frac{\beta\tau}{\gamma}\right) - \gamma \int_0^\tau \|\Delta c\|_{L^2}^2\, ds \leq \|c_0 - \bar{c}\|_{L^2}^2\, (1 - 2\mu\tau).
$$

Since $1 - 2\mu\tau \leq e^{-2\mu\tau}$, this proves (2.3) as desired. □

We now turn to Lemma 2.3, where the time integral of $\|\Delta c\|_{L^2}^2$ is assumed small. In this case, by definition of τ_2^*, the linear terms halve the variance of c in time τ_2^*. If τ_2^* is small enough, then we show that the nonlinear terms cannot increase the variance too much in this time interval.

Proof of Lemma 2.3 For notational convenience, we use $\mathcal{S}_{s,t}$ to denote $\mathcal{S}_{s,t}^{u,2}$, the solution operator in Definition 1.1 with $\alpha = 2$. As before, we also use τ_2^* to denote $\tau_2^*(u, \gamma)$. For simplicity, and without loss of generality, we will again assume $t_0 = 0$.

By Duhamel's principle, we know

$$
c(\tau_2^*) - \bar{c} = \mathcal{S}_{0,\tau_2^*}(c_0 - \bar{c}) + \int_0^{\tau_2^*} \mathcal{S}_{s,\tau_2^*}(\Delta(c^3(s) - c(s)))\, ds.
$$

By definition of $\tau_2^* = \tau_2^*(u, \gamma)$, and the fact that $\mathcal{S}_{s,t}^{u,\alpha}$ is an L^2-contraction, we have

$$
\|c(\tau_2^*) - \bar{c}\|_{L^2} \leq \frac{B}{2} + \int_0^{\tau_2^*} \|\Delta(c^3 - c)\|_{L^2}\, ds,
\tag{2.13}
$$

where $B \overset{\text{def}}{=} \|c_0 - \bar{c}\|_{L^2}$. We now estimate the second term on the right of (2.13). First, note

$$
\begin{aligned}
\|\Delta(c^3 - c)\|_{L^2} &= \|6c|\nabla c|^2 + 3c^2 \Delta c - \Delta c\|_{L^2} \\
&\leq 6\|c - \bar{c}\|_{L^\infty} \|\nabla c\|_{L^4}^2 + 6|\bar{c}|\|\nabla c\|_{L^4}^2 + 3\|c - \bar{c}\|_{L^\infty}^2 \|\Delta c\|_{L^2} \\
&\quad + 6|\bar{c}|\|c - \bar{c}\|_{L^\infty}\|\Delta c\|_{L^2} + (3\bar{c}^2 + 1)\|\Delta c\|_{L^2}. \quad (2.14)
\end{aligned}
$$

By the Gagliardo–Nirenberg inequality, we know

$$
\|c - \bar{c}\|_{L^\infty} \leq C\|\Delta c\|_{L^2}^{d/4}\|c - \bar{c}\|_{L^2}^{1-d/4},
$$
$$
\|\nabla c\|_{L^4} \leq C\|\Delta c\|_{L^2}^{(4+d)/8}\|c - \bar{c}\|_{L^2}^{(4-d)/8},
$$

for some dimensional constant C. Here, and subsequently, we assume C is a purely dimensional constant that may increase from line to line. Substituting this in (2.14) when $d = 2$, we find

$$
\begin{aligned}
\|\Delta(c^3 - c)\|_{L^2} &\leq C\|\Delta c\|_{L^2}^2 \|c - \bar{c}\|_{L^2} \\
&\quad + C|\bar{c}|\|\Delta c\|_{L^2}^{3/2}\|c - \bar{c}\|_{L^2}^{1/2} + (3\bar{c}^2 + 1)\|\Delta c\|_{L^2} \\
&\leq C(1 + \bar{c}^2)(1 + \|c - \bar{c}\|_{L^2})(\|\Delta c\|_{L^2} + \|\Delta c\|_{L^2}^2). \quad (2.15)
\end{aligned}
$$

If we choose T_0' small enough to ensure $T_0' < \gamma \ln \beta$, then (2.1), (2.4b), (2.13) and (2.15) yield
$$
\|c(\tau_2^*) - \bar{c}\|_{L^2} \leq \frac{B}{2} + \frac{C_{\beta,\mu}\tau_2^*}{\gamma^2}(1 + \bar{c}^2)(1 + B^2)B.
$$

Here, $C_{\beta,\mu}$ is a constant that only depends on β, μ that may increase from line to line. Now choosing

$$
T_0' = \min\left\{ \frac{\gamma^2}{4C_{\beta,\mu}(1 + \bar{c}^2)(1 + B^2)}, \ \gamma \ln \beta, \ \frac{1}{4\mu} \right\} \quad (2.16)
$$

we see that whenever $\tau_2^* \leq T_0'$, we must have

$$
\|c(\tau_2^*) - \bar{c}\|_{L^2} \leq \frac{3B}{4} \leq (1 - \mu\tau_2^*)B \leq e^{-\mu\tau_2^*}\|c_0 - \bar{c}\|_{L^2},
$$

as claimed. Clearly, the choice of T_0' above is decreasing in B, finishing the proof. $\quad \square$

2.4 Variance Decay in 3D (Lemma 2.4)

To prove variance decay in 3D, we first need an H^1 bound. For the remainder of this subsection, we assume $d = 3$.

Lemma 2.5 *Define the* free energy, $\mathcal{E}$, *by*

$$\mathcal{E}(t) \overset{def}{=} \frac{1}{4} \int_{\mathbb{T}^3} (c^2 - 1)^2 \, dx + \frac{1}{2}\gamma \int_{\mathbb{T}^3} |\nabla c|^2 \, dx.$$

Then, for any $t_0, \tau \geq 0$ we have

$$\|\nabla c(t_0 + \tau)\|_{L^2}^2 \leq \frac{2\mathcal{E}(t_0)}{\gamma} + \frac{\|\nabla u\|_{L^\infty}}{2\pi^2 \gamma} e^{\tau/\gamma} \|c(t_0) - \bar{c}\|_{L^2}^2. \tag{2.17}$$

Proof Without loss of generality assume, $t_0 = 0$. Multiplying (1.2) by $c^3 - c - \gamma \Delta c$ and integrating over $\mathbb{T}^3$, we have

$$\partial_t \mathcal{E} + \langle u \cdot \nabla c, \ c^3 - c - \gamma \Delta c \rangle = -\|\nabla (c^3 - c - \gamma \Delta c)\|_{L^2}^2. \tag{2.18}$$

Since u is divergence-free,

$$|\langle u \cdot \nabla c, \ c^3 - c - \gamma \Delta c \rangle| = |\langle u \cdot \nabla c, \gamma \Delta c \rangle| \leq \gamma \|\nabla u\|_{L^\infty} \|\nabla c\|_{L^2}^2.$$

Use this in (2.18), integrate in time, then use Poincaré's inequality to get

$$\int_0^\tau \|\nabla (c^3 - c - \gamma \Delta c)\|^2 \, ds + \mathcal{E}(\tau) \leq \mathcal{E}(0) + \gamma \|\nabla u\|_{L^\infty} \int_0^\tau \|\nabla c\|_{L^2}^2 \, ds$$

$$\leq \mathcal{E}(0) + \frac{\gamma \|\nabla u\|_{L^\infty}}{4\pi^2} \int_0^\tau \|\Delta c\|_{L^2}^2 \, ds. \tag{2.19}$$

Time-integrating (2.11) and using (2.12), we find

$$\gamma \int_0^\tau \|\Delta c\|_{L^2}^2 \, ds \leq \frac{1}{\gamma} \int_0^\tau \|c - \bar{c}\|_{L^2}^2 \, ds + \|c(t_0) - \bar{c}\|_{L^2}^2 \leq e^{\tau/\gamma} \|c_0 - \bar{c}\|_{L^2}^2. \tag{2.20}$$

Finally, we substitute (2.20) in (2.19) to obtain

$$\tfrac{1}{2}\gamma \|\nabla c(\tau)\|_{L^2}^2 \leq \mathcal{E}(\tau) \leq \mathcal{E}(0) + \frac{\|\nabla u\|_{L^\infty}}{4\pi^2} e^{\tau/\gamma} \|c_0 - \bar{c}\|_{L^2}^2,$$

which immediately implies (2.17) as claimed. ∎

We now prove Lemma 2.4.

Proof of Lemma 2.4 As before, we assume without loss of generality that $t_0 = 0$. In the 3D case, we will express $c(2\tau_2^*)$ using Duhamel's principle. However, for reasons that will be explained below, we need to use a starting time of $t_1 \in [0, \tau_2^*]$, which might not be 0. Note that for any $t_1 \in [0, \tau_2^*]$, we have

$$c(2\tau_2^*) - \bar{c} = \mathcal{S}_{t_1, 2\tau_2^*}(c(t_1) - \bar{c}) + \int_{t_1}^{2\tau_2^*} \mathcal{S}_{s, 2\tau_2^*}(\Delta(c^3 - c)) \, ds.$$

Since $2\tau_2^* - t_1 \geq \tau_2^*$, the above implies

$$\|c(2\tau_2^*) - \bar{c}\|_{L^2} \leq \tfrac{1}{2}\|c(t_1) - \bar{c}\|_{L^2} + \int_{t_1}^{2\tau_2^*} \|\Delta(c^3 - c)\| \, ds. \tag{2.21}$$

To bound the first term on the right, we note that if $2T_1' \leq \gamma \ln \beta$, then (2.1) implies

$$\|c(t_1) - \bar{c}\|_{L^2} \leq \sqrt{\beta} B, \tag{2.22}$$

where $B \overset{\text{def}}{=} \|c_0 - \bar{c}\|_{L^2}$.

To bound the second term on the right-hand side, recall the Gagliardo–Nirenberg interpolation inequalities in 3D guarantee

$$\|c - \bar{c}\|_{L^\infty} \leq C\|\nabla c\|_{L^2}^{1/2}\|\Delta c\|_{L^2}^{1/2},$$
$$\|\nabla c\|_{L^4} \leq C\|\nabla c\|_{L^2}^{1/4}\|\Delta c\|_{L^2}^{3/4}.$$

Expanding $\|\Delta(c^3 - c)\|_{L^2}$ as in (2.14), and using these inequalities, we see

$$\begin{aligned}
\|\Delta(c^3 - c)\|_{L^2} &\leq C\|\Delta c\|_{L^2}^2\|\nabla c\|_{L^2} + C|\bar{c}|\|\Delta c\|_{L^2}^{3/2}\|\nabla c\|_{L^2}^{1/2} \\
&\quad + (3\bar{c}^2 + 1)\|\Delta c\|_{L^2}, \\
&\leq C(1 + \bar{c}^2)(1 + \|\nabla c\|_{L^2})(\|\Delta c\|_{L^2} + \|\Delta c\|_{L^2}^2).
\end{aligned} \tag{2.23}$$

The difference from the 2D case is precisely at this step, as the above estimate does not allow us to bound the second term on the right of (2.21) using (2.8) and (2.1) alone. Indeed, to bound this term, we now need a time-uniform bound on $\|\nabla c\|_{L^2}$, in combination with (2.8) and (2.1). Unfortunately, the only such bounds we can obtain depend on u, and thus, our criterion in 3D involves both $\|\nabla u\|_{L^\infty}$ and τ_2^*.

To carry out the details, note first that by Chebyshev's inequality and (2.8) we can choose $t_1 \in [0, \tau_2^*]$ so that

$$\|\Delta c(t_1)\|_{L^2}^2 \leq \frac{2\beta + 4\gamma\mu}{\gamma^2} B^2. \tag{2.24}$$

Using the Gagliardo–Nirenberg inequality and (2.24), we note that the free energy $\mathcal{E}$ at time t_1 can be bounded by

$$\begin{aligned}
\mathcal{E}(t_1) &\leq \tfrac{1}{4}\|c(t_1)\|_{L^4}^4 + \tfrac{1}{2}\gamma\|\nabla c(t_1)\|_{L^2}^2 + \tfrac{1}{4} \\
&\leq 2\|c(t_1) - \bar{c}\|_{L^4}^4 + \tfrac{1}{2}\gamma\|\nabla c\|_{L^2}^2 + 2\bar{c}^4 + \tfrac{1}{4} \\
&\leq C\|c(t_1) - \bar{c}\|_{L^2}^{5/2}\|\Delta c(t_1)\|_{L^2}^{3/2} + \frac{\gamma}{8\pi^2}\|\Delta c(t_1)\|_{L^2}^2 + 2\bar{c}^4 + \tfrac{1}{4} \\
&\leq \frac{C\beta^{5/4}(2\beta + 4\gamma\mu)^{3/4}}{\gamma^{3/2}}B^4 + \frac{\beta + 2\gamma\mu}{4\pi^2\gamma}B^2 + 2\bar{c}^4 + \tfrac{1}{4}
\end{aligned}$$

$$\leq \frac{C_{\beta,\mu}B^4}{\gamma^2} + 2(\bar{c}^4 + 1).$$

Thus, for any time $t \in [t_1, 2\tau^*]$, we use Lemma 2.5 and obtain

$$\|\nabla c(t)\|_{L^2}^2 \leq \frac{2\mathcal{E}(t_1)}{\gamma} + \frac{\|\nabla u\|_{L^\infty}}{2\pi^2\gamma} e^{(t-t_1)/\gamma} \|c(t_1) - \bar{c}\|_{L^2}^2$$

$$\leq \frac{C_{\beta,\mu}}{\gamma^3}(1 + \|\nabla u\|_{L^\infty})(1 + \bar{c}^4)(B^4 + 1). \tag{2.25}$$

The use of (2.8), (2.23) and (2.25) in (2.21) yields

$$\|c(2\tau_2^*) - \bar{c}\|_{L^2} \leq \frac{\sqrt{\beta}B}{2} + (1 + \bar{c}^2)\left(1 + \frac{C_{\beta,\mu}}{\gamma^{3/2}}(1 + \|\nabla u\|_{L^\infty})^{1/2}(1 + \bar{c}^2)(1 + B^2)\right)$$

$$\cdot \left(\frac{C_{\beta,\mu}}{\gamma^2}B^2 + \frac{C_{\beta,\mu}}{\gamma}B\right)\tau_2^*$$

$$\leq \frac{\sqrt{\beta}B}{2} + \frac{C_{\beta,\mu}\tau_2^*}{\gamma^{7/2}}(1 + \|\nabla u\|_{L^\infty})^{1/2}(1 + \bar{c}^4)(1 + B^3)B. \tag{2.26}$$

Thus, if we choose

$$T_1' \stackrel{\text{def}}{=} \min\left\{\left(\frac{3}{4} - \frac{\sqrt{\beta}}{2}\right)\frac{\gamma^{7/2}}{(1 + B^3)(1 + \bar{c}^4)C_{\beta,\mu}}, \frac{\gamma \ln \beta}{2}, \frac{1}{8\mu}\right\}, \tag{2.27}$$

then our assumption (2.7) and the bound (2.26) imply (2.9) as claimed. Note that, since we have previously assumed $\beta \leq 2$, the choice of T_1' will be strictly positive. Finally, the fact that T_1' is decreasing in $\|c_0 - \bar{c}\|_{L^2}$ follows directly from (2.27). $\qquad\square$

3 The Dissipation Time of Mixing Flows

In this section, we prove Proposition 1.4. Since working on closed Riemannian manifolds introduces almost no added complexity, we will prove Proposition 1.4 in this setting. Let M be a d-dimensional, smooth, closed Riemannian manifold, with metric normalized so that $\text{vol}(M) = 1$. Let Δ denote the Laplace–Beltrami operator on M, and $u \in L^\infty([0, \infty); W^{1,\infty}(M))$ be a divergence-free vector field. We begin by recalling the definition of *weakly mixing* and *strongly mixing* that we use.

Definition 3.1 Let $h\colon [0, \infty) \to (0, \infty)$ be a continuous decreasing function that vanishes at ∞. Given $\phi_0 \in \dot{L}^2(M)$, let ϕ denote the solution of

$$\partial_t \phi + u(t, x) \cdot \nabla \phi = 0, \tag{3.1}$$

on M, with initial data ϕ_0.

(1) We say u is *weakly mixing* with rate function h if for every $\phi_0, \psi \in \dot{H}^1(M)$ and every $s, T \geq 0$ we have

$$\left(\frac{1}{T} \int_0^T |\langle \phi(s+t), \psi \rangle|^2 \, dt \right)^{1/2} \leq h(T) \|\phi(s)\|_{H^1} \|\psi\|_{H^1}.$$

(2) We say u is *strongly mixing* with rate function h if for every $\phi_0, \psi \in \dot{H}^1(M)$ and every $s, t \geq 0$ we have

$$|\langle \phi(s+t), \psi \rangle| \leq h(t) \|\phi(s)\|_{H^1} \|\psi\|_{H^1}.$$

The use of H^1 norms in Definition 3.1 is purely for convenience and is motivated by Lin et al. (2011), Thiffeault (2012), and Feng and Iyer (2019). The traditional choice in the dynamical systems literature is to use C^1 norms instead. This difference, however, is not significant as varying the norms used in Definition 3.1 only changes the mixing rate function (see, for instance, "Appendix A" in Feng and Iyer (2019)).

In Feng and Iyer (2019), Feng (2019), the authors estimated the dissipation time $\tau_1^*(u, \gamma)$ in terms of the weak (or strong) mixing rate function h. With minor modifications, their work can be modified to give the following estimate for τ_2^*.

Theorem 3.2 *Let $u \in L^\infty([0, \infty); C^2(M))$ be a divergence-free vector field, and $h \colon [0, \infty) \to (0, \infty)$ be a continuous decreasing function that vanishes at ∞.*

(1) *There exists constants $C_1, C_2 > 0$ such that if u is weakly mixing with rate function h, then for all sufficiently small γ we have*

$$\tau_2^*(u, \gamma) \leq t_* + C_1 \|u\|_{C^2} t_*^2. \tag{3.2}$$

Here, t^* is the unique solution of

$$\gamma \|u\|_{C^2} t_*^2 = C_2 \left(h(t_*/\sqrt{2}) \right)^{8/(4+d)}. \tag{3.3}$$

(2) *There exists constants $C_1, C_2 > 0$ such that if u is strongly mixing with rate function h, then for all sufficiently small γ, we have (3.2), where t_* is the unique solution of*

$$\gamma \|u\|_{C^2} t_*^2 = C_2 h^2(t_*/2\sqrt{2}). \tag{3.4}$$

The proof of Theorem 3.2 is very similar to that in Feng (2019, Chapter 4), and we provide a sketch in "Appendix A." We now prove Proposition 1.4 using Theorem 3.2.

Proof of Proposition 1.4 Rescaling time by a factor of A, we immediately see that

$$\tau_2^*(u_A, \gamma) = \frac{1}{A} \tau_2^* \left(v, \frac{\gamma}{A} \right). \tag{3.5}$$

For the first assertion in Proposition 1.4, we assume v is weakly mixing with rate function h. Using (3.2) and (3.5), we see that

$$\tau_2^*(u_A, \gamma) \leq \frac{1}{A} \left(t_*(A) + C_1 \|v\|_{C^2} t_*^2(A) \right), \tag{3.6}$$

where $t_*(A)$ solves

$$\frac{\gamma}{A}\|v\|_{C^2}\, t_*^2(A) = C_2\, (h(\tfrac{t_*(A)}{\sqrt{2}}))^{8/(4+d)}. \tag{3.7}$$

Clearly, this implies $t_*(A) \to \infty$ as $A \to \infty$. Since h vanishes at ∞, this in turn implies that $t_*^2(A)/A \to 0$ as $A \to \infty$. Consequently, the right-hand side of (3.6) vanishes as $A \to \infty$, proving the first assertion of Proposition 1.4.

For the second assertion, we assume v is strongly mixing with rate function h satisfying (1.9). In this case, Theorem 3.2 and (3.5) imply (3.6) still holds, provided $t_*(A)$ is defined by

$$\frac{\gamma}{A}\|v\|_{C^2}\, t_*^2(A) = C_2\, h^2\left(\frac{t_*(A)}{2\sqrt{2}}\right). \tag{3.8}$$

Note that this still implies $t_*(A) \to \infty$ as $A \to \infty$. Using this along with (1.9), we see that

$$\frac{t_*^2(A)}{A} \le \frac{\varepsilon}{t_*^2(A)}$$

for any $\varepsilon > 0$, and all sufficiently large A. Using this in (3.6) yields $A^{1/2}\tau_2^*(u_A, \gamma) \to 0$ as $A \to \infty$, concluding the proof. $\qquad\square$

4 Relationship Between τ_1^* and τ_2^* (Lemma 1.5)

In this section, we prove Lemma 1.5 bounding $\tau_2^*(u, \gamma)$ in terms of $\tau_1^*(u, \gamma)$. Throughout we fix $u \in L^\infty([0, \infty); C^2(\mathbb{T}^d))$, and assume θ is a solution of (1.3) with $\alpha = 2$ and mean-zero initial data $\theta_0 \in \dot{L}^2(\mathbb{T}^d)$. As before, we abbreviate $\tau_\alpha^*(u, \gamma)$ to τ_α^*.

The proof of Lemma 1.5 is similar to that of Theorem 1.2 in 3D. We divide the analysis into two cases: the first where the time average of $\|\Delta\theta\|_{L^2}^2$ is large (Lemma 4.1), and the second where the time average of $\|\Delta\theta\|_{L^2}^2$ is small (Lemma 4.2). Lemma 1.5 will be proven after these two lemmas.

Lemma 4.1 *If for some $t_0 \ge 0$, $\lambda, \tau > 0$, we have*

$$\frac{1}{\tau}\int_{t_0}^{t_0+\tau}\|\Delta\theta\|_{L^2}^2\, \mathrm{d}s \ge \lambda\|\theta(t_0)\|_{L^2}^2, \tag{4.1}$$

then

$$\|\theta(t_0 + \tau)\|_{L^2} \le e^{-\lambda\gamma\tau}\|\theta(t_0)\|_{L^2}. \tag{4.2}$$

Proof Multiplying (1.3) by θ and integrating, we obtain

$$\|\theta(t_0 + \tau)\|_{L^2}^2 = \|\theta(t_0)\|_{L^2}^2 - 2\gamma\int_{t_0}^{t_0+\tau}\|\Delta\theta\|_{L^2}^2\, \mathrm{d}s.$$

Inequalities (4.1) and $1 - x \le e^{-x}$ yield (4.2) as desired. $\qquad\square$

Lemma 4.2 *There exists an explicit dimensional constant C_1 such that if*

$$\lambda \overset{def}{=} \frac{1}{4\gamma\tau_1^*(20C_1\|u\|_{C^2}\tau_1^* + 11)},$$

and for some $t_0 \geq 0$ we have

$$\frac{1}{2\tau_1^*}\int_{t_0}^{t_0+2\tau_1^*}\|\Delta\theta\|_{L^2}^2\,ds \leq \lambda\|\theta(t_0)\|_{L^2}^2, \tag{4.3}$$

then (4.2) *still holds at time $\tau = 2\tau_1^*$.*

Proof Without loss of generality assume $t_0 = 0$. By Chebyshev's inequality, there exists $t_1 \in [0, \tau_1^*]$ such that

$$\|\Delta\theta(t_1)\|_{L^2}^2 \leq 2\lambda\|\theta_0\|_{L^2}^2. \tag{4.4}$$

Since

$$\partial_t\theta + u\cdot\nabla\theta - \gamma\Delta\theta = -\gamma\Delta^2\theta - \gamma\Delta\theta,$$

Duhamel's principle implies

$$\theta(2\tau_1^*) = \mathcal{S}_{t_1,2\tau_1^*}^{u,1}\theta(t_1) - \gamma\int_{t_1}^{2\tau_1^*}\mathcal{S}_{s,2\tau_1^*}^{u,1}(\Delta^2\theta(s) + \Delta\theta(s))\,ds,$$

where $\mathcal{S}$ is the solution operator from Definition 1.1. Since $2\tau_1^* - t_1 \geq \tau_1^*$, and $\mathcal{S}$ is an L^2 contraction, then Poincaré's inequality gives

$$\|\theta(2\tau_1^*)\|_{L^2} \leq \frac{\|\theta_0\|_{L^2}}{2} + 2\gamma\int_{t_1}^{2\tau_1^*}\|\Delta^2\theta\|_{L^2}\,ds. \tag{4.5}$$

To estimate the second term on the right, we multiply (1.3) with $\alpha = 2$ by $\Delta^2\theta$ and integrate in space to obtain

$$\tfrac{1}{2}\partial_t\|\Delta\theta\|_{L^2}^2 + \gamma\|\Delta^2\theta\|_{L^2}^2 \leq C_1\|u\|_{C^2}\|\Delta\theta\|_{L^2}^2,$$

for some explicit dimensional constant C_1. Integration in time together with (4.3) and (4.4) yields

$$2\gamma\int_{t_1}^{2\tau_1^*}\|\Delta^2\theta\|_{L^2}^2\,ds \leq \lambda(4C_1\|u\|_{C^2}\tau_1^* + 2)\|\theta_0\|_{L^2}^2.$$

Using this in (4.5), we have

$$\|\theta(2\tau_1^*)\|_{L^2} \leq \left(\tfrac{1}{2} + 2\left(\gamma\tau_1^*\lambda(2 + 4C_1\|u\|_{C^2}\tau_1^*)\right)^{1/2}\right)\|\theta_0\|_{L^2}.$$

By our choice of λ, this implies

$$\|\theta(2\tau_1^*)\|_{L^2} \leq (1 - 2\lambda\gamma\tau_1^*)\|\theta_0\|_{L^2} \leq e^{-2\lambda\gamma\tau_1^*}\|\theta_0\|_{L^2},$$

finishing the proof. □

The proof of Lemma 1.5 follows quickly from Lemmas 4.1 and 4.2.

Proof of Lemma 1.5 Iterating Lemmas 4.1 and 4.2 repeatedly, we see that for any $t_0 \geq 0$ and $n \in \mathbb{N}$ we have

$$\|\theta(t_0 + 2n\tau_1^*)\|_{L^2} \leq e^{-2n\lambda\gamma\tau_1^*}\|\theta(t_0)\|_{L^2}.$$

Thus, we must have $\tau_2^* \leq (\ln 2)/(\lambda\gamma)$, from which (1.10) follows. □

Appendix A. Dissipation Time Bounds of Mixing Vector Fields

In this section, we prove Theorem 3.2. As in Sect. 3, we assume here that M is a smooth, closed, Riemannian manifold with volume 1, and Δ is the Laplace–Beltrami operator on M. We also fix a divergence-free vector field $u \in L^\infty([0, \infty); C^2(M))$, and let θ be the solution to the advection–hyperdiffusion equation (1.3) with $\alpha = 2$ on the manifold M, with mean-zero initial data $\theta_0 \in \dot{L}^2(M)$.

The idea behind the proof of Theorem 3.2 is to divide the analysis into two cases. When $\|\Delta\theta\|_{L^2}/\|\theta\|_{L^2}$ is large, the energy inequality implies $\|\theta\|_{L^2}$ decays rapidly. On the other hand, when $\|\Delta\theta\|_{L^2}/\|\theta\|_{L^2}$ is small, we use the mixing assumption on u to show that $\|\theta\|_{L^2}$ still decays rapidly. The outline of the proof is the same as that of Theorem 1.2; however, the proof of the second case is substantially different. We begin by stating two lemmas handling each of the above cases.

Lemma A.1 *The solution θ satisfies the energy inequality*

$$\partial_t \|\theta\|_{L^2}^2 = -2\gamma\|\Delta\theta\|_{L^2}^2. \tag{A.1}$$

Consequently, if for some $c_0 > 0$ we have

$$\|\Delta\theta(t)\|_{L^2}^2 \geq c_0\|\theta(t)\|_{L^2}^2, \quad \text{for all } 0 \leq t \leq t_0,$$

then

$$\|\theta(t)\|_{L^2}^2 \leq e^{-2\gamma c_0 t}\|\theta_0\|_{L^2}^2, \quad \text{for all } 0 \leq t \leq t_0. \tag{A.2}$$

Lemma A.2 *Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of the Laplacian, where each eigenvalue is repeated according to its multiplicity. Suppose u is weakly mixing with rate function h. There exists positive, finite dimensional constants $\tilde{C}, \tilde{c}$ such that for all*

γ sufficiently small the following holds: If λ_N is an eigenvalue of the Laplace–Beltrami operator such that [1]

$$h^{-1}\left(\frac{1}{\tilde{c}\lambda_N^{(d+4)/4}}\right) \leq \frac{1}{\tilde{C}\lambda_N\sqrt{\gamma}\|u\|_{C^2}^{1/2}}, \tag{A.3}$$

and if

$$\|\Delta\theta_0\|_{L^2}^2 < \lambda_N^2\|\theta_0\|_{L^2}^2 \tag{A.4}$$

holds, then we have

$$\|\theta(t_0)\|_{L^2}^2 \leq \exp\left(-\frac{\gamma\lambda_N^2 t_0}{4}\right)\|\theta_0\|_{L^2}^2, \tag{A.5}$$

at a time t_0 given by

$$t_0 \stackrel{def}{=} h^{-1}\left(\frac{1}{\tilde{c}\lambda_N^{(d+4)/4}}\right). \tag{A.6}$$

If instead *u* is strongly mixing, then the analog of Lemma A.2 is as follows.

Lemma A.3 *Suppose u is strongly mixing with rate function h. There exists a finite dimensional $\tilde{C} > 0$ such that for all γ sufficiently small the following holds: If λ_N is an eigenvalue of the Laplace–Beltrami operator such that*

$$2h^{-1}\left(\frac{1}{2\lambda_N}\right) \leq \frac{1}{\tilde{C}\lambda_N\sqrt{\gamma}\|u\|_{C^2}^{1/2}}, \tag{A.7}$$

and if (A.4) *holds, then* (A.5) *holds at a time t_0 given by*

$$t_0 \stackrel{def}{=} 2h^{-1}\left(\frac{1}{2\lambda_N}\right). \tag{A.8}$$

Finally, for the proof of Theorem 3.2 we need Weyl's Lemma [see, for instance, (Minakshisundaram and Pleijel 1949)], which describes the asymptotic growth of the eigenvalues of the Laplace–Beltrami operator.

Lemma A.4 (Weyl's Lemma). *Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of the Laplacian, where each eigenvalue is repeated according to its multiplicity. We have*

$$\lambda_j \approx \frac{4\pi\,\Gamma(\frac{d}{2}+1)^{2/d}}{\mathrm{vol}(M)^{2/d}}\,j^{2/d}, \tag{A.9}$$

asymptotically as $j \to \infty$.

Momentarily postponing the proof of Lemmas A.1–A.3, we prove Theorem 3.2.

---

[1] When γ is sufficiently small such a λ_N is guaranteed to exist.

Proof of Theorem 3.2 For the first assumption, we assume u is weakly mixing with rate function h. Let $\tilde{c}, \tilde{C}$ be the constants from Lemma A.2. Note that the intermediate value theorem readily implies the existence of a unique $\lambda_* > 0$ such that

$$h^{-1}\left(\frac{1}{\tilde{c}\lambda_*^{(d+4)/4}}\right) = \frac{1}{\tilde{C}\lambda_*\sqrt{\gamma}\|u\|_{C^2}^{1/2}}. \tag{A.10}$$

Further, it is easy to see that $\lambda_* \to \infty$ as $\gamma \to 0$. Thus, for all sufficiently small γ, Weyl's lemma implies $\lambda_{j+1} - \lambda_j = o(\lambda_j)$ as $j \to \infty$. Hence, for all sufficiently large λ_*, one can always find N large enough such that

$$\frac{\lambda_*^2}{2} \leq \lambda_N^2 \leq \lambda_*^2. \tag{A.11}$$

Now choosing $c_0 = \lambda_N^2$ and repeatedly applying Lemmas A.1 and Lemma A.2, we obtain an increasing sequence of times (t'_k), such that $t'_k \to \infty$, $t'_{k+1} - t'_k \leq t_0$, and

$$\|\theta_s(t'_k)\|_{L^2}^2 \leq \exp\left(-\frac{\gamma\lambda_N^2 t'_k}{4}\right)\|\theta_0\|_{L^2}^2.$$

This immediately implies

$$\tau_2^*(u, \gamma) \leq \frac{8\ln 2}{\gamma\lambda_N^2} + t_0. \tag{A.12}$$

Choosing

$$t_* \stackrel{\text{def}}{=} \frac{\sqrt{2}}{\tilde{C}\lambda_*\sqrt{\gamma}\|u\|_{C^2}^{1/2}},$$

and using (A.10), (A.11), and (A.12) yields (3.2) as claimed.

The proof of the second assertion of Theorem 3.2 is almost identical to that of the first assertion. The only change required is to replace Lemma A.2 with A.3. $\qquad\square$

It remains to prove Lemmas A.1–A.3.

Proof of Lemma A.1 Multiplying (1.3) by θ, integrating over M, and using the fact that u is divergence-free immediately yield (A.1). The second assertion of Lemma A.1 follows from this and Gronwall's lemma. $\qquad\square$

For Lemmas A.2 and A.3, we will need a standard result estimating the difference between θ and solutions to the inviscid transport equation.

Lemma A.5 *Let ϕ be the solution of (3.1) with initial data θ_0. There exists a dimensional constant C_d such that for all $t \geq 0$, we have*

$$\|\theta(t) - \phi(t)\|_{L^2}^2 \leq \sqrt{2\gamma t}\,\|\theta_0\|_{L^2}\left(C_d\,\|u\|_{C^2}\int_0^t \|\Delta\theta\|_{L^2}^2\,\mathrm{d}s + \|\Delta\theta_0\|_{L^2}^2\right)^{1/2}. \tag{A.13}$$

Proof Subtracting (1.3) and (3.1) shows

$$\partial_t(\theta - \phi) + u \cdot \nabla(\theta - \phi) + \gamma \Delta^2 \theta = 0.$$

Multiplying this by $\theta(t) - \phi(t)$ and integrating over space and time give

$$\|\theta(t) - \phi(t)\|_{L^2}^2 = -2\gamma \int_0^t \int_M (\theta - \phi)\Delta^2\theta \, dx \, ds \le 2\gamma \|\theta_0\|_{L^2} \int_0^t \|\Delta^2\theta\|_{L^2} \, ds. \tag{A.14}$$

On the other hand, multiplying (1.3) by $\Delta^2\theta$ and integrating over M give

$$\partial_t \|\Delta\theta\|_{L^2}^2 + 2\langle u \cdot \nabla\theta, \Delta^2\theta\rangle + 2\gamma \|\Delta^2\theta\|_{L^2}^2 = 0.$$

Integrating the middle term by parts, using the fact that u is divergence-free, and integrating in time yields

$$2\gamma \int_0^t \|\Delta^2\theta\|_{L^2}^2 \, ds \le C_d \|u\|_{C^2} \int_0^t \|\Delta\theta\|_{L^2}^2 \, ds + \|\Delta\theta_0\|_{L^2}^2,$$

for some dimensional constant C_d. Substituting this in (A.14) and using the Cauchy–Schwartz inequality give (A.13) as claimed. $\qquad\square$

We now prove Lemma A.2.

Proof of Lemma A.2 We claim that our choice of λ_N and t_0 will guarantee

$$\int_0^{t_0} \|\Delta\theta(s)\|_{L^2}^2 \, ds \ge \frac{\lambda_N^2 t_0 \|\theta_0\|_{L^2}^2}{8}. \tag{A.15}$$

Once this is established, integrating (A.1) in time immediately yields (A.5).

Thus, to prove Lemma A.2, we only need to prove (A.15). Suppose, for contradiction, the inequality (A.15) does not hold. Letting $P_N : \dot{L}^2(M) \to \dot{L}^2(M)$ denote the orthogonal projection onto the span of the first N eigenfunctions of the Laplace–Beltrami operator, we observe

$$\frac{\lambda_N^2 t_0 \|\theta_0\|_{L^2}^2}{8} > \int_0^{t_0} \|\Delta\theta(s)\|_{L^2}^2 \, ds \ge \lambda_N^2 \int_{t_0/2}^{t_0} \|(I - P_N)\theta(s)\|_{L^2}^2 \, ds$$

$$\ge \frac{\lambda_N^2}{2} \int_{t_0/2}^{t_0} \|(I - P_N)\phi(s)\|_{L^2}^2 \, ds - \lambda_N^2 \int_{t_0/2}^{t_0} \|(I - P_N)(\theta(s) - \phi(s))\|_{L^2}^2 \, ds$$

$$\ge \frac{\lambda_N^2 t_0}{4} \|\theta_0\|_{L^2}^2 - \frac{\lambda_N^2}{2} \int_{t_0/2}^{t_0} \|P_N\phi(s)\|_{L^2}^2 \, ds - \lambda_N^2 \int_0^{t_0} \|\theta(s) - \phi(s)\|_{L^2}^2 \, ds. \tag{A.16}$$

We will now bound the last two terms in (A.16).

For the last term in (A.16), we use Lemma A.5 to obtain

$$
\int_0^{t_0} \|\theta(s) - \phi(s)\|_{L^2}^2 \, ds
$$

$$
\leq \int_0^{t_0} \sqrt{2\gamma s} \, \|\theta_0\|_{L^2} \left(C_d \|u\|_{C^2} \int_0^s \|\Delta\theta(t)\|_{L^2}^2 \, dt + \|\Delta\theta_0\|_{L^2}^2 \right)^{1/2} ds
$$

$$
\leq C\sqrt{\gamma} \, t_0^{3/2} \|\theta_0\|_{L^2} \left(\|u\|_{C^2} \int_0^{t_0} \|\Delta\theta(t)\|_{L^2}^2 \, dt + \|\Delta\theta_0\|_{L^2}^2 \right)^{1/2}
$$

$$
\leq C\sqrt{\gamma} \, t_0^{3/2} \lambda_N \|\theta_0\|_{L^2}^2 \left(\|u\|_{C^2} t_0 + 1 \right)^{1/2}. \tag{A.17}
$$

For the last inequality above, we used our assumption that the inequality (A.15) does not hold.

To estimate the second term on the right of (A.16), let e_j denote the eigenfunction of the Laplace–Beltrami operator corresponding to the eigenvalue λ_j. Now

$$
\int_{t_0/2}^{t_0} \|P_N\phi(s)\|_{L^2}^2 \, ds \leq \sum_{j=1}^N \int_0^{t_0} |\langle \phi(s), e_j \rangle|^2 \, ds \leq t_0 h^2(t_0) \|\phi_0\|_{H^1}^2 \sum_{j=1}^N \lambda_j.
$$

Using Weyl's lemma (A.9) and the assumption (A.4), we see

$$
\int_{t_0/2}^{t_0} \|P_N\phi(s)\|_{L^2}^2 \, ds \leq C t_0 h^2(t_0) \|\phi_0\|_{L^2}^2 \lambda_N^{(d+4)/2}, \tag{A.18}
$$

for some constant $C = C(M)$.

We now let C_1 be the larger of the constants appearing in (A.17) and (A.18). Using these two inequalities in (A.16) shows

$$
\frac{1}{8} > \frac{1}{4} - C_1 \lambda_N \sqrt{\gamma t_0} \left(1 + t_0 \|u\|_{C^2} \right)^{1/2} - C_1 \lambda_N^{(d+4)/2} h^2(t_0). \tag{A.19}
$$

If we choose $\tilde{c} \geq \sqrt{16 C_1}$, then by equation (A.6) the last term on the right is at most $1/16$. Next, when γ is sufficiently small we will have $t_0 \|u\|_{C^2} \geq 1$. Thus, if $\tilde{C} \geq 16\sqrt{2} C_1$ and λ_N is the largest eigenvalue for which (A.3) holds, then the second term above is also at most $1/16$. This implies $1/8 > 1/8$, which is the desired contradiction. $\qquad\square$

The proof of Lemma A.3 is very similar to that of Lemma A.2.

Proof of Lemma A.3 Follow the proof of Lemma A.2 until (A.18). Now, to estimate the second term on the right of (A.16), the strongly mixing property of u gives

$$
\int_{t_0/2}^{t_0} \|P_N\phi(s)\|_{L^2}^2 \, ds \leq \lambda_N \int_{t_0/2}^{t_0} \|\phi(s)\|_{H^{-1}}^2 \, ds \leq \lambda_N \int_{t_0/2}^{t_0} h^2(s) \|\theta_0\|_{H^1}^2 \, ds
$$

$$
\leq \frac{t_0}{2} \lambda_N h^2(t_0/2) \|\Delta\theta_0\|_{L^2} \|\theta_0\|_{L^2} \leq \frac{t_0}{2} \lambda_N^2 h^2(t_0/2) \|\theta_0\|_{L^2}^2. \tag{A.20}
$$

Above, the last inequality followed from interpolation and the assumption (A.4).

Now let C_1 be the constant appearing in (A.17). Using (A.17) and (A.20) in (A.16) implies

$$\frac{1}{8} > \frac{1}{4} - C_1 \lambda_N \sqrt{\gamma t_0} \left(1 + t_0 \|u\|_{C^2}\right)^{1/2} - \frac{1}{4} \lambda_N^2 h^2(t_0/2).$$

If t_0 is defined by (A.8), then the last term above is at most $1/16$. Moreover, if $\tilde{C} = 2^{9/2} C_1$ and λ_N is the largest eigenvalue of the Laplace–Beltrami operator satisfying (A.7), then the second term above is also at most $1/16$. This again forces $1/8 > 1/8$, which is our desired contradiction. $\qquad\square$

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