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A Loewner Matrix Based Convex Optimization Approach to Finding Low Rank Mixed Time/Frequency Domain Interpolants

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Abstract— **We consider the problem of finding the lowest order stable rational transfer function that interpolates a set of given noisy time and frequency domain data points. Our main result shows that exploiting results from rational interpolation theory allows for recasting this problem as minimizing the rank of a matrix constructed from the frequency domain data (the Loewner matrix) along with the Hankel matrix of time domain data, subject to a semidefinite constraint that enforces stability and consistency between the time and frequency domain data. These results are applied to a practical problem: identifying a system from noisy measurements of its time and frequency responses. The proposed method is able to obtain stable low order models using substantially smaller matrices than those reported earlier and consequently in a fraction of the computation time.**

I. Introduction

Many practical problems involve finding the lowest order stable transfer function that interpolates a set of given time and frequency data points. Examples include designing low order (stabilizing) controllers such that the closed loop system satisfies some given performance specifications and control oriented identification [1], [2], [3]. Interpolation with mixed data was addressed in [4], where it was shown that the problem reduces to a convex semi-definite program, a result that subsumes the celebrated Nevanlinna-Pick interpolation for frequency-domain data and the Carathéodory-Fejér interpolation for time-domain data as special cases (see Corollaries 1 and 2 in [2]). Furthermore this result provided a parameterization of all such interpolants in the form of a Linear Fractional Transformation (LFT) of a free contraction $Q(z) \in \mathcal{H}_\infty$. In principle, one could try to use these additional degrees of freedom to search for minimum order interpolants. Unfortunately, due to the LFT dependence on Q, this problem, if posed in the frequency domain, is non-convex. This fact coupled with the infinite dimensional nature of Q leads to an exceedingly hard optimization problem. As an alternative [5] proposed to solve the problem in the time domain by minimizing the rank of a truncated Hankel matrix formed by considering the impulse response over a sufficiently large horizon of all interpolants generated by Q, obtained by replacing the latter by the corresponding Integral Quadratic Constraint (IQC)[1]. This approach leads to a convex optimization formulation and has been successfully used to solve non-trivial problems. However, it suffers from the need to consider large time-horizons (with the resulting increase in computational complexity) to guarantee that the corresponding truncated Hankel matrix is indeed a good approximation to the actual, infinite dimensional one.

Motivated by these difficulties, in this paper we propose an alternative approach to stable low order interpolation with mixed data that exploits the connection between the order of a rational interpolant and the rank of the associated Loewner matrix. Our main result shows that minimum order interpolants can be obtained by minimizing the rank of the Loewner matrix constructed from the frequency response of all stable interpolants subject to the generalized interpolation conditions that guarantee stability and consistency between the frequency and time domain data. This approach leads to a substantial computational complexity reduction vis-à-vis algorithms based only on generalized interpolation. In the second portion of the paper we apply the proposed interpolation framework to the problem of control oriented (or set-membership) identification from mixed time/frequency domain data. Unlike the existing (e.g., subspace) techniques, our approach guarantees stable results and provides hard bounds on the identification errors. Also, it leads directly to low order models, as opposed to the existing control oriented identification methods [3] where the order of the model is equal to the total number of data points used in the identification, necessitating a model reduction step before these models can be used for controller synthesis. Unfortunately, the resulting reduced order model may no longer interpolate the experimental data within the noise level.

The paper is organized as follows. Section II presents some required background results on generalized interpolation theory and Loewner matrices and formally states the problem under consideration. Section III contains our main result, showing that this problem can be reduced to a semi-definite program by exploiting the tools presented in Section II, combined with the usual nuclear norm relaxation of rank, described in section IV. Section V applies the proposed framework to control oriented identification of a very lightly damped cantilevered beam. Finally section VI summarizes

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[1]Note that the simpler approach of just minimizing the rank of the Hankel matrix of the system subject to the constraint that its response interpolates the given data within the noise level, but without imposing the additional conditions from [6] cannot guarantee that the resulting system is stable. See for instance Example 1 in [5]

our results and points out to directions for further research.

II. Preliminaries

A. Notation

ℓ_1 denotes space of absolutely summable sequences. $G(z)$ is z transform of the sequence $g \in \ell_1$. $G(z) \doteq \sum_{i=0}^{\infty} g_i z^i$; $G(z)$ is analytic *inside* $\mathcal{D}$. $\mathcal{L}_\infty$ is Lebesgue space of complex valued functions bounded on the unit circle, equipped with the norm $\|G(z)\|_\infty \doteq \text{ess sup}_{|z|=1}|G(z)|$. $\mathcal{H}_\infty$ denotes subspace of functions in $\mathcal{L}_\infty$ analytic inside the unit disk $\mathcal{D}$, $\mathcal{H}_{\infty,\rho}$ denotes subspace of functions in $\mathcal{H}_\infty$ analytic inside the disk of radius $\rho > 1$, equipped with the norm $\|G(z)\|_{\infty,\rho} \doteq \sup_{|z|<\rho}|G(z)|$ (e.g. exponentially stable systems with a stability margin of $\rho - 1$). $\mathcal{H}^K_{\infty,\rho}$ denotes K-ball in $\mathcal{H}_{\infty,\rho}$, e.g. $\mathcal{H}^K_{\infty,\rho} \doteq \{G \in \mathcal{H}_{\infty,\rho} : \|G\|_{\infty,\rho} \leq K\}$.

B. Loewner Matrices

Rational functions represent a natural way of describing linear dynamics in frequency domain. The Lagrange basis offer numerical computation friendly choice [7] for representing them. For a polynomial $P(z)$ of order n, the Lagrange basis is the set $\ell_i(z) = \prod_{k\neq i}^{n+1}(z - z_k), i = 1, \ldots n+1$. Given polynomial values p_i at $n+1$ points z_i, the polynomial written in these bases is $P(z) = \sum_i p_i \ell_i(z)/\ell_i(z_i)$. A rational transfer function of order n can therefore be written as:

$$G(z) = \frac{\sum_i^{n+1} \frac{b_i}{z-z_i}}{\sum_i^{n+1} \frac{a_i}{z-z_i}} \tag{1}$$

Given $N = 2n+1$ samples $\{w(i)\}$ of the function $G(z)$ at unique points $z_1, z_2, \ldots, z_N$, one can pose the estimation problem of estimating coefficients a_i and b_i. If there is no noise, at the measurement points we have $G(z(i)) = w_i = b_i/a_i$. Partition the available measurements into two groups $\{z_i^a, w_i^a\}$ and $\{z_i^b, w_i^b\}$ containing n + 1 and n samples respectively. Use the first set as interpolation nodes in Equation (1), and evaluate the transfer function at the measurement points z_i^b of the second set to get n equations for $G(z_i^b)$. These equations can be arranged in a matrix form:

$$\mathbf{L}(z^a, z^b, w^a, w^b)\mathbf{a} = 0 \tag{2}$$

where:

$$\mathbf{L}(z^a, z^b, w^a, w^b) = \begin{bmatrix} \frac{w_1^b - w_1^a}{z_1^b - z_1^a} & \frac{w_1^b - w_2^a}{z_1^b - z_2^a} & \cdots & \frac{w_1^b - w_{n+1}^a}{z_1^b - z_{n+1}^a} \\ \frac{w_2^b - w_1^a}{z_2^b - z_1^a} & \frac{w_2^b - w_2^a}{z_2^b - z_2^a} & \cdots & \frac{w_2^b - w_{n+1}^a}{z_2^b - z_{n+1}^a} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{w_n^b - w_1^a}{z_n^b - z_1^a} & \frac{w_n^b - w_2^a}{z_n^b - z_2^a} & \cdots & \frac{w_n^b - w_{n+1}^a}{z_n^b - z_{n+1}^a} \end{bmatrix} \tag{3}$$

and $\mathbf{a} = [a_1, a_2, \ldots, a_{n+1}]^T$ is the unknown coefficient vector. $\mathbf{L}$ is a matrix of size $n \times (n+1)$ and is called the *Loewner matrix*. This matrix plays a fundamental role in determining the form of rational interpolant (see [7], [8]). We note some of the useful properties here:

1) The solution for $\mathbf{a}$ lies in the right null space of $\mathbf{L}$. Given $\mathbf{a}$, b_i can be determined by linear least squares.
2) For proper rational functions, the interpolant has a state space representation of the form $G(z) = C(\bar{z}I - A)^{-1}B$, where rank$(A) = n$. Furthermore, $\mathbf{L}$ can be expressed in state-space terms as $\mathbf{L} = -\mathcal{O}\mathcal{R}$, where $\mathcal{O}$ is the generalized observability matrix associated with the samples of the second subset $\{z^b\}$:

$$\mathcal{O} = \begin{bmatrix} C(\bar{z}_1^b I - A)^{-1} \\ C(\bar{z}_2^b I - A)^{-1} \\ \vdots \\ C(\bar{z}_{N_2}^b I - A)^{-1} \end{bmatrix} \tag{4}$$

Similarly, $\mathcal{R}$ is the generalized reachability matrix associated with the samples $\{z^a\}$:

$$\mathcal{R} = [(\bar{z}_1^a I - A)^{-1}B, (\bar{z}_2^a I - A)^{-1}B, \ldots, (\bar{z}_{N_1}^a I - A)^{-1}B] \tag{5}$$

This is similar to the familiar Hankel matrix composed of impulse response coefficients which can also be expressed as $\mathbf{H}_N = \mathcal{O}_N \mathcal{R}_N$ where:

$$\mathcal{O}_N = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{bmatrix}, \quad \mathcal{R}_N = [B, AB, \ldots, A^{N-1}B] \tag{6}$$

are the finite observability and reachability matrices and N represents the number of impulse response samples. Thus $\mathbf{L}$ can be viewed as a frequency domain counterpart of the $\mathbf{H}_N$. The advantage of using $\mathbf{L}$ over $\mathbf{H}_N$ is that it encapsulates the infinite-horizon Hankel matrix $\mathbf{H}_\infty$ while using a matrix of a finite size. Furthermore, the frequency samples composing $\mathbf{L}$ need not be uniformly spaced allowing significant data compression; we can pick a higher density of samples in the frequency bands of high modal density while using relatively fewer samples elsewhere. In contrast, $\mathbf{H}_N$ requires uniform sampling often leading to a large value of N for accurately capturing the system order information. *Example: Singular values of Hankel and Loewner matrices*: Consider a lightly damped system: $G(z) = z^3/(1 + 0.3z + 0.9z^2)$. The frequency grid [0.1, 0.5, 1, 2, 2.5, 3] Hz and its negative counterpart is chosen for frequency response computation while the impulse response is generated for 56 samples. Both responses are corrupted with 20dB noise. A Hankel matrix of size 27 and a Loewner matrix of size 6 were computed for the data. The first 6 singular values of $\mathbf{H}_{56}$ and $\mathbf{L}$ are shown in Figure 1. The singular values of the Loewner matrix indicate the true order more clearly than the Hankel matrix.
3) Lowest order interpolant: If the number of samples N is larger than $2n+1$, one can split the dataset into two subsets such that the Loewner matrix is roughly square, with each subset containing more than n samples. The rank of this matrix is equal to the order of (proper) rational function $G(z)$, as can be verified from the state space representation. This fact can be utilized to

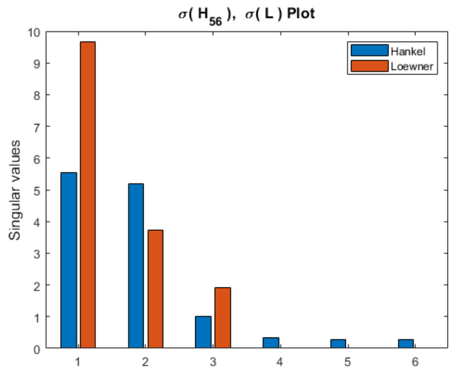

Fig. 1. Singular values of Hankel and Loewner matrices. Blue: Hankel matrix of 56-sample impulse response. Red: Loewner matrix of 12 frequency points.

determine the order of the system, provided data has no noise. In presence of noise, a low order interpolant can be found by minimizing the rank of a sufficiently large Loewner matrix subject to other constraints imposed by the modeling requirements.

C. A Generalized Interpolation Framework

Next we recall a result from [4], [2] establishing a necessary and sufficient condition for the existence of a function in $\mathcal{H}^K_{\infty,\rho}$ that interpolates a given set of time and frequency domain points.

Theorem 1: Given N_f frequency–domain data points $(z_i, w_i), |z_i| < \rho, \;\; i = 1, \ldots, N_f$, and N_t impulse response samples $h_k, \;\; k = 1, \ldots, N_t$, there exists $G \in \mathcal{H}^K_{\infty,\rho}$ that interpolates the frequency domain data (i.e. $G(z_i) = w_i$) and such that $G(z) = h_1 + h_2 z + h_3 z^2 + \cdots + h_{N_t} z^{N_t} + \cdots$ if and only if the following inequality holds:

$$\mathbf{Z} \doteq \begin{bmatrix} \mathbf{M}_0^{-1} & \frac{1}{K}\mathbf{X} \\ \frac{1}{K}\mathbf{X}^T & \mathbf{M}_0 \end{bmatrix} \succeq 0 \tag{7}$$

where

$$\mathbf{M}_0 = \begin{bmatrix} \mathbf{Q} & S_0 R^{-2} \\ R^{-2} S_0^H & R^{-2} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} \mathbf{W} & 0 \\ 0 & \mathcal{T} \end{bmatrix}$$

$$R = \mathrm{diag}[1, \rho, \rho^2, \ldots, \rho^{N_t - 1}]$$
$$\mathbf{Q} = \left[\frac{\rho^2}{\rho^2 - \overline{z_i} z_j}\right]_{ij}, \; i, j = 1, 2, \ldots, N_f \tag{8}$$

$$S_0 = [\overline{z_i^j}]_{ij}, \; i, j = 1, 2, \ldots, N_f$$

$$\mathbf{W} = \mathrm{diag}[w_1, w_2, \ldots, w_{N_f}] \tag{9}$$

(10)

$$\mathcal{T} = \begin{bmatrix} h_1 & h_2 & \ldots & h_{N_t} \\ 0 & h_1 & \ldots & h_{N_{t-1}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & h_1 \end{bmatrix}. \tag{11}$$

Moreover, if $\mathbf{Z}$ is rank deficient, then the interpolant is unique. When $\mathbf{Z} \succ 0$, the solution is not unique and all the interpolants can be written as Linear Fractional Transformation (LFT) of a free parameter $Q(z) \in \mathcal{H}^1_{\infty,\rho}$ as follows:

$$F(z) \;\; = \;\; \frac{T_{11}(z)Q(z) + T_{12}(z)}{T_{21}(z)Q(z) + T_{22}(z)} \tag{12}$$

where the transfer matrices $T_{i,j}$ depend only on the problem data (an explicit expression for these matrices can be found for instance in [9]). In particular, if the free parameter $Q(z)$ is chosen as a constant, then the model order is less than or equal to $N_f + N_t$.

D. Problem Formulation

The interpolation problems of interest in this paper can be stated as follows:

Problem 1 (Exact minimum order interpolation): Given N_f frequency–domain data points $(z_i, w_i), |z_i| < \rho, \;\; i = 1, \ldots, N_f$, N_t time–domain input-output data points $(u_i, y_i), \;\; i = 1, \ldots, N_t$, and positive real numbers K, ρ find the minimum McMillan degree stable transfer function $G(z) \in \mathcal{H}^K_{\infty,\rho}$ that interpolates the given data, or show that none exists.

For system identification applications, it is of interest to consider the following "noisy" version of the problem above:

Problem 2 (Noisy minimum order interpolation): Given

(1) N_f frequency–domain data points $(z_i, w_i), |z_i| < \rho, \;\; i = 1, \ldots, N_f$, and N_t time–domain data points $(u_i, y_i), \;\; i = 1, \ldots, N_t$
(2) positive real numbers $K, \rho, \epsilon_t, \epsilon_f$

find the minimum McMillan degree stable transfer function $G(z) \in \mathcal{H}^K_{\infty,\rho} \doteq g_1 + g_2 z + \ldots g_{N_t} z^{N_t} + ..$ such that

$$\begin{aligned} |y_i - (\mathbf{g} * \mathbf{u})_i| \;\; &\leq \epsilon_t, \; i = 1, \ldots, N_t \\ |w_i - G(z_i)| \;\; &\leq \epsilon_f \; i = 1, \ldots, N_f \end{aligned} \tag{13}$$

where $\mathbf{g}$ is the impulse response vector, $\mathbf{u}$ the input vector and $*$ is the convolution operator.

III. Main Results

In this section we present the main results of the paper showing that Problem 1 and 2 can be reduced to minimizing the rank of a matrix that is affine on the optimization variables subject to convex semi-definite constraints. This result will be exploited to relax these problems to a convex optimization by following the commonly used approach of replacing rank by a weighted nuclear norm.

Proposition 1: Problem 1 admits a unique solution with McMillan degree r if and only if:

1) The $N_t > 2r + 1$ length vector $\mathbf{h}$ of real numbers $h_i, \;\; i = 1, \ldots N_t$ and $N_f \geq 2(r + 1)$ pairs $(z_i, w_i), \;\; i = 1, \ldots, N_f$ satisfy (7), where $y_i = (\mathbf{h} * \mathbf{u})_i$,
2) The Pick matrix $\mathbf{P} \doteq \begin{bmatrix} \mathbf{Q}^{-1} & \frac{1}{K}\mathbf{W} \\ \frac{1}{K}\mathbf{W}^* & \mathbf{Q} \end{bmatrix}$, where $\mathbf{Q}$ and $\mathbf{W}$ are defined in (8) and (9), respectively, is rank deficient

3) rank $[L_{N_f}(z_i, w_i)] = r$ and every $r \times r$ submatrix of L is full rank, and
4) rank $[H_{N_t}(h)] \leq r$,

where $L_{N_f}(.,.)$ and $H_{N_t}(.)$ denote the (square) Loewner and Hankel matrix formed using the frequency and time domain data respectively.

Proof: (Sufficiency) Since h_i and (z_i, w_i) satisfy (7), from Theorem 1 there exist a transfer function $G \in \mathcal{H}^K_{\infty,\rho}$ that interpolates the time and frequency domain data points. Further, since $\mathbf{P}$ is rank deficient, G is the only function in $\mathcal{H}^K_{\infty,\rho}$ that interpolates the given frequency domain data points (see e.g. Theorem 2.3.4 in [3]). Since the corresponding Hankel matrix has rank at most r, it follows that $deg(G) \leq r$. Finally, from the fact the condition 3) above implies that there exists a unique function G_r of degree r that interpolates the given frequency domain data points, it follows that $G = G_r$. Necessity follows from the fact that if Problem 1 admits a solution with McMillan degree r, then the corresponding Hankel and Loewner matrices have rank r. Moreover, since by assumption this solution is unique, then $\mathbf{P}$ is rank deficient and all $r \times r$ submatrices of L have full rank. ∎

Next, we use the result above to solve Problem 2.

Proposition 2: Problem 2 admits a unique solution with McMillan degree r if and only if there exist N_t real numbers $g_i, i = 1, \ldots, N_t$, and N_f complex numbers $\hat{w}_i, i = 1, \ldots, N_f$ such that the pairs g_i and $(z_i, \hat{w}_i)$ satisfy the conditions in Theorem 1 and Proposition 1 and such that

$$\begin{aligned} |y_i - (\mathbf{g} * \mathbf{u})_i| &\leq \epsilon_t, \ i = 1, \ldots, N_t \\ |w_i - \hat{w}_i| &\leq \epsilon_f \ i = 1, \ldots, N_f \end{aligned} \tag{14}$$

Proof: Follows immediately by noting that, from Proposition 1, there exists a transfer function $G_r(z) \in \mathcal{H}^K_{\infty,\rho}$ with $deg(G_r) = r$, impulse response coefficients g_i and frequency response $\hat{w}_i$ at the frequencies z_i. Hence G_r solves Problem 2. ∎

Remark 1: From Proposition 2 it follows that Problem 2 can be recast as a constrained rank minimization of the form:

$$\begin{aligned} &\underset{z,h}{\text{minimize}} \quad \max\{\text{rank}(\mathbf{L}(z,h), \text{rank}(\mathbf{H}(z,h))\} \\ &\text{subject to} \\ &\qquad\qquad \text{feasibility constraints (7)} \\ &\qquad\qquad \text{rank}(\mathbf{P}) < 2 * N_f \end{aligned} \tag{15}$$

We note the following:

- The role of the feasibility conditions in this context is to deliver noise-free sequences w_i, h_i.
- In practice, we found it useful to impose a DC gain bound on the interpolant as an added constraint in the formulation of (15).
- Often there are many stable interpolants that satisfy (7). Hence the search for minimal order interpolant can be successful without imposing rank deficiency of the Pick matrix. The stability of the interpolants is not guaranteed, but in practice is true for the lowest order model. This is the approach adopted for the practical example discussed in V.
- In case where we only have the frequency response samples, the problem reduces to rank minimization of the Loewner matrix subject to Nevanlinna-Pick feasibility conditions; in particular, $\boldsymbol{Z} \succeq 0$ is equivalent to the (rescaled) Pick matrix P being positive semidefinite:

$$P_{i,j} = \frac{1 - \frac{1}{K^2} h_i \overline{h_j}}{1 - \frac{1}{\rho^2} z_i \overline{z_j}}, \ i,j = 0, 1, \ldots, N_f - 1 \tag{16}$$

  In the feasibility conditions of Equations (7), we then replace $\mathbf{M}_0$ with $\mathbf{Q}$ and $\mathbf{X}$ with $\mathbf{W}$. Also, we drop the time-domain noise bound.
- Similarly, this procedure also applies when only the time domain samples are available. We then minimize the rank of the Loewner matrix, computed over some suitable frequency grid (for example, 0 to Nyquist, linearly spaced). In this case, we drop the frequency response noise bound.
- Note also, in passing, that if we use $\rho = 1$ and ignore frequency domain considerations, the classical Carathéodory-Fejér feasibility conditions are retrieved.

IV. A Convex Relaxation

Since rank minimization is computationally NP-hard, a convex relaxation of the problem above is obtained by using an iteratively reweighted trace minimization heuristics [10], summarized in Algorithm 1.

Algorithm 1 Reweighted $\|.\|_*$ based rank minimization

Initialize: $k = 0, \mathbf{W}_y(0) = \mathbf{I}, \mathbf{W}_z(0) = \mathbf{I}$, δ_o small
repeat
  Solve
  $$\min_{\mathbf{X}^{(k)}, \mathbf{Y}^{(k)}, \mathbf{Z}^{(k)}} \text{Trace} \begin{bmatrix} \mathbf{W}_y^{(k)} \mathbf{Y}^{(k)} & 0 \\ 0 & \mathbf{W}_z^{(k)} \mathbf{Z}^{(k)} \end{bmatrix}$$
  subject to: $\begin{bmatrix} \mathbf{Y}^{(k)} & \mathbf{L}^{(k)} \\ \mathbf{L^T}^{(k)} & \mathbf{Z}^{(k)} \end{bmatrix} \succeq 0$
  $\mathcal{L}^{(k)} \in \mathcal{S}$
where $\mathcal{S}$ is the feasible set in (15).
  Decompose $\mathbf{L}^{(k)} = \mathbf{U}\mathbf{D}\mathbf{V}^T$.
  Set $\delta \leftarrow \min[\text{diag}(\mathbf{D})] + \delta_0$.
  Set $\mathbf{W}_y^{(k+1)} \leftarrow \left(\mathbf{Y}^{(k)} + \delta \mathbf{I}\right)^{-1}$
  Set $\mathbf{W}_z^{(k+1)} \leftarrow \left(\mathbf{Z}^{(k)} + \delta \mathbf{I}\right)^{-1}$
  Set $k \leftarrow k + 1$.
until a convergence criterion is reached. **return** $\mathbf{L}^{(k)}$

V. Example: Lightly damped cantilever beam

Consider a flexible structure in Figure 2. The structure is a two degree of freedom mass-beam system consisting of two discrete masses M_1 and M_2 supported by cantilever beams, excited by the vibratory motion of a shaker table. The first mass is connected to the shaker table, which excites the mechanical system by vibrating up and down,

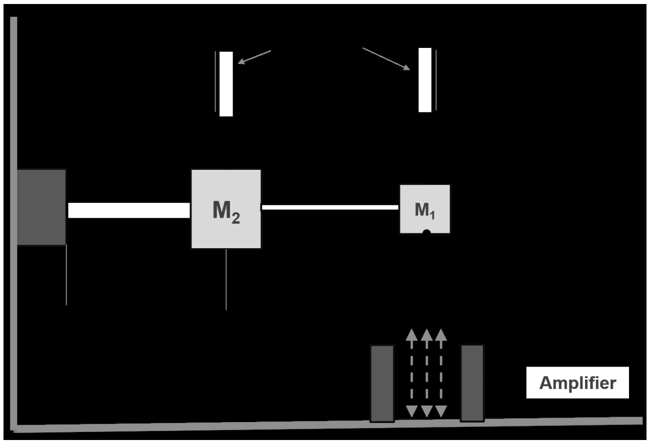

Fig. 2. Lightly damped system used to test life extending control.

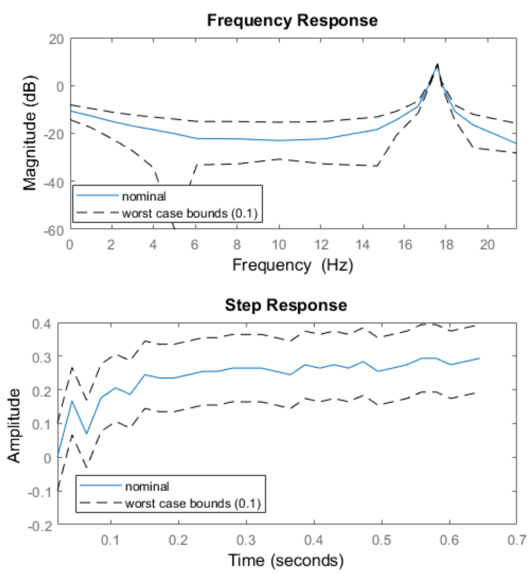

Fig. 3. Measured data. Top: frequency response magnitude with 0.1 bound. Bottom: time (step) response with 0.1 bound.

through a flexible pivot. The displacement y_1 caused by the shaker table is measured using a linear variable differential transformer (LVDT) sensor located at the midpoint of the mass M_1. To obtain the frequency response measurements, the system was driven by a peak-to-peak 0.5 V sinusoidal signal, with frequency ranging from 1 to 21 Hz. The time domain data samples where obtained by a step test. In both cases the outputs were sampled at 0.0215 seconds.

The accuracy requirement was $\epsilon^t = \epsilon^f = 0.1$. See Figure 2 which shows the acceptable bounds on the measured data curves. Note that a bound implies that any system within the shaded region meets the control design requirements. Our objective is to find a system within this region of the smallest possible order. Unlike the approach of [5], K and ρ were treated as tuning parameters adjusted to achieve the accuracy goals.

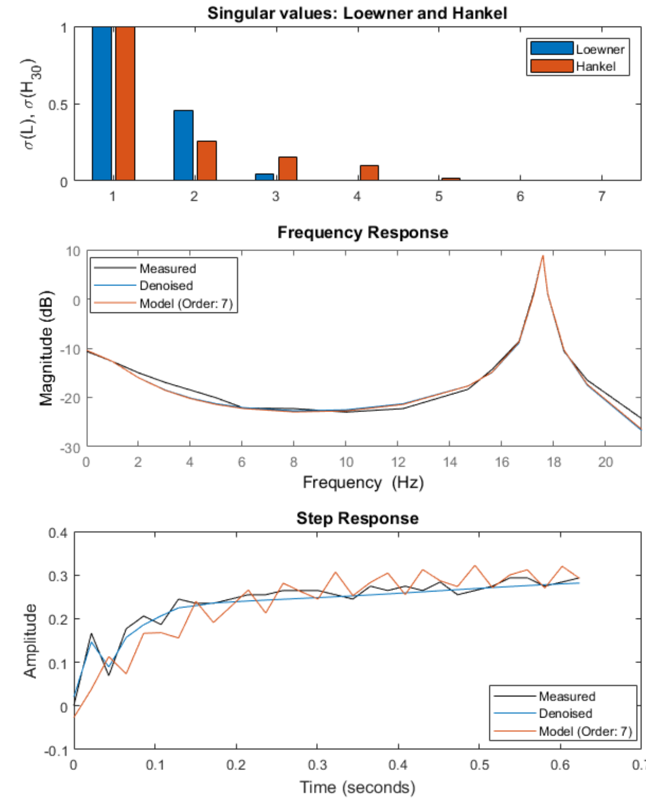

Fig. 4. Identification using both time and frequency response data. Top: singular values (normalized) of the Loewner and Hankel matrices. Middle: frequency response fit. Bottom: Step response fit. Measured, denoised and order 7 approximation are shown. $K = 125$, $\rho = 1.0001$.

A. Using Both Time- and Frequency-Domain Data

Running the trace minimization heuristics for about 2 iterations yields a Loewner matrix whose singular values are shown in the top axes of Figure 4. The order chosen was 7 which includes contribution of 3 rather small singular values which were required to meet the prescribed bounds.

A seventh order system was re-estimated to fit the denoised frequency response $\boldsymbol{h}$. The frequency response and the step response of the resulting model are shown in the middle and bottom axes of Figure 4, overlaid on their measured and denoised values. The denoised values are what are delivered as solution of Problem 2 with no restriction on model order. The worst case errors for the frequency response and the step response, after 2 iterations, are 0.07 and 0.11 respectively. Table V-A compares the accuracy of the proposed method to those reported in [9] and, more recently, in [5]. SNNAM refers to the Structured Nuclear Norm ADMM Minimization method of [5]. The errors are shown for data points at which the identification was performed. The computation time is about 10 seconds per iteration for the indicated sample size, which is orders of magnitude faster than the SNNAM algorithm, whose computation time depends upon the impulse response horizon and has been reported to be greater than 15 minutes for all acceptable horizons.

Method	Number of Samples	Model Order	Max Error: Time	Max Error: Frequency
ℓ_1 [9]	40	3	0.07	2.92
$\mathcal{H}_{\infty,\rho}$ [9]	39	39	0.39	0.056
Mixed [9]	15+29	19	0.15	0.39
SNNAM [5]	14+35	6	0.03	0.53
Proposed	21+30	7	0.11	0.07

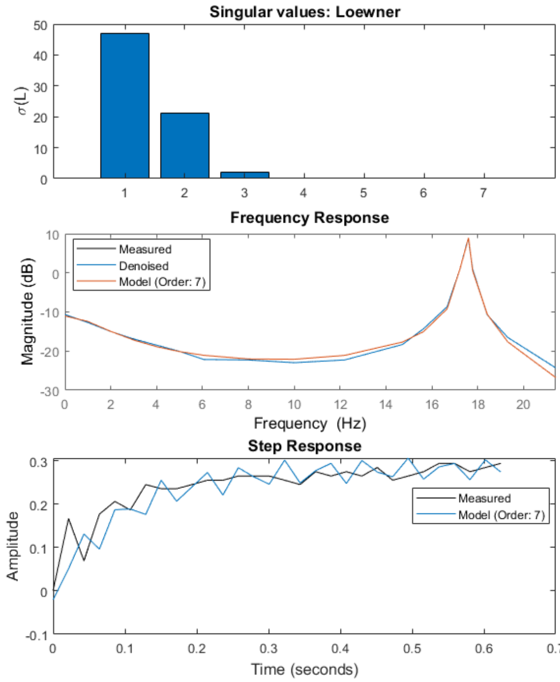

Fig. 5. Identification using frequency response data only. Top: singular values of the Loewner matrix. Middle: frequency response fit. Bottom: step response fit.

B. Using Only Frequency-Domain Data

If we only use the frequency response data, a seventh order system is determined as shown in Figure 5. The worst case errors for the frequency response and the step response are 0.09 and 0.11 respectively. The fit to time data is also generated even though it was not used for estimation.

C. Using Only Time-Domain Data

If we only use the time-domain data, the time and frequency domain responses of the estimated fourth order system are shown in Figure 6. The worst case errors for the frequency response and the time-domain output signal are 2.89 and 0.02 respectively. The time-domain data does not contain enough excitation to discern the resonance at 17.6 Hz leading to a poor fit in frequency domain.

VI. Conclusions

We proposed a method for finding the lowest order interpolant that is consistent with the prior information and fits the time- and frequency–domain data in the worst case sense. The method exploits the rank-revealing properties of a Loewner matrix composed of frequency response samples to suggest a low order interpolant. This way, our approach

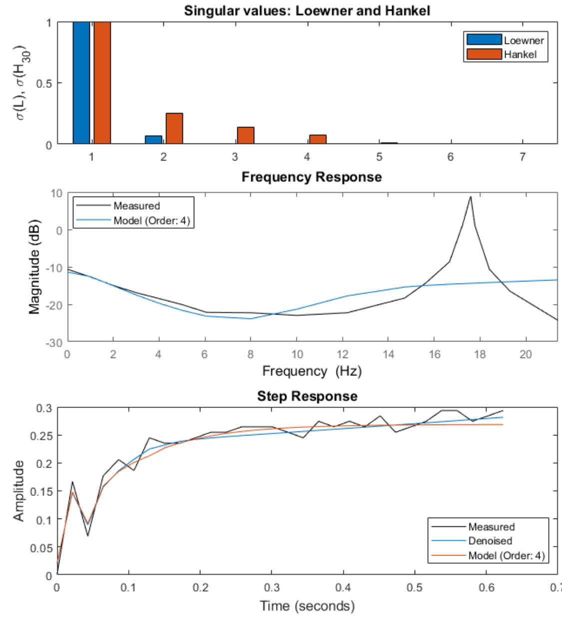

Fig. 6. Identification using time-domain data only. Top: singular values (normalized) of the Loewner and Hankel matrices. Middle: frequency response fit. Bottom: Step response fit.

extends the familiar Hankel matrix rank analysis for order determination to frequency domain data. The interpolant model was also shown to be stable in case the Pick matrix is rank deficient. In practice we have found that lowest order interpolants are often stable even when the Pick matrix is full rank.

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