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Combinatorics of Generalized Exponents

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We give a purely combinatorial proof of the positivity of the stabilized forms of the generalized exponents associated to each classical root system. In finite type A_{n-1}, we rederive the description of the generalized exponents in terms of crystal graphs without using the combinatorics of semistandard tableaux or the charge statistic. In finite type C_n, we obtain a combinatorial description of the generalized exponents based on the so-called distinguished vertices in crystals of type A_{2n-1}, which we also connect to symplectic King tableaux. This gives a combinatorial proof of the positivity of Lusztig t-analogs associated to zero-weight spaces in the irreducible representations of symplectic Lie algebras. We also present three applications of our combinatorial formula and discuss some implications to relating two type C branching rules. Our methods are expected to extend to the orthogonal types.

1 Introduction

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of rank n and G its corresponding Lie group. The group G acts on the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$, and it was proved by Kostant [18] that $S(\mathfrak{g})$ factors as $S(\mathfrak{g}) = H(\mathfrak{g}) \otimes S(\mathfrak{g})^G$, where $H(\mathfrak{g})$ is the harmonic part of $S(\mathfrak{g})$. The

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generalized exponents of $\mathfrak{g}$, as defined by Kostant [18], are the polynomials appearing as the coefficients in the expansion of the graded character of $H(\mathfrak{g})$ in the basis of the Weyl characters. It was shown by Hesselink [10] that these polynomials coincide, in fact, with the Lusztig t-analogs [29] of zero-weight multiplicities in the irreducible finite-dimensional representations of $\mathfrak{g}$. In particular, they have nonnegative integer coefficients, because they are affine Kazhdan–Lusztig polynomials (see [32, 36]). Note that the zero-weight Lusztig t-analogs are the most complex ones.

For $\mathfrak{g} = \mathfrak{sl}_n$, the generalized exponents admit a nice combinatorial description in terms of the Lascoux–Schützenberger charge statistic on semistandard tableaux of zero weight [24]. This statistic is defined via the cyclage operation on tableaux, which is based on the Schensted insertion scheme. This combinatorial description extends, in fact, to any Lusztig t-analog of type A_{n-1}, that is possibly associated to a nonzero weight (also called Kostka polynomials). So we have a purely combinatorial proof of the positivity of their coefficients. It was also established in [31] that the Lusztig t-analogs in type A_{n-1} are one-dimensional sums, that is, some graded multiplicities related to finite-dimensional representations of quantum groups of affine type $A_{n-1}^{(1)}$. Another interpretation of the charge statistic in terms of crystals of type A_{n-1} was given later by Lascoux *et al.* in [23].

Despite many efforts during the past 3 decades, no general combinatorial proof of the positivity of the Lusztig t-analogs is known beyond type A. Nevertheless, such proofs have been obtained in some particular cases. Notably, a combinatorial description of the generalized exponents associated to small representations was given in [13] and [14] for any root system. In [28], it was established that some Lusztig t-analogs of classical types equal one-dimensional sums for affine quantum groups, which generalizes the result of [31]. Nevertheless, the two families of polynomials do not coincide beyond type A. In [25] and [26], charge statistics based on cyclage on Kashiwara–Nakashima (KN) tableaux were defined for classical types, yielding the desired positivity for particular Lusztig t -analogs. It is worth mentioning that, in type C_n, a version of the mentioned statistic [25] permits conjecturally to describe all the Lusztig t-analogs in this case.

In [4], Brylinsky obtained an algebraic proof of the positivity of any Lusztig t-analog based on the filtration by a central idempotent of $\mathfrak{g}$. For classical types, this filtration stabilizes [7, 27], which yields stabilized versions of these polynomials. They are formal series in the variable t that, in many respects, are more tractable as their finite-rank counterparts.

The goal of this paper is two-fold. First we give a combinatorial description of the stabilized version of the generalized exponents and a proof of their positivity by using the combinatorics of type $A_{+\infty}$ crystal graphs. This can be regarded as a generalization of results in [23] for the weight zero, and in fact we were able to rederive the latter without any reference to the charge statistic or the combinatorics of semistandard tableaux. Our description is in terms of the so-called distinguished vertices in crystal of type $A_{+\infty}$, but we show that these vertices are in natural bijection with some generalizations of symplectic King tableaux, which makes the link with stable Lusztig t-analog more natural. Next, we provide a complete combinatorial proof of the positivity of the generalized exponents in the non-stable C_n case. Observe there that the non-stable case is much more involved than the stable one, essentially because we need a combinatorial description of the non-Levi branching from $\mathfrak{gl}_{2n}$ to $\mathfrak{sp}_{2n}$, which is complicated in general. Here we use in a crucial way recent duality results by Kwon [19, 20] giving a crystal interpretation of the previous branching and a combinatorial model relevant to its study. We also rely on the complex combinatorics of the bijections realizing the symmetries of type A Littlewood–Richardson (LR) coefficients: the combinatorial R-matrix and the conjugation symmetry map; both have many different realizations in the literature. We strongly expect to extend our approach to orthogonal types as soon as all the results of [20] will be available for the non-Levi orthogonal branchings.

The paper is organized as follows. In Section 2, we recall the definition of the generalized exponents and show that, for classical types, they satisfy important relations in the ring of formal series in t deduced from Cauchy and Littlewood identities. In Section 3, we briefly rederive the combinatorial description of the generalized exponents in type A_{n-1} obtained in [23] without using the results of [24] on the charge. Section 4 is devoted to the combinatorial description of the stabilized form of the generalized exponents in terms of distinguished tableaux, which we define and study here. Our approach also permits us to extend our results to multivariable versions of the generalized exponents as done in [23] for type A. In Sections 5 and 6, we give the promised combinatorial description of the generalized exponents in type C_n by using distinguished tableaux adapted to the finite rank n. In Section 5 we do this based on the type C_n branching rule due to Sundaram [34], whereas in Section 6 we use Kwon's branching rule; the latter leads to a more explicit description, including one in terms of the symplectic King tableaux [17]. In Section 7, we give three applications of the description in Section 6: (1) analyzing the growth of the generalized exponents of type C_n with respect to the rank n; (2) proving a conjecture related to the construction of the type C_n charge in [25]; (3) determining the smallest power of t in a generalized exponent (note that the largest one is well known). The 3rd result turns out to be quite subtle, and it illustrates the combinatorial complexity of these polynomials. Finally, in Section 8 we raise a question about the possible relationship between the Sundaram and Kwon branching rules.

2 Generalized Exponents

2.1 Background

Let $\mathfrak{g}_n$ be a simple Lie algebra over $\mathbb{C}$ of rank n with triangular decomposition

$$\mathfrak{g}_n = \bigoplus_{\alpha\in R_+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha\in R_+} \mathfrak{g}_{-\alpha}\,,$$

so that $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}_n$ and R_+ is its set of positive roots. The root system $R = R_+ \sqcup (-R_+)$ of $\mathfrak{g}_n$ is realized in a real Euclidean space E with inner product $(\cdot,\cdot)$. For any $\alpha \in R$, we write $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ for its coroot. Let $S \subset R_+$ be the subset of simple roots and Q_+ the $\mathbb{Z}_+$-cone generated by S. The set P of integral weights for $\mathfrak{g}_n$ satisfies $(\beta,\alpha^\vee) \in \mathbb{Z}$ for any $\beta \in P$ and $\alpha \in R$. We write $P_+ = \{\beta \in P \mid (\beta,\alpha^\vee) \geq 0 \text{ for any } \alpha \in S\}$ for the cone of dominant weights of $\mathfrak{g}_n$ and denote by $\omega_1,\ldots,\omega_n$ its fundamental weights. Let W be the Weyl group of $\mathfrak{g}_n$ generated by the reflections s_α with $\alpha \in S$, and write ℓ for the corresponding length function.

The graded character of the symmetric algebra $S(\mathfrak{g}_n)$ of $\mathfrak{g}_n$ is defined by

$$\mathrm{char}_t(S(\mathfrak{g}_n)) = \prod_{\delta \text{ weight of } \mathfrak{g}_n} \frac{1}{1-te^\delta} = \frac{1}{(1-t)^n}\prod_{\alpha\in R}\frac{1}{1-te^\alpha}\,.$$

By a classical theorem due to Kostant, the graded character of the harmonic part of the symmetric algebra $S(\mathfrak{g}_n)$ satisfies

$$\mathrm{char}_t(H(\mathfrak{g}_n)) = \frac{\prod_{i=1}^n(1-t^{d_i})}{(1-t)^n}\prod_{\alpha\in R}\frac{1}{1-te^\alpha} = \prod_{i=1}^n(1-t^{d_i})\mathrm{char}_t(S(\mathfrak{g}_n)),$$

where we have $d_i = m_i + 1$, for $i = 1,\ldots,n$, and $m_1,\ldots,m_n$ are the (classical) exponents of $\mathfrak{g}_n$. On the other hand, it is known (see [10]) that $\mathrm{char}_t(H(\mathfrak{g}_n))$ coincides with the

Hall–Littlewood polynomial Q'_0, namely, we have

$$\mathrm{char}_t(H(\mathfrak{g}_n)) = Q'_0 = W_0(t) \prod_{\alpha \in R} \frac{1}{1 - te^{\alpha}} = \sum_{\lambda \in P_+} K^{\mathfrak{g}_n}_{\lambda,0}(t) s^{\mathfrak{g}_n}_{\lambda} ,$$

where

$$W_0(t) = \sum_{w \in W} t^{\ell(w)} ,$$

and $s^{\mathfrak{g}_n}_{\lambda}$ is the Weyl character associated to the finite-dimensional irreducible representation $V(\lambda)$ of $\mathfrak{g}_n$ with highest weight λ. In particular, we have the identity

$$W_0(t) = \prod_{i=1}^{n} \frac{1 - t^{d_i}}{1 - t} .$$

The polynomials $K^{\mathfrak{g}_n}_{\lambda,0}(t)$ are the generalized exponents of $\mathfrak{g}_n$, and they coincide with the Lusztig t-analogs associated to the zero-weight subspaces in the representations $V(\lambda)$. We thus have

$$K^{\mathfrak{g}_n}_{\lambda,0}(t) = \sum_{w \in W} (-1)^{\ell(w)} P_t(w(\lambda + \rho) - \rho) ,$$

where ρ is half the sum of the positive roots, and P_t is the t-Kostant partition function defined by

$$\prod_{\alpha \in R_+} \frac{1}{1 - te^{\alpha}} = \sum_{\beta \in Q_+} P_t(\beta) e^{\beta}.$$

The classical exponents $m_1, \dots, m_n$ correspond to the adjoint representation of $\mathfrak{g}_n$, namely, we have

$$K^{\mathfrak{g}_n}_{\widetilde{\alpha},0}(t) = \sum_{i=1}^{n} t^{m_i} ,$$

where $\widetilde{\alpha}$ is the highest root in R_+.

2.2 Classical types

Recall the following values of the classical exponents in types $A - D$:

type X	exponents
A_{n-1}	$1, 2, \ldots, n-1$
B_n	$1, 3, \ldots, 2n-1$
C_n	$1, 3, \ldots, 2n-1$
D_n	$1, 3, \ldots, 2n-3,$ and $n-1$.

In classical types, $\mathrm{char}_t(S(\mathfrak{g}_n))$ is easy to compute. Let $\mathcal{P}_n$ be the set of partitions with at most n parts and $\mathcal{P}$ be the set of all partitions. The rank of the partition γ is defined as the sum of its parts and is denoted by $|\gamma|$.

In type A_{n-1}, we start from the Cauchy identity

$$\prod_{1\le i,j\le n} \frac{1}{1-tx_iy_j} = \sum_{\gamma\in\mathcal{P}_n} t^{|\gamma|} s_\gamma(x) s_\gamma(y).$$

Here $s_\nu(x)$ stands for the ordinary Schur function in the variables $x_1, \ldots, x_n$. By setting $y_i = \frac{1}{x_i}$ for any $i = 1, \ldots, n$, and by considering the images of the symmetric polynomials in $R^{A_{n-1}} = \mathrm{Sym}[x_1, \ldots, x_n]/(x_1 \cdots x_n - 1)$, we get

$$\mathrm{char}_t(S(\mathfrak{sl}_n)) = (1-t) \sum_{\gamma\in\mathcal{P}_n} t^{|\gamma|} s_\gamma(x) s_\gamma(x^{-1}) = (1-t) \sum_{\gamma\in\mathcal{P}_n} t^{|\gamma|} s_\gamma s_{\gamma^*} = \tag{1}$$

$$= (1-t) \sum_{\gamma\in\mathcal{P}_n} t^{|\gamma|} \sum_{\lambda\in\mathcal{P}_{n-1}} c^\lambda_{\gamma,\gamma^*} s_\lambda(x).$$

Here $\gamma^* = -w_\circ(\gamma)$, where $w_\circ$ is the permutation of maximal length in S_n, and we use the same notation for a symmetric polynomial and its image in $R^{A_{n-1}}$. Recall also that the partitions of $\mathcal{P}_{n-1}$ are in one-to-one correspondence with the dominant weights of $\mathfrak{sl}_n$. More precisely, we associate to the dominant weight $a_1\omega_1 + \cdots + a_{n-1}\omega_{n-1}$ the partition $\lambda = (1^{a_1}, \ldots, (n-1)^{a_{n-1}})'$, where μ' denotes the conjugate of μ. It is also worth mentioning here that two partitions in $\mathcal{P}_n$ whose conjugates have the same parts less than n correspond to the same dominant weight of $\mathfrak{sl}_n$. So the coefficients $c^\lambda_{\gamma,\gamma^*}$ are not properly LR coefficients, but only tensor multiplicities corresponding to the decomposition of $V(\gamma) \otimes V(\gamma^*)$ into irreducible components. Similarly $s_\lambda(x)$ is not properly a Schur polynomial but belongs to $R^{A_{n-1}}$ (see Section 3).

For any positive integer m, define $\mathcal{P}_m^{(2)}$ as the set of partitions of the form 2κ with $\kappa \in \mathcal{P}_m$ and $\mathcal{P}_m^{(1,1)}$ as the subset of $\mathcal{P}_m$ containing the partitions of the form $(2\kappa)'$ with $\kappa \in \mathcal{P}$. Moreover, we denote by $s_\lambda^{\mathfrak{so}_{2n+1}}$, $s_\lambda^{\mathfrak{sp}_{2n}}$, and $s_\lambda^{\mathfrak{so}_{2n}}$ the irreducible characters corresponding to the highest weight λ, for the Lie algebras of types B_n, C_n, and D_n, respectively.

In type B_n, we start from the Littlewood identity [21]

$$\prod_{1\leq i<j\leq 2n+1} \frac{1}{1-ty_iy_j} = \sum_{\nu\in\mathcal{P}_{2n+1}^{(1,1)}} t^{|\nu|/2} s_\nu(y),$$

and we specialize $y_{2n+1} = 1, y_{2i-1} = x_i$, and $y_{2i} = \frac{1}{x_i}$, for any $i = 1, \ldots, n$. This gives

$$\mathrm{char}_t(S(\mathfrak{so}_{2n+1})) = \sum_{\nu\in\mathcal{P}_{2n+1}^{(1,1)}} t^{|\nu|/2} \sum_{\lambda\in\mathcal{P}_n} c_\nu^\lambda(\mathfrak{so}_{2n+1}) s_\lambda^{\mathfrak{so}_{2n+1}},$$

where $c_\nu^\lambda(\mathfrak{so}_{2n+1})$ is the branching coefficient corresponding to the restriction from $\mathfrak{gl}_{2n+1}$ to $\mathfrak{so}_{2n+1}$. Similarly, we can consider the identities

$$\prod_{1\leq i<j\leq 2n} \frac{1}{1-ty_iy_j} = \sum_{\nu\in\mathcal{P}_{2n}^{(1,1)}} t^{|\nu|/2} s_\nu(y) \quad \text{and} \quad \prod_{1\leq i\leq j\leq 2n} \frac{1}{1-ty_iy_j} = \sum_{\nu\in\mathcal{P}_{2n}^{(2)}} t^{|\nu|/2} s_\nu(y).$$

They permit to write

$$\mathrm{char}_t(S(\mathfrak{sp}_{2n})) = \sum_{\nu\in\mathcal{P}_{2n}^{(2)}} t^{|\nu|/2} \sum_{\lambda\in\mathcal{P}_n} c_\nu^\lambda(\mathfrak{sp}_{2n}) s_\lambda^{\mathfrak{sp}_{2n}} \quad \text{and}$$

$$\mathrm{char}_t(S(\mathfrak{so}_{2n})) = \sum_{\nu\in\mathcal{P}_{2n}^{(1,1)}} t^{|\nu|/2} \sum_{\lambda\in\mathcal{P}_n} c_\nu^\lambda(O_{2n}) s_\lambda^{O_{2n}}.$$

Note that here we considered the character $s_\lambda^{O_{2n}}$ of the $O(2n)$-module $V^{O(2n)}(\lambda)$ parametrized by the partition λ. When $\lambda_n = 0$, we have $s_\lambda^{O_{2n}} = s_\lambda^{\mathfrak{so}_{2n}}$. Nevertheless, when $\lambda_n > 0$, $V^{O(2n)}(\lambda)$ decomposes as the sum of two irreducible $SO(2n)$-modules whose highest weights correspond via the Dynkin diagram involution ι flipping the nodes $n-1$ and n. For $1 \leq i \leq n-1$, define a_i as the number of columns of height i in λ, and $a_n = 2\lambda_n + a_{n-1}$. We then have

$$s_\lambda^{O_{2n}} = s_{\omega(\lambda)}^{\mathfrak{so}_{2n}} + s_{\iota(\omega(\lambda))}^{\mathfrak{so}_{2n}}, \tag{2}$$

where $\omega(\lambda) = \sum_{i=1}^{n} a_i\omega_i$.

Since we have

$$\mathrm{char}_t(H(\mathfrak{g}_n)) = \frac{\prod_{i=1}^{n}(1-t^{d_i})}{(1-t)^n}\prod_{\alpha\in R}\frac{1}{1-te^{\alpha}} = \prod_{i=1}^{n}(1-t^{d_i})\mathrm{char}_t(S(\mathfrak{g}_n)),$$

we can write

$$\frac{1}{\prod_{i=1}^{n}(1-t^{d_i})}\mathrm{char}_t(H(\mathfrak{g}_n)) = \sum_{\lambda\in P_+}\frac{K_{\lambda,0}(t)}{\prod_{i=1}^{n}(1-t^{d_i})}s_\lambda^{\mathfrak{g}_n} = \mathrm{char}_t(S(\mathfrak{g}_n)). \tag{3}$$

So we get the following simple expressions for the formal series $\frac{K_{\lambda,0}(t)}{\prod_{i=1}^{n}(1-t^{d_i})}$.

Proposition 2.1. We have the following identities:

(1) In type A_{n-1}, for any $\lambda \in \mathcal{P}_{n-1}$, we have

$$\frac{K_{\lambda,0}^{\mathfrak{sl}_n}(t)}{\prod_{i=1}^{n}(1-t^{i})} = \sum_{\gamma\in\mathcal{P}_n} t^{|\gamma|}c_{\gamma,\gamma^*}^{\lambda}. \tag{4}$$

The factor $(1-t)$ in (1) gives the missing "$d_i = 1$" in type A_{n-1}.

(2) In type B_n, for any $\lambda \in \mathcal{P}_n$, we have

$$\frac{K_{\lambda,0}^{\mathfrak{so}_{2n+1}}(t)}{\prod_{i=1}^{n}(1-t^{2i})} = \sum_{\nu\in\mathcal{P}_{2n+1}^{(1,1)}} t^{|\nu|/2}c_\nu^\lambda(\mathfrak{so}_{2n+1}).$$

Here the partition λ can have an odd rank.

(3) In type C_n, for any $\lambda \in \mathcal{P}_n$, we have

$$\frac{K_{\lambda,0}^{\mathfrak{sp}_{2n}}(t)}{\prod_{i=1}^{n}(1-t^{2i})} = \sum_{\nu\in\mathcal{P}_{2n}^{(2)}} t^{|\nu|/2}c_\nu^\lambda(\mathfrak{sp}_{2n}).$$

(4) In type D_n, for any $\lambda \in \mathcal{P}_n$, we have

$$\frac{K_{\lambda,0}^{O(2n)}(t)}{(1-t^n)\prod_{i=1}^{n-1}(1-t^{2i})} = \sum_{\nu\in\mathcal{P}_{2n}^{(1,1)}} t^{|\nu|/2}c_\nu^\lambda(O_{2n}).$$

For type D_n, the dominant weights appearing in (3) are not necessarily partitions, whereas this is the case in Assertion 4 of the previous proposition. So here we have in fact to write

$$\sum_{\omega\in P_+}\frac{K^{\mathfrak{so}_{2n}}_{\omega,0}(t)}{\prod_{i=1}^n(1-t^{d_i})}s^{\mathfrak{so}_{2n}}_{\omega}=\sum_{\lambda\in P_+,\lambda\in\mathcal{P}_{n-1}}\frac{K^{\mathfrak{so}_{2n}}_{\lambda,0}(t)}{\prod_{i=1}^n(1-t^{d_i})}s^{O_{2n}}_{\lambda}+$$
$$+\sum_{\omega\in P_+,\omega\notin\mathcal{P}_{n-1}}\frac{K^{\mathfrak{so}_{2n}}_{\omega,0}(t)s^{\mathfrak{so}_{2n}}_{\omega}+K^{\mathfrak{so}_{2n}}_{\iota(\omega),0}(t)s^{\mathfrak{so}_{2n}}_{\iota(\omega)}}{\prod_{i=1}^n(1-t^{d_i})}$$
$$=\sum_{\lambda\in\mathcal{P}_n}\frac{K^{O(2n)}_{\lambda,0}(t)}{\prod_{i=1}^n(1-t^{d_i})}s^{O_{2n}}_{\lambda},$$

where

$$K^{O(2n)}_{\lambda,0}(t)=K^{\mathfrak{so}_{2n}}_{\omega(\lambda),0}(t)=K^{\mathfrak{so}_{2n}}_{\iota(\omega(\lambda)),0}(t)$$

for any partition $\lambda\in\mathcal{P}_n\setminus\mathcal{P}_{n-1}$ and $\omega(\lambda)$ defined as in (2).

The notation $K^{\mathfrak{sl}_n}_{\lambda,0}(t)$ is a little unusual in type A_{n-1}, where the polynomials $K^{\mathfrak{sl}_n}_{\lambda,0}(t)$ coincide with the Kostka polynomials, which are usually labeled by pairs of partitions with the same rank (i.e., by using the weights of $\mathfrak{gl}_n$ rather than those of $\mathfrak{sl}_n$). When $K^{\mathfrak{sl}_n}_{\lambda,0}(t)\neq 0$, the rank of λ should in particular be a multiple of n. Also the sum in the right-hand side of Assertion 1 is in fact infinite. Indeed, to the weight λ correspond an infinite number of partitions, since adding columns of height n to a Young diagram does not modify the corresponding weight of $\mathfrak{sl}_n$.

We have then by a theorem of Lascoux and Schützenberger [24]

$$K^{\mathfrak{sl}_n}_{\lambda,0}(t)=\sum_{T\in SST(\lambda)_0}t^{\mathrm{ch}_n(T)},$$

where $SST(\lambda)_0$ is the set of semistandard tableaux labeled by letters of $\{1<\cdots<n\}$ of weight $\mu=(a,\ldots,a)=0$ (i.e., each letter i appear a times in T) where $a=|\lambda|/n$, and $\mathrm{ch}_n(T)$ is the charge statistic evaluated on T. Recall that this charge statistic is defined by rather involved combinatorial operation such as cyclage on tableaux.

2.3 Stable versions

When the ranks of the classical root systems considered go to infinity, the previous relations simplify. In particular, for n sufficiently large, we have the following stable branching formulas [12]:

$$c_\nu^\lambda(\mathfrak{so}_{2n+1}) = \sum_{\delta\in\mathcal{P}} c^\nu_{\lambda,2\delta}, \quad c_\nu^\lambda(\mathfrak{sp}_{2n}) = \sum_{\delta\in\mathcal{P}} c^\nu_{\lambda,(2\delta)'}, \quad \text{and} \quad c_\nu^\lambda(\mathfrak{so}_{2n}) = \sum_{\delta\in\mathcal{P}} c^\nu_{\lambda,2\delta}.$$

Observe that, for $\mathfrak{g} = \mathfrak{so}_{2n+1}$, this implies in particular that $c_\nu^\lambda(\mathfrak{so}_{2n+1}) = 0$ when the ranks of λ and ν do not have the same parity, which is false in general. Thus, we get the relations

$$\frac{K^{B_\infty}_{\lambda,0}(t)}{\prod_{i=1}^\infty(1-t^{2i})} = \sum_{\nu\in\mathcal{P}^{(1,1)}}\sum_{\delta\in\mathcal{P}^{(2)}} t^{|\nu|/2} c^\nu_{\lambda,\delta} \quad \text{in type } B_\infty \text{ when } |\lambda| \text{ is even},$$
$$\frac{K^{C_\infty}_{\lambda,0}(t)}{\prod_{i=1}^\infty(1-t^{2i})} = \sum_{\nu\in\mathcal{P}^{(2)}}\sum_{\delta\in\mathcal{P}^{(1,1)}} t^{|\nu|/2} c^\nu_{\lambda,\delta} \quad \text{in type } C_\infty,$$
$$\frac{K^{D_\infty}_{\lambda,0}(t)}{\prod_{i=1}^\infty(1-t^{2i})} = \sum_{\nu\in\mathcal{P}^{(1,1)}}\sum_{\delta\in\mathcal{P}^{(2)}} t^{|\nu|/2} c^\nu_{\lambda,\delta} \quad \text{in type } D_\infty.$$

In particular, this gives

$$K^{B_\infty}_{\lambda,0}(t) = K^{D_\infty}_{\lambda,0}(t) \quad \text{and} \quad K^{B_\infty}_{\lambda,0}(t) = K^{C_\infty}_{\lambda',0}(t). \tag{5}$$

All these stabilized forms are in fact formal power series in t equal to zero when the rank of λ is odd (see [27]). The previous identities permit to restrict to the study of the stabilized formal series $K^{C_\infty}_{\lambda,0}(t)$ when λ runs over the set of partitions with even rank. We are going to see that stabilized form of the generalized exponents are easier to handle than their finite-rank counterparts. Observe also that stabilized versions of Lusztig t-analogs [27] exist in general (i.e., for nonzero weights) in connection with the stabilization of the Brylinski filtration. Finally, in type A, the Kostka polynomials $K^{\mathfrak{sl}_n}_{\lambda,0}(t)$ stabilize to zero when n becomes greater than the rank of λ.

3 Charge in Type A_{n-1} and Crystal Graphs

We are now going to explain how the interpretation of the charge for zero-weight tableaux in terms of crystals obtained in [23] naturally emerges from (4), without any reference to cyclage. In particular, we obtain a direct proof of the positivity of the polynomials $K_{\lambda,0}^{A_{n-1}}(t)$; for simplicity, we drop the superscript A_{n-1}. We refer the reader to [11, 16] for complements on Kashiwara crystal basis theory, including the standard notation.

Step 1: Observe that $c^{\lambda}_{\gamma,\gamma^*} = c^{\lambda}_{\kappa,\kappa^*}$ for γ,κ in $\mathcal{P}_n$ whose conjugates differ only by their parts equal to n. So by decomposing each $\kappa \in \mathcal{P}_n$ as $\kappa = (\gamma, n^m)$, we get

$$\sum_{\kappa\in\mathcal{P}_n} t^{|\kappa|} c^{\lambda}_{\kappa,\kappa^*} = \sum_{\gamma\in\mathcal{P}_{n-1}} t^{|\kappa|} c^{\lambda}_{\gamma,\gamma^*} \sum_{m=0}^{+\infty} (t^n)^m = \frac{1}{1-t^n} \sum_{\gamma\in\mathcal{P}_{n-1}} t^{|\kappa|} c^{\lambda}_{\gamma,\gamma^*}.$$

Therefore, (4) can be rewritten in the form

$$\frac{K_{\lambda,0}(t)}{\prod_{i=1}^{n-1}(1-t^i)} = \sum_{\gamma\in\mathcal{P}_{n-1}} t^{|\gamma|} c^{\lambda}_{\gamma,\gamma^*}, \tag{6}$$

where now all the partitions are in one-to-one correspondence with weights of $\mathfrak{sl}_n$.

Step 2: Recall that $R^{A_{n-1}}$ is endowed with the scalar product $\langle\cdot,\cdot\rangle$ defined by

$$\langle f, g\rangle = [f a_\rho \overline{g a_\rho}\,]_0,$$

where $a_\rho = \prod_{\alpha\in R_+}(x^{\alpha/2} - x^{-\alpha/2})$ in $R^{A_{n-1}}$, the bar involution sends x^{μ} to $x^{-\mu}$ and for any $u \in R^{A_{n-1}}$, $[u]_0$ is the constant term in u . We then have $\langle s_\lambda, s_\mu\rangle = \delta_{\lambda,\mu}$. It follows that the adjoint of the multiplication by s_λ in $R^{A_{n-1}}$ for this scalar product is the multiplication by s_{λ^*}. This gives

$$c^{\lambda}_{\gamma,\gamma^*} = \langle s_\gamma s_{\gamma^*}, s_\lambda\rangle = \langle s_\gamma, s_\gamma s_\lambda\rangle = c^{\gamma}_{\gamma,\lambda}.$$

Step 3: For any $\lambda \in \mathcal{P}_{n-1}$, write $B(\lambda)$ for the crystal graph of the irreducible $\mathfrak{sl}_n$-module of highest weight λ. Let b_λ be the highest weight vertex of $B(\lambda)$. For any vertex $b \in B(\lambda)$, set

$$\boldsymbol{\varepsilon}(b) = \sum_{i=1}^{n-1} \varepsilon_i(b)\omega_i \in \mathcal{P}_{n-1}.$$

Also given κ,δ in $\mathcal{P}_{n-1}$, write $\kappa \le \delta$ when $\delta - \kappa$ is a dominant weight. We know that

$$c^{\gamma}_{\gamma,\lambda} = \mathrm{card}\left\{b_\gamma \otimes b \in B(\gamma)\otimes B(\lambda) \mid \mathrm{wt}(b) = 0 \quad \text{and} \quad \boldsymbol{\varepsilon}(b) \le \gamma\right\}.$$

So we have in fact $c^{\gamma}_{\gamma,\lambda} = \mathrm{card}(B_\lambda(\gamma))$, where

$$B_\lambda(\gamma) = \{b \in B(\lambda) \mid \mathrm{wt}(b) = 0 \quad \text{and} \quad \boldsymbol{\varepsilon}(b) \le \gamma\}.$$

Now for any $b \in B(\lambda)_0$, that is, $b \in B(\lambda)$ such that $\mathrm{wt}(b) = 0$, set

$$S(b) = \{\gamma \in \mathcal{P}_{n-1} \mid \boldsymbol{\varepsilon}(b) \leq \gamma\}.$$

We have in fact

$$S(b) = \boldsymbol{\varepsilon}(b) + \mathcal{P}_{n-1},$$

that is, $\gamma \in S(b)$ if and only if there exists $\kappa \in \mathcal{P}_{n-1}$ such that $\gamma = \boldsymbol{\varepsilon}(b) + \kappa$.
Step 4: Write

$$\sum_{\gamma \in \mathcal{P}_{n-1}} t^{|\gamma|} c^{\lambda}_{\gamma,\gamma^*} = \sum_{b \in B(\lambda)_0} \sum_{\gamma \in S(b)} t^{|\gamma|} = \sum_{b \in B(\lambda)_0} \sum_{\kappa \in \mathcal{P}_{n-1}} t^{|\kappa| + |\boldsymbol{\varepsilon}(b)|} =$$
$$= \sum_{b \in B(\lambda)_0} t^{|\boldsymbol{\varepsilon}(b|} \sum_{\kappa \in \mathcal{P}_{n-1}} t^{|\kappa|} = \frac{1}{\prod_{i=1}^{n-1}(1-t^i)} \sum_{b \in B(\lambda)_0} t^{|\boldsymbol{\varepsilon}(b|}.$$

In view of (6), this gives the following result.

Theorem 3.1. For any partition $\lambda \in \mathcal{P}_{n-1}$, we have

$$K_{\lambda,0}(t) = \sum_{b \in B(\lambda)_0} t^{|\boldsymbol{\varepsilon}(b)|}.$$

Remark 3.2. Since $b \in B(\lambda)_0$, we have for any $i = 1, \ldots, n-1$ that $\varepsilon_i(b) = \varphi_i(b)$. Moreover,

$$|\boldsymbol{\varepsilon}(b)| = \sum_{i=1}^{n-1} i\varepsilon_i(b),$$

so the previous expression of $K_{\lambda,0}(t)$ is the same as that obtained in [23]. The interesting point is that it emerges directly from our computations and does not use the definition of the charge (as in [23]) given by Lascoux and Schützenberger in terms of cyclage of tableaux or indices on letters of words.

Remark 3.3. This also permits us to recover the multivariable version, defined by

$$\frac{K_{\lambda,0}(t_1, \ldots, t_{n-1})}{\prod_{i=1}^{n-1}(1-t_i)} = \sum_{\gamma \in \mathcal{P}_{n-1}} \prod_{i=1}^{n-1} t_i^{a_i(\gamma)} c^{\lambda}_{\gamma,\gamma^*},$$

where $\gamma = \sum_{i=1}^{n-1} a_i(\gamma)\omega_i$. Namely, we have

$$K_{\lambda,0}(t_1,\ldots,t_{n-1}) = \sum_{b\in B(\lambda)_0} \prod_{i=1}^{n-1} t_i^{\varepsilon_i(b)}.$$

4 Stabilized Generalized Exponents and Crystal Graphs of Type $A_{+\infty}$

We are going to explain the way in which the formula

$$\frac{K_{\lambda,0}^{C_\infty}(t)}{\prod_{i=1}^{\infty}(1-t^{2i})} = \sum_{\nu\in\mathcal{P}^{(2)}} \sum_{\delta\in\mathcal{P}^{(1,1)}} t^{|\nu|/2} c_{\lambda,\delta}^{\nu} \quad \text{in type } C_\infty$$

can be obtained from the combinatorics of crystals of type $A_{+\infty}$, which leads to a combinatorial proof of the positivity of the stabilized generalized exponents (or stabilized Lusztig t-analogs). In particular, this will provide a combinatorial description of $K_{\lambda,0}^{C_\infty}(t)$, and thus a similar description of $K_{\lambda,0}^{B_\infty}(t)$ and $K_{\lambda,0}^{D_\infty}(t)$, by (5). Furthermore, this will give a flavor of the methods we will employ in the non-stable type C_n case.

4.1 Crystal of type $A_{+\infty}$

Recall that crystals of type $A_{+\infty}$ are those associated to the infinite Dynkin diagram

$$\overset{1}{\circ} - \overset{2}{\circ} - \overset{3}{\circ} \cdots.$$

The partitions label the dominant weights of $\mathfrak{sl}_{+\infty}$. If we denote by $(\omega_i)_{\geq 1}$ the sequence of fundamental weights of $\mathfrak{sl}_{+\infty}$, we have for any partition $\lambda \in \mathcal{P}$

$$\lambda = \sum_i a_i\omega_i,$$

where a_i is the number of columns with height i in the Young diagram of λ.

To each partition λ corresponds the crystal $B(\lambda)$ of the irreducible infinite-dimensional representation of $\mathfrak{sl}_{+\infty}$ parametrized by λ. A classical model for $B(\lambda)$ is that of semistandard tableaux of shape λ on the infinite alphabet $\mathbb{Z}_{>0} = \{1 < 2 < 3 < \cdots\}$. Given $b \in B(\lambda)$, we define

$$\boldsymbol{\varepsilon}(b) = \sum_{i=1}^{+\infty} \varepsilon_i(b)\omega_i \quad \text{and} \quad \boldsymbol{\varphi}(b) = \sum_{i=1}^{+\infty} \varphi_i(b)\omega_i,$$

where both sums are in fact finite. The weight of $b \in B(\lambda)$ then verifies $\mathrm{wt}(b) = \boldsymbol{\varphi}(b) - \boldsymbol{\varepsilon}(b)$.

4.2 Combinatorial preliminaries

In the sequel we consider the order $\leq$ on $\mathcal{P}$ such that $\lambda \leq \mu$ if and only if $\mu - \lambda \in P_+^\infty$, that is, $\mu - \lambda$ decomposes in the basis of the ω_i's with nonnegative integer coefficients.

The partitions in $\mathcal{P}^{(2)}$ (resp. in $\mathcal{P}^{(1,1)}$) are those that can be tiled with horizontal (resp. vertical) dominoes. Equivalently, a partition κ belongs to $\mathcal{P}^{(2)}$ (resp. $\mathcal{P}^{(1,1)}$) if and only if the number of columns (resp. rows) of fixed height (resp. length) is even. So

$$\kappa \in \mathcal{P}^{(2)} \iff \kappa = \sum_i 2a_i\omega_i \quad \text{and} \quad \kappa \in \mathcal{P}^{(1,1)} \iff \kappa = \sum_i a_i\omega_{2i}.$$

Set $\mathcal{P}^{\boxplus} = \mathcal{P}^{(2)} \cap \mathcal{P}^{(1,1)}$. It follows that

$$\kappa \in \mathcal{P}^{\boxplus} \iff \kappa = \sum_i 2a_i\omega_{2i},$$

that is, λ decomposes in terms of the fundamental weights ω_{2i} with even coefficients. In the general case of a partition $\kappa \in \mathcal{P}$ written as

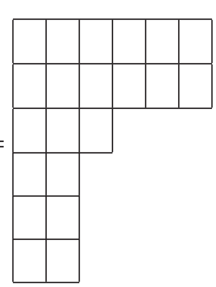
$$\kappa = \sum_i a_i\omega_i,$$

we define

$$\kappa_{\boxplus} = \sum_i (a_{2i} - (a_{2i} \bmod 2))\omega_{2i} \quad \text{and} \quad \kappa^{\boxplus} = \kappa - \kappa_{\boxplus} = \sum_i a_{2i+1}\omega_{2i+1} + \sum_i (a_{2i} \bmod 2)\omega_{2i}.$$

So $\kappa_{\boxplus}$ and $\kappa^{\boxplus}$ are partitions and $\kappa_{\boxplus} \in \mathcal{P}^{\boxplus}$.

Example 4.1. Consider

$\kappa =$  .

Then

$$\kappa_{\boxplus} = \ydiagram{4,4,2,2,2,2} \quad \text{and} \quad \kappa^{\boxplus} = \ydiagram{2,2,1}.$$

We denote by $P^{\infty}_{(2)}$ and $P^{\infty}_{(1,1)}$ the sublattices of $P = \bigoplus_{i\geq 1} \mathbb{Z}\omega_i$ defined by

$$P^{\infty}_{(2)} = \bigoplus_{i\geq 1} 2\mathbb{Z}\omega_i \quad \text{and} \quad P^{\infty}_{(1,1)} = \bigoplus_{i\geq 1} \mathbb{Z}\omega_{2i}.$$

Observe that $P^{\infty}_{(2)} \cap \mathcal{P} = \mathcal{P}_{(2)}$ and $P^{\infty}_{(1,1)} \cap \mathcal{P} = \mathcal{P}_{(1,1)}$. We have also

$$P^{\infty}_{\boxplus} = P^{\infty}_{(2)} \cap P^{\infty}_{(1,1)} = \bigoplus_{i\geq 1} 2\mathbb{Z}\omega_{2i} \quad \text{and} \quad \mathcal{P}^{\boxplus} = \mathcal{P} \cap P^{\infty}_{(2)} \cap P^{\infty}_{(1,1)}.$$

We define the order $\leq_{\boxplus}$ on $\mathcal{P}$ by

$$\lambda \leq_{\boxplus} \mu \Longleftrightarrow \mu - \lambda \in \mathcal{P}^{\boxplus}.$$

4.3 A combinatorial description of the series $K^{C_\infty}_{\lambda,0}(t)$

Definition 4.2. Consider a partition μ. A vertex $b \in B(\lambda)$ is called μ-distinguished if there exists $(\nu, \delta) \in \mathcal{P}^{(2)} \times \mathcal{P}^{(1,1)}$ such that

$$\boldsymbol{\varphi}(b) = \nu - \mu \quad \text{and} \quad \boldsymbol{\varepsilon}(b) = \delta - \mu.$$

Definition 4.3. Let $D(\lambda)$ be the set of all vertices in $B(\lambda)$ that are μ-distinguished for at least a partition μ.

Clearly, if b is μ-distinguished, then b is $(\mu + \kappa)$-distinguished for any $\kappa \in \mathcal{P}^{\boxplus}$ (change $(\nu, \delta) \in \mathcal{P}^{(2)} \times \mathcal{P}^{(1,1)}$ to $(\nu + \kappa, \delta + \kappa) \in \mathcal{P}^{(2)} \times \mathcal{P}^{(1,1)}$). For any $b \in D(\lambda)$, set

$$S_b = \{\mu \in \mathcal{P} \mid b \text{ is } \mu\text{-distinguished}\}.$$

Lemma 4.4. The set S_b has the form

$$S_b = \mu_b + \mathcal{P}^{\boxplus},$$

and μ_b is minimal for $\leq_{\boxplus}$ such that b is μ_b-distinguished. Moreover, for any $\mu \in S_b$, we have $\mu_b = \mu^{\boxplus}$.

Proof. It suffices to show that S_b contains a unique element μ_b minimal for the order $\leq_{\boxplus}$, since each element μ of S_b can then be written in the form $\mu = \mu_b + \kappa$ with κ in $\mathcal{P}^{\boxplus}$. So consider μ and μ' two elements in S_b minimal for $\leq_{\boxplus}$ in S_b. Write

$$\mu = \sum_i a_i \omega_i \quad \text{and} \quad \mu' = \sum_i a'_i \omega_i.$$

Since μ is minimal in S_b, we must have $a_i \in \{0,1\}$ for any even i. Indeed, if $a_i \geq 2$ for an even integer $i \geq 1$, we could consider $\mu^\flat = \mu - 2\omega_i \in \mathcal{P}$. Since $\boldsymbol{\varphi}(b) = \nu - \mu \geq 0$, we can consider $\nu^\flat = \nu - 2\omega_i \in \mathcal{P}_{(2)}$. Similarly, we have $\boldsymbol{\varepsilon}(b) = \delta - \mu \geq 0$, so $\delta^\flat = \delta - 2\omega_i \in \mathcal{P}_{(1,1)}$ (as i is even). Finally, we obtain a contradiction since

$$\boldsymbol{\varphi}(b) = \nu^\flat - \mu^\flat \quad \text{and} \quad \boldsymbol{\varepsilon}(b) = \delta^\flat - \mu^\flat,$$

so $\mu^\flat <_{\boxplus} \mu$ belongs to S_b.

We prove similarly that $a'_i \in \{0,1\}$ for any even i. Now we can use that μ and μ' belong to S_b, which implies that

$$\mu = \mu' \bmod P_{(2)}^{+\infty} \quad \text{and} \quad \mu = \mu' \bmod P_{(1,1)}^{+\infty}.$$

Since $P_{(2)}^{+\infty} \cap P_{(1,1)}^{+\infty} = P_{\boxplus}^{+\infty} = \sum_{i \text{ even}} 2\mathbb{Z}\omega_i$, we get in fact

$$\mu = \mu' \bmod P_{\boxplus}^{+\infty}.$$

This imposes that $a_i = a'_i$ for any odd i and $a_i = a'_i$ mod 2 for any even i. But we have seen that for any even i, both a_i and a'_i belong to $\{0,1\}$. So we obtain finally that $a_i = a'_i$ for any i even also. This permits us to conclude that $\mu = \mu'$ and S_b admits a unique minimal element for $\leq_{\boxplus}$. ∎

The following proposition makes more explicit the structure of the distinguished tableaux.

Proposition 4.5. Let b be a vertex of $B(\lambda)$ with $\lambda \in \mathcal{P}$. Then b is distinguished if and only if

(1) $\varepsilon_i(b) = 0$ for any odd i,

(2) $\varphi_i(b)$ is even for any odd i.

Moreover, we then have $\mu_b = \sum_i (\varphi_{2i}(b) \bmod 2)\omega_{2i} =: \varphi(b) \bmod 2$.

Proof. If $\varepsilon_i(b) = 0$ for any odd i, then $\varepsilon(b)$ belongs to $\mathcal{P}^{(1,1)}$, and thus $\varepsilon(b) + \mu$ belongs to $\mathcal{P}^{(1,1)}$ for any μ in $\mathcal{P}^{(1,1)}$. Since $\boldsymbol{\varphi}(b)$ is even for any odd i, we will have that $\boldsymbol{\varphi}(b) + \mu$ belongs to $\mathcal{P}^{(2)}$ for any μ in $\mathcal{P}^{(1,1)}$ such that the coefficients of ω_i with i even in the expansions of μ and $\boldsymbol{\varphi}(b)$ have the same parity. Finally, b is μ-distinguished for any such μ.

Conversely, assume there exists μ in $\mathcal{P}$ such that $\varepsilon(b) + \mu \in \mathcal{P}^{(1,1)}$ and $\varphi(b) + \mu \in \mathcal{P}^{(2)}$. Since the coefficients of ω_i with i odd in the expansion of $\varepsilon(b) + \mu$ are equal to 0, and both $\varepsilon(b)$ and μ are dominant weights, we must have that they belong in fact to $\mathcal{P}^{(1,1)}$. Therefore, the condition $\varphi(b) + \mu \in \mathcal{P}^{(2)}$ implies that $\varphi_i(b)$ is even for any odd i.

To determine μ_b, we have to choose μ minimal for the order $<_{\boxplus}$. Since $\varepsilon_i(b) = 0$ for any odd i, we have in fact to choose μ minimal for the order $<_{\boxplus}$ so that $\varphi(b) + \mu \in \mathcal{P}^{(2)}$. This imposes that $\mu_b = \sum_i (\varphi_{2i}(b) \bmod 2)\omega_{2i}$. ■

Proposition 4.6. We have

$$\sum_{\nu \in \mathcal{P}^{(2)}} \sum_{\delta \in \mathcal{P}^{(1,1)}} t^{|\nu|/2} c^{\nu}_{\lambda,\delta} = \sum_{b \in D(\lambda)} t^{|\boldsymbol{\varphi}(b)+\mu_b|/2} \sum_{\kappa \in \mathcal{P}^{\boxplus}} t^{|\kappa|/2}.$$

Proof. Recall that $c^{\nu}_{\lambda,\delta} = \operatorname{card}\{b \in B(\lambda) \mid \boldsymbol{\varepsilon}(b) \le \delta \text{ and } \boldsymbol{\varphi}(b) = \boldsymbol{\varepsilon}(b) + \nu - \delta\}$. For a fixed $b \in B(\lambda)$, the idea is to gather all the pairs $(\nu, \delta) \in \mathcal{P}^{(2)} \times \mathcal{P}^{(1,1)}$ such that $b_\delta \otimes b$ is of highest weight ν. This is equivalent to saying that b is μ-distinguished with $\mu = \nu - \boldsymbol{\varphi}(b) = \delta - \boldsymbol{\varepsilon}(b)$. So we get

$$\sum_{\nu \in \mathcal{P}^{(2)}} \sum_{\delta \in \mathcal{P}^{(1,1)}} t^{|\nu|/2} c^{\nu}_{\lambda,\delta} = \sum_{b \in D(\lambda)} \sum_{\mu \in S_b} t^{|\boldsymbol{\varphi}(b)+\mu|/2}.$$

Now by Lemma 4.4, we can write $\boldsymbol{\varphi}(b) + \mu = \boldsymbol{\varphi}(b) + \mu_b + \kappa$, where $\kappa = \mu - \mu_b$ belongs to $\mathcal{P}^{\boxplus}$. This gives

$$\sum_{\nu \in \mathcal{P}^{(2)}} \sum_{\delta \in \mathcal{P}^{(1,1)}} t^{|\nu|/2} c^{\nu}_{\lambda,\delta} = \sum_{b \in D(\lambda)} t^{|\boldsymbol{\varphi}(b)+\mu_b|/2} \sum_{\kappa \in \mathcal{P}^{\boxplus}} t^{|\kappa|/2}.$$

■

Theorem 4.7. We have

$$K_{\lambda,0}^{C_\infty}(t) = \sum_{b\in D(\lambda)} t^{|\varphi(b)+\mu_b|/2}.$$

Proof. It suffices to observe that

$$\sum_{\kappa\in\mathcal{P}^{\boxplus}} t^{|\kappa|/2} = \frac{1}{\prod_{i=1}^{\infty}(1-t^{2i})}.$$

■

We can get similarly a multivariable version. For any $b \in D(\lambda)$, set

$$\varphi(b) + \mu_b = \sum_i 2a_i(b)\omega_i \in \mathcal{P}^{(2)},$$

and assign to each fundamental weight ω_i a formal variable t_i. The decomposition

$$\nu = \varphi(b) + \mu_b + \kappa \text{ with } \kappa \in \mathcal{P}^{\boxplus}$$

will give the multivariable version. First let $\boldsymbol{t} = (t_1, t_2, \dots t_n, \dots)$ be the sequence of formal variables $t_i, i \geq 1$. If one prefers, one can also consider each t_i as a real number in $[0, a]$ with $a < 1$. For any $\beta \in P^{+\infty}$ such that $\beta = \sum_i \beta_i\omega_i$, set $\boldsymbol{t}^\beta = \prod_{i\geq 1} t_i^{\beta_i}$.

Theorem 4.8. Define the multivariable formal series $K_{\lambda,0}^{C_\infty}(\boldsymbol{t})$ by

$$\frac{K_{\lambda,0}^{C_\infty}(\mathbf{t})}{\prod_{i=1}^{\infty}(1-t_{2i})} = \sum_{\nu\in\mathcal{P}^{(2)}} \sum_{\delta\in\mathcal{P}^{(1,1)}} \boldsymbol{t}^{\frac{1}{2}\nu} c_{\lambda,\delta}^{\nu}.$$

Then we have

$$K_{\lambda,0}^{C_\infty}(\boldsymbol{t}) = \sum_{b\in D(\lambda)} \boldsymbol{t}^{\frac{1}{2}(\varphi(b)+\mu_b)}.$$

Remark 4.9. Multivariable generalized exponents defined via the Joseph–Letzter filtration already appear in the literature (see [5]).

4.4 Distinguished tableaux and zero-weight King-type tableaux

We are now going to explain how the distinguished tableaux we introduced previously to describe the stable generalized exponents are in natural bijection with zero-weight tableaux very close to King tableaux. We will in fact consider the sets $T_{C_\infty}(\lambda)$ of semistandard tableaux of shape λ on the infinite-ordered alphabet $\{1 < \overline{1} < 2 < \overline{2} < \cdots\}$. There will be no condition on the position of the barred letters here, contrary to the definition of King tableaux.

We start by discussing the structure of the distinguished tableaux. Recall the notation of Section 4.3. For any distinguished vertex b in $D(\lambda)$, set

$$\boldsymbol{\theta}(b) = \boldsymbol{\varphi}(b) + \mu_b,$$

and let $\theta_j(b)$ be the coefficient of ω_j in the expansion of $\boldsymbol{\theta}(b)$. Since $\boldsymbol{\theta}(b)$ is a dominant weight for $\mathfrak{sl}_\infty$, it can be regarded as a partition. Recall also that $|\lambda|$ is even, says $|\lambda| = 2\ell$. In the sequel of this section, we shall assume that $B(\lambda)$ is realized as the set of semistandard tableaux on the infinite-ordered alphabet $\mathbb{Z}_{>0}$. For any integer $i \geq 1$, a reverse lattice skew tableau on $\{2i-1, 2i\}$ is a semistandard filling of a skew Young diagram with columns of height at most 2 by letters $2i-1$ and $2i$ whose Japanese reading is a lattice word (i.e., in each left factor the number of letters $2i$ is less than or equal to that of letters $2i-1$).

Example 4.10. Assume $i = 2$. Then

$$\begin{array}{cccccccccccccc}
 & & & & & & & & & & \boxed{3} & \boxed{3} & \boxed{3} & \boxed{3} \\
 & & & & & & \boxed{3} & \boxed{3} & \boxed{4} & \boxed{4} & \boxed{4} & & & \\
 & & & \boxed{3} & \boxed{3} & \boxed{4} & & & & & & & & \\
\boxed{3} & \boxed{3} & \boxed{4} & \boxed{4} & & & & & & & & & &
\end{array} \tag{7}$$

is a reverse lattice skew tableau on $\{3, 4\}$.

The following proposition is a reformulation of Proposition 4.5.

Proposition 4.11. A semistandard tableau T of shape λ is distinguished if and only if for any integer $i \geq 1$, the skew tableau obtained by keeping only the letters $2i-1$ and $2i$ in T is a reverse lattice tableau, and the rows of $\boldsymbol{\theta}(T)$ have even lengths.

We now explain the correspondence between distinguished tableaux and zero-weight King-type tableaux.

Observe that a tableau T in $T_{C_\infty}(\lambda)$ of weight zero is a juxtaposition of skew tableaux of weight 0 on $\{i, \overline{\imath}\}$ obtained by keeping only the letters i and $\overline{\imath}$. So to obtain a bijection between the set of distinguished tableaux of shape λ and the subset $T^0_{C_\infty}(\lambda) \subset T_{C_\infty}(\lambda)$ of zero-weight tableaux, it suffices to describe a bijection between the set of reverse lattice tableaux on $\{2i-1, 2i\}$ of given shape and weight in $2\omega_i\mathbb{Z}_{\geq 0}$, and the set of skew tableaux on $\{i, \bar{\imath}\}$ with weight 0. Now recall that we have the structure of a $U_q(\mathfrak{sl}_2)$-crystal on the set of all skew semistandard tableaux of fixed skew shape both on $\{2i - 1, 2i\}$ and $\{i, \bar{\imath}\}$. By replacing each letter $2i-1$ by i and each letter $2i$ by $\bar{\imath}$, we get a crystal isomorphism f. The distinguished tableaux correspond to the highest weight vertices of weight in $2\omega_i\mathbb{Z}_{\geq 0}$ for the $\{2i-1, 2i\}$-structure, whereas the tableaux of weight 0 give the vertices of weight 0 in the $\{i, \bar{\imath}\}$-crystal structure. By observing that only $U_q(\mathfrak{sl}_2)$-crystals with highest weight in $2\omega_i\mathbb{Z}_{\geq 0}$ admit a vertex of weight 0, which is then unique, we obtain that the map $\mathcal{C}$ that associates to each zero-weight vertex in the $\{i, \bar{\imath}\}$-crystal structure its highest weight vertex in the $\{2i-1, 2i\}$-crystal structure is the bijection we need. More precisely, the map $\mathcal{C}$ (resp. its inverse) is obtained as usual: we start by encoding in the reading of each $\{i, \bar{\imath}\}$-tableau (resp. of each $\{2i-1, 2i\}$-tableau) the letters i by $+$ and the letters $\bar{\imath}$ by $-$ (resp. the letters $2i-1$ by $+$ and the letters $2i-1$ by $-$), and next by recursively deleting all the factors $+-$, thus obtaining a reduced word of the form $-^m+^m$ (resp. $+^{2m}$). It then suffices to change the m letters $\bar{\imath}$ corresponding to the m surviving symbols $-$ into i and to apply the isomorphism f^{-1} (resp. change m letters $2i-1$ corresponding to the rightmost m surviving symbols $+$ into $2i$ and apply the isomorphism f).

Example 4.12. The skew tableau of weight 0 on $\{2, \bar{2}\}$ corresponding to (7) is

$$\begin{ytableau} \none & \none & \none & 2 & 2 & \overline{2} \\ 2 & 2 & \overline{2} & \overline{2} \end{ytableau} \qquad \begin{ytableau} \none & \none & \none & \none & 2 & 2 & 2 & \overline{2} \\ 2 & \overline{2} & \overline{2} & \overline{2} & \overline{2} \end{ytableau} .$$

In the sequel, we shall abuse the notation and identify the two crystal structures corresponding up to the isomorphism f.

Remarks 4.13.

(1) It seems not immediate to read $\boldsymbol{\theta}$ directly on zero-weight tableaux. The simplest way to do this is to start from a tableau $T \in T^0_{C_\infty}(\lambda)$ and compute its associated highest weight tableau $H(T)$ for the $U_q(\mathfrak{sl}_2 \oplus \cdots \oplus \mathfrak{sl}_2)$-structure obtained by considering only the action of the crystal operators indexed by

odd integers. So we get

$$K_{\lambda,0}^{C_\infty}(\boldsymbol{t}) = \sum_{T \in T^0_{C_\infty}(\lambda)} \boldsymbol{t}^{\frac{|\boldsymbol{\theta}(H(T))|}{2}}.$$

(2) Let $K_{C_\infty}(\lambda)$ be the set of King tableaux on the infinite-ordered alphabet $\{1 < \overline{1} < 2 < \overline{2} < \cdots\}$. Recall that $T \in T_{C_\infty}(\lambda)$ belongs to $K_{C_\infty}(\lambda)$ when, for any $i = 1, \ldots, n$, the letters in row i are greater than or equal to i. Since the number of barred letters can only decrease when we compute $H(T)$, the tableaux T and $H(T)$ either both belong to $K_{C_\infty}(\lambda)$ or belong to $T_{C_\infty}(\lambda) \setminus K_{C_\infty}(\lambda)$. Nevertheless, the set $K^0_{C_\infty}(\lambda)$ of King tableaux of type C_∞ and zero weight is only strictly contained in $T^0_{C_\infty}(\lambda)$ due to the constraints on the rows. In particular, we have

$$K_{\lambda,0}^{C_\infty}(\boldsymbol{t}) \neq \sum_{T \in K^0_{C_\infty}(\lambda)} \boldsymbol{t}^{\frac{1}{2}\boldsymbol{\theta}(H(T))}$$

in general, and the finite rank t-analog thus cannot be obtained from the statistic $\boldsymbol{\theta}$ and King tableaux of zero weight and type C_n.

Example 4.14. Assume $\lambda = (1, 1)$. Then we get

$$T^0_{C_\infty}(\lambda) = \left\{ \begin{array}{|c|}\hline k \\ \hline \overline{k} \\ \hline \end{array} \mid k \in \mathbb{Z}_{\geq 1} \right\} \quad \text{and} \quad K^0_{C_\infty}(\lambda) = \left\{ \begin{array}{|c|}\hline k \\ \hline \overline{k} \\ \hline \end{array} \mid k \in \mathbb{Z}_{\geq 2} \right\}.$$

This gives

$$H\left(\begin{array}{|c|}\hline k \\ \hline \overline{k} \\ \hline \end{array} \right) = \begin{array}{|c|}\hline 2k-1 \\ \hline 2k \\ \hline \end{array} \quad \text{and} \quad \boldsymbol{\varphi}\left(\begin{array}{|c|}\hline 2k-1 \\ \hline 2k \\ \hline \end{array} \right) = \omega_{2k} \text{ for any } k \geq 1.$$

Therefore,

$$\boldsymbol{\theta}\left(\begin{array}{|c|}\hline 2k-1 \\ \hline 2k \\ \hline \end{array} \right) = 2\omega_{2k} \text{ for any } k \geq 1.$$

Finally

$$K_{\lambda,0}^{C_\infty}(\boldsymbol{t}) = \sum_{k \geq 1} t_{2k} \text{ and } K_{\lambda,0}^{C_\infty}(t) = \sum_{k \geq 1} t^{2k} = \frac{t^2}{1 - t^2}.$$

5 Type C_n Generalized Exponents via the Sundaram–LR Tableaux

5.1 Sundaram description of the coefficients $c_\nu^\lambda(\mathfrak{sp}_{2n})$

Recall that in type C_n, the equality $c_\nu^\lambda(\mathfrak{sp}_{2n}) = \sum_{\delta\in\mathcal{P}^{(1,1)}} c^\nu_{\lambda,\delta}$ only holds when $\nu \in \mathcal{P}_n$, in which case we have in fact

$$c_\nu^\lambda(\mathfrak{sp}_{2n}) = \sum_{\delta\in\mathcal{P}_n^{(1,1)}} c^\nu_{\lambda,\delta}.$$

In the general case of a partition $\nu \in \mathcal{P}_{2n}$, we have by a result of Sundaram (see [33, Corollary 3.12])

$$c_\nu^\lambda(\mathfrak{sp}_{2n}) = \sum_{\delta\in\mathcal{P}^{(1,1)}} \widehat{c}^\nu_{\lambda,\delta},$$

where $\widehat{c}^\nu_{\lambda,\delta}$ is the number of Sundaram–LR tableaux, that is, the number of LR tableaux of shape ν/λ and weight δ filled with letters in $\{1,\ldots,2n\}$ such that each odd letter $2i+1$ appears no lower (English convention) than row $(n+i)$ in ν (the rows being numbered from top to bottom). Observe that for any partition κ in $\mathcal{P}_{2n}^{\boxplus}$, a Sundaram–LR tableau of shape ν/λ and weight δ can be easily turned into a Sundaram–LR tableau of shape $(\nu+\kappa)/\lambda$ and weight $\delta+\kappa$ by adding letters i in rows i, which does not violate the Sundaram condition.

5.2 LR tableaux and crystals

Given ν, λ, μ three partitions, the LR coefficient $c^\nu_{\lambda\mu}$ is equal to the cardinality of the four following sets:

(1) the set of LR tableaux of shape ν/λ and weight μ,
(2) the set of LR tableaux of shape ν/μ and weight λ,
(3) the set of vertices $b \in B(\lambda)$ such that $\boldsymbol{\varepsilon}(b) \le \mu$,
(4) the set of vertices $b' \in B(\mu)$ such that $\boldsymbol{\varepsilon}(b') \le \lambda$.

Now there exist bijections between all these sets. Given an LR tableau τ of shape ν/μ and weight λ, we obtain the corresponding tableau $\mathbf{T}(\tau)$ in $B(\lambda)$, called companion tableau, by placing in the k-th row of the Young diagram λ the numbers of the rows of τ containing an entry k.

Example 5.1. For

$$\tau = \begin{array}{ccccc} & & & 1 & 1 \\ & & 1 & 2 & 2 \\ & 1 & 3 & & \\ & 2 & 4 & & \\ 1 & 3 & 5 & & \end{array} \quad \text{we get } \mathbf{T}(\tau) = \begin{array}{ccccc} 1 & 1 & 2 & 3 & 5 \\ 2 & 2 & 4 & & \\ 3 & 5 & & & \\ 4 & & & & \\ 5 & & & & \end{array}.$$

Now we can proceed as in Section 4 by first determining the subset of $\widehat{D}(\lambda) \subset B^{\mathfrak{gl}_{2n}}(\lambda)$ coming from Sundaram–LR tableaux ($D(\lambda)$ would correspond to all the LR tableaux as in the previous section). To do this we proceed as follows:

(1) Start with a Sundaram–LR tableau of shape ν/λ and weight δ, and determine its associated tableau $\mathbf{T}(\tau)$ of shape δ and entries in $\{1,\ldots,2n\}$.

(2) Observe that $T_\lambda \otimes \mathbf{T}(\tau)$ is of highest weight ν in $B(\lambda) \otimes B(\delta)$.

(3) Compute the combinatorial R-matrix, and obtain $\widehat{\mathbf{T}}(\tau)$ in $B(\lambda)$ such that $T_\lambda \otimes \mathbf{T}(\tau) \Longleftrightarrow T_\delta \otimes \widehat{\mathbf{T}}(\tau)$. Here we can choose the version of the combinatorial R-matrix given by the Henriques–Kamnitzer commutor [9, 15], which has several concrete realizations; see Section 8 for more details.

(4) Finally, define $\widehat{D}(\lambda)$ as the subset of tableaux $T \in D(\lambda)$ for which there exists $(\nu,\delta) \in \mathcal{P}_{2n}^{(2)} \times \mathcal{P}_{2n}^{(1,1)}$ and τ a Sundaram–LR tableau of shape ν/λ and weight δ such that $T = \widehat{\mathbf{T}}(\tau)$.

Now, we have

$$\sum_{\kappa \in \mathcal{P}_{2n}^{\boxplus}} t^{|\kappa|/2} = \frac{1}{\prod_{i=1}^{n}(1-t^{2i})},$$

since $\mathcal{P}_{2n}^{\boxplus}$ is obtained by dilating by a factor 2 the set $\mathcal{P}_n$ (i.e., each square becomes a $\boxplus$). By using similar arguments (here, we need to use that for any $\kappa \in \mathcal{P}_{\boxplus}^{(2n)}$, one can produce a Sundaram–LR tableau of shape $(\nu+\kappa)/\lambda$ and weight $\delta+\kappa$ starting from any Sundaram–LR tableau of shape ν/λ and weight δ) to those of Section 4, we obtain the following result.

Theorem 5.2. We have

$$K_{\lambda,0}^{C_n}(t) = \sum_{b \in \widehat{D}(\lambda)} t^{|\varphi(b)+\mu_b|/2}.$$

6 Type C_n Generalized Exponents via the Kwon Model

In this section, we refine the results in Sections 4 and 4.4 to the finite type C_n, based on Kwon's model for the corresponding branching coefficients [19, 20]. We also need to use a combinatorial map realizing the conjugation symmetry of LR coefficients. It turns out that Kwon's model, the version of the conjugation symmetry map used here, and the distinguished tableaux in Section 4.3 fit together in a beautiful way. This allows us to express the related statistic in terms of a natural combinatorial labeling of the vertices of weight 0 in the corresponding type C_n crystal of highest weight λ, namely the corresponding tableaux due to King [17]. In this way, we obtain a more explicit result than the one in Section 5 in terms of LR–Sundaram tableaux.

6.1 The LR conjugation symmetry

Consider partitions $\lambda \in \mathcal{P}_n$ and $\delta, \nu \in \mathcal{P}_m$ with $n \le m$. We will exhibit combinatorially the equality of LR coefficients $c^{\nu}_{\lambda,\delta} = c^{\nu'}_{\lambda',\delta'}$. Throughout, we denote by δ^{rev} the reverse of δ, namely $\delta^{\mathrm{rev}} = (\delta^{\mathrm{rev}}_1 \le \ldots \le \delta^{\mathrm{rev}}_m)$, where we add leading 0s if necessary.

Let $\mathrm{LR}^{\nu}_{\lambda,\delta}$ denote the set of LR tableaux T of shape λ and content ν/δ; in other words, $T \in B_m(\lambda)$ and $H_\delta \otimes T$ is a highest weight element of weight ν, where H_δ denotes the Yamanouchi tableau of shape δ. We will construct a bijection $T \mapsto T'$ between $\mathrm{LR}^{\nu}_{\lambda,\delta}$ and $\mathrm{LR}^{\nu'}_{\lambda',\delta'}$, where T' is viewed as an element of $B_{\ell(\nu')}(\lambda')$. The construction has the following three steps:

- **Step 1.** Apply the Schützenberger evacuation [6] (realizing the Lusztig involution) to T within the crystal $B_m(\lambda)$, and obtain $S(T) \in B_m(\lambda)$.
- **Step 2.** Transpose the tableau $S(T)$ and denote the resulting filling of λ' by $S(T)^{\mathrm{tr}}$.
- **Step 3.** For each $i = 1, \ldots, m$, consider in $S(T)^{\mathrm{tr}}$ the vertical strip of i's, and replace these entries, scanned from northeast to southwest, with $\delta^{\mathrm{rev}}_i + 1$, $\delta^{\mathrm{rev}}_i + 2$, ..., respectively.

Example 6.1. Let $n = 3$, $m = 4$, $\lambda = (4,3,1)$, $\nu = (5,4,4,2)$, $\delta = (3,3,1)$, and $\delta^{\mathrm{rev}} = (0,1,3,3)$. Consider the following tableau of shape ν/δ and content λ whose reverse row word is a lattice permutation, and its associated companion tableau $T \in B_4(\lambda)$:

$$\begin{array}{|c|c|c|c|c|}
\hline
 & & & 1 & 1 \\
\hline
 & & & 2 \\
\cline{1-4}
 & 1 & 2 & 3 \\
\cline{1-4}
1 & 2 \\
\cline{1-2}
\end{array}\ , \qquad T = \begin{array}{|c|c|c|c|}
\hline
1 & 1 & 3 & 4 \\
\hline
2 & 3 & 4 \\
\cline{1-3}
3 \\
\cline{1-1}
\end{array}\ .$$

The tableau $S(T) \in B_4(\lambda)$ and $S(T)^{\mathrm{tr}}$ are

$$S(T) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}\,, \qquad S(T)^{\mathrm{tr}} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 1 & 3 \\ \cline{1-2} 2 & 4 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}\,.$$

Step 3 above produces

$$T' = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 2 & 4 \\ \cline{1-2} 3 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$$

in $B_5(\lambda')$. One can then check that the same procedure maps T' back to T.

Theorem 6.2. The above map $T \mapsto T'$ is a bijection between $\mathrm{LR}^{\nu}_{\lambda,\delta}$ and $\mathrm{LR}^{\nu'}_{\lambda',\delta'}$.

Proof. A bijection realizing the conjugation symmetry of the LR coefficients was given on the skew LR tableaux (of shape ν/δ and content λ) as the map ρ_3 in [1]. It is not hard to show that on the companion tableau it is described by the above algorithm. The key fact involved here is that the crystal action of the longest permutation in S_m on the skew LR tableau corresponds to the Schützenberger involution applied to the companion tableau. This fact is well known to experts and is based on the so-called "double crystal graph structure" on biwords [22]. According to this, the action of crystal operators on words corresponds to jeu de taquin slides on two-row tableaux, where the latter are involved in the construction of the Schützenberger involution; see also [1, 6] for more details. ■

Remarks 6.3.

(1) It is easy to see that, if we change m in the above construction, Step 1 is different, but the final result is the same.

(2) It was shown in [6] and [1] that the above map coincides with the maps constructed by Hanlon–Sundaram [8], White [38], and Benkart–Sottile–Stroomer [3]. In fact, Benkart–Sottile–Stroomer also give a characterization of their map based on Knuth and dual Knuth equivalences. Furthermore, the inverse of the conjugation symmetry map is described by the same procedures [8].

6.2 Kwon's model

In this section we describe Kwon's *spin model* for crystals of classical type [19, 20], which is also used to express certain branching coefficients and leads to an interesting branching duality.

We start with the Lie algebra $\mathfrak{sp}_{2n}$, with the corresponding long simple root being indexed by 0. Consider a dominant weight $\lambda \in \mathcal{P}_n$, and let $\Lambda^{\mathfrak{sp}}(\lambda) := n\Lambda_0^{\mathfrak{sp}} + \lambda'_1\varepsilon_1 + \lambda'_2\varepsilon_2 + \ldots$, where $\Lambda_0^{\mathfrak{sp}}$ is the 0-fundamental weight for $\mathfrak{sp}_\infty$. Kwon first constructs a combinatorial model for the crystal $B(\mathfrak{sp}_\infty, \Lambda^{\mathfrak{sp}}(\lambda))$, which we now briefly describe.

The model is built on a certain family $\mathbf{T}^{\mathfrak{sp}}(\lambda, n)$ formed by sequences $\mathbf{T} := C_1C_2\ldots C_{2n}$ of fillings with positive integers of column shapes. These sequences satisfy the following conditions:

(1) each pair $C_{2i-1}C_{2i}$ is a semistandard Young tableau (SSYT) of shape $(\lambda_i + \delta^{\mathrm{rev}}_{2i-1}, \delta^{\mathrm{rev}}_{2i})'$, denoted by T_i, where δ is some partition in $\mathcal{P}^{(1,1)}_{2n}$, which means that $\delta_{2i-1} = \delta_{2i}$ for $i = 1, \ldots, n$;

(2) each pair (T_i, T_{i+1}) satisfies certain compatibility conditions, see [19, Definition 3.2].

For each $i \geq 0$, Kwon defines crystal operators $\widetilde{e}_i$, $\widetilde{f}_i$ on the set of pairs of columns described in (1) above and then extends them to $\mathbf{T}^{\mathfrak{sp}}(\lambda, n)$ via the usual tensor product rule. With this structure, it is proved that $\mathbf{T}^{\mathfrak{sp}}(\lambda, n)$ is isomorphic to the crystal $B(\mathfrak{sp}_\infty, \Lambda^{\mathfrak{sp}}(\lambda))$.

Following [19], we introduce further notation related to the above objects. The left and right columns of T_i defined above are denoted by T_i^L and T_i^R, respectively. The bottom part of T_i^L of height λ_i is denoted by T_i^{tail}; the remaining top part together with T_i^R, which forms an SSYT of rectangular shape $(\delta^{\mathrm{rev}}_{2i-1}, \delta^{\mathrm{rev}}_{2i})'$, is by denoted T_i^{body}. In the filling $\mathbf{T}$ the columns are arranged such that $\mathbf{T}^{\mathrm{body}} := (T_1^{\mathrm{body}}, \ldots, T_n^{\mathrm{body}})$ is a filling of the shape $(\delta')^\pi$ denoting the rotation of δ' by 180 degrees. Kwon also uses the notation $\mathbf{T}^{\mathrm{tail}} := (T_1^{\mathrm{tail}}, \ldots, T_n^{\mathrm{tail}})$, which is a filling of the shape λ'. As usual, content$(\mathbf{T})$ is defined as the sequence $(c_1, c_2, \ldots)$, where c_i is the number of entries i in $\mathbf{T}$. We identify $\mathbf{T}$ with its column word, denoted by word$(\mathbf{T})$, which is obtained by reading the columns from right to left and from top to bottom. Let $L(\mathbf{T})$ be the maximal length of a weakly decreasing subword of word$(\mathbf{T})$.

Lemma 6.4. [19] If $L(\mathbf{T}) \leq n$, then we have

(1) $\mathbf{T}^{\mathrm{body}}$ is an SSYT of shape $(\delta')^\pi$ for some $\delta \in \mathcal{P}^{(1,1)}_{2n}$, and $\mathbf{T}^{\mathrm{tail}}$ is an SSYT of shape λ';

(2) $\mathbf{T} \equiv \mathbf{T}^{\mathrm{body}} \otimes \mathbf{T}^{\mathrm{tail}}$, where $\equiv$ denotes the usual (type A) plactic equivalence.

Now fix a partition $\nu \in \mathcal{P}_{2n}$. Consider the set $\mathrm{LR}_\nu^\lambda(\mathfrak{sp}_{2n})$ of type A highest weight elements $\mathbf{T}$ in $\mathbf{T}^{\mathfrak{sp}}(\lambda, n)$ with content($\mathbf{T}$) $= \nu'$; in other words, we have $\widetilde{e}_i(\mathbf{T}) = 0$ for all $i > 0$.

Theorem 6.5. [19] The cardinality of $\mathrm{LR}_\nu^\lambda(\mathfrak{sp}_{2n})$ is equal to the branching coefficient $c_\nu^\lambda(\mathfrak{sp}_{2n})$.

Considering $\mathbf{T}$ in $\mathrm{LR}_\nu^\lambda(\mathfrak{sp}_{2n})$, we have by definition $\mathbf{T} \equiv H_{\nu'}$. Thus, in the special case $\nu \in \mathcal{P}_n$, it follows from Lemma 6.4 that $\mathbf{T}^{\mathrm{body}} \equiv H_{\delta'}$ and $\mathbf{T}^{\mathrm{tail}} \in \mathrm{LR}_{\lambda',\delta'}^{\nu'}$, for some $\delta \in \mathcal{P}_{2n}^{(1,1)}$. Here and throughout, we use implicitly the fact that the crystal operators preserve the plactic equivalence. Based on the above facts, the following result is proved.

Theorem 6.6. [19] Assume $\nu \in \mathcal{P}_n$. The map $\mathbf{T} \mapsto \mathbf{T}^{\mathrm{tail}}$ is a bijection

$$\mathrm{LR}_\nu^\lambda(\mathfrak{sp}_{2n}) \longrightarrow \bigsqcup_{\delta \in \mathcal{P}_{2n}^{(1,1)}} \mathrm{LR}_{\lambda',\delta'}^{\nu'}.$$

As $c_{\lambda',\delta'}^{\nu'} = c_{\lambda,\delta}^{\nu}$, Theorem 6.6 gives a simple combinatorial realization of the well-known stable branching rule [12] (for $\nu \in \mathcal{P}_n$):

$$c_\nu^\lambda(\mathfrak{sp}_{2n}) = \sum_{\delta \in \mathcal{P}_{2n}^{(1,1)}} c_{\lambda,\delta}^{\nu}.$$

Without the assumption $L(\mathbf{T}) \leq n$, Lemma 6.4 fails, that is, $\mathbf{T}^{\mathrm{body}}$ and $\mathbf{T}^{\mathrm{tail}}$ are no longer SSYT of the corresponding shapes. Kwon addresses this complication in [20, Section 5], by first mapping $\mathbf{T} = C_1 C_2 \ldots C_{2n}$ to a new filling $\overline{\mathbf{T}}$. The construction is based on jeu de taquin on successive columns, which is used to perform the following operations in the indicated order:

- move λ_2 entries from column C_3 to the 2nd column;
- move λ_3 entries from column C_5 to the 3rd column (past the 4th column in between);
- continue in this fashion, and end by moving λ_n entries from column C_{2n-1} to the n-th column (past the columns in between).

It is easy to see that the above operations can always be performed. The shape of the filling $\overline{\mathbf{T}}$ is a skew Young diagram, obtained by gluing λ' to the bottom of $(\delta')^\pi$, such that their 1st columns are aligned (we view $(\delta')^\pi$ as a diagram with $2n$ columns, where possibly the leading ones have length 0). The fillings of shapes λ' and $(\delta')^\pi$ are denoted by $\overline{\mathbf{T}}^{\text{tail}}$ and $\overline{\mathbf{T}}^{\text{body}}$, respectively. We have an analog of Lemma 6.4.

Lemma 6.7. [20] The following hold:

(1) $\overline{\mathbf{T}}^{\text{body}}$ is an SSYT of shape $(\delta')^\pi$ for some $\delta \in \mathcal{P}_{2n}^{(1,1)}$, and $\overline{\mathbf{T}}^{\text{tail}}$ is an SSYT of shape λ';

(2) $\mathbf{T} \equiv \overline{\mathbf{T}} \equiv \overline{\mathbf{T}}^{\text{body}} \otimes \overline{\mathbf{T}}^{\text{tail}}$.

The difficulty lies in the 1st part of this lemma, whose proof is highly technical. The 2nd part follows from the 1st one simply by noting that jeu de taquin is compatible with the plactic equivalence and that the row and column words of a skew SSYT are placticly equivalent.

In [20, Remark 5.6] it is observed that, if $L(\mathbf{T}) \leq n$ (in particular, if $\mathbf{T} \in \mathrm{LR}_\nu^\lambda(\mathfrak{sp}_{2n})$ and $\nu \in \mathcal{P}_n$), then we have $\overline{\mathbf{T}}^{\text{body}} = \mathbf{T}^{\text{body}}$ and $\overline{\mathbf{T}}^{\text{tail}} = \mathbf{T}^{\text{tail}}$, so Lemma 6.4 is a special case of Lemma 6.7. In fact, we can show that the mentioned equalities also hold for the elements of $\mathrm{LR}_\nu^\lambda(\mathfrak{sp}_{2n})$, for any $\nu \in \mathcal{P}_{2n}$. This leads to the following generalization of Theorem 6.6. To state it, we define $\overline{\mathrm{LR}}_{\lambda',\delta'}^{\nu'}$ to be the subset of $\mathrm{LR}_{\lambda',\delta'}^{\nu'}$ consisting of fillings S with the following property: denoting the 1st row of S by $(r_1 \leq \ldots \leq r_p)$, for $p \leq n$, we have

$$r_i > \delta_{2i-1}^{\text{rev}} = \delta_{2i}^{\text{rev}} \qquad \text{for } i = 1, \ldots, p. \tag{8}$$

Let $\overline{c}_{\lambda,\delta}^{\nu}$ be the cardinality of $\overline{\mathrm{LR}}_{\lambda',\delta'}^{\nu'}$.

Theorem 6.8. Consider $\mathbf{T}$ in $\mathrm{LR}_\nu^\lambda(\mathfrak{sp}_{2n})$, and let $(\delta')^\pi$ be the shape of $\mathbf{T}^{\text{body}}$.

(1) We have $\mathbf{T}^{\text{body}} \equiv H_{\delta'}$ and $\mathbf{T}^{\text{tail}} \in \overline{\mathrm{LR}}_{\lambda',\delta'}^{\nu'}$.

(2) The map $\mathbf{T} \mapsto \mathbf{T}^{\text{tail}}$ is an injection

$$\mathrm{LR}_\nu^\lambda(\mathfrak{sp}_{2n}) \hookrightarrow \bigsqcup_{\delta \in \mathcal{P}_{2n}^{(1,1)}} \mathrm{LR}_{\lambda',\delta'}^{\nu'},$$

and its image is $\bigsqcup_{\delta \in \mathcal{P}_{2n}^{(1,1)}} \overline{\mathrm{LR}}_{\lambda',\delta'}^{\nu'}$.

Proof. Consider the filling $\overline{\mathbf{T}}$ obtained from $\mathbf{T}$ via the procedure described above. Since $\mathbf{T} \equiv H_{\nu'}$, it follows that $\overline{\mathbf{T}}^{\text{body}} \equiv H_{\delta'}$ and $\overline{\mathbf{T}}^{\text{tail}} \in \mathrm{LR}_{\lambda',\delta'}^{\nu'}$, by Lemma 6.7. Thus, the i-th column of the SSYT $\overline{\mathbf{T}}^{\text{body}}$ is $(1 < 2 < \ldots < \delta_i^{\text{rev}})$, for $i = 1, \ldots, 2n$.

The procedure $\mathbf{T} \mapsto \overline{\mathbf{T}}$, which is based on jeu de taquin on successive columns, is reversible. We claim that this reverse procedure $\overline{\mathbf{T}} \mapsto \mathbf{T}$ simply slides the columns of $\overline{\mathbf{T}}^{\text{tail}}$ horizontally (i.e., restricts to horizontal jeu de taquin moves) from positions $1, \ldots, n$ within $\overline{\mathbf{T}}$ to positions $1, 3, \ldots, 2n-1$, respectively, while the columns of $\overline{\mathbf{T}}^{\text{body}}$ do not move (recall that the columns of $\overline{\mathbf{T}}^{\text{tail}}$ and $\mathbf{T}^{\text{tail}}$ within $\overline{\mathbf{T}}$ and $\mathbf{T}$ have their top entries on the same row). This means that $\overline{\mathbf{T}}^{\text{body}} = \mathbf{T}^{\text{body}}$ and $\overline{\mathbf{T}}^{\text{tail}} = \mathbf{T}^{\text{tail}}$. Therefore, the map $\mathbf{T} \mapsto \mathbf{T}^{\text{tail}}$ is the desired injection. Moreover, the image of this map is contained in $\bigsqcup_{\delta \in \mathcal{P}_{2n}^{(1,1)}} \overline{\mathrm{LR}}_{\lambda', \delta'}^{\nu'}$ because the columns of $\mathbf{T}$ are strictly increasing.

The proof of the above claim is based on the following fact. Consider columns $(1 < 2 < \ldots < k < c_1 < \ldots < c_s)$ and $(1 < 2 < \ldots < l)$ with $k \leq l$, and assume that we can move s entries from the 1st one to the 2nd one via jeu de taquin. To do this, we start by aligning the two columns such that they form a skew SSYT, and this can be done by placing $k \leq l$ in the same row. We claim that $c_1 > l$, which implies that the resulting columns are $(1 < 2 < \ldots < k)$ and $(1 < 2 < \ldots < l < c_1 < \ldots < c_s)$, as needed. Indeed, if $c_1 \leq l$, then $k \leq l-1$, $k-1 \leq l-2$, etc., so we can align the two initial columns such that all the mentioned pairs are in the same rows. But then at most $s-1$ entries can move from the 1st column to the 2nd one, which is a contradiction.

It remains to prove that any filling $S \in \overline{\mathrm{LR}}_{\lambda', \delta'}^{\nu'}$ is in the image of the given map. Consider the SSYT whose i-th column is $(1 < 2 < \ldots < \delta_i^{\text{rev}})$, and glue the columns of S to the bottom of the columns of the former in positions $1, 3, 5, \ldots$. It is easy to check that the resulting filling $\mathbf{T}$ satisfies the conditions in [19, Definition 3.2], so $\mathbf{T} \in \mathbf{T}^{\mathfrak{sp}}(\lambda, n)$. Now observe that the procedure $\mathbf{T} \mapsto \overline{\mathbf{T}}$ consists of sliding the columns of S within $\mathbf{T}$ horizontally, as far left as possible, which means that $\overline{\mathbf{T}}^{\text{body}} = \mathbf{T}^{\text{body}}$ and $\overline{\mathbf{T}}^{\text{tail}} = \mathbf{T}^{\text{tail}}$. By Lemma 6.7 (2), it follows that $\mathbf{T} \equiv H_{\delta'} \otimes S$. The latter is a highest weight element, as $S \in \mathrm{LR}_{\lambda', \delta'}^{\nu'}$, and this implies $\mathbf{T} \in \mathrm{LR}_{\nu}^{\lambda}(\mathfrak{sp}_{2n})$. ■

By combining Theorems 6.5 and 6.8, we obtain a simple combinatorial description of the branching coefficient $c_{\nu}^{\lambda}(\mathfrak{sp}_{2n})$ in full generality.

Corollary 6.9. We have

$$c_{\nu}^{\lambda}(\mathfrak{sp}_{2n}) = \sum_{\delta \in \mathcal{P}_{2n}^{(1,1)}} \overline{c}_{\lambda, \delta}^{\nu}.$$

6.3 Generalized exponents in terms of distinguished tableaux

The goal is to derive a finite rank analog of the results in Section 4.3, that is, for type C_n.

We use the same notation, except that everything now happens in finite rank. Thus, we require $\lambda \in \mathcal{P}_n$. We denote the underlying type A_{2n-1} crystal by $B_{2n}(\lambda)$, and the set of distinguished tableaux contained in it by $D_{2n}(\lambda)$. The latter is defined like in Definition 4.3 and is characterized by the analogs of the two conditions in Proposition 4.5. Recall the set S_b, whose analog is defined for any $b \in B_{2n}(\lambda)$ by

$$S_{b,n} := \left\{\mu \in \mathcal{P}_{2n} \,:\, \boldsymbol{\varphi}(b) + \mu \in \mathcal{P}_{2n}^{(2)}\,,\ \boldsymbol{\varepsilon}(b) + \mu \in \mathcal{P}_{2n}^{(1,1)}\right\}.$$

Note that the above conditions on μ simply mean that

$$b \in \mathrm{LR}^{\nu}_{\lambda,\delta} \quad \text{for } \delta := \boldsymbol{\varepsilon}(b) + \mu \in \mathcal{P}_{2n}^{(1,1)},\ \nu := \boldsymbol{\varphi}(b) + \mu \in \mathcal{P}_{2n}^{(2)}.$$

The analog of the weight μ_b, denoted $\mu_{b,n}$, is constructed as in Proposition 4.5 (2)

$$\mu_{b,n} := \sum_{i=1}^{n-1} (\varphi_{2i}(b) \mod 2)\, \omega_{2i}\,. \tag{9}$$

With this notation, we have the analog of Lemma 4.4, namely,

$$S_{b,n} := \begin{cases} \mu_{b,n} + \mathcal{P}_{2n}^{\boxplus} & \text{if } b \in D_{2n}(\lambda) \\ \emptyset & \text{otherwise}\,. \end{cases} \tag{10}$$

We also need some new notation. Let $D^*_{2n}(\lambda)$ be defined by "swapping" the conditions characterizing $D_{2n}(\lambda)$ in Proposition 4.5; namely, $D^*_{2n}(\lambda)$ consists of $b \in B_{2n}(\lambda)$ such that

(C1) $\varphi_i(b) = 0$ for any odd i;

(C2) $\varepsilon_i(b)$ is even for any odd i.

Let $\overline{D}^*_{2n}(\lambda)$ be the subset of $D^*_{2n}(\lambda)$ consisting of those SSYT satisfying the following flag condition:

(C3) the entries in row i are at least $2i-1$, for $i = 1, \ldots, n$.

Finally, we define the analogs of $\boldsymbol{\varepsilon}(b)$, $\boldsymbol{\varphi}(b)$, and of $\mu_{b,n}$ in (9) by

$$\boldsymbol{\varepsilon}^*(b) := \sum_{i=1}^{2n-1} \varepsilon_{2n-i}(b)\,\omega_i\,, \quad \boldsymbol{\varphi}^*(b) := \sum_{i=1}^{2n-1} \varphi_{2n-i}(b)\,\omega_i\,, \quad \mu^*_{b,n} := \sum_{i=1}^{n-1} (\varepsilon_{2n-2i}(b) \mod 2)\,\omega_{2i}.$$

Now recall Lusztig's involution S on the crystal $B_{2n}(\lambda)$. This is realized by Schützenberger's evacuation [6] and is known to commute with the crystal operators as follows:

$$\widetilde{e}_i S = S\widetilde{f}_{2n-i}\,, \qquad \widetilde{f}_i S = S\widetilde{e}_{2n-i}. \tag{11}$$

It is then clear that S maps $D_{2n}(\lambda)$ to $D^*_{2n}(\lambda)$. It also follows that we have

$$\varepsilon_i(S(b)) = \varphi_{2n-i}(b)\,, \qquad \varphi_i(S(b)) = \varepsilon_{2n-i}(b), \tag{12}$$

and therefore

$$\boldsymbol{\varepsilon}(b) = \boldsymbol{\varphi}^*(S(b))\,, \qquad \boldsymbol{\varphi}(b) = \boldsymbol{\varepsilon}^*(S(b))\,, \qquad \mu_{b,n} = \mu^*_{S(b),n}. \tag{13}$$

We start with the analog of Proposition 4.6.

Theorem 6.10. We have

$$\sum_{\nu\in\mathcal{P}^{(2)}_{2n}}\ \sum_{\delta\in\mathcal{P}^{(1,1)}_{2n}} t^{|\nu|/2}\,\overline{c}^{\nu}_{\lambda,\delta} = \sum_{b\in\overline{D}^*_{2n}(\lambda)} t^{\left|\boldsymbol{\varepsilon}^*(b)+\mu^*_{b,n}\right|/2} \sum_{\kappa\in\mathcal{P}^{\boxplus}_{2n}} t^{|\kappa|/2}.$$

The proof of this theorem is based on the following lemma. To state it, let us recall the LR conjugation symmetry map in Section 6.1. Following the notation used there, we set $m = 2n$, and given fixed λ we denote by σ_δ the bijection from $\mathrm{LR}^{\nu}_{\lambda,\delta}$ to $\mathrm{LR}^{\nu'}_{\lambda',\delta'}$; note that this map uses δ in a crucial way, in Step 3 of its construction.

Lemma 6.11. Consider $b \in \mathrm{LR}^{\nu}_{\lambda,\delta}$ with $\delta \in \mathcal{P}^{(1,1)}_{2n}$. The SSYT $\sigma_\delta(b)$ satisfies condition (8) with respect to δ if and only if $S(b)$ satisfies condition (C3). So in fact, the 1st condition is independent of δ.

Proof. Let us denote the 1st column of $S(b)$ by $(c_1 < \ldots < c_p)$, where $p \le n$. By the construction of the map σ_δ in Section 6.1, condition (8) for $\sigma_\delta(b)$ simply means

$$\delta^{\mathrm{rev}}_{c_1} \ge \delta^{\mathrm{rev}}_1 = \delta^{\mathrm{rev}}_2\,,\ \ldots\,,\ \delta^{\mathrm{rev}}_{c_p} \ge \delta^{\mathrm{rev}}_{2p-1} = \delta^{\mathrm{rev}}_{2p}\,.$$

We need to show that this is equivalent to

$$c_1 \geq 1\,,\ \ldots\,,\ c_p \geq 2p-1.$$

The implication ($\Leftarrow$) is clear since $\delta^{\mathrm{rev}} = (\delta^{\mathrm{rev}}_1 = \delta^{\mathrm{rev}}_2 \leq \delta^{\mathrm{rev}}_3 = \delta^{\mathrm{rev}}_4 \leq \ldots)$, while ($\Rightarrow$) is only clear if the weak inequalities defining δ are strict.

Assuming that ($\Rightarrow$) fails, pick the largest i such that $\delta^{\mathrm{rev}}_{c_i} = \delta^{\mathrm{rev}}_{2i-1}$ and $c_i < 2i-1$, where clearly $i \geq 2$; we call such an index i bad. Let us assume first that $c_i = 2i-2$, so $\delta^{\mathrm{rev}}_{2i-2} = \delta^{\mathrm{rev}}_{2i-1}$. Since $b \in \mathrm{LR}^{\nu}_{\lambda,\delta}$, we have $\boldsymbol{\varepsilon}(b) \leq \delta$, so by (13) we deduce $\varphi_{2i-2}(S(b)) = 0$. This rules out $i = p$, as well as $i < p$ and $c_{i+1} \geq 2i$, because in these cases $\widetilde{f}_{2i-2}(S(b)) \neq 0$, by the usual bracketing rule for crystal operators, see for example, [11]. It follows that $c_{i+1} = 2i-1$, but this contradicts $c_{i+1} \geq 2(i+1)-1$, which holds by the maximality of i. Thus, we must have $c_i \leq 2i-3$.

Assuming $i > 2$, the index $i-1$ must also be bad, because otherwise we would have

$$2(i-1)-1 \leq c_{i-1} < c_i \leq 2i-3\,.$$

By repeating the above argument with i replaced by $i-1$, we deduce $c_{i-1} \leq 2i-5$. We repeat the previous reasoning for the indices $i-2, i-3, \ldots, 2$ and conclude $c_2 \leq 1$. This leads to the contradiction $1 \leq c_1 < c_2 \leq 1$, which concludes the proof. ■

Proof of Theorem 6.10. We define the following subset of $S_{b,n}$:

$$\overline{S}_{b,n} := \{\mu \in S_{b,n}\ :\ \sigma_\delta(b) \text{ satisfies (8) with respect to } \delta\}\,, \quad \text{where } \delta := \boldsymbol{\varepsilon}(b) + \mu.$$

Letting

$$\overline{D}_{2n}(\lambda) := \{b \in D_{2n}(\lambda)\ :\ S(b) \text{ satisfies condition (C3)}\}\,,$$

we observe that its image under S is precisely $\overline{D}^{*}_{2n}(\lambda)$. By (10) and Lemma 6.11, we have

$$\overline{S}_{b,n} := \begin{cases} \mu_{b,n} + \mathcal{P}^{\boxplus}_{2n} & \text{if } b \in \overline{D}_{2n}(\lambda) \\ \emptyset & \text{otherwise}\,. \end{cases} \tag{14}$$

We now follow the approach in the proof of Proposition 4.6. This gives

$$\sum_{\nu\in\mathcal{P}_{2n}^{(2)}}\sum_{\delta\in\mathcal{P}_{2n}^{(1,1)}} t^{|\nu|/2}\,\overline{c}_{\lambda,\delta}^{\nu} = \sum_{b\in D_{2n}(\lambda)}\sum_{\mu\in\overline{S}_{b,n}} t^{|\varphi(b)+\mu|/2}$$
$$= \sum_{b\in\overline{D}_{2n}(\lambda)} t^{|\varphi(b)+\mu_{b,n}|/2}\sum_{\kappa\in\mathcal{P}_{2n}^{\boxplus}} t^{|\kappa|/2}$$
$$= \sum_{b\in\overline{D}_{2n}^{*}(\lambda)} t^{\left|\varepsilon^{*}(b)+\mu_{b,n}^{*}\right|/2}\sum_{\kappa\in\mathcal{P}_{2n}^{\boxplus}} t^{|\kappa|/2}.$$

Here the 2nd equality follows from (14), while the 3rd one follows by translating all the parameters from $\overline{D}_{2n}(\lambda)$ to $\overline{D}_{2n}^{*}(\lambda)$ via (13). ■

We now derive the analog of Theorem 4.7, and also of Theorem 3.1 in type A. Observe first we can write more explicitely for any vertex $b\in B_{2n}(\lambda)$

$$\left|\varepsilon^{*}(b)+\mu_{b,n}^{*}\right|/2 = \sum_{i=1}^{2n-1}(2n-i)\left\lceil\frac{\varepsilon_i(b)}{2}\right\rceil.$$

Theorem 6.12. We have

$$K_{\lambda,0}^{C_n}(t) = \sum_{b\in\overline{D}_{2n}^{*}(\lambda)} t^{\mathrm{ch}_{C_n}(b)},$$

where

$$\mathrm{ch}_{C_n}(b) = \sum_{i=1}^{2n-1}(2n-i)\left\lceil\frac{\varepsilon_i(b)}{2}\right\rceil.$$

Proof. The proof is immediately based on Corollary 6.9 and Proposition 2.1 (3). Indeed, it suffices to observe that

$$\sum_{\kappa\in\mathcal{P}_{2n}^{\boxplus}} t^{|\kappa|/2} = \frac{1}{\prod_{i=1}^{n}(1-t^{2i})}.$$

■

6.4 From distinguished tableaux to King tableaux

We follow a similar approach to that in Section 4.4. The goal is to transfer the results to a natural labeling of the vertices of weight 0 in the type C_n crystal of highest weight λ, via a bijection with $\overline{D}_{2n}^{*}(\lambda)$. Such a natural labeling is given by the King tableaux of weight 0 [18]. Recall that the King tableaux of type C_n are just semistandard tableaux of shape λ in the alphabet $\{1 < \overline{1} < 2 < \overline{2} < \ldots < n < \overline{n}\}$, with the additional flag condition that the entries in each row i are greater than or equal to i. The set of such tableaux of weight 0 will be denoted by $K^0_{C_n}(\lambda)$.

Consider a tableau b in $\overline{D}^*_{2n}(\lambda)$, and let $N_i(b)$ denote the number of entries equal to i. Note first that conditions (C1) and (C2) in Section 6.3 can be phrased as the following more explicit ones, for $i = 1, \ldots, n$:

(C1′) the subword of the Japanese reading of the tableau b formed by $2i-1$ and $2i$ has the property that in each right factor the number of $2i-1$ is less than or equal to the number of $2i$;

(C2′) $N_{2i}(b) - N_{2i-1}(b)$ is a (nonnegative) even integer.

Condition (C2) is also equivalent to the fact that the rows of $\boldsymbol{\theta}^*_n(b) := \boldsymbol{\varepsilon}^*(b) + \mu^*_{b,n}$ have even lengths.

Given b as above, we will map it to a King tableau in $K^0_{C_n}(\lambda)$. Letting $k_i := N_{2i}(b) - N_{2i-1}(b)$, we apply the crystal operator $\widetilde{e}^{k_i/2}_{2i-1}$ to b, for $i = 1, \ldots, n$. Note that these operators commute, and in fact they correspond to a $U_q(\mathfrak{sl}_2 \oplus \ldots \oplus \mathfrak{sl}_2)$-crystal structure, cf. Section 4.4. Afterwards, we replace the entries $2i-1$ and $2i$ with i and $\overline{\imath}$, respectively, for each i. It is easy to see that the resulting filling has weight 0 and that the flag condition (C3) turns into the similar condition for King tableaux. So the result is in $K^0_{C_n}(\lambda)$.

Moreover, this map has an inverse. Indeed, given a King tableau T, we first replace the entries i and $\overline{\imath}$ with $2i-1$ and $2i$, respectively. Then we map the resulting filling to the lowest weight element with respect to the corresponding $U_q(\mathfrak{sl}_2 \oplus \ldots \oplus \mathfrak{sl}_2)$-crystal structure. It is easy to see that the resulting filling is in $\overline{D}^*_{2n}(\lambda)$. For obvious reasons, we denote this map by $T \mapsto L(T)$.

Based on the above discussion, Theorem 6.12 can be rephrased as follows.

Theorem 6.13. We have

$$K^{C_n}_{\lambda,0}(t) = \sum_{T \in K^0_{C_n}(\lambda)} t^{\mathrm{ch}_{C_n}(L(T))},$$

where

$$\mathrm{ch}_{C_n}(L(T)) = \sum_{i=1}^{2n-1} (2n-i) \left\lceil \frac{\varepsilon_i(L(T))}{2} \right\rceil .$$

Remarks 6.14.

(1) As noted in Remark 4.13 (1), there does not seem to be a simple way to express the related statistic above directly in terms of T. However, the map $T \mapsto L(T)$ is a simple one.

(2) Theorem 6.13 shows that it is more natural to define a statistic for computing the Kostka–Foulkes polynomial on King tableaux, rather than on the other important set of symplectic tableaux, namely the KN tableaux [11]. A natural question is whether the statistic above can be translated to the KN tableaux via the bijection in [33] and moreover if one recovers in this way the charge statistic constructed in [25] (which conjecturally computes the Kostka–Foulkes polynomials); we will be investigating this question in the future.

We have the following analog of Theorem 4.8, cf. also Remark 4.13, related to the expression of the multivariable generalization of $K^{C_n}_{\lambda,0}(t)$, denoted $K^{C_n}_{\lambda,0}(\boldsymbol{t})$. Like in the infinite case, the related combinatorial expression follows immediately from the (finite type) combinatorics worked out above. Note that the discrepancy mentioned in Assertion 2 of Remark 4.13 has now been corrected by passing from the set of distinguished tableaux $D_{2n}(\lambda)$ to its image $D^*_{2n}(\lambda)$ under the Schützenberger involution.

Theorem 6.15. Define the multivariable polynomial $K^{C_n}_{\lambda,0}(\boldsymbol{t})$ by

$$\frac{K^{C_n}_{\lambda,0}(\boldsymbol{t})}{\prod_{i=1}^{n}(1-t_{2i})} = \sum_{\nu\in\mathcal{P}^{(2)}_{2n}} \sum_{\delta\in\mathcal{P}^{(1,1)}_{2n}} \boldsymbol{t}^{\frac{1}{2}\nu} c^{\nu}_{\lambda,\delta}.$$

Then we have

$$K^{C_n}_{\lambda,0}(\boldsymbol{t}) = \sum_{T\in K^0_{C_n}(\lambda)} \boldsymbol{t}^{\boldsymbol{\theta}^*_n(L(T))/2},$$

where

$$\boldsymbol{t}^{\boldsymbol{\theta}^*_n(L(T))/2} = \prod_{i=1}^{2n-1} t_{2n-i}^{\lceil \varepsilon_i(L(T))/2\rceil}.$$

We will now continue Example 4.14.

Example 6.16. Assume $\lambda = (1, 1)$ in type C_n. Then we get

$$K^0_{C_n}(\lambda) = \left\{ \begin{array}{|c|}\hline k \\ \hline \bar{k} \\ \hline \end{array} \mid k = 2, \ldots, n \right\}.$$

This gives

$$L\left(\begin{array}{|c|}\hline k \\ \hline \bar{k} \\ \hline \end{array}\right) = \begin{array}{|c|}\hline 2k-1 \\ \hline 2k \\ \hline \end{array} \quad \text{and} \quad \boldsymbol{\varepsilon}^*\left(\begin{array}{|c|}\hline 2k-1 \\ \hline 2k \\ \hline \end{array}\right) = \omega_{2(n-k+1)} \text{ for any } k = 2, \ldots, n.$$

Therefore,

$$\boldsymbol{\theta}^*_n\left(\begin{array}{|c|}\hline 2k-1 \\ \hline 2k \\ \hline \end{array}\right) = 2\omega_{2(n-k+1)} \text{ for any } k = 2, \ldots, n.$$

Finally,

$$K^{C_n}_{\lambda,0}(\mathbf{t}) = \sum_{k=2}^{n} t_{2(n-k+1)} = \sum_{k=1}^{n-1} t_{2k} \quad \text{and} \quad K^{C_n}_{\lambda,0}(t) = \sum_{k=1}^{n-1} t^{2k} = \frac{t^2 - t^{2n}}{1 - t^2}.$$

7 Three Applications

In this section, we present three applications of Theorem 6.13

7.1 Growth of generalized exponents

First we analyze the growth of the generalized exponents of type C_n with respect to the rank n.

The (weight 0) symplectic King tableaux of type C_n embed into those of type C_{n+1} by changing the entries $k, \bar{k}$ to $k+1, \overline{k+1}$, for all k, respectively. Moreover, it is easy to see that this map preserves the statistic in Theorem 6.13. So we obtain the following result, which to our knowledge is new.

Theorem 7.1. For any integer n and any partition λ with at most n parts, we have $K^{C_{n+1}}_{\lambda,0}(t) - K^{C_n}_{\lambda,0}(t) \in \mathbb{Z}_{\geq 0}[t]$.

7.2 Reducing a type C generalized exponent to one of type A

We now prove a conjecture of the 1st author [25]. This conjecture is the 1st step in the construction of the type C_n charge statistic in [25] and proves the conjecture that this charge computes the corresponding Kostka–Foulkes polynomials in the case of column shapes; see Remark 6.14 (2).

We now label the Dynkin diagram of type C_n such that the special node is n. Consider the fundamental weight ω_{2p}, where $p \in \{1,\ldots,\lfloor n/2 \rfloor\}$. All the zero-weight vertices in the crystal $B(\omega_{2p})$ belong to the same type A_{n-1} component, which has highest weight $\gamma_p := \varepsilon_1 + \ldots + \varepsilon_p - \varepsilon_{n-p+1} - \ldots - \varepsilon_n$, where ε_i are the coordinate vectors in $\mathbb{R}^n$. In type A_{n-1}, this weight corresponds to the partition $(1^{n-2p}, 2^p)$.

Theorem 7.2. We have

$$K^{C_n}_{\omega_{2p},0}(t) = K^{A_{n-1}}_{\gamma_p,0}(t^2).$$

Before proving this theorem, we need to describe the KN tableaux for some column shape (1^k) [13], which index the vertices of the type C_n crystal $B(\omega_k)$.

Definition 7.3. A column-strict filling $C = (c_1 < \ldots < c_k)$ with entries in $\{1 < \ldots < n < \overline{n} < \ldots < \overline{1}\}$ is a KN column if there is no pair $(z, \overline{z})$ of letters in C such that

$$z = c_p\,, \qquad \overline{z} = c_q\,, \qquad q - p \leq k - z.$$

We will need a different definition of KN columns, which was proved to be equivalent to the one above in [33].

Definition 7.4. Let C be a column and $I = \{x_1 > \ldots > x_r\}$ the set of unbarred letters z such that the pair $(z, \overline{z})$ occurs in C. The column C can be split when there exists a set of r unbarred letters $J = \{y_1 > \ldots > y_r\} \subset \{1,\ldots,n\}$ such that

- y_1 is the greatest letter in $\{1,\ldots,n\}$ satisfying $y_1 < x_1$, $y_1 \notin C$, and $\overline{y_1} \notin C$,
- for $i = 2,\ldots,r$, the letter y_i is the greatest one in $\{1,\ldots,n\}$ satisfying $y_i < \min(y_{i-1}, x_i)$, $y_i \notin C$, and $\overline{y_i} \notin C$.

In this case, we say that x_i is paired with y_i, and we write

- lC for the column obtained by changing x_i into y_i in C for each letter $x_i \in I$, and by reordering if necessary;
- rC for the column obtained by changing $\overline{x_i}$ into $\overline{y_i}$ in C for each letter $x_i \in I$, and by reordering if necessary.

The pair (lC, rC) will be called a split column.

Example 7.5. The following is a KN column of height 5 in type C_n for $n \geq 5$, together with the corresponding split column

$$C = \begin{array}{|c|}\hline 4 \\ \hline 5 \\ \hline \overline{5} \\ \hline \overline{4} \\ \hline \overline{3} \\ \hline \end{array}\,, \qquad (lC, rC) = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline \overline{5} & \overline{3} \\ \hline \overline{4} & \overline{2} \\ \hline \overline{3} & \overline{1} \\ \hline \end{array}\,.$$

We used the fact that $I = \{5 > 4\}$, so $J = \{2 > 1\}$.

For the definition of the crystal operators on KN columns via the well-known bracketing rule, we refer to [11].

Proof of Theorem 7.2. We use the King tableaux for computing $K^{C_n}_{\omega_{2p},0}(t)$ via Theorem 6.13. Meanwhile, $K^{A_{n-1}}_{\gamma_p,0}(t)$ is computed based on an analog of Theorem 3.1, namely

$$K^{A_{n-1}}_{\lambda,0}(t) = \sum_{b \in B(\lambda)_0} t^{\sum_{i=1}^{n-1}(n-i)\varepsilon_i(b)}, \tag{15}$$

which is referred to [23]. For this computation, we use the crystal structure on the type A_{n-1} component of highest weight γ_p of $B(\omega_{2p})$, which contains the zero-weight KN tableaux.

First we need a bijection between the zero-weight King tableaux and KN tableaux of shape (1^{2p}). Let $C_K = (c_1 < \overline{c_1} < \ldots < c_p < \overline{c_p})$ be such a King tableau, which means that $c_i \geq 2i-1$ and $\overline{c_i} \geq 2i$, for $i = 1, \ldots, p$; but these conditions are equivalent to $c_i \geq 2i$. Let $C_{KN} = (d_1 < \ldots < d_p < \overline{d_p} < \ldots < \overline{d_1})$ be a zero-weight KN column, where we note the different order used on the alphabet $\{1, \ldots, n, \overline{n}, \ldots, \overline{1}\}$. The condition in Definition 7.4 implies that $d_i \geq 2i$ for any i, because $d_1, \ldots d_i$ need to be paired with distinct entries strictly less than d_i, which are also different from $d_1, \ldots, d_{i-1}$. One can check that the reciprocal is also true. Thus, the desired bijection maps C_K to C_{KN} with $d_i = c_i$, which we now assume.

Now let us calculate the exponent of the variable t corresponding to C_K in $K^{C_n}_{\omega_{2p},0}(t)$, as given by Theorem 6.13. First we replace c_i by $2c_i - 1$ and $\overline{c_i}$ by $2c_i$, obtaining a column C'_K. Note that this is both a highest and lowest weight element with respect to the corresponding $U_q(\mathfrak{sl}_2 \oplus \ldots \oplus \mathfrak{sl}_2)$-crystal structure, so $L(C_K) = C'_K$.

Let $P := \{c_i \in C_K \mid c_i - 1 \notin C_K\}$. Note that the only type A raising crystal operators that can be applied to C'_K are $\widetilde{e}_{2p-2}$ for $p \in P$, and each can be applied only once. Thus, for each $p \in P$, we get a contribution of $2(n-p+1)$ to the mentioned exponent of t.

Finally, let us calculate the exponent of t corresponding to C_{KN} in $K^{A_{n-1}}_{\gamma_p,0}(t)$, as mentioned above, based on (15). Let $C^+_{KN} := (c_1 < \ldots < c_p)$. Observe first that

$$\varepsilon_{p-1}(C_{KN}) = \varepsilon_{p-1}(C^+_{KN}) = \begin{cases} 1 & \text{if } p \in P \\ 0 & \text{otherwise}. \end{cases}$$

This means that, for each $p \in P$, we get a contribution of $n-p+1$ to the mentioned exponent of t. This concludes the proof. ■

Remark 7.6. Theorem 7.2 also permits us to establish the conjecture of [25] for Lusztig t-analogs associated to any fundamental weight. Indeed, each such fundamental weight is indexed by a column partition $\lambda = (1^k) = \omega_k$ with $k \leq n$ and the possible corresponding dominant weights yielding nonzero polynomials have the form $\mu = (1^a) = \omega_a$ where $k-a$ is a nonnegative even integer. We then have

$$K^{C_n}_{\omega_k,\omega_a}(t) = K^{C_{n-a}}_{\omega_{k-a},0}(t).$$

This follows in fact from a more general row removal property of Lusztig t-analogs of type C_n. Assume that λ and μ are two partitions such that $\lambda_1 = \mu_1$ then

$$K^{C_n}_{\lambda,\mu}(t) = K^{C_{n-1}}_{\lambda^\flat,\mu^\flat}(t),$$

where $\lambda^\flat$ and $\mu^\flat$ are the partitions obtained by removing the part $\lambda_1 = \mu_1$ in λ and μ, respectively. This can be proved directly from the very definition of $K^{C_n}_{\lambda,\mu}(t)$ in terms of partition function or by using the Morris type recurrence formula established in [25].

7.3 The smallest power of t in $K^{C_n}_{\lambda,0}(t)$

The largest power of t in $K^{C_n}_{\lambda,0}(t)$ is well known to be $\langle \lambda, \rho^\vee \rangle$, where $\rho^\vee$ is half the sum of the positive coroots. Furthermore, it is also known that the smallest power is greater than or equal to $|\lambda|/2$. See [26, 27]. As the 3rd application of our formula for $K^{C_n}_{\lambda,0}(t)$, we will determine this smallest power.

Let $\lambda \in \mathcal{P}_n$ be such that $|\lambda|$ is even, and write $\lambda = \sum_{i=1}^{n} a_i \, \omega_{n+1-i}$. Define

$$s_k := \sum_{i=1}^{k} a_i \,, \qquad b_i := \begin{cases} a_i + 1 & \text{if } a_i \text{ odd and } s_i \text{ odd} \\ a_i - 1 & \text{if } a_i \text{ odd and } s_i \text{ even} \\ a_i & \text{if } a_i \text{ even} \,. \end{cases}$$

Also let $s_0 := 0$ and $S := s_n$.

Theorem 7.7. The smallest power of t in $K^{C_n}_{\lambda,0}(t)$, for $|\lambda|$ even, is

$$\frac{1}{2} \sum_{i=1}^{n} (n+1-i) b_i = \frac{|\lambda|}{2} + \frac{1}{2} \sum_{i \,:\, a_i \text{ odd}} (-1)^{s_i - 1} (n+1-i). \tag{16}$$

We start by sketching the idea of the proof, whose details can be found in the next section. Based on Theorem 6.12, we need to find the filling $\sigma \in \overline{D}^*_{2n}(\lambda)$ that minimizes

$$\mathrm{ch}_{C_n}(\sigma) = \sum_{i=1}^{2n-1} (2n-i) \left\lceil \frac{\varepsilon_i(\sigma)}{2} \right\rceil .$$

We will first minimize $\mathrm{ch}_{C_n}(\sigma)$ for fillings σ of the row shape (S) with $1, \ldots, 2n$, subject to certain conditions. Namely, let Σ be the set of all $\sigma = (\sigma_1 \leq \ldots \leq \sigma_S)$ satisfying

$$\sigma_i \leq n + k \,, \qquad \text{for } s_{k-1} < i \leq s_k, \text{ and } k = 1, \ldots, n \,. \tag{17}$$

Note that this condition is a necessary one for the 1st row of a filling of λ with $1, \ldots, 2n$. Let us also define the sequence $c_1, \ldots, c_n$ by setting $c_i := b_i$, except for the case in which, for the largest i with a_i odd, we have s_i odd, in which case $c_i := a_i$ (and $b_i := a_i + 1$). Note that $a_1 + \ldots + a_n = c_1 + \ldots + c_n = S$.

Lemma 7.8. We have

$$\min_{\sigma \in \Sigma} \mathrm{ch}_{C_n}(\sigma) = \frac{1}{2} \sum_{i=1}^{n} (n+1-i) b_i,$$

and the minimum is attained for $\sigma^{\mathrm{row}}_{\min} := ((n+1)^{c_1} (n+2)^{c_2} \ldots (2n)^{c_n})$.

Now consider $\sigma \in \overline{D}^*_{2n}(\lambda)$. By the usual bracketing rule for crystal operators, see for example [11], it is easy to see that all entries $i \geq 2$ in the 1st row of σ contribute to $\varepsilon_{i-1}(\sigma)$. Thus, it suffices to construct $\sigma_{\min} \in \overline{D}^*_{2n}(\lambda)$ whose 1st row is $\sigma^{\mathrm{row}}_{\min}$, and for which no entry i below the 1st row contributes to $\varepsilon_{i-1}(\sigma_{\min})$. This is achieved with one mild failure of the last property; nevertheless, we always have $\mathrm{ch}_{C_n}(\sigma_{\min}^{\mathrm{row}}) = \mathrm{ch}_{C_n}(\sigma_{\min})$, which is all that is needed.

Algorithm 7.10 describes the construction of $\sigma_{\min}$. In order to state it, we need some definitions and related results. Let $k_1 < k_2 < \ldots < k_p$ be the indexes i for which a_i is odd. We pair them from left to right as (k_1, k_2), (k_3, k_4), ..., where k_p is unpaired if p is odd. Given such a pair (k, k'), we say that all the columns in λ of heights $n+1-i$ with $k \leq i \leq k'$ form a block. This block is called odd or even, depending $k' - k$ being odd or even, respectively. A subblock of columns is formed by all columns of the same height in a given block. If p is odd, we say that all columns of height at most $n+1-k_p$ form an incomplete block.

We call a column of λ special if it is the 1st one in a subblock, without being the 1st one of the corresponding block. Note that, if the 1st row of λ is filled with the entries of $\sigma_{\min}^{\mathrm{row}}$, then a column is special if and only if its top entry is strictly smaller than the maximum possible, namely $n+i$ if $n+1-i$ is the corresponding column height. We call a special column odd if its top entry has the same parity as the column height. Note that this condition on a special column is equivalent to the bottom entry being odd (hence the name), when the column is filled with consecutive entries starting from the top one.

Lemma 7.9.

(1) The number of odd blocks is even unless p and $n+1-k_p$ are odd.

(2) The number of odd special columns in a block is odd or even, depending on the block being odd or even, respectively.

(3) The total number of odd special columns is even unless p and $n+1-k_p$ are odd.

Algorithm 7.10. Construction of $\sigma_{\min}$.

Step 1: Fill the 1st row of λ with the entries of $\sigma_{\min}^{\mathrm{row}}$.

Step 2: Fill all columns except the odd special ones with consecutive entries starting from the top entry.

Step 3: Fill the odd special columns, considered from right to left, as follows:

- If the last entry of the previously filled odd special column (assuming it exists) is $2i-1$, then the current one will contain $2i$, but not $2i-1$.
- Assume that p and $n+1-k_p$ are odd, in which case the top entry in each column of height $n+1-k_p$ is $n+k_p$. Let $i := (n+k_p)/2$. In this case, the rightmost odd special column will contain $2i$, but not $2i-1$.

- With the above rules in place, fill the current special column by considering consecutive entries starting from the top one.

Lemma 7.11. The filling $\sigma_{\min}$ belongs to $\overline{D}^{*}_{2n}(\lambda)$. Furthermore, no entry i below the 1st row contributes to $\varepsilon_{i-1}(\sigma_{\min})$ with one exception: if p and $n+1-k_p$ are odd, then one entry $n+k_p$ below the 1st row contributes to $\varepsilon_{n+k_p-1}(\sigma_{\min})$. In addition, we always have $\mathrm{ch}_{C_n}(\sigma^{\mathrm{row}}_{\min}) = \mathrm{ch}_{C_n}(\sigma_{\min})$.

Proof of Theorem 7.7. By Lemma 7.11, $\mathrm{ch}_{C_n}(\sigma_{\min})$ is given by the expression in (16). Lemma 7.8 guarantees that this is the minimum of the charge over $\overline{D}^{*}_{2n}(\lambda)$. Thus, Theorem 7.7 is proved. ■

In conclusion, all that is left is to prove Lemmas 7.8, 7.9, and 7.11, which is done in Section 7.4.

We will now give an example of the construction of $\sigma_{\min}$. We will also exhibit a 2nd filling, with the same shape and 1st row as $\sigma_{\min}$, which also satisfies the properties in Lemma 7.11. This will have the same charge as $\sigma_{\min}$, which shows that the coefficient of the smallest power of t in $K^{C_n}_{\lambda,0}(t)$ can be strictly larger than 1.

Example 7.12. Let $n = 5$ and $\lambda = (7,6,5,3,1)$. The sequence (a_i) is $(1,2,2,1,1)$, and thus there is a single block, which is odd and consists of all the columns of λ except the last one. The special columns are the 2nd, the 4th, and the 6th; they are all odd. The sequences (b_i) and (c_i) are $(2,2,2,0,2)$ and $(2,2,2,0,1)$, respectively, while $\sigma^{\mathrm{row}}_{\min} = (6,6,7,7,8,8,10)$. The filling $\sigma_{\min}$ and a different one with the same charge are

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 6&6&7&7&8&8&10\\\hline 7&7&8&8&9&10\\\cline{1-6} 8&8&9&\mathbf{9}&10\\\cline{1-5} 9&\mathbf{10}&10\\\cline{1-3} 10\\\cline{1-1}\end{array}\;, \qquad \begin{array}{|c|c|c|c|c|c|c|}\hline 6&6&7&7&8&8&10\\\hline 7&7&8&8&9&\mathbf{9}\\\cline{1-6} 8&8&9&\mathbf{10}&10\\\cline{1-5} 9&10&10\\\cline{1-3} 10\\\cline{1-1}\end{array}\;.$$

It is straightforward to check that the above fillings satisfy the properties in Lemma 7.11; in particular, the highlighted entries are bracketed in the usual procedure for applying crystal operators. Thus, both of these fillings have charge $|\lambda|/2 + 1/2(5-2+1) = 11 + 2 = 13$.

Below is a different filling $\sigma_{\min}$, which illustrates another aspect of Algorithm 7.10.

6	6	9	**9**
7	**10**	10	
8			
9			
10			

.

7.4 The proof of the lemmas in Section 7.3

The terminology and notation in the previous section will be used. We start with Lemma 7.8. We first define the following moves $\sigma \to \sigma'$ on sequences $\sigma = (i^{m_i})_{i=1,\ldots,2n}$ in Σ, assuming that σ' is still in Σ:

(1) $(\ldots, i^{k+2}, \ldots) \to (\ldots, i^k, i+1, i+1, \ldots)$;
(2) $(\ldots, i^{2k+1}, \ldots) \to (\ldots, i^{2k}, i+1, \ldots)$;
(3) $(\ldots, i^{2k-1}, j^{l+1}, \ldots) \to (\ldots, i^{2k}, j^l, \ldots)$;
(4) $(\ldots, i^{2k}, j^{2l-1}, \ldots) \to (\ldots, i^{2k-1}, j^{2l}, \ldots)$.

It is not hard to see that in all cases we have

$$\mathrm{ch}_{C_n}(\sigma) \geq \mathrm{ch}_{C_n}(\sigma'); \tag{18}$$

moreover, in case (4) we always have equality. Indeed, note that

$$\mathrm{ch}_{C_n}(\sigma) = \sum_{i=2}^{2n} (2n+1-i) \left\lceil \frac{m_i}{2} \right\rceil;$$

based on this, it suffices to observe that if we insert an entry $i > 1$ into a sequence σ, the charge increases by $2n+1-i$, if m_i is even, and does not change, otherwise.

It is helpful to visualize the (1)–(4) using the following representation of a sequence $\sigma = (i^{m_i})_{i=1,\ldots,2n}$ in Σ as a lattice path from $(0,0)$ to $(S, 2n)$ with steps $(1,0)$ and $(0,1)$. The horizontal segments in this path are

$$(m_1 + \ldots + m_{i-1}, i) \to (m_1 + \ldots + m_i, i)\,, \quad \text{for } m_i > 0,\ i = 1, \ldots, 2n.$$

Note that condition (17) defining Σ simply means that this path stays weakly below the similar path from $(0,0)$ to $(S,2n)$, whose horizontal segments are

$$(s_{i-1}, n+i) \to (s_i, n+i)\,, \quad \text{for } a_i > 0,\, i = 1, \dots, n.$$

The latter path will be called the upper-bound path. We will also consider the path corresponding to $\sigma^{\mathrm{row}}_{\mathrm{min}}$, which will be called the target path.

Now recall that $k_1 < k_2 < \ldots < k_p$ are the indexes i for which a_i is odd, which are paired (k_1, k_2), (k_3, k_4), etc. For each such pair (k, k'), we consider the subpath of the upper-bound path between the horizontal segments with $y = n + k$ and $y = n + k'$, inclusive, plus the vertical segment after the last horizontal one; we call it an odd subpath. If p is odd, the subpath between the horizontal segment with $y = n + k_p$ and the end of the path is called an incomplete odd subpath. The subpaths obtained by removing the odd ones are called even.

Now note that the upper-bound path and the target one are closely related. Namely, every even subpath of the former coincides with a corresponding subpath of the latter, and so does the incomplete odd subpath (if any). Moreover, for every odd subpath of the former, there is a corresponding one in the latter whose vertical segments are translations by $(1,0)$ of the vertical segments of the former; the exceptions are the last vertical segments in the two paths, which coincide. Thus, we can also divide the target path into even, odd, and incomplete odd subpaths.

Proof of Lemma 7.8. In terms of the above visualization, and based on (18), it suffices to show that any path that is weakly below the upper-bound path (including the latter) can be related to the target path by applying the moves (1)–(4). This can be done as follows, using a sequence of intermediate paths. See Example 7.13 for an illustration of this procedure.

Step 1: By applying the moves (1) and (2), from southwest to northeast in the current path, we obtain a path in which every vertical segment coincides with the corresponding one of the upper-bound path, or with its translation by $(1,0)$; moreover, for the 1st vertical segment (starting at $(0,0)$), the 1st statement holds. Divide the obtained path into even, odd, and incomplete odd subpaths.

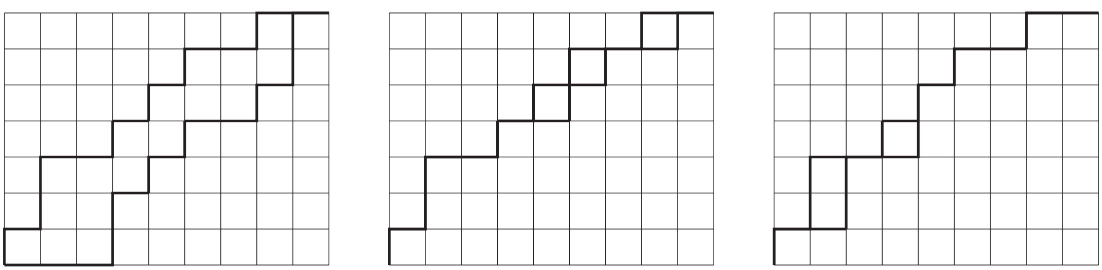
Steps 2–4: These steps are applied to the subpaths of the path in Step 1, considered from southwest to northeast. As a result, each subpath will coincide with the corresponding one of the target path.

Step 2: The move (2) is applied to an even subpath.

Step 3: The moves (2) and (3) are applied to an odd subpath.

Step 4: The moves (2) and (4) are applied to the incomplete odd subpath (if any). ■

Example 7.13. Let $n = 7$, and let the sequence (a_i) be $(1, 0, 2, 1, 1, 2, 2)$. We have $\sigma^{\text{row}}_{\min} = (8, 8, 10, 10, 12, 13, 13, 14, 14)$, which corresponds to the target path, while the upper-bound path corresponds to $(8, 10, 10, 11, 12, 13, 13, 14, 14)$. Both of these paths consist of an odd subpath and an incomplete odd subpath. Consider $\sigma = (7, 7, 7, 9, 10, 11, 11, 12, 14)$. Its corresponding path is represented in the 1st diagram below, whose bottom left corner has coordinates $(0, 7)$, while the upper-bound path appears in all three diagrams. The 2nd diagram represents the result of Step 1 in the above algorithm; move (1) was applied six times, while move (2) twice. The last diagram contains the target path, which is obtained from the path in the 2nd diagram via Step 3 followed by Step 4. In Step 3, move (3) was applied twice, and after that move (2) once; in Step 4, move (4) was applied twice (from northeast to southwest).



We conclude by proving Lemmas 7.9 and 7.11.

Proof of Lemma 7.9. It is not hard to see that the number of boxes in an odd (resp. even) block is odd (resp. even); in addition, if p is odd, the number of boxes in the incomplete block is even unless $n + 1 - k_p$ is odd (recall that this number represents the height of the 1st column in the incomplete block). Based on this and the fact that $|\lambda|$ is even, the 1st statement is immediate.

Now let us consider a block corresponding to a pair (k, k'), that is, it contains all columns of heights $n + 1 - i$ for $k \le i \le k'$. A special column is odd or even depending on the difference between its height and the height of the previous column being odd or even. But the sum of all these numbers is the difference between the heights of the 1st and last columns in the block, namely $(n + 1 - k) - (n + 1 - k') = k' - k$. The 2nd statement now follows from the fact that the parity of a block is determined by $k' - k$. The 3rd statement is an immediate consequence of the 1st two. ■

Proof of Lemma 7.11. It is not hard to see that the filling $\sigma_{\min}$ is a semistandard Young tableau satisfying the flag condition (C3) in Section 6.3. Indeed, for semistandardness, observe first that if the 1st row of λ is $\sigma_{\min}^{\text{row}}$ and we fill all columns with consecutive entries starting from the top one, we clearly obtain a semistandard tableau. To obtain $\sigma_{\min}$, the only change we need is a certain increase in the entries of every other odd special column starting with the leftmost one. But in each case the entries of the next column are the largest possible, so the weakly increasing condition for rows is still verified.

To complete the proof, it suffices to check the following properties for the column word of $\sigma_{\min}$, the 1st two of which rely on the usual bracketing rule for crystal operators [11]:

(P1) After bracketing $(2i, 2i-1)$, there is no unbracketed $2i-1$; also, there is no unbracketed $2i$ below the 1st row with one exception: a single $2i = n + k_p$ if p and $n+1-k_p$ are odd.

(P2) For any pair $(2i+1, 2i)$, there is no unbracketed $2i+1$ below the 1st row.

(P3) Each even entry in the 1st row occurs an even number of times with one exception: $n + k_p$ if p and $n+1-k_p$ are odd.

Property (P3) is immediate from the construction of $\sigma_{\min}^{\text{row}}$. By analyzing Algorithm 7.10, observe that the columns of $\sigma_{\min}$ have the following structure (the notation $\widehat{m}$ indicates the absence of the element m in a sequence):

- Every other odd special column starting with the 2nd leftmost one is of the form $(j, j+1, \ldots, 2i-2, 2i-1)$.
- Every other odd special column starting with the leftmost one is of the form: $(j, j+1, \ldots, 2i-2, \widehat{2i-1}, 2i, 2i+1, \ldots, 2l)$ with $i \le l$, or $(j, j+1, \ldots, 2l-1, 2l, 2i)$ with $l < i$.
- A non-odd special column is of the form $(j, j+1, \ldots, 2n-1, 2n)$.

In particular, the 2nd fact follows from the 1st two rules in Step 3 of Algorithm 7.10. Based on these facts, we can describe each bracketing for the column word of $\sigma_{\min}$, which will prove (P1) and (P2).

Let us first bracket $(2i, 2i-1)$ and ignore all such pairs coming from the same column of $\sigma_{\min}$. The remaining subword in these letters starts with a set of pairs $(2i, 2i-1)$ coming from successive odd special columns and ends with an even number of $2i$; the latter are all in the 1st row, with the one exception indicated in (P1) above, which corresponds to the number of odd special columns being odd (the entry $2i$ below the 1st row is in the rightmost odd special column). Here we applied Lemma 7.9 (3). Now let us bracket $(2i+1, 2i)$ and again ignore all such pairs coming from the same column of $\sigma_{\min}$. The remaining subword in these letters consists of a sequence of $2i$ followed by a sequence of $2i+1$, where all the elements of the latter are in the 1st row. Indeed, no column can contain $2i+1$ below the 1st row but no $2i$ above it. ■

8 Comparing the Sundaram and Kwon Branching Rules

The work in Sections 5 and 6 raises the question whether the Sundaram and Kwon branching rules (mentioned in those sections) are, in fact, equivalent. Based on the results above, we discuss what this equivalence entails and we present an example that provides evidence for an affirmative answer.

We consider the branching coefficient $c_\nu^\lambda(\mathfrak{sp}_{2n})$, for fixed $\lambda \in \mathcal{P}_n$ and $\nu \in \mathcal{P}_{2n}$. The Sundaram rule says that $c_\nu^\lambda(\mathfrak{sp}_{2n})$ is the number of Sundaram–LR tableaux of shape ν/λ and content δ, for some $\delta \in \mathcal{P}_{2n}^{(1,1)}$. By Corollary 6.9 and Lemma 6.11, the same coefficient is expressed as the number of LR tableaux T in $LR_{\lambda,\delta}^\nu$ for which $S(T)$ satisfies the flag condition (C3) in Section 6.3, where $\delta \in \mathcal{P}_{2n}^{(1,1)}$ (recall the notation in Section 6.1).

To relate the two types of tableaux, we need to consider the composition of the following maps:

$$\{\text{LR tableaux of shape } \nu/\lambda, \text{ content } \delta\} \xRightarrow{\text{companion}} LR_{\delta\lambda}^\nu \xRightarrow{R\text{-matrix}} LR_{\lambda,\delta}^\nu \overset{S}{\Rightarrow} S(LR_{\lambda,\delta}^\nu). \quad (19)$$

For the combinatorial R-matrix, we use the Henriques–Kamnitzer commutor [9, 15], which has several other realizations, cf. [2] and the references therein. Note that the Henriques–Kamnitzer commutor was defined in terms of the Schützenberger involution, which connects it to the last map in (19), namely the Schützenberger involution in the crystal $B_{2n}(\lambda)$.

The main question is whether the composition (19) bijects the tableaux mentioned above, coming from the Sundaram and Kwon branching rules. The example below suggests an affirmative answer.

Example 8.1. Consider $n = 3$, $\lambda = (2, 1, 1)$ and $\nu = (5, 4, 3, 3, 3, 2)$, with $c_\nu^\lambda(\mathfrak{sp}_6) = 1$. There are three LR tableaux of shape ν/λ for which the corresponding δ is in $\mathcal{P}_{2n}^{(1,1)}$. We indicate them below, together with the result of applying the maps in (19).

(1) $\delta = (3, 3, 3, 3, 2, 2)$.

$$\begin{array}{|c|c|c|c|c|} \hline & & 1 & 1 & 1 \\ \hline & 2 & 2 & 2 \\ \cline{1-4} & 3 & 3 \\ \cline{1-3} 3 & 4 & 4 \\ \cline{1-3} 4 & 5 & 5 \\ \cline{1-3} 6 & 6 \\ \cline{1-2} \end{array} \xrightarrow{\text{companion}} \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & 3 & 4 \\ \hline 4 & 4 & 5 \\ \hline 5 & 5 \\ \cline{1-2} 6 & 6 \\ \cline{1-2} \end{array} \xrightarrow{R\text{-matrix}} \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \overset{S}{\Longrightarrow} \begin{array}{|c|c|} \hline 2 & 6 \\ \hline 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array} .$$

(2) $\delta = (4, 4, 2, 2, 2, 2)$.

$$\begin{array}{|c|c|c|c|c|} \hline & & 1 & 1 & 1 \\ \hline & 2 & 2 & 2 \\ \cline{1-4} & 3 & 3 \\ \cline{1-3} \mathbf{1} & 4 & 4 \\ \cline{1-3} 2 & 5 & 5 \\ \cline{1-3} 6 & 6 \\ \cline{1-2} \end{array} \xrightarrow{\text{companion}} \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 4 \\ \hline 2 & 2 & 2 & 5 \\ \hline 3 & 3 \\ \cline{1-2} 4 & 4 \\ \cline{1-2} 5 & 5 \\ \cline{1-2} 6 & 6 \\ \cline{1-2} \end{array} \xrightarrow{R\text{-matrix}} \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \overset{S}{\Longrightarrow} \begin{array}{|c|c|} \hline 2 & 6 \\ \hline 3 \\ \cline{1-1} \mathbf{4} \\ \cline{1-1} \end{array} .$$

(3) $\delta = (4, 4, 3, 3, 1, 1)$.

$$\begin{array}{|c|c|c|c|c|} \hline & & 1 & 1 & 1 \\ \hline & 1 & 2 & 2 \\ \cline{1-4} & 2 & 3 \\ \cline{1-3} 2 & 3 & 4 \\ \cline{1-3} \mathbf{3} & 4 & 5 \\ \cline{1-3} 4 & 6 \\ \cline{1-2} \end{array} \xrightarrow{\text{companion}} \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & 4 \\ \hline 3 & 4 & 5 \\ \cline{1-3} 4 & 5 & 6 \\ \cline{1-3} 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array} \xrightarrow{R\text{-matrix}} \begin{array}{|c|c|} \hline 1 & 5 \\ \hline 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array} \overset{S}{\Longrightarrow} \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \mathbf{2} \\ \cline{1-1} 6 \\ \cline{1-1} \end{array} .$$

Note that in case (1) the 1st tableau is a Sundaram–LR tableau, while the last one satisfies condition (C3) mentioned above. However, both of these properties fail in cases (2) and (3); the entries causing these failures are shown in bold.

Remark 8.2. Kwon's rule also works in orthogonal types, whereas there is no Sundaram-type rule in this case. For symplectic types, there is also the rule conjectured by Naito–Sagaki [30], which was proved via its relation to the Sundaram rule in [35], cf. also [37].

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