A Circuit-based Approach to the Synthesis of 2-D Omega Materials

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***Abstract*—In one dimension, a circuit network equivalence has been established for omega materials. However, 2-D circuit-based or transmission-line metamaterials have previously been restricted to magnetic and electric responses. This paper proposes 2-D circuit-based omega materials using asymmetric circuits. A general formulation is provided in terms of ABCD-parameters and an example is shown using π-networks. These metamaterials could enable the design of compact beamformers, power dividers, and other microwave devices.**

I. Introduction

In the early 2000's circuit-based or transmission-line (TL) metamaterials were introduced [1]. Following the introduction of 1-D TL metamaterials, 2-D TL metamaterials followed [2]. Subsequently, [3] demonstrated that TL metamaterials can provide access to a wide range of anisotropic and tensor material properties. In [4], bianisotropic materials with an omega response were reported using asymmetric 1-D circuit networks. To see the equivalence between circuit networks and omega media, let's consider the dispersion relation and wave impedance for a lossless and reciprocal 1-D omega medium,

$$-\gamma^2 = \omega^2(\mu\epsilon + a^2) \tag{1}$$

$$\eta^{\pm} = \frac{\omega\mu}{-j\gamma^{\pm} - \omega a} \tag{2}$$

where the permeability, μ, and permittivity, ϵ, are purely real, the magneto-electric coupling, a, is purely imaginary, and $\gamma^{\pm} = \alpha \pm j\beta$. Performing Bloch analysis on a periodic network with period d, composed of lossless and reciprocal networks represented by ABCD-matrices, yields the following dispersion relation and Bloch impedance

$$-\sinh^2 \gamma_B^{\pm} d = -BC + \left(j\frac{D-A}{2}\right)^2 \tag{3}$$

$$Z_B^{\pm} = \frac{-jB}{-j\sinh\gamma_B^{\pm} d - j\frac{D-A}{2}} \tag{4}$$

where A and D are purely real, B and C are purely imaginary, and $\gamma_B^{\pm} = \alpha_B \pm j\beta_B$. By comparing (1) with (3) and (2) with (4), in the homogeneous limit, reveals the following equivalency between the asymmetric network ($A \neq D$) and an omega medium: $\mu = -jB/(\omega d)$, $\epsilon = -jC/(\omega d)$, and $a = j(D-A)/(2\omega d)$. Although these expressions are different from the form presented in [4] they are equivalent.

With a circuit analogue in 1-D, the question remains whether there is a circuit analogue to a 2-D omega medium. This work aims to address this question. A general 2-D periodic circuit network composed of asymmetric branches is proposed, and taken in the homogeneous limit, to establish a circuit model equivalent of a 2-D reciprocal and lossless omega medium.

II. 2-D Omega Materials

A. Field Theory

In bianisotropic media, the constitutive relations are,

$$\overline{D} = \overline{\overline{\epsilon}} \cdot \overline{E} + \overline{\overline{a}} \cdot \overline{H} \tag{5}$$

$$\overline{B} = \overline{\overline{\mu}} \cdot \overline{H} + \overline{\overline{b}} \cdot \overline{E} \tag{6}$$

In a medium with an omega response, the magneto-electric terms, $\overline{\overline{a}}$ and $\overline{\overline{b}}$, are anti-symmetric. Further, if it is lossless $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$ are purely real and $\overline{\overline{a}}$ and $\overline{\overline{b}}$ are purely imaginary. Additionally, reciprocity requires $\overline{\overline{\epsilon}} = \overline{\overline{\epsilon}}^T$, $\overline{\overline{\mu}} = \overline{\overline{\mu}}^T$, and $\overline{\overline{b}} = -\overline{\overline{a}}^T$. Consider a TE plane wave propagating in such a medium with wave vector, $\overline{k}$, in the x-z plane i.e. $\overline{E} = E\hat{y}$ and $\overline{H} = H_x\hat{x} + H_z\hat{z}$. Under these constraints, the dispersion relation and wave impedances along the two principal axes are,

$$\frac{{\gamma_x^{\pm}}^2 + (\omega a_{zy})^2}{-\omega\mu_{zz}} + \frac{{\gamma_z^{\pm}}^2 + (\omega a_{xy})^2}{-\omega\mu_{xx}} = \omega\epsilon \tag{7}$$

$$\eta_x = \frac{E_y}{H_z} = \frac{\omega\mu_{zz}}{-j\gamma_x^{\pm} + \omega a_{zy}} \tag{8}$$

$$\eta_z = -\frac{E_y}{H_x} = \frac{\omega\mu_{xx}}{-j\gamma_z^{\pm} - \omega a_{xy}} \tag{9}$$

where $\gamma_x^{\pm} = \alpha_x \pm j\beta_x$ and $\gamma_z^{\pm} = \alpha_z \pm j\beta_z$.

B. Circuit Theory

The 2-D circuit network, shown in Fig. 1, can be analyzed as a network composed of ABCD matrices [2]. In order to mitigate spatial dispersion within the unit cell, the individual circuit networks are assumed to be electrically small. Using 1-D periodic analysis, the electrical length of the individual networks can be expressed as $\cosh(\gamma_{B_{1,2}}\frac{d}{2}) = (A_{1,2} + D_{1,2})/2$. Therefore, requiring the circuit networks to be electrically small is equivalent to mandating that $A_{1,2} + D_{1,2} \approx 2$. Using this approximation and the analysis procedure presented in [2], the dispersion relation and Bloch impedances for the network in Fig. 1 become,

$$\frac{\sinh^2\frac{\gamma_{Bx}^{\pm} d}{2} + \left(j\frac{A_1 - D_1}{2}\right)^2}{-jB_1} + \frac{\sinh^2\frac{\gamma_{Bz}^{\pm} d}{2} + \left(j\frac{D_2 - A_2}{2}\right)^2}{-jB_2} = j(C_1 + C_2) \tag{10}$$

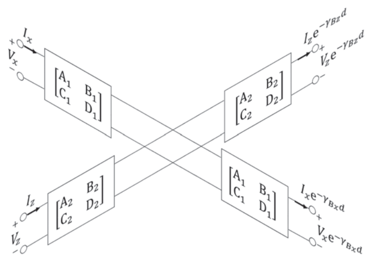

Fig. 1. ABCD matrix representation of two-dimensional periodic circuit network.

$$Z_{Bx}^{\pm} = \frac{-jB_1}{-j\tanh\frac{\gamma_{Bx}^{\pm} d}{2} + j\frac{A_1 - D_1}{2}} \tag{11}$$

$$Z_{Bz}^{\pm} = \frac{-jB_2}{-j\tanh\frac{\gamma_{Bz}^{\pm} d}{2} - j\frac{D_2 - A_2}{2}} \tag{12}$$

where, $\gamma_{Bx}^{\pm} = \alpha_{Bx} \pm j\beta_{Bx}$ and $\gamma_{Bz}^{\pm} = \alpha_{Bz} \pm j\beta_{Bz}$, and $A_{1,2}$, $B_{1,2}$, $C_{1,2}$, and $D_{1,2}$ satisfy the reciprocal and lossless conditions for an ABCD matrix mentioned earlier.

In the homogeneous limit ($\alpha_{Bx,z}d, \beta_{Bx,z}d \ll 1$), there exists a one-to-one relationship between (7)–(9) and (10)–(12), and the 2-D network behaves like an omega medium with material parameters,

$$\mu_{xx} = -j\frac{2B_2}{\omega d}, \quad \mu_{zz} = -j\frac{2B_1}{\omega d}, \quad \epsilon = -j\frac{2(C_1 + C_2)}{\omega d}$$
$$a_{xy} = j\frac{D_2 - A_2}{\omega d}, \quad a_{zy} = j\frac{A_1 - D_1}{\omega d} \tag{13}$$

III. EXAMPLE

To verify the equivalency presented in the previous section, simulated isofrequency contours for the circuit network of Fig. 1 will be compared to the analogous omega medium computed using (7). One way to implement the proposed 2-D network is with asymmetric π-networks for each of the branches shown in Fig. 1, resulting in the unit cell shown in Fig. 2.

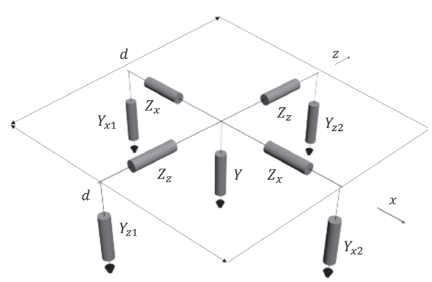

Fig. 2. Unit cell used to implement a 2-D omega medium composed of asymmetric π-networks for each branch. ($Y = Y_{x1} + Y_{x2} + Y_{z1} + Y_{z2}$)

Using (13) and assuming an isotropic permittivity, $C_1 = C_2$, the equivalent material properties in terms of the circuit elements become: $\omega\mu_{xx}d = -2jZ_z$, $\omega\mu_{zz}d = -2jZ_x$, $\omega\epsilon d = -4j(Y_{x1} + Y_{x2} + Y_{x1}Y_{x2}Z_x)$, $\omega a_{xy}d = jZ_z(Y_{z1} - Y_{z2})$, $\omega a_{zy}d = jZ_x(Y_{x2} - Y_{x1})$, and $Y_{x1} + Y_{x2} + Y_{x1}Y_{x2}Z_x = Y_{z1} + Y_{z2} + Y_{z1}Y_{z2}Z_z$. Generally this unit cell produces a dispersive omega medium. However, by choosing the magneto-electric coupling to satisfy $|a_{xy}| \ll \sqrt{\mu_{xx}\epsilon}$ and $|a_{zy}| \ll \sqrt{\mu_{zz}\epsilon}$ the π-network becomes a low-pass network, thereby reducing the effects of dispersion below cutoff. Let's consider a medium with the following material parameters: $\epsilon = 2\epsilon_0$, $\mu_{xx} = 5\mu_0$, $\mu_{zz} = 7\mu_0$, $a_{xy} = -a_{yx} = j0.1\sqrt{\mu_0\epsilon_0}$, and $a_{zy} = -a_{yz} = -j0.2\sqrt{\mu_0\epsilon_0}$. Choosing a maximum phase delay of $\pi/5$ rad at $f_0 = 10$ GHz ($d = 0.8$ mm) the lumped elements of the unit cell in Fig. 2 become ($\omega_0 = 2\pi f_0$),

$$Z_x = j\omega_0 L_x = j222\Omega, \quad Z_z = j\omega_0 L_z = j159\Omega$$
$$Y_{x1} = j\omega_0 C_{x1} = j36.6\mu S, \quad Y_{x2} = j\omega_0 C_{x2} = j188\mu S$$
$$Y_{z1} = j\omega_0 C_{z1} = j59.3\mu S, \quad Y_{z2} = j\omega_0 C_{z2} = j166\mu S \tag{14}$$

Isofrequency contours were computed for the unit cell using Keysight's Advanced Design System (ADS) at 8, 9, and 10 GHz. The results are shown in Fig. 3. There is close agreement between the simulated isofrequency contours and the analogous omega medium's isofrequency contours.

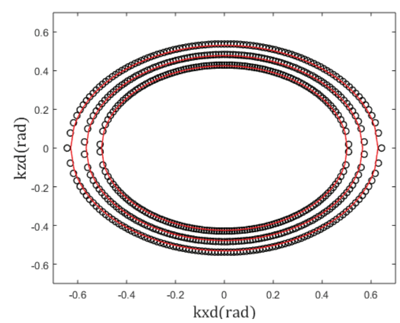

Fig. 3. Analytical (field theory) and simulated (using inductors and capacitors in the unit cell shown in Fig. 2) isofrequency contours at 8, 9, and 10 GHz. The black circles are simulated and the red solid lines are analytical results.

IV. CONCLUSION

In this paper, a circuit approach for synthesizing 2-D omega materials was presented. The proposed method is valid for electrically small circuit networks with asymmetries confined to the branches of the unit cell. For validation, the propagation characteristics of a lumped element unit cell emulating an omega medium were simulated. These results were in close agreement with those of an analogous omega medium.

ACKNOWLEDGEMENT

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