Impact of an elliptical Fermi surface deformation on the energy of a spinless two-dimensional electron gas

Orion Ciftja
Department of Physics, Prairie View A&M University, Prairie View, Texas 77446, USA

(Dated: May 9, 2019)

We study the effects of an elliptical Fermi surface deformation on the energy of a two-dimensional electron gas. We consider a standard model for fully spin-polarized (spinless) electrons embedded in a uniformly charged neutralizing jellium background and treat the system in the thermodynamic (bulk) limit. We calculate exactly the energy changes of the system as a function of a parameter that gauges the elliptic deformation of the Fermi surface. The results obtained give insight on various scenarios and competing tendencies that may arise in such a system in presence of some form of internal anisotropy originating from factors such as electron's anisotropic effective mass, anisotropic interaction potential or both.

PACS numbers: 73.43.Cd, 73.20.Dx, 73.21.La.

I. INTRODUCTION

One of the most important concepts in condensed matter physics which is very useful to describe free or weakly-interacting fermions is that of the Fermi surface. The Fermi surface separates the occupied quantum states in reciprocal space from the unoccupied ones at zero temperature[1]. It is generally assumed that the Fermi surface of a three-dimensional (3D) system is a sphere for fermions with isotropic mass if the fermions are free (non-interacting) or when the interaction between them is isotropic in uniform space. The possibility of a deformed Fermi surface is real if any of the conditions above is violated. For instance, an ellipsoidal deformation of the Fermi surface was recently experimentally observed in dipolar Fermi gases of erbium (Er) atoms[2]. The origin of the Fermi surface ellipsoidal deformation in this system is the combined effect of the anisotropic dipole-dipole interaction potential and the Pauli exclusion principle as predicted by several theoretical studies focused on dipolar Fermi gases[3–8]. Based on these findings, one would reasonably argue that the counterpart of this phenomenon in a two-dimensional (2D) fermion system (for instance, a 2D electron system) would be directly connected to an elliptical deformation of the Fermi surface in 2D reciprocal space.

When it comes to an uniform electron gas system[9], the standard approach is to assume that the electrons have an isotropic mass. Depending on the dimensionality of space, the electrons would fill a spherical Fermi surface in 3D or a circular Fermi disk in 2D. In the case of a two-dimensional electron gas (2DEG) the non-interacting electrons would occupy all the quantum plane wave states, $|\vec{k}| \leq k_F$ inside a circular Fermi disk in reciprocal space where k_F is the Fermi wave number. It is also common to assume that all the interactions between various point charges involve the usual isotropic Coulomb interaction potential. However, in reality, there are various experimental situations where the conditions above may be violated and, as a result, the occupied Fermi surface in reciprocal space may not be exactly circular. For instance, it is very common for systems of electrons in semiconductors to posses an effective anisotropic mass[10]. This occurance would immediately tend to induce an elliptical Fermi surface deformation. Also, the interactions between the electrons may not be purely of Coulomb type. Therefore, we deem of current interest to study in detail how the energy of a 2DEG system is affected under the assumption of a deformed Fermi surface. The simplest deformation of an originally circular Fermi surface would be an elliptical deformation.

One of the key objectives of this work is to get a clear insight of the magnitude of the energies involved with a Fermi surface deformation in a bulk 2DEG system at a constant electron number density. Since a large number of results are available for the case of a 2DEG with circular Fermi surface[11–16], we aim to obtain exact analytical expressions of various forms of energy that are associated with such an elliptical deformation of the Fermi surface. The results obtained tell us how the Fermi surface deformation affects the system in general, its kinetic energy, its potential energy and other quantities of interest. For simplicity, we consider only a fully spin-polarized (spinless) system of electrons described by an anti-symmetrized Slater determinant wave function of plane waves[17]. We adopt the usual jellium approximation in order to enforce the overall charge neutrality of the system.

II. THEORY

To start with, we consider a 2DEG system of N electrons with isotropic mass, m moving in a uniformly charged background. The positive charge fills a 2D region, R in the shape of a square:

$$R : \left\{ -\frac{L}{2} \leq x, y \leq +\frac{L}{2} \right\} . \tag{1}$$

We assume a spinless system of electrons. The electron number density is written as $\rho_0 = N/A$ where the area of the square region is $A = L^2$. In the thermodynamic limit, $N \to \infty$, $A \to \infty$ ($L \to \infty$) but ρ_0 remains constant and finite. The formalism and notation is similar to that for a finite spinless 2DEG ensemble[18,19] but with the difference that: (i) The current model is for an infinite 2DEG system and (ii) The current model assumes that the Fermi surface of the infinite 2DEG is elliptically deformed. The Hamiltonian of the system is given by

$$\hat{H} = \hat{T} + \hat{U} \ , \tag{2}$$

where

$$\hat{T} = \sum_{i=1}^{N} \frac{|\hat{\vec{p}}_i|^2}{2\,m} \ , \tag{3}$$

is the kinetic energy operator and

$$\hat{U} = \hat{U}_{ee} + \hat{U}_{eb} + \hat{U}_{bb} \ , \tag{4}$$

is the potential energy operator that incorporates the electron-electron (ee), electron-background (eb) and background-background (bb) interactions. One can verify that:

$$\hat{U}_{ee} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i}^{N} \hat{v}_{ee}(|\vec{r}_i - \vec{r}_j|) \ , \tag{5}$$

$$\hat{U}_{eb} = -\rho_0 \sum_{i=1}^{N} \int_R d^2 r' \, \hat{v}(|\vec{r}_i - \vec{r}\,'|) \ , \tag{6}$$

and

$$\hat{U}_{bb} = \frac{\rho_0^2}{2} \int_R d^2 r \int_R d^2 r' \ \hat{v}(|\vec{r} - \vec{r}\,'|) \ , \tag{7}$$

where $\hat{v}(|\vec{r}_i - \vec{r}_j|) = k_e\, e^2/|\vec{r}_i - \vec{r}_j|$ would represent the interaction energy between two electrons with charge, $-e$ ($e > 0$) at a distance $r_{ij} = |\vec{r}_i - \vec{r}_j|$ and k_e is Coulomb's electric constant. We describe the ground state of the 2DEG by a normalized Slater determinant wave function of plane waves $\phi_{\vec{k}}(\vec{r}) = e^{i\,\vec{k}\,\vec{r}}/\sqrt{A}$ (periodic boundary conditions are imposed):

$$\Psi = \frac{1}{\sqrt{N!}} \, Det \left\{ \phi_{\vec{k}_1}(\vec{r}_1), \ldots, \phi_{\vec{k}_N}(\vec{r}_N) \right\} \ , \tag{8}$$

where states with $\vec{k} = (k_x, k_y)$ fill a reciprocal space region with elliptical Fermi surface. Such an elliptical Fermi surface is represented by the domain/region $D_{\vec{k}}$ in reciprocal space:

$$D_{\vec{k}} : \left\{ \frac{k_x^2}{k_a^2} + \frac{k_y^2}{k_b^2} \leq 1 \right\} \ . \tag{9}$$

The total energy of the system, $T+U$ which is the expectation value of the Hamiltonian with respect to the Slater determinant wave function for a system with an elliptical Fermi surface can be calculated through well-established procedures[20]. The total kinetic energic is written as:

$$T = \sum_{\{D_{\vec{k}}\}} \frac{\hbar^2\, |\vec{k}|^2}{2\,m} \ , \tag{10}$$

where the sum is over all $\vec{k}$-states in the elliptical region given by Eq.(9). The total potential energy of the system can be written as:

$$U = -\frac{1}{2} \int_R d^2 r_1 \int_R d^2 r_2 \, |\rho(\vec{r}_1, \vec{r}_2)|^2 \, v(r_{21}) \ , \tag{11}$$

where R represents the square region of integration in real space in the thermodynamic limit of $L \to \infty$ and

$$\rho(\vec{r}_1, \vec{r}_2) = \sum_{j=1}^{N} \phi_{\vec{k}_j}(\vec{r}_1)^* \, \phi_{\vec{k}_j}(\vec{r}_2) = \frac{1}{A} \sum_{\{D_{\vec{k}}\}} e^{i\,\vec{k}\,(\vec{r}_2 - \vec{r}_1)} \ , \tag{12}$$

represents the one-particle density matrix for this case. The quantity in Eq.(11) is known as the exchange potential energy and originates from Pauli's exclusion principle. The energies in Eq.(10) and Eq.(11) should be calculated in the thermodynamic limit for $\vec{k}$-states contained inside an elliptically deformed Fermi surface.

III. RESULTS

A. Kinetic energy

The kinetic energy per particle, $t = T/N$ for a spinless 2DEG with elliptical Fermi surface is calculated in Appendix. A and the result reads:

$$t = \frac{\hbar^2}{2\,m} \frac{\left(k_a^2 + k_b^2\right)}{4} \ . \tag{13}$$

One way to describe the elliptical deformation of the Fermi surface is to introduce a real positive parameter, α and write:

$$k_a = \alpha\, k_F \quad ; \quad k_b = \frac{k_F}{\alpha} \ , \tag{14}$$

where $k_a\, k_b = k_F^2 = 4\,\pi\,\rho_0$ and k_F represents the value of the Fermi wave number for a circular Fermi surface. The choice made in Eq.(14) guarantees the constraint of a constant number of states (constant area in $\vec{k}$-space). For this choice of notation, the kinetic energy per particle denoted as $t(\alpha)$ to point out its α-dependence, reads:

$$t(\alpha) = \frac{\epsilon_F}{4} \left(\alpha^2 + \frac{1}{\alpha^2} \right) \tag{15}$$

where

$$\epsilon_F = \frac{\hbar^2\, k_F^2}{2\,m} \ , \tag{16}$$

is the Fermi energy for a circular Fermi surface.

B. Potential energy

In order to calculate the potential energy, one starts from the expression for the one-particle density matrix in Appendix B. As shown in Appendix C, one may write the potential energy per particle, $u = U/N$ as:

$$u = -2\,\rho_0 \int_R d^2 r_{21} \left[\frac{J_1\left(\sqrt{k_a^2 x_{21}^2 + k_b^2 y_{21}^2}\right)}{\left(\sqrt{k_a^2 x_{21}^2 + k_b^2 y_{21}^2}\right)} \right]^2 v(r_{21}) \ , \tag{17}$$

where the integral is carried out over an infinite 2D region of space and $v(r_{21}) = k_e\, e^2/\sqrt{x_{21}^2 + y_{21}^2}$ is a Coulomb interaction potential. Based on the choice made in Eq.(14), one writes $k_a^2\, x_{21}^2 = k_F^2\, \alpha^2\, x_{21}^2$ and, similarly, $k_b^2\, y_{21}^2 = k_F^2\, y_{21}^2/\alpha^2$ This approach suggests a strategy to calculate the expression in Eq.(17) in terms of new variables defined as:

$$x' = \alpha\, x_{21} \quad ; \quad y' = \frac{y_{21}}{\alpha} \ , \tag{18}$$

which, in a 2D polar system of coordinates, are written as:

$$x' = r' \cos(\varphi') \quad ; \quad y' = r' \sin(\varphi') \ . \tag{19}$$

This choice of variables leads to the following expression for the potential energy per particle:

$$u = -2\,\rho_0 \int_0^\infty dr'\, r' \left[\frac{J_1\left(k_F\, r'\right)}{\left(k_F\, r'\right)} \right]^2 \int_0^{2\pi} d\varphi' v(r', \varphi') \ . \tag{20}$$

The original Coulomb interaction potential takes the following form in the new system of coordinates:

$$v(r', \varphi') = \frac{k_e\, e^2}{r'} \, \frac{1}{\sqrt{\frac{\cos^2(\varphi')}{\alpha^2} + \alpha^2 \sin^2(\varphi')}} \ . \tag{21}$$

By substituting from Eq.(21) to Eq.(20) one obtains:

$$u = -2\,\rho_0\, k_e\, e^2 \int_0^\infty dr' \left[\frac{J_1\left(k_F\, r'\right)}{\left(k_F\, r'\right)} \right]^2 \int_0^{2\pi} d\varphi' \frac{1}{\sqrt{\frac{\cos^2(\varphi')}{\alpha^2} + \alpha^2 \sin^2(\varphi')}} \ . \tag{22}$$

The integration over the radial variable can be done with the following integral formula:

$$\int_0^\infty dx \left[\frac{J_1(x)}{x} \right]^2 = \frac{4}{3\,\pi} \ . \tag{23}$$

It is shown in Appendix. D that the integral over the angular variable leads to an expression that involves complete elliptic integrals of the first kind. Using the fact that $\rho_0 = k_F^2/(4\pi)$ one can finally calculate the potential energy per electron, denoted as $u(\alpha)$ which reads:

$$u(\alpha) = -\frac{8}{3\pi^2} k_F\, k_e\, e^2\, F(\alpha) \ , \tag{24}$$

where $F(\alpha)$ is an auxiliary function of the form:

$$F(\alpha) = \frac{1}{\alpha} K\left(m = 1 - \frac{1}{\alpha^4} \right) \ . \tag{25}$$

Note that $F(\alpha = 1) = \pi/2$ and $K(m)$ is a complete elliptic integral of the first kind with parameter m.

IV. DISCUSSION AND CONCLUSIONS

For the way it is defined, a value of $\alpha = 1$ represents a circular Fermi surface. One can verify that the results in Eq.(15) and Eq.(24) do, indeed, reproduce the values of the kinetic energy and potential energy per particle for the case of a spinless 2DEG with circular Fermi surface:

$$t(\alpha = 1) = \frac{\epsilon_F}{2} \quad ; \quad u(\alpha = 1) = -\frac{4}{3\pi}\, k_F\, k_e\, e^2 \ . \tag{26}$$

One can write the Fermi energy for a circular Fermi surface as:

$$\epsilon_F = \frac{\hbar^2\, k_F^2}{2\, m} = (k_F\, a_B)^2\, \frac{k_e\, e^2}{2\, a_B} \ , \tag{27}$$

where $a_B = \hbar^2/(k_e\, m\, e^2)$ is the Bohr radius for Hydrogen atom. Using similar transformations, one writes:

$$k_F k_e e^2 = 2\, (k_F\, a_B)\, \frac{k_e\, e^2}{2\, a_B} \ , \tag{28}$$

where $k_e\, e^2/(2\, a_B) = 1 Ry$ is a standard unit of energy. With help from Eq.(27) and Eq.(28) one writes:

$$t(\alpha) = \frac{(k_F\, a_B)^2}{4} \left(\alpha^2 + \frac{1}{\alpha^2} \right) \frac{k_e\, e^2}{2\, a_B} \ . \tag{29}$$

and

$$u(\alpha) = -\frac{16}{3\,\pi^2} (k_F\, a_B)\, F(\alpha)\, \frac{k_e e^2}{2\, a_B} \ . \tag{30}$$

It is customary to write the electron number density for a 2DEG as:

$$\rho_0 = \frac{1}{\pi\, (r_s\, a_B)^2} \ , \tag{31}$$

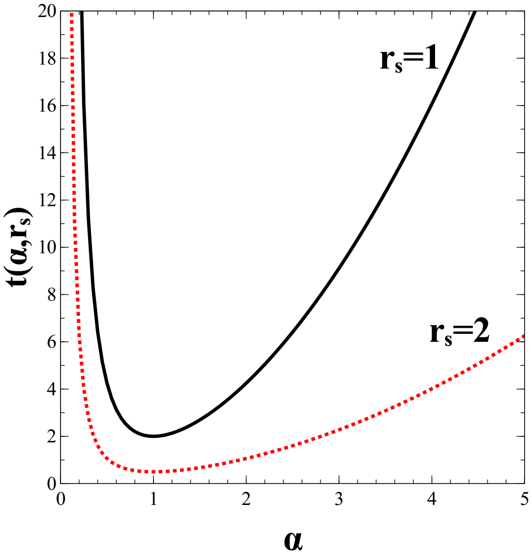

FIG. 1: Kinetic energy per particle, $t(\alpha, r_s)$ in units of $k_e e^2/(2\,a_B)$ as a function of parameter, α for values $r_s = 1$ and $r_s = 2$.

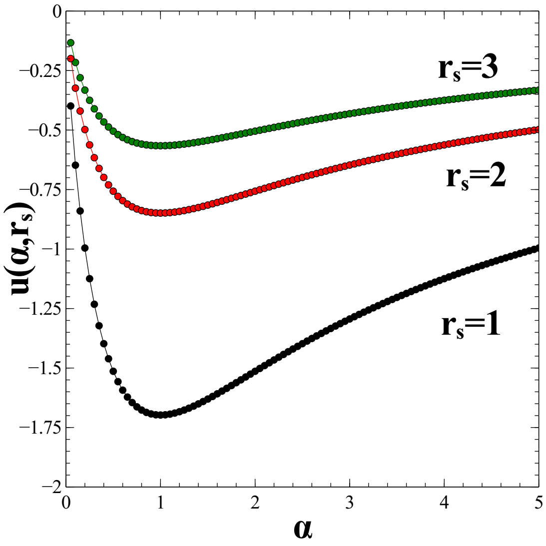

FIG. 2: Potential energy per particle, $u(\alpha, r_s)$ in units of $k_e e^2/(2\,a_B)$ as a function of parameter, α for values $r_s = 1$, $r_s = 2$ and $r_s = 3$.

where r_s is the Wigner-Seitz radius. In this case we have:

$$(k_F\, r_s\, a_B)^2 = 4 \;, \tag{32}$$

which means that $(k_F\, a_B)^2 = 4/r_s^2$ and $(k_F\, a_B) = 2/r_s$. As a result, we can express the kinetic energy and potential energy per particle solely as a function of parameter α and r_s:

$$t(\alpha, r_s) = \frac{1}{r_s^2}\left(\alpha^2 + \frac{1}{\alpha^2}\right)\frac{k_e\, e^2}{2\, a_B} \;, \tag{33}$$

and

$$u(\alpha, r_s) = -\frac{32}{3\pi^2}\,\frac{1}{r_s}\,F(\alpha)\,\frac{k_e\, e^2}{2\, a_B} \;. \tag{34}$$

Note that, from now on, we denote the kinetic and potential energy per particle, respectively as, $t(\alpha, r_s)$ and $u(\alpha, r_s)$. In Fig. 1 we plot the kinetic energy per particle, $t(\alpha, r_s)$ as a function of parameter, α for values of $r_s = 1$ and $r_s = 2$. In Fig. 2 we plot the potential energy per particle, $u(\alpha, r_s)$ as a function of parameter, α for selected values of r_s. The total energy per particle of the system is:

$$\epsilon(\alpha, r_s) = t(\alpha, r_s) + u(\alpha, r_s) \;, \tag{35}$$

where the parameter α measures the degree of elliptical deformation of the Fermi surface and r_s gauges the density of the 2DEG system. Based on the results previously derived one has:

$$\epsilon(\alpha, r_s) = \left[\frac{1}{r_s^2}\left(\alpha^2 + \frac{1}{\alpha^2}\right) - \frac{32}{3\pi^2}\,\frac{1}{r_s}\,F(\alpha)\right]\frac{k_e e^2}{2\, a_B} \;. \tag{36}$$

where $F(\alpha)$ is given from Eq.(25). The total energy per particle is plotted in Fig. 3 as a function of α for selected values of r_s. In Fig. 4 we show the energy difference, $\epsilon(\alpha, r_s) - \epsilon(\alpha = 1, r_s)$ as a function of α for various values of r_s. As expected, a 2DEG with circular Fermi surface has a lower energy than its elliptically deformed counterpart for any value $\alpha \neq 1$ when electrons have an isotropic mass and they interact via an isotropic Coulomb interaction potential.

Expansion of kinetic energy per particle around $\alpha = 1$ gives the leading terms:

$$\frac{t(\alpha, r_s)}{t(\alpha = 1, r_s)} = 1 + 2\,(\alpha - 1)^2 - 2\,(\alpha - 1)^3 + O[\alpha - 1]^4 \;. \tag{37}$$

Similarly, an expansion of potential energy per particle around $\alpha = 1$ reads:

$$\frac{u(\alpha, r_s)}{|u(\alpha = 1, r_s)|} = -1 + \frac{1}{4}\,(\alpha - 1)^2 - \frac{1}{4}\,(\alpha - 1)^3 + O[\alpha - 1]^4 \;. \tag{38}$$

Note the use of the absolute value, $|u(\alpha = 1, r_s)|$ in the denominator of the quantity in Eq.(38).

Both kinetic and potential energy per particle increase the same way for small elliptical deformations of the Fermi surface with the leading term being proportional to $(\alpha - 1)^2$ for $\alpha \approx 1$. Note that $t(\alpha, r_s)$ grows faster than $|u(\alpha, r_s)|$ (compare the proportionality constant of 2 versus 1/4) for $\alpha \approx 1$ if one assumes that the density of the 2DEG system is such that $t(\alpha = 1, r_s) \approx |u(\alpha = 1, r_s)|$. At such density, kinetic energy dominates over potential energy. At other situations, the dominant term is determined by the specific density of the system, namely the value of parameter $k_F\, a_B$ (or equivalently r_s).

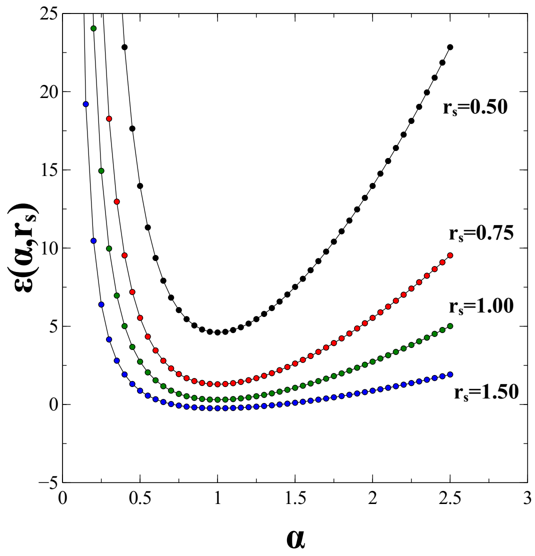

FIG. 3: Total energy per particle in units of $k_e e^2/(2\,a_B)$ as a function of parameter, α for several values of r_s.

Obviously, all the prior conclusions are based on the assumption of a Coulomb interaction potential between electrons. One would expect a reduced influence of the potential energy term if screening of the exchange interaction is considered, as is normally the case in metallic systems. For instance, let us consider a screened Coulomb potential of Yukawa form:

$$v(\lambda, r_{21}) = \frac{k_e\, e^2}{r_{21}}\, \exp\left(-\lambda\, r_{21}\right) \ . \tag{39}$$

The low density regime is the one where, typically, the potential energy dominates the kinetic energy for the case of a bare Coulomb potential. We expect this behavior to change when screening is present since the exponential damping factor in Eq.(39) will reduce the potential energy to a negligible value at distances greater than order $1/\lambda$. Quantitatively speaking, the counterpart to the expression for potential energy in Eq.(22) becomes a very complicated function of screening parameter λ for the case of a Yukawa potential since the separation of integrals over a radial part and an angular part does not seem feasible:

$$u(\lambda) = -2\,\rho_0\, k_e\, e^2 \int_0^\infty dr' \left[\frac{J_1\left(k_F\, r'\right)}{\left(k_F\, r'\right)}\right]^2 \int_0^{2\pi} d\varphi' \frac{\exp\left[-\lambda\, r' \sqrt{\frac{\cos^2(\varphi')}{\alpha^2} + \alpha^2 \sin^2(\varphi')}\right]}{\sqrt{\frac{\cos^2(\varphi')}{\alpha^2} + \alpha^2 \sin^2(\varphi')}} \ . \tag{40}$$

Returning to the results for a bare Coulomb potential, the insights gained suggest some interesting scenarios that might emerge. They also allow us to make qualitative predictions of what might happen in situations where the 2DEG system has some internal anisotropy present. The simplest scenario of a 2DEG possesing some internal anisotropy would be that of a spinless 2DEG consisting of electrons with mass anisotropy, $m_x \neq m_y$ $(m_{x,y} > 0)$ interacting via a standard isotropic Coulomb interaction potential. Within this framework, one can always define some effective mass, $m_0 = \sqrt{m_x\, m_y}$. For simplicity, one may introduce a mass anisotropy parameter, $\beta > 0$ so that:

$$m_x = \beta\, m_0 \quad ; \quad m_y = \frac{m_0}{\beta} \quad ; \quad m_x\, m_y = m_0^2 \ . \tag{41}$$

One can calculate that the kinetic energy per particle for a spinless 2DEG with a given elliptical Fermi surface deformation and anisotropic mass of this type reads:

$$\frac{1}{4}\,\frac{\hbar^2\, k_F^2}{2\, m_0}\left(\frac{\alpha^2}{\beta} + \frac{\beta}{\alpha^2}\right) . \tag{42}$$

On the other hand, the potential energy per particle will not depend on the mass anisotropy parameter, β since the anisotropic mass does not enter the expression in Eq.(24). This means that, in this case, the kinetic energy term favors an elliptical deformation of the Fermi surface with a value of parameter $\alpha = \sqrt{\beta}$ representing the minimum of kinetic energy. On the other hand, the potential energy term for an isotropic Coulomb potential tends to favor a circular Fermi surface since any elliptical deformation of the Fermi surface (coming from the anisotropic mass of the electrons) increases the potential energy. The issue of which one of the two opposing energy terms dominates is not that straightforward because the outcome will depend very sensitively on the density of the system.

The considerations above suggest a variety of interesting possible scenarios that we can envision for a 2DEG with anisotropic mass and Coulomb interaction potential. Generally speaking, one expects the potential energy term to dominate at relatively low density. Thus, for the case of anisotropic mass at relatively low density, it is reasonable to predict that even a 2DEG with nearly circular Fermi surface might be energetically more favorable than the one with elliptical Fermi surface (an elliptical Fermi surface minimizes the kinetic energy in the case of an anisotropic mass but a circular Fermi sur-

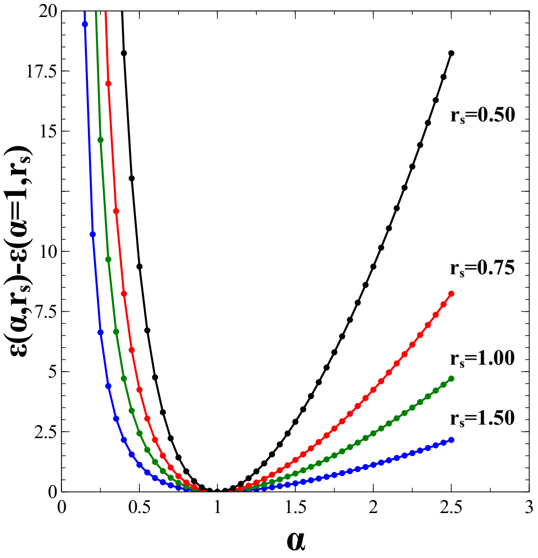

FIG. 4: The difference of the total energy per particle, $\epsilon(\alpha, r_s) - \epsilon(\alpha = 1, r_s)$ in units of $k_e e^2/(2\,a_B)$ as a function of α for various r_s.

face is favored by the isotropic Coulomb interaction, the dominant term at low density). At all other higher densities, one expects a 2DEG with elliptical Fermi surface to be favored at an optimum deformation α. This optimal value of the elliptical deformation may not necessarily be identical or close to $\alpha = \sqrt{\beta}$, which represents the value that minimizes the kinetic energy for the case of a spinless 2DEG system of electrons with mass anisotropy. Obviously, much more complex scenarios are possible when we have : (i) isotropic mass and anisotropic interaction or (ii) anisotropic mass and anisotropic interaction.

For example, let's elaborate further on the above-mentioned first scenario that applies to a 2DEG of particles with an effective isotropic mass but interacting with an anisotropic interaction potential. Such would the case of an anisotropic Coulomb interaction potential that has been considered recently in the context of studies of a 2DEG system in the quantum Hall regime[22,23] which reads:

$$v_\gamma(x_{ij}, y_{ij}) = \frac{k_e\, e^2}{\sqrt{\frac{x_{ij}^2}{\gamma^2} + \gamma^2\, y_{ij}^2}}\ , \tag{43}$$

where $\gamma \geq 1$ is the interaction anisotropy parameter and $\vec{r}_{ij} = (x_{ij}, y_{ij})$ is the separation distance vector between particles i and j. Note that $v_{\gamma=1}(x_{ij}, y_{ij})$ is the standard isotropic Coulomb interaction potential. For the same magnitude of the separation distance between two electrons, repulsion along x direction (when $y_{ij} = 0$) is stronger than repulsion along y direction (when $x_{ij} = 0$) when $\gamma > 1$. Using the same mathematical approach as before, one can calculate that the potential energy per particle for a 2DEG with circular Fermi surface in the case of the anisotropic Coulomb interaction potential of Eq.(43) will be given by the quantity $u(\gamma)$ where the function $u(\alpha)$ is defined in Eq.(24). This means that an elliptically deformed Fermi surface with a deformation parameter $\alpha = 1/\gamma$ represents the optimally deformed elliptical Fermi surface that minimizes the potential energy. However, as discussed earlier, the optimum value of the parameter α which minimizes the potential energy does not coincide with the optimum value of such parameter that minimizes the kinetic energy (such value is trivially $\alpha = 1$ for a 2DEG with isotropic mass). Additionally, the density of the system determines whether it is the potential or the kinetic energy that dominates the total energy of the system. This means that it will be the interplay of several factors that will decide the optimum elliptical deformation of the Fermi surface. In conclusion, the results obtained in this work provide useful insights on several rich scenarios that emerge when various anisotropic factors play a role on the stability of a spinless 2DEG system.

Acknowledgments

This research was supported in part by National Science Foundation (NSF) Grant No. DMR-1705084.

APPENDIX A: KINETIC ENERGY

We consider a 2DEG with elliptical Fermi surface in the thermodynamic limit where $\rho_0 = N/A$ is finite while $N \to \infty$ and $A \to \infty$. The total number of spinless electrons for such a case can be written as:

$$N = \frac{A}{(2\pi)^2}\pi\, k_a\, k_b\ . \tag{A1}$$

This result leads to an expression for the uniform particle density as:

$$\rho_0 = \frac{N}{A} = \frac{k_a\, k_b}{4\,\pi}\ . \tag{A2}$$

The total kinetic energy of the system in the thermodynamic limit is written as:

$$T = \frac{A}{(2\pi)^2}\frac{\hbar^2}{2\,m} \iint\limits_{D_{\vec{k}}:\left\{\frac{k_x^2}{k_a^2}+\frac{k_y^2}{k_b^2}\leq 1\right\}} dk_x\, dk_y\, \left(k_x^2 + k_y^2\right)\ , \tag{A3}$$

where $D_{\vec{k}}$ is the region bounded by the ellipse in $\vec{k}$-space. The following integration formula applies:

$$\iint\limits_{D_{\vec{r}}:\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1\right\}} dx\, dy\, \left(x^2 + y^2\right) = \frac{\pi\, a\, b}{4}\left(a^2 + b^2\right)\ . \tag{A4}$$

Use of Eq.(A4) for the expression in Eq.(A3) in conjunction with Eq.(A1) leads to the following result for the total kinetic energy:

$$T = \frac{N}{2}\,\frac{\hbar^2}{2\,m}\,\frac{\left(k_a^2 + k_b^2\right)}{2}\ . \tag{A5}$$

As a result, the kinetic energy per particle reads:

$$t = \frac{T}{N} = \frac{\hbar^2}{2\,m}\,\frac{\left(k_a^2 + k_b^2\right)}{4}\ . \tag{A6}$$

APPENDIX B: ONE-PARTICLE DENSITY MATRIX

In the thermodynamic limit, one can write the original quantity in Eq.(12) in integral form as:

$$\rho(\vec{r}_1, \vec{r}_2) = \frac{1}{(2\pi)^2} \iint\limits_{D_{\vec{k}}:\left\{\frac{k_x^2}{k_a^2}+\frac{k_y^2}{k_b^2}\le 1\right\}} dk_x\, dk_y\, e^{i\,\vec{k}\,(\vec{r}_2-\vec{r}_1)}\ , \tag{B1}$$

where $D_{\vec{k}}$ is the region bounded by the ellipse in $\vec{k}$-space. One can prove the following integration formula:

$$\iint\limits_{D_{\vec{k}}:\left\{\frac{k_x^2}{k_a^2}+\frac{k_y^2}{k_b^2}\le 1\right\}} dk_x dk_y\, e^{\pm i\,\vec{k}\,\vec{r}} = 2\,\pi\,k_a\,k_b\,\frac{J_1\left(\sqrt{k_a^2\,x^2 + k_b^2\,y^2}\right)}{\sqrt{k_a^2\,x^2 + k_b^2\,y^2}}\ , \tag{B2}$$

where $J_1(x)$ is a Bessel function of the first kind. By using the result in Eq.(B2) one can write the quantity in Eq.(B1) as:

$$\rho(\vec{r}_1, \vec{r}_2) = \frac{k_a\,k_b}{2\pi}\,\frac{J_1\left(\sqrt{k_a^2\,x_{21}^2 + k_b^2\,y_{21}^2}\right)}{\sqrt{k_a^2\,x_{21}^2 + k_b^2\,y_{21}^2}}\ , \tag{B3}$$

where $\vec{r}_{21} = \vec{r}_2 - \vec{r}_1 = (x_{21}, y_{21})$. One can write the above expression in a more compact form as:

$$\rho(\vec{r}_1, \vec{r}_2) = 2\,\rho_0\,\frac{J_1\left(\sqrt{k_a^2\,x_{21}^2 + k_b^2\,y_{21}^2}\right)}{\sqrt{k_a^2\,x_{21}^2 + k_b^2\,y_{21}^2}}\ , \tag{B4}$$

where $\rho_0 = k_a k_b/(4\,\pi)$ is the uniform density.

APPENDIX C: POTENTIAL ENERGY

We substitute the expression for $\rho(\vec{r}_1, \vec{r}_2)$ given from Eq.(B4) into the expression for the exchange potential energy in Eq.(11) to obtain:

$$U = -2\,\rho_0^2 \int_R d^2r_1 \int_R d^2r_2 \left[\frac{J_1\left(\sqrt{k_a^2\,x_{21}^2 + k_b^2\,y_{21}^2}\right)}{\sqrt{k_a^2\,x_{21}^2 + k_b^2\,y_{21}^2}}\right]^2 v(r_{21}) \tag{C1}$$

The area of region R tends to infinity in the thermodynamic limit and as a result one has:

$$U = -2\,\rho_0\,N \int_R d^2r_{21} \left[\frac{J_1\left(\sqrt{k_a^2\,x_{21}^2 + k_b^2\,y_{21}^2}\right)}{\sqrt{k_a^2\,x_{21}^2 + k_b^2\,y_{21}^2}}\right]^2 v(r_{21})\ . \tag{C2}$$

More explicitly one writes the potential energy per particle in the thermodynamic limit as:

$$u = \frac{U}{N} = -2\,\rho_0 \int_R d^2r_{21} \left[\frac{J_1\left(\sqrt{k_a^2 x_{21}^2 + k_b^2 y_{21}^2}\right)}{\left(\sqrt{k_a^2 x_{21}^2 + k_b^2 y_{21}^2}\right)}\right]^2 v(r_{21})\ , \tag{C3}$$

where

$$v(r_{21}) = \frac{k_e\,e^2}{\sqrt{x_{21}^2 + y_{21}^2}} \tag{C4}$$

is a standard Coulomb interaction potential.

APPENDIX D: COMPLETE ELLIPTIC INTEGRALS

It is a straightforward problem in mathematics to prove that

$$\int_0^{2\pi} \frac{d\varphi}{\sqrt{a^2\,\cos^2(\varphi) + b^2\,\sin^2(\varphi)}} = \int_0^{2\pi} \frac{d\varphi}{\sqrt{a^2\,\sin^2(\varphi) + b^2\,\cos^2(\varphi)}}\ . \tag{D1}$$

With some care, one can prove that the integrals in Eq.(D1) can be written as:

$$\int_0^{2\pi} \frac{d\varphi}{\sqrt{a^2\,\cos^2(\varphi) + b^2\,\sin^2(\varphi)}} = \frac{4}{b}\,K\left(m = 1 - \frac{a^2}{b^2}\right)\ , \tag{D2}$$

where $K(m)$ is a complete elliptic integral of the first kind with parameter, m. The complete elliptic integrals of the first and second kind are, respectively, defined as

$$K(m) = \int_0^{\pi/2} \frac{d\,\theta}{\sqrt{1 - m\,\sin^2(\theta)}} \ , \tag{D3}$$

and

$$E(m) = \int_0^{\pi/2} d\,\theta \,\sqrt{1 - m\,\sin^2(\theta)} \ . \tag{D4}$$

We remark that other notations are available in the literature and the reader should be cautious[21]. At this juncture, one uses Eq.(D2) to calculate

$$\int_0^{2\pi} d\varphi' \frac{1}{\sqrt{\frac{\cos^2(\varphi')}{\alpha^2} + \alpha^2 \sin^2(\varphi')}} = \frac{4}{\alpha}\, K\left(m = 1 - \frac{1}{\alpha^4}\right) \ , \tag{D5}$$

where we remind the reader that the $\alpha > 0$ and real.

[1] S. B. Dugdale, Phys. Scr. **91**, 053009 (2016).

[2] K. Aikawa, S. Baier, A. Frisch, M. Mark, C. Ravensbergen, and F. Ferlaino, Science **345**, 1484 (2014).

[3] T. Miyakawa, T. Sogo, and H. Pu, Phys. Rev. A **77**, 061603 (2008).

[4] B. M. Fregoso and E. Fradkin, Phys. Rev. Lett. **103**, 205301 (2009).

[5] B. M. Fregoso, K. Sun, E. Fradkin, and B. L. Lev, New J. Phys. **11**, 103003 (2009).

[6] D. Baillie and P. Blakie, Phys. Rev. A **86**, 023605 (2012).

[7] C-K. Chan, C. Wu, W-C. Lee, and S. Das Sarma, Phys. Rev. A **81**, 023602 (2010).

[8] C. Zhao, L. Jiang, X. Lu, W. M. Liu, X. Zou, and H. Pu, Phys. Rev. A **81**, 063642 (2010).

[9] P. M. W. Gill and P-F. Loos, Theor Chem Acc. **131**, 1069 (2012).

[10] O. Ciftja, V. Livingston, and E. Thomas, Am. J. Phys. **85**, 359 (2017).

[11] B. Tanatar and D. M. Ceperley, Phys. Rev. B **39**, 5005 (1989).

[12] Y. Kwon, D. M. Ceperley, and R. M. Martin, Phys. Rev. B **48**, 12037 (1993).

[13] C. Attaccalite, S. Moroni, P. Gori-Giorgi, and G. B. Bachelet, Phys. Rev. Lett. **88**, 256601 (2002); Phys. Rev. Lett. **91**, 109902(E) (2003).

[14] P. Gori-Giorgi, S. Moroni, and G. B. Bachelet, Phys. Rev. B **70**, 115102 (2004).

[15] D. Ceperley, Phys. Rev. B **18**, 3126 (1978).

[16] F. H. Zong, C. Lin, and D. M. Ceperley, Phys. Rev. E **66**, 036703 (2002).

[17] J. C. Slater, Phys. Rev. **34**, 1293 (1929).

[18] O. Ciftja, B. Sutton, and A. Way, AIP Advances **3**, 052110 (2013).

[19] O. Ciftja, Physica B **458**, 92 (2015).

[20] B. Bernu, F. Delyon, M. Holzmann, and L. Baguet, Phys. Rev. B **84**, 115115 (2011).

[21] O. Ciftja, Res. Phys. **7**, 1674 (2017).

[22] H. Wang, R. Narayanan, X. Wan, and F. Zhang, Phys. Rev. B **86**, 035122 (2012).

[23] O. Ciftja, Phys. Rev. B **95**, 075410 (2017).