Electrostatic interaction energy between two coaxial parallel uniformly charged disks

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We calculate exactly the electrostatic interaction energy between two coaxial parallel uniformly charged infinitely thin disks. For the sake of generality, it is assumed that the disks have different radii and contain different amounts of total electric charge. The results derived when expressed in one-dimensional integral form are very suitable for numerical calculations. An explicit analytical formula is provided for the electrostatic interaction energy between two disks with same radius in terms of a family of special functions known as complete elliptic integrals. The results obtained are applicable to various electrostatic models including the case of a circular parallel plate capacitor.

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I. INTRODUCTION

Coulomb's law in physics states that the electrostatic force between two charged particles is proportional to the product of the amount of charge of the particles divided by the square of the separation distance between them. The interaction energy between any two charged particles is derived from Coulomb's law using well known procedures. The interaction energy between particles is negative if the particles are oppositely charged. However, the interaction energy between them is positive, if the particles are similarly charged. The magnitude of the interaction energy between any two charged particles diverges when their separation distance tends to zero. The model of interacting charged particles works well as long as the real interacting charged bodies have a size that is much smaller than the distance separating them and, as a result, they can be treated approximately as particles (point charges). However, for any other situation, the interaction energy between any two charged bodies has to be calculated by generalizing the fundamental laws of electrostatics.

The calculation of the electrostatic interaction energy between any two charged bodies is very important in the field of electrostatics. This problem is also crucial to understand many related biological and/or soft condensed matter systems that contain charged structures as their main constituents. For example, electrostatic interactions dictate the physical properties of charged cylindrical structures in electrolyte solutions[1]. Such systems can be commonly described as charged cylindrical surfaces embedded in an electrolyte solution of positive and negative ions[2]. Depending on their charge, some ions in the electrolyte solution are attracted by the charged surfaces and some are repelled. This process routinely leads to the creation of a capacitor-like electric double layer[3,4] around the charged surface. This effect is crucial to understand the behavior of a wide range of systems containing charged bodies in electrolyte solutions[5].

Exact analytic results are generally not possible if the shape of the bodies and the charge distribution density in them is arbitrary. Analytic results might be possible only in the case of regular bodies that possess some symmetry and when the charge distribution is either known or assumed to be uniform[6–8]. The cases where the equilibrium charge distribution is known are very few. Finding the equlibrium charge distribution even in a regular body such as a one-dimensional (1D) straight wire with finite length is not simple[9–12]. Therefore, one is routinely forced to adopt approximations. One of the most common approximations made in electrostatics is that of assuming a uniform charge distribution over a surface in the case of two-dimensional (2D) structures or over a volume for the case of three-dimensional (3D) charge distributions. It has been found for the case of an infinitely thin 2D disk that the assumption of a uniform charge distribution over the surface leads to a final result for the Coulomb self-energy that is quite close to the exact value obtained by using the known exact expression for the equilibrium charge distribution[13].

While finding the equilibrium charge distribution for a system of two charged disks remains an unsolved analytical problem[14–17] we believe that the assumption of uniform charge distribution is a good one and we adopt it in this work. Therefore, the problem that we face is the calculation of the electrostatic interaction energy between two coaxial parallel uniformly charged infinitely thin disks at some arbitrary separation distance between their centers. This problem is relevant to many science and engineering disciplines given that disks, cylinders and cylindrical shells are some of the most common objects found in various electronic devices. For example, two oppositely charged disk plates would be the key ingredients of a circular parallel plate capacitor that represents the counterpart to the more common square parallel plate capacitor[18]. Similarly, two oppositely charged concentric cylindrical shells[19] are part of cylindrical capacitors, and so on.

One way to calculate the electrostatic interaction energy between two charged bodies is to first calculate the electrostatic potential created by one of them[20–23] at some arbitrary point in space and then consider the effect of this potential on the other body. The mathematical treatment of this problem is also closely related to that of the calculation of the Coulomb self-energy of a given

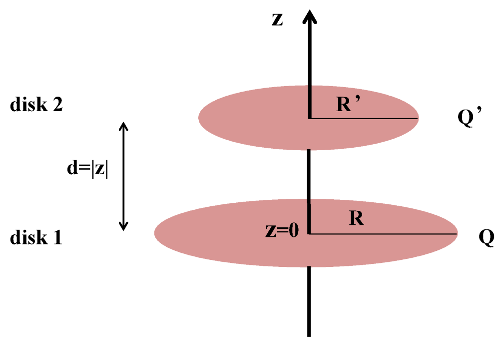

FIG. 1: Schematic view of a system of two coaxial parallel uniformly charged disks. The two disks have respective radii, R and R'. The separation distance between the centers of the disks is denoted as $d = |z| \geq 0$. The two disks contain respective charges, Q and Q' that are uniformly spread.

body[24–31]. In this work, we show that an exact analytic expression is possible for the electrostatic interaction energy of a system of two coaxial parallel uniformly charged infinitely thin disks at an arbitrary separation distance between their centers.

II. MODEL AND THEORY

The system that we study consists of two coaxial parallel uniformly charged infinitely thin disks with respective radii, R and R'. Each of the disks contains, respectively, a total arbitrary charge, Q and Q' uniformly distributed over the surface. The respective uniform surface charge densities of the two disks are:

$$\sigma = \frac{Q}{\pi R^2} \quad ; \quad \sigma' = \frac{Q'}{\pi R'^2} \ . \tag{1}$$

The system of coordinates is suitably chosen so that the two disks lie parallel to each other and perpendicular to the z-direction. The first disk denoted as disk 1 has charge Q and lies in the $z = 0$ plane. The other disk denoted as disk 2 contains a charge Q' and lies at some arbitrary z plane. The origin of the system of coordinates is at the center of disk 1. The separation distance between the two coaxial parallel disks is denoted as $d = |z| \geq 0$. The system is shown in Fig. 1.

We adopt the following notation for the 3D position vector, $\vec{r} = \vec{\rho} + \vec{k}\, z$ where $\vec{\rho} = \vec{i}\, x + \vec{j}\, y$ represents a 2D vector. The vectors $\vec{i}, \vec{j}, \vec{k}$ are unit vectors for the x, y, z directions of a Cartesian system of coordinates. For the given choice of the coordinative system, the elementary charges in the two coaxial disks are confined to the corresponding domains:

$$\Omega : \{0 \leq |\vec{\rho}| \leq R \ ; \ z = 0\} \quad ; \quad \Omega' : \{0 \leq |\vec{\rho}| \leq R' \ ; \ z\} \ . \tag{2}$$

It is assumed that any two arbitrary elementary charges, dQ and dQ' interact with each other via a standard Coulomb interaction potential, $k_e\, dQ\, dQ'/|\vec{r} - \vec{r}\,'|$ where k_e is Coulomb's electric constant and $|\vec{r} - \vec{r}\,'|$ is the separation distance between the pair of elementary charges.

The electrostatic potential created by a uniformly charged disk with radius, R containing a total charge, Q lying in the $z = 0$ plane (with its center coinciding with the origin of the system of coordinates) has been calculated in Ref.[32] and is given by the following expression:

$$V(\rho, z) = 2\, k_e\, Q \int_0^\infty dk\, J_0(k\, \rho)\, \frac{J_1(k\, R)}{(k\, R)}\, e^{-k\, |z|} \ , \tag{3}$$

where $J_m(x)$ are Bessel functions of the first kind of integral m-th order and k is a dummy variable (not to be confused with the unit vector $\vec{k}$ of z-axis). The problem has cylindrical symmetry, thus, the value of the electrostatic potential depends only on ρ and z (not the angle).

The corresponding expression for the electric field created by the disk can be obtained from Eq.(3) via the well-known formula:

$$\vec{E}(\rho, z) = -\vec{\nabla}\, V(\rho, z) \ , \tag{4}$$

where $\vec{\nabla}$ is the 3D gradient operator. Note that the electric field reflects the cylindrical symmetry of the potential. It is a straightforward exercise to calculate the vector components of the electric field in a cylindrical system of coordinates by using the expression of the electrostatic potential in Eq.(3). For instance, the z component of the electric field, $E_z(\rho, z)$ at an arbitrary location above the plane of the uniformly charged disk ($z > 0$) can be written as:

$$E_z(\rho, z) = \frac{2\, k_e\, Q}{R} \int_0^\infty dk\, J_0(k\, \rho)\, J_1(k\, R)\, e^{-k\, z} \quad ; \quad z > 0 \ . \tag{5}$$

In Fig. 2 we plot $E_z(\rho, z)$ above the disk ($z > 0$) as a function of z/R for two selected values of the dimensionless parameter, ρ/R. The electric field is given in units of $k_e\, Q/R^2$.

The total electrostatic potential created by the system of the two uniformly charged disks can be calculated via the superposition principle. In Fig. 3 we plot the resulting total electrostatic potential along the z direction for two cases: (i) $Q' = Q; R' = R$ (filled circles) and (ii) $Q' = -Q; R' = R$ (filled squares). It is assumed that disk 1 (with charge Q and radius R) is located at $z = 0$ while disk 2 is located at $z = R$. Distances along the z direction are given as a function of z/R. The electrostatic potential is given in units of $k_e\, Q/R$.

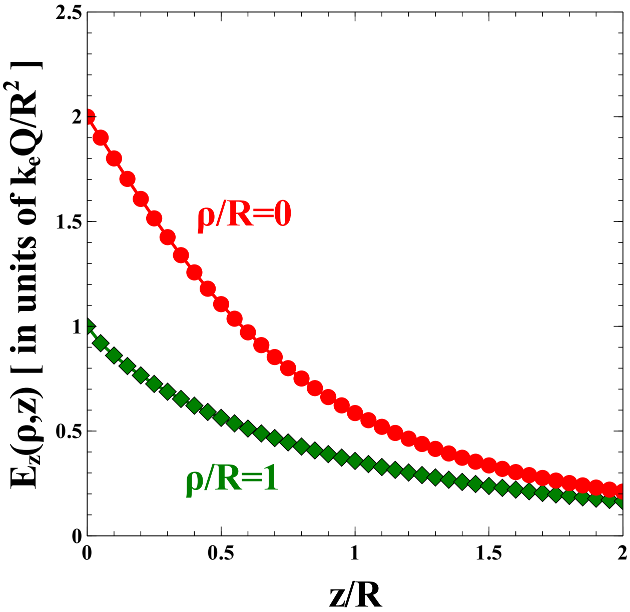

FIG. 2: Plot of the z component of the electric field created by a uniformly charged disk with radius, R and total charge, Q lying on the $z = 0$ plane with its center at the origin. The quantity $E_z(\rho, z)$ is calculated at points above the plane of the disk ($z > 0$) as a function of z/R for values of $\rho/R = 0$ (filled circles) and $\rho/R = 1$ (filled diamonds). The electric field is expressed in units of $k_e\,Q/R^2$.

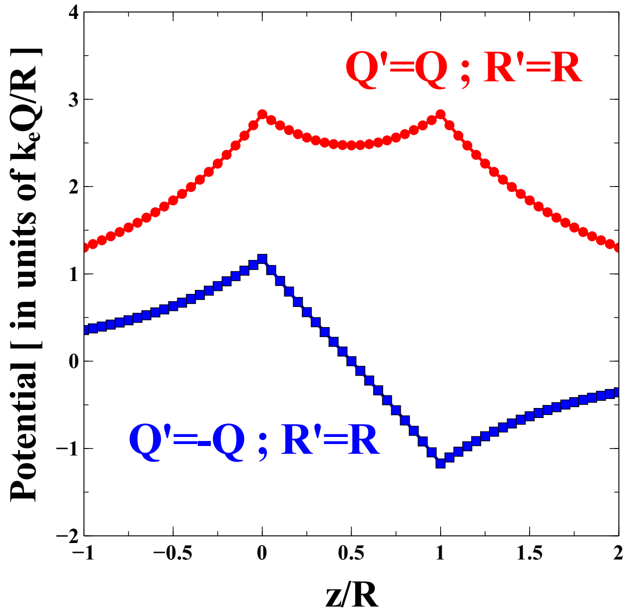

FIG. 3: Plot of the total electrostatic potential along the z direction created by the system of the two uniformly charged disks. Two cases are considered: (i) $Q' = Q; R' = R$ (filled circles) and (ii) $Q' = -Q; R' = R$ (filled squares). Disk 1 with charge Q and radius R has its center located at $z = 0$ while disk 2 with charge Q' and radius R' has its center located at $z = R$. The potential is expressed in units of $k_e\,Q/R$.

The integral expression in Eq.(3) is prefered since it allows one to obtain easily the result for the electrostatic potential of a point charge in the $R \to 0$ limit starting from the well known limit formula:

$$\lim_{x\to 0} \frac{J_1(x)}{x} = \frac{1}{2} \; . \tag{6}$$

With help from the expression in Eq.(3), the electrostatic interaction energy between the two coaxial parallel uniformly charged disks can be written as:

$$U_{R\,R'}(z) = \int_{\Omega'} dQ'\, V(\rho, z) \; , \tag{7}$$

where $dQ' = \sigma'\, d^2\rho$ and Ω' is the domain of integration over the surface of disk 2. The next step is to substitute the expression of $V(\rho, z)$ from Eq.(3) into Eq.(7). The integration over variable ρ involves a standard table integral, for example see pg. 278 of Ref.[33]:

$$\int dx\, J_0(x) = x\, J_1(x) \; . \tag{8}$$

After some algebra and a rearrangement of terms we may write the final result as:

$$U_{R\,R'}(z) = 4\,k_e\,Q\,Q' \int_0^\infty dk\, \frac{J_1(k\,R)}{(k\,R)}\, \frac{J_1(k\,R')}{(k\,R')}\, e^{-k\,|z|} \; . \tag{9}$$

Note that the expression in Eq.(9) is invariant if one exchanges R with R' (and Q with Q'). The expression in Eq.(9) allows one to recover the result for the interaction energy of two point charges in a straightforward way by using the formula for the limit in Eq.(6):

$$U_{R\to 0\,R'\to 0}(z) = \frac{k_e\,Q\,Q'}{|z|} \; . \tag{10}$$

Let's now consider the case when the two disks have the same radius:

$$R' = R \; . \tag{11}$$

This is the common situation that arises in the case of a circular parallel plate capacitor. Substituting for $R' = R$ in Eq.(9) leads to the following result:

$$U_{R\,R}(z) = 4\,k_e\,Q\,Q' \int_0^\infty dk \left[\frac{J_1(k\,R)}{(k\,R)}\right]^2 e^{-k\,|z|} \; . \tag{12}$$

From now on, let's assume that $R \neq 0$. An immediate result obtained from Eq.(12) is the value of the electrostatic interaction energy at $z = 0$ which is:

$$U_{R\,R}(z = 0) = \frac{16}{3\,\pi}\, \frac{k_e\,Q\,Q'}{R} \; . \tag{13}$$

The result in Eq.(13) can be easily verified by recalling that:

$$\int_0^\infty dx \left[\frac{J_1(x)}{x}\right]^2 = \frac{4}{3\,\pi} \ . \qquad (14)$$

The expression for the electrostatic interaction energy at an arbitrary z can be suitably written as:

$$U_{R\,R}(z) = \frac{4\,k_e\,Q\,Q'}{R} \int_0^\infty du \left[\frac{J_1(u)}{u}\right]^2 e^{-\frac{|z|}{R}\,u} \ , \qquad (15)$$

where $u = k\,R$ is a convenient dummy variable. Let's introduce a new dimensionless variable:

$$a = \frac{|z|}{R} \geq 0 \quad ; \quad R \neq 0 \ , \qquad (16)$$

that is real and non-negative. With a slight change of notation, one writes the interaction energy as $U_{RR}(a)$ and expresses it as:

$$U_{R\,R}(a) = \frac{4\,k_e\,Q\,Q'}{R}\, f(a) \ , \qquad (17)$$

where

$$f(a) = \int_0^\infty du \left[\frac{J_1(u)}{u}\right]^2 e^{-a\,u} \ , \qquad (18)$$

is an auxiliary function. The integral representing the auxiliary function in Eq.(18) can be calculated analytically. The final expression for the auxiliary function $f(a)$ may be written as:

$$f(a) = -\frac{a}{2} + \frac{a}{6\,\pi}\left[(4-a^2)\,E\left(-\frac{4}{a^2}\right) + (4+a^2)\,K\left(-\frac{4}{a^2}\right)\right] \ , \qquad (19)$$

where $K(m)$ and $E(m)$ are, respectively, complete elliptic integrals of the first and second kind:

$$K(m) = \int_0^{\pi/2} \frac{d\,\theta}{\sqrt{1 - m\,\sin^2(\theta)}} \ , \qquad (20)$$

and

$$E(m) = \int_0^{\pi/2} d\,\theta\,\sqrt{1 - m\,\sin^2(\theta)} \ . \qquad (21)$$

The notation adopted above follows that of Ref.[34] (see pgs. 587-607). For real values of parameter, m (where it is assumed that $m \leq 1$), the values of the complete elliptic integrals $K(m)$ and $E(m)$ are real. One should be careful when using formulas involving complete elliptic integrals since different notations are widely used in the literature[35].

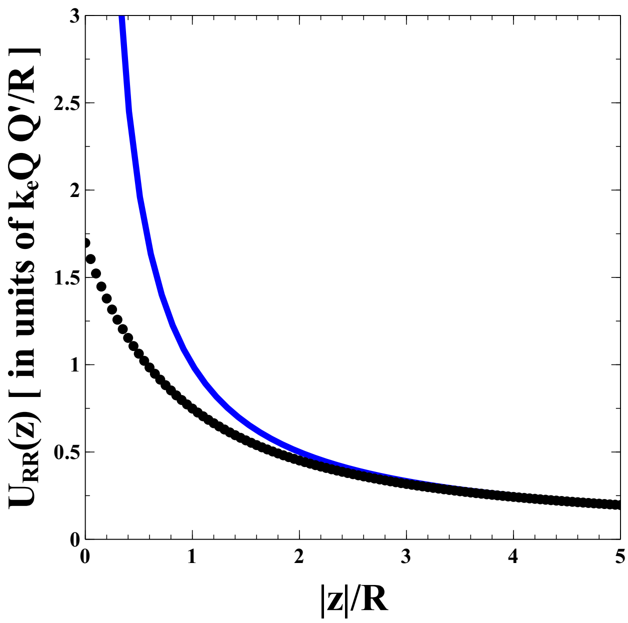

FIG. 4: Electrostatic interaction energy of a pair of coaxial parallel uniformly charged disks as a function of $|z|/R$ where $|z|$ is the separation distance between the centers of the disks and R is the radius of the disks. The energy is expressed in units of $k_e\,Q\,Q'/R$ (filled circles). The result is compared to a Coulomb interaction potential for two point charges, Q and Q' (solid line).

III. DISCUSSION AND CONCLUSIONS

We calculated exactly the electrostatic interaction energy between two coaxial parallel uniformly charged disks with arbitrary radii containing an arbitrary amount of charge. The final result for the physically interesting case of two coaxial disks with same radius (but each containing an arbitrary amount of charge) is given in general terms as an analytic function of the arbitrary radius and separation distance between the centers of the two disks. The final expression is written in convenient form in terms of complete elliptic integral functions of the first and second kind. In Fig. 4 we plot the electrostatic interaction energy between a pair of coaxial parallel uniformly charged disks, $U_{R\,R}(z)$ in energy units of $k_e\,Q\,Q'/R$ as a function of $|z|/R$. The Coulomb interaction potential for two point charges, Q and Q' at a separation distance, z is also plotted in same energy units, $U_C(z) = \frac{1}{|z|/R}\,\frac{k_e\,Q\,Q'}{R}$. One notes that the magnitude of the Coulomb potential, $U_C(z)$ is always larger than the magnitude of $U_{R\,R}(z)$ for any given finite separation distance. Hence, it is clear that the interaction potential between the coaxial parallel uniformly charged disks is softer than the corresponding Coulomb potential for a pair of point charges. This means that, in this case, the quantity $U_C(z) - U_{R\,R}(z)$ does not show any sort of Lennard-Jones (LJ) potential features in striking difference with the case of a pair of identical coplanar uniformly charged disks[36].

The theoretical model considered in this work can be generalized to study the interaction energy between any arbitrary numbers of coaxial parallel uniformly charged disks. For example, let's consider an arbitrary system of $N \geq 2$ identical uniformly charged disks (all with same radius, R) which are coaxial and parallel to each other. The disks are localized at positions, $z_1, \ldots, z_N$ and each contains, respectively, charges, $Q_1, \ldots, Q_N$. The total electrostatic energy of such a system can be calculated as:

$$U(z_1, \ldots, z_N) = \sum_{j>i}^{N} U_{R\,R}(z_j - z_i) \ , \tag{22}$$

where $|z_j - z_i|$ represents the separation distance between the center of disk j at z_j with respect to disk i at z_i and $U_{R\,R}(z)$ is given from Eq.(15) or equivalently from Eq.(17), Eq.(18) and Eq.(19). The expression in Eq.(22) reduces to the result in Eq.(15) for the $N = 2$ case if one assumes that $z_1 = 0$, $Q_1 = Q$, $z_2 = z$ and $Q_2 = Q'$.

The general expression obtained through the present approach reproduces the known results for special cases. For example, the expression in Eq.(13) is directly related to the calculation of the Coulomb self-energy of a uniformly charged disk. More precisely, one can obtain the Coulomb self-energy of a uniformly charged disk from Eq.(13) by assuming that $Q = Q'$ in the expression in Eq.(13) and then dividing the resulting quantity by 2. The known expression for the Coulomb self-energy of a uniformly charged disk with radius R containing a total charge Q is:

$$U_{disk} = \frac{8}{3\,\pi} \, \frac{k_e \, Q^2}{R} . \tag{23}$$

Knowledge of the Coulomb self-energy of a uniformly charged disk is crucial to analyze various models in condensed matter physics. For example, such a quantity represents the background-background energy term for studies of the quantum Hall effect in systems of electrons in a disk geometry[37–42].

The disk configuration setup studied in this work is often encountered in many scientific disciplines ranging from nanoscience[43] to biological systems[44]. A possible application of this model would be in studies of the electrostatic interaction between coaxial parallel uniformly charged disk structures embedded into an electrolyte solution of mobile ions. For such a case, one may use the same numerical approach as that for the counterpart study of parallel charged cylinders in an electrolyte solution[1]. It is expected that the interplay of many factors key to describe the interaction of charged bodies in an electrolyte solution may lead to interesting scenarios similar to the ones seen for two parallel charged cylinders which were recently studied via a modified nonlinear Poisson-Boltzmann equation[1].

It is also worthwhile mentioning that the final results obtained in this work can be useful to mathematical studies. Exact expressions when available can be used as a reference to gauge the accuracy of various computational tools used to solve numerically electrostatic problems. The current result for the electrostatic interaction energy of two coaxial parallel uniformly charged disks may be directly applied to coaxial systems consisting of circular plates of this nature that arise in various fields, for instance to the case of a circular parallel plate capacitor[45].

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