Results for charged disks with different forms of surface charge density

Orion Ciftja*

Department of Physics, Prairie View A&M University, Prairie View, Texas 77446, USA

(Dated: January 17, 2020)

The equilibrium surface charge density of a charged disk is strikingly different from that of a uniformly charged disk. This dissimilarity is reflected on the corresponding Coulomb electrostatic potentials that they create. However, it is shown in this work that the Coulomb self-energy of these two differently charged disks is not much influenced by the pronounced difference that exists between their corresponding surface charge density distributions.

PACS numbers: 41.20.Cv, 73.43.Cd, 73.43.-f.

Keywords: Electrostatics; Theory and modeling; Quantum Hall effects.

Finding the equilibrium surface charge density of a two-dimensional (2D) body (which results in the body's surface becoming an equipotential surface) is a very difficult problem that cannot be solved analytically except for very few cases of regular bodies [1]. One such special case is that of an isolated, infinitely thin, flat, cicular conducting disk of radius, R containing a total charge, Q, for simplicity, assumed to be positive. The equilibrium surface charge density in such a case is:

$$\sigma(\rho) = \frac{Q}{2\,\pi\,R}\,\frac{1}{\sqrt{R^2-\rho^2}} \quad ; \quad 0 \le \rho \le R \ , \tag{1}$$

where $\rho = \sqrt{x^2+y^2} \ge 0$ represents the 2D radial distance from the center of the disk, (x, y) are 2D position coordinates and R is the radius of the disk. Note the special values:

$$\sigma(\rho = 0) = \frac{1}{2}\,\frac{Q}{\pi\,R^2} \ , \tag{2}$$

and

$$\sigma(\rho \to R^-) = +\infty \ . \tag{3}$$

On the other hand, if one assumes that the disk is uniformly charged, the surface charge density reads:

$$\sigma_0(\rho) = \frac{Q}{\pi\,R^2} \quad ; \quad 0 \le \rho \le R \ . \tag{4}$$

As shown in Fig. 1, there is a striking difference between these two charge distributions. On this regard, it suffices to note that the equilibrium surface charge density in Eq.(1) diverges for points on the edge of the disk as pointed out in Eq.(3). Expressions for the electrostatic potential created by a charged disk with either equilibrium surface charge density or uniform charge density are readily available in the literature. For simplicity, we

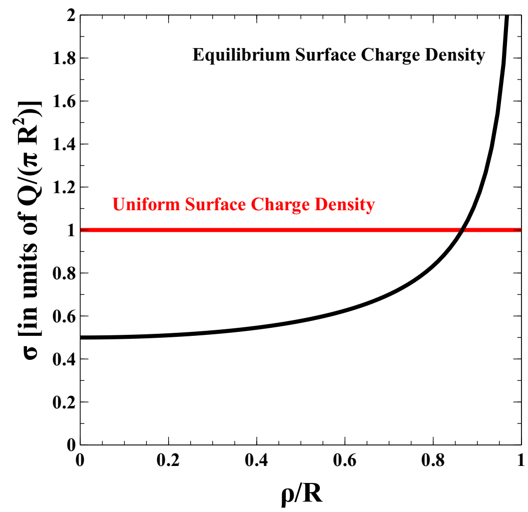

FIG. 1. Surface charge density of a charged disk for the cases of equilibrium surface charge distribution (the disk is an equipotential surface) and uniform surface charge distribution. The radial distance from the center of the disk, ρ is given in units of R. The surface charge density is given in units of $Q/(\pi\,R^2)$ where Q is the total charge contained in the disk and R is the radius of the disk.

write them in integral form for points on the plane of the disk. The electrostatic potential $V(\rho)$ corresponding to the equilibrium surface charge density, $\sigma(\rho)$ may be given as:

$$V(\rho) = k_e\,Q \int_0^{\infty} dk\, J_0(k\,\rho)\,\frac{\sin(k\,R)}{(k\,R)} \ , \tag{5}$$

while $V_0(\rho)$ corresponding to $\sigma_0(\rho)$ is:

$$V_0(\rho) = 2\,k_e\,Q \int_0^{\infty} dk\, J_0(k\,\rho)\,\frac{J_1(k\,R)}{(k\,R)} \ . \tag{6}$$

In the expressions above, k_e is Coulomb's electric constant, k is a dummy variable and $J_n(x)$ are Bessel functions of order $n = 0$ and 1. One can easily verify that the expression in Eq.(5) leads to an equipotential on the

* ogciftja@pvamu.edu

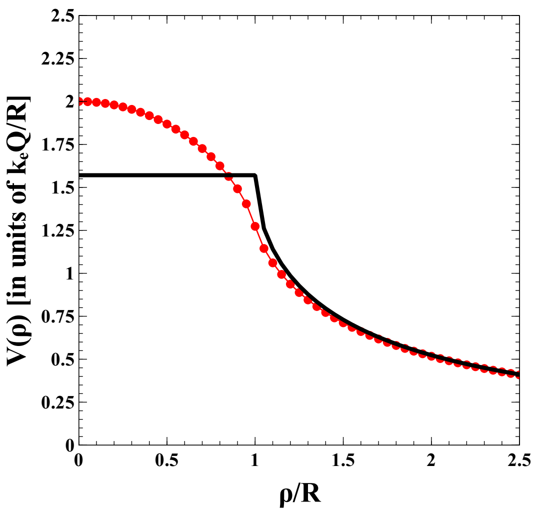

FIG. 2. Electrostatic potential on the plane of the disk for the cases of equilibrium surface charge density (solid line) and uniform surface charge distribution (dotted line). The radial distance from the center of the disk, ρ is given in units of R. The electrostatic potential on the plane of the disk, $V(\rho)$ is given in units of $k_e\,Q/R$ where k_e is Coulomb's electric constant, Q is the total charge contained in the disk and R is the radius of the disk.

surface of the disk:

$$V(0 \leq \rho \leq R) = \frac{\pi}{2}\,\frac{k_e\,Q}{R} \ . \tag{7}$$

The integral expressions for the electrostatic potential provided in Eq.(5) and Eq.(6) are very convenient since they allow one to obtain the Coulomb potential for a point charge in the $R \to 0$ limit by relying on standard mathematical formulas such as: $\lim_{x\to 0} \sin(x)/x = 1$, $\lim_{x\to 0} J_1(x)/x = 1/2$ and $\int_0^{\infty} dx\, J_0(x) = 1$. More explicit exact analytical expressions for the electrostatic potential can be obtained after completing the integration [2, 3]. As seen from Fig. 2, the electrostatic potential on the plane of the charged disk for the case of an equilibrium surface charge density profoundly differs from that resulting from a uniform surface charge density.

Based on these observations, one is tempted to believe that the Coulomb self-energy for a charged disk with an equilibrium surface charge density distribution, U should be quite different from that of a uniformly charged disk, denoted as U_0. These quantities can be readily calculated from the following expression:

$$U = \frac{1}{2} \int_{Disk} d^2\rho\, \sigma(\rho)\, V(\rho) \ , \tag{8}$$

where $d^2\rho$ represents an elementary disk area and the integration is carried out over the disk's surface. Obviously, U_0 is calculated from Eq.(8) by using the corresponding expressions for $\sigma_0(\rho)$ and $V_0(\rho)$. The details of the calculations are left to the reader. The final results are:

$$U = \frac{\pi}{4}\,\frac{k_e\,Q^2}{R} \approx 0.78540\,\frac{k_e\,Q^2}{R} \ , \tag{9}$$

and

$$U_0 = \frac{8}{3\,\pi}\,\frac{k_e\,Q^2}{R} \approx 0.84883\,\frac{k_e\,Q^2}{R} \ . \tag{10}$$

A comparison between values U and U_0 indicates that, somehow unexpectedly, the relative difference between them, $|(U-U_0)/U|$ is small less than 10 % in percentage.

To conclude, in this work we show that the Coulomb self-energy of a uniformly charged disk does not differ much from its counterpart with an equilibrium surface charge density distribution. This observation seems to suggest that a similar conclusion may hold also for a square/rectangular geometry where, as far as we know, there are no known exact results for the equilibrium surface charge density. As a result the assumption of a uniform surface charge density (whether out of necessity or because of its simplicity) for various 2D shapes [4–7] can be seen as legitimate in all models where our main concern is the Coulomb self-energy of the object. For example, a uniformly charged disk or square plate is a key ingredient to models that deal with 2D systems of electrons since such a body represents the neutralizing background in a jellium approximation setup [8, 9]. Similarly, the assumption of a uniformly charged disk is used by many researchers to represent the neutralizing background in studies of quantum Hall systems of electrons in a disk geometry [10–17].

ACKNOWLEDGEMENTS

This research was supported in part by National Science Foundation (NSF) Grant No. DMR-1705084.

[1] J. D. Jackson, Classical Electrodynamics, Third Edition, John Wiley & Sons (1999).
[2] O. Ciftja and I. Hysi, Appl. Math. Lett. **24**, 1919 (2011).
[3] O. Ciftja, Res. Phys. **7**, 1674 (2017).
[4] O. Ciftja, Adv. Math. Phys. **2016**, 7207536 (2016).
[5] O. Ciftja, T. Rasco, G. Vargas, and S. Dunn, Phys. Status Solidi C **9**, 1337 (2012).
[6] O. Ciftja, RSC Adv. **9**, 7849 (2019).
[7] O. Ciftja, L. Escamilla, and R. Mills, Adv. Condens. Matter Phys. **2015**, 851356 (2015).
[8] O. Ciftja, Physica B **458**, 92 (2015).
[9] O. Ciftja, Phys. Scr. **94**, 105806 (2019).
[10] R. Morf and B. I. Halperin, Phys. Rev. B **33**, 2221 (1986).
[11] J. Xia, J. Math. Phys. **40**, 150 (1999).

[12] G. Dev and J. K. Jain, Phys. Rev. B **45**, 1223 (1992).
[13] O. Ciftja, J. Math. Phys. **52**, 122105 (2011).
[14] O. Ciftja, C. Ozurumba, and F. Ujeyah, J. Low. Temp. Phys. **170**, 166 (2013).
[15] O. Ciftja, J. Kor. Phys. Soc. **62**, 1550 (2013).
[16] O. Ciftja, Phys. Rev. B **95**, 075410 (2017).
[17] O. Ciftja, J. Phys. Chem. Solids **130**, 256 (2019).