A uniformly charged circular disk with an anisotropic Coulomb interaction potential

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Abstract

We consider a uniformly charged circular disk containing elementary charges that interact with an anisotropic Coulomb interaction potential. Such an anisotropic Coulomb interaction potential has been recently considered within the framework of a two-dimensional system of electrons in the quantum Hall regime. The anisotropy of the potential is introduced by a phenomenological parameter that can be tuned continuously as to incorporate the standard isotropic Coulomb interaction potential as a special case. The energy of the uniformly charged circular disk is calculated exactly for the given anisotropic interaction potential as a function of the anisotropy parameter. The results obtained are related to electrostatic problems that involve uniformly charged flat elliptical plates and may be useful to understand finite two-dimensional systems of electrons in the quantum Hall regime in which all point charges interact with the anisotropic Coulomb interaction potential presently considered.

Keywords: Uniformly charged body, Anisotropic Coulomb potential, Electrostatic energy, Coulomb self-energy

1. Introduction

Generally speaking, the electron gas model is one of the most important theoretical models in condensed matter physics and has been studied under a variety of different conditions [1, 2, 3, 4]. In particular, a two-dimensional electron gas (2DEG) in a perpendicular magnetic field is fundamental to the understanding of very important novel phenomena discovered during the last few decades such as integer quantum Hall effect [5, 6] and fractional quantum Hall effect [7, 8, 9, 10, 11]. Although various two-dimensional (2D) geometries are possible, a very popular choice in the field of quantum Hall studies is that of a disk geometry. For such a selection, one assumes that the positive neutralizing charge is uniformly spread on a circular disk region. A common assumption for many quantum Hall studies is to consider point charges that interact with a standard Coulomb interaction potential. Obviously, a Coulomb interaction potential is isotropic in the sense that the interaction energy of any pair of point charges depends only on their relative separation distance.

However, recent quantum Hall studies have emphasized the role played by various sources of anisotropy including anisotropic behavior introduced in the system via a specific anisotropic Coulomb interaction potential [12, 13, 14]. Such an anisotropic interaction potential may be viewed as some sort of distortion of the common Coulomb potential along the respective x and y coordinates. The degree of anisotropy of this distorted anisotropic Coulomb potential is tuned via a phenomenological parameter that can be varied in a way as to include the isotropic Coulomb interaction potential as a special case. While one may limit the use of such an anisotropic Coulomb interaction potential only to electron-electron interactions, the completeness of a given electron gas model requires that all the point charges of the system (including the neutralizing background) should interact similarly via the same interaction potential.

Based on these considerations, one of the most basic questions posed for any theoretical work on systems containing charge is that of the calculation of the energy of a given charged jellium background [15, 16]. Differently from earlier cases involving a uniformly charged jellium background, the new twist in the current work is the assumption that the elementary charges of the system interact with an anisotropic Coulomb interaction potential. In a nutshell, the problem posed is how to calculate the total energy of a uniformly charged circular disk in which the constituent elementary charges do not interact with a conventional isotropic Coulomb interaction potential.

As expected, presence of an anisotropic interaction makes the calculation of the energy a rather difficult task. However, it is shown in this work that an exact analytical result is possible if one implements a specific solution strategy that relies on a convenient mathematical transformation of coordinates. The current work is a necessary preliminary step to obtain the total energy of any given finite system of electrons in the quantum Hall regime under the assumption that all point charges of the system interact with the same anisotropic Coulomb interaction potential. From a more general point of view, an analytical result of this nature may have its own interest from the perspective

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of electrostatics [17, 18, 19] or electronics [20, 21] since it can be used in more systematic studies of charged systems that employ such a particular form of the anisotropic interaction potential between point charges.

2. Model

We consider a uniformly charged circular disk with surface area, πR^2 where R is the radius of the disk. A total charge Q is uniformly distributed over such an area. The result for the electrostatic self-energy of the body under consideration does not depend on the sign of Q, therefore, it does not matter whether charge Q is positive or negative. For simplicity, from now on, we consider Q to be positive. Typically, a positive uniformly charged background also represents the neutralizing background in an electron gas model, therefore, the choice $Q > 0$ is reasonable from this point of view, too. The uniform surface charge density may be written as:

$$\sigma = \frac{Q}{\pi R^2} \ . \tag{1}$$

We assume an anisotropic Coulomb interaction potential between two point charges q_i and q_j that has the form:

$$v_\gamma(\vec{r}_i - \vec{r}_j) = \frac{k_e \, q_i \, q_j}{\sqrt{\frac{(x_i - x_j)^2}{\gamma^2} + \gamma^2 \, (y_i - y_j)^2}} \ , \tag{2}$$

where k_e is Coulomb's electric constant, $\vec{r}_i = (x_i, y_i)$ is the 2D position vector for the charge q_i, $\vec{r}_j = (x_j, y_j)$ is the 2D position vector for the charge q_j and $\gamma > 0$ represents a positive phenomenological anisotropy parameter whose value $\gamma \neq 1$ leads to an anisotropic interaction. When the anisotropy parameter has the value $\gamma = 1$, the interaction potential in Eq.(2) becomes:

$$v_{\gamma=1}(\vec{r}_i - \vec{r}_j) = \frac{k_e \, q_i \, q_j}{|\vec{r}_i - \vec{r}_j|} \ , \tag{3}$$

which represents the standard isotropic Coulomb interaction potential for a pair of point charges.

We start the calculation of the electrostatic energy of the body by considering two elementary charges, $dq_1 = \sigma \, d^2 r_1$ and $dq_2 = \sigma \, d^2 r_2$ at respective positions $\vec{r}_1$ and $\vec{r}_2$ interacting via the potential $v_\gamma(\vec{r}_1 - \vec{r}_2)$ and write the electrostatic energy of the circular disk as:

$$U(\gamma) = \frac{k_e \, \sigma^2}{2} \int_{\pi R^2} d^2 r_1 \int_{\pi R^2} d^2 r_2 \frac{1}{\sqrt{\frac{(x_1 - x_2)^2}{\gamma^2} + \gamma^2 \, (y_1 - y_2)^2}} \tag{4}$$

where $d^2 r_i = dx_i \, dy_i$ (i=1,2) is an elementary surface area and πR^2 is the domain of integration which is a circular disk with area πR^2.

The expression for $U(\gamma)$ can also be written as:

$$U(\gamma) = \frac{\sigma}{2} \int_{\pi R^2} d^2 r \, V_\gamma(\vec{r}) \ , \tag{5}$$

where

$$V_\gamma(\vec{r}) = k_e \, \sigma \int_{\pi R^2} d^2 r' \frac{1}{\sqrt{\frac{(x - x')^2}{\gamma^2} + \gamma^2 \, (y - y')^2}} \ , \tag{6}$$

represents the electrostatic potential created by the whole uniformly charged circular disk at some 2D position $\vec{r}$ on the disk surface. At this juncture, it would be interesting to carry out a simple analysis of the behavior of the electrostatic potential at the surface of the uniformly charged circular disk. For convenience, let's try to understand the dependence of the electrostatic potential as a function of parameter γ by chosing the center of the disk as a case study. To this effect, let us calculate:

$$V_\gamma(\vec{r} = 0) = k_e \, \sigma \int_{\pi R^2} d^2 r' \frac{1}{\sqrt{\frac{x'^2}{\gamma^2} + \gamma^2 \, y'^2}} \ . \tag{7}$$

By switching to 2D polar coordinates, $x' = r' \cos(\theta')$ and $y' = r' \sin(\theta')$ one obtains:

$$V_\gamma(\vec{r} = 0) = k_e \, \sigma \, R \int_0^{2\pi} d\theta' \frac{1}{\sqrt{\frac{\cos^2(\theta')}{\gamma^2} + \gamma^2 \, \sin^2(\theta')}} \ . \tag{8}$$

From here, it is easy to see the result:

$$V_{\gamma=1}(\vec{r} = 0) = 2 \pi \, k_e \, \sigma \, R = \frac{2 \, k_e \, Q}{R} \ , \tag{9}$$

which represents the value of the electrostatic potential at the center of a uniformly charged circular disk for the case of a standard Coulomb interaction potential between point charges [22]. Based on the results from Eq.(8) and Eq.(9), one can write quite generally that:

$$\frac{V_\gamma(\vec{r} = 0)}{V_{\gamma=1}(\vec{r} = 0)} = \frac{1}{2\pi} \int_0^{2\pi} d\theta' \frac{1}{\sqrt{\frac{\cos^2(\theta')}{\gamma^2} + \gamma^2 \, \sin^2(\theta')}} \ . \tag{10}$$

A careful analytical analysis of the quantity on the right-hand-side of Eq.(10) allows us to conclude that:

$$V_\gamma(\vec{r} = 0) \leq V_{\gamma=1}(\vec{r} = 0) \ , \tag{11}$$

where the equality happens only for $\gamma = 1$. This means that the value of $V_{\gamma \neq 1}(\vec{r} = 0)$ (for $Q > 0$) is smaller than the value of its counterpart for $\gamma = 1$. Recall that $\gamma = 1$ represents the case of a standard isotropic Coulomb interaction potential. This analysis seems to suggest that any anisotropic deformation of the Coulomb interaction potential of the form considered in this work leads to a "softening" for $\gamma \neq 1$ values.

3. Results and Discussion

The calculation of the electrostatic energy in Eq.(4) involves a non-trivial four-dimensional integral. Furthermore, the integrand function depends separately on the x

and y coordinates when $\gamma \neq 1$. The calculation of the integral in Eq.(4) is simplified if we introduce the following new coordinates:

$$x'_i = \frac{x_i}{\gamma} \quad ; \quad y'_i = \gamma\, y_i \quad ; \quad i = 1, 2 \ . \tag{12}$$

Note that this transformation leads to: $dx_i\, dy_i = dx'_i\, dy'_i$. One can verify that the domain of integration for the new primed coordinates becomes an ellipse:

$$D \ : \ \left\{ \frac{x'^2}{a^2} + \frac{y'^2}{b^2} \leq 1 \right\} , \tag{13}$$

where the semimajor and the semiminor axis of the ellipse, respectively, are:

$$a = \frac{R}{\gamma} \quad ; \quad b = \gamma\, R \ . \tag{14}$$

Within the framework of the new primed coordinative system, one has:

$$U(\gamma) = \frac{k_e\, \sigma^2}{2} \int_D d^2 r'_1 \int_D d^2 r'_2 \ \frac{1}{|\vec{r}_1{}' - \vec{r}_2{}'|} \ , \tag{15}$$

where $D \ : \ \left\{ \frac{x'^2}{a^2} + \frac{y'^2}{b^2} \leq 1 \right\}$ is the elliptical domain of integration given by Eq.(13). The integral in the right-hand-side of Eq.(15) is calculated with help from the following result displayed in Eq.(A.12) of Appendix A:

$$\begin{aligned} I(a,b) \quad &= \quad \int_D d^2 r_1 \int_D d^2 r_2 \ \frac{1}{|\vec{r}_1 - \vec{r}_2|} \\ &= \frac{32}{3}\, (a\, b)^2\, \frac{1}{b}\, K\left(m = 1 - \frac{a^2}{b^2} \right) \ , \end{aligned} \tag{16}$$

where $K(m) = \int_0^{\pi/2} \frac{d\,\theta}{\sqrt{1 - m\, \sin^2(\theta)}}$ represents the complete elliptic integral of the first kind with parameter m. This notation for the complete elliptic integral of the first kind is the one adopted in Ref. [23, 24]. Other notations for such a function are widely available in the literature, therefore readers should be careful when using readily available formulae. Some properties of the complete elliptic integral of the first kind are given in Appendix B.

Using the result, $\pi\, a\, b = \pi\, R^2$ which is easily obtained from Eq.(14), using the formula in Eq.(16) and after carrying out some straighforward transformations, one writes the energy in Eq.(15) as:

$$U(\gamma) = \frac{16}{3\, \pi^2}\, \frac{k_e\, Q^2}{R}\, F(\gamma) \ , \tag{17}$$

where $F(\gamma)$ is an auxiliary function:

$$F(\gamma) = \frac{1}{\gamma}\, K\left(m = 1 - \frac{1}{\gamma^4} \right) \ . \tag{18}$$

One can verify that the maximum value of this auxiliary function is obtained for $\gamma = 1$ and is given by:

$$F(\gamma = 1) = \frac{\pi}{2} \ . \tag{19}$$

From the general formula in Eq.(17), one sees that:

$$U(\gamma = 1) = \frac{8}{3\, \pi}\, \frac{k_e\, Q^2}{R} \ . \tag{20}$$

The result in Eq.(20) represents the correct value for the Coulomb self-energy of a uniformly charged circular disk with total charge Q and radius R for the case of a standard Coulomb interaction potential [25].

Having reached this point, a keen reader may immediately notice that the quantity $U(\gamma)$ in Eq.(15) represents an explicit expression for the electrostatic energy of a uniformly charged infinitely thin elliptical disk for point charges that interact with a common Coulomb interaction potential. This means that the presented results can be directly utilized within the framework of electrostatic studies that deal with explicit energy calculations for charged elliptical plates where various methods are used to study thin flat elliptical plates with arbitrary charge distributions [26]. The results derived in this work are also related to certain problems involving elliptical cracks [27]. Analytical results of the nature derived here can also serve as powerful tools to assess the accuracy of numerical codes that are used for integral calculations. In particular, the availability of analytical results for the energy allows one to gauge the accuracy of boundary element integration codes in a non-trivial geometry like that of a charged elliptical infinitely thin disk [28]. Furthermore, the present work is also related to the understanding of the physics of charged-particle beam systems with elliptic cross-sections. In particular, electron beams of elongated elliptic cross-sections (or "sheet" beams) have long generated great interest in vacuum electronics [29, 30].

4. Conclusions

To conclude, we considered the problem of the electrostatic energy of a uniformly charged circular disk containing elementary charges that interact with an anisotropic Coulomb interaction potential of a specific form recently considered in the context of quantum Hall effect studies [12, 13]. The energy of the system was calculated exactly in closed form in terms of an auxiliary function that is related to a complete elliptic integral of the first kind. This auxiliary function depends only on the interaction anisotropy parameter denoted as γ. The result for $\gamma = 1$ reproduces the expected value for the Coulomb self-energy of a uniformly charged circular disk for the case of a standard isotropic Coulomb interaction.

The expression in Eq.(15) deserves some special attention. It turns out that one can interpret such a result as representing the Coulomb self-energy (when elementary charges interact with a standard isotropic Coulomb interaction potential) of a uniformly charged elliptical plate bound by the domain, $x^2/a^2 + y^2/b^2 \leq 1$. Knowing that the auxiliary function $F(\gamma)$ has a maximum at $\gamma = 1$ leads us to conclude that, for the case of a standard Coulomb interaction potential, any deformation of a uniformly charged circular disk into a uniformly charged elliptical plate with same surface area leads to a decrease of the electrostatic energy.

This observation is relevant to the ever-growing technology of devices that incorporate charged-particle beam systems as constituents. Until recently, the majority of manufactured systems have utilized charged-particle beams of circular cross-section as components since systems with circular symmetry are easier to fabricate. However, it is clear even from the simple electrostatic considerations of this work, that charged-particle beams (electron beams) of elongated elliptic cross-sections would be better suited for many applications since elliptic beam distributions have a lower self-energy of assembly than circular beam distributions. This setup has already led to a better efficiency of operation in various systems [31, 32, 33, 34].

The resut for the energy derived in this work can be used to study systems of few-N electrons in a disk geometry [35, 36, 37, 38] that interact with the presently considered phenomenological anisotropic Coulomb interaction potential. Studies of this nature are currently being implemented in the context of 2D systems of electrons in the quantum Hall regime and may reveal subtle anisotropic effects. For example, it is known that under realistic experimental conditions, quantum Hall systems may be anisotropic. One natural source for this behavior is the anisotropic dielectric tensor, which in turn leads to an anisotropic Coulomb interaction potential of the same form as the potential in Eq.(2) where the directions of x and y are along the two principal axes of the dielectric tensor [12]. It is expected that the properties of a quantum Hall state with an anisotropic interaction may be different from those of a conventional liquid Hall state in which the particles interact with a standard isotropic Coulomb potential. It is also worthwhile noting that the same anisotropic Coulomb interaction potential studied in this work arises as an effective interaction when calculating the energy of a 2DEG with an elliptical Fermi surface deformation [39].

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Appendix A. Integral $I(a,b)$

We want to calculate the following integral:

$$I(a,b) = \int_D d^2r_1 \int_D d^2r_2 \; \frac{1}{|\vec{r}_1 - \vec{r}_2|} \; , \qquad \text{(A.1)}$$

where $\vec{r}_i = (x_i, y_i)$ (i=1,2) are 2D position vectors and D represents an elliptical domain of integration:

$$D \;\; : \;\; \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \right\} . \qquad \text{(A.2)}$$

We use a wellknown 2D Fourier transformation formula to write:

$$\frac{1}{|\vec{r}_1 - \vec{r}_2|} = \int \frac{d^2k}{(2\,\pi)^2} \; e^{-i\,\vec{k}\,(\vec{r}_1 - \vec{r}_2)} \; \frac{2\,\pi}{k} \; , \qquad \text{(A.3)}$$

where $\vec{k}$ is a 2D vector, $k = |\vec{k}| \ge 0$ is its magnitude and $i = \sqrt{-1}$ is the imaginary unit. One substitutes the expression from Eq.(A.3) to Eq.(A.1) to obtain:

$$I(a,b) = \int \frac{d^2k}{(2\,\pi)^2} \; \frac{2\,\pi}{k} \int_D d^2r_1 \, e^{-i\,\vec{k}\,\vec{r}_1} \int_D d^2r_2 \, e^{+i\,\vec{k}\,\vec{r}_2} \; . \text{(A.4)}$$

By changing to new variables, $u = x_i/a$ and $v = y_i/b$, one can calculate that:

$$\int_D d^2r_1 \, e^{-i\,\vec{k}\,\vec{r}_1} = \int_D d^2r_2 \, e^{+i\,\vec{k}\,\vec{r}_2} = (2\,\pi\,a\,b) \, \frac{J_1\left(\sqrt{a^2\,k_x^2 + b^2\,k_y^2}\,\right)}{\sqrt{a^2\,k_x^2 + b^2\,k_y^2}} \; , \qquad \text{(A.5)}$$

where $J_1(x)$ is a Bessel function of the first order. With help from the result in Eq.(A.5), the quantity in Eq.(A.4) reads:

$$I(a,b) = (2\pi\,a\,b)^2 \int \frac{d^2k}{(2\pi)^2} \, \frac{2\pi}{k} \left[\frac{J_1\left(\sqrt{a^2k_x^2 + b^2k_y^2}\right)}{\sqrt{a^2k_x^2 + b^2k_y^2}} \right]^2 . \text{(A.6)}$$

The integral in Eq.(A.6) can be better handled by switching to 2D polar coordinates:

$$k_x = k\,\cos(\theta) \quad ; \quad k_y = k\,\sin(\theta) \; . \qquad \text{(A.7)}$$

After some simple algebraic manipulations, one has:

$$I(a,b) = 2\,\pi\,(a\,b)^2 \; \times \int_0^\infty dk \int_0^{2\,\pi} d\theta \, \left\{ \frac{J_1\left[k\,\sqrt{a^2\,\cos^2(\theta) + b^2\,\sin^2(\theta)}\right]}{k\,\sqrt{a^2\,\cos^2(\theta) + b^2\,\sin^2(\theta)}} \right\}^2 \text{(A.8)}$$

We initially calculate the integral over the variable k. Although, at first sight, such an integration looks complicated it can be done rather easily after noting that the expression can be reduced to the following form:

$$\int_0^\infty dk \left\{ \frac{J_1\left[\alpha(\theta)\,k\right]}{\alpha(\theta)\,k} \right\}^2 = \frac{4}{3\,\pi} \, \frac{1}{\alpha(\theta)} \; , \qquad \text{(A.9)}$$

where $\alpha(\theta) = \sqrt{a^2\,\cos^2(\theta) + b^2\,\sin^2(\theta)}$ represents a θ-dependent function that does not depend on k. By using the result in Eq.(A.9), one has:

$$I(a,b) = \frac{8}{3}\,(a\,b)^2 \int_0^{2\,\pi} d\theta \, \frac{1}{\sqrt{a^2\,\cos^2(\theta) + b^2\,\sin^2(\theta)}} \; . \text{(A.10)}$$

After some careful trigonometric transformations one can prove that:

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{a^2\,\cos^2(\theta)+b^2\,\sin^2(\theta)}} = 2\int_0^{\pi} \frac{d\theta}{\sqrt{a^2\,\cos^2(\theta)+b^2\,\sin^2(\theta)}} = 4\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\,\sin^2(\theta)+b^2\,\cos^2(\theta)}} \ . \qquad (A.11)$$

The integral appearing in the right-hand-side of Eq.(A.11) can be casted in the form of a complete elliptic integral of the first kind. By using the formula in Eq.(B.3) one may write the final result as:

$$I(a,b) = \frac{32}{3}\,(a\,b)^2\,\frac{1}{b}\,K\left(m = 1-\frac{a^2}{b^2}\right) \ , \qquad (A.12)$$

where $K(m)$ is a complete elliptic integral of the first kind with parameter, m. A brief description of the complete elliptic integral of the first kind and its properties is provided in Appendix B. Based on the identity from Eq.(B.4), one notes that the quantity in Eq.(A.12) can also be written as:

$$I(a,b) = \frac{32}{3}\,(a\,b)^2\,\frac{1}{a}\,K\left(m = 1-\frac{b^2}{a^2}\right) \ . \qquad (A.13)$$

Appendix B. Complete elliptic integral of the first kind

The complete elliptic integral of the first kind [23, 24] with parameter m is defined as:

$$K(m) = \int_0^{\pi/2} \frac{d\,\theta}{\sqrt{1-m\,\sin^2(\theta)}} \ . \qquad (B.1)$$

Special limiting values of this function are:

$$K(m=0) = \frac{\pi}{2} \quad ; \quad K(m\to 1)\to\infty \ . \qquad (B.2)$$

At this point, we remark that the reader should exercise a lot of care when consulting the body of literature dealing with complete elliptic integrals since other notations are widely available. For our specific case, after using straightforward trigonometric transformations, one sees that:

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\,\sin^2(\theta)+b^2\,\cos^2(\theta)}} = \frac{1}{b}\,K\left(m = 1-\frac{a^2}{b^2}\right) . \qquad (B.3)$$

Among a multitude of formulas for complete elliptic integrals of the first kind we mention the following one:

$$\frac{1}{a}\,K\left(m = 1-\frac{b^2}{a^2}\right) = \frac{1}{b}\,K\left(m = 1-\frac{a^2}{b^2}\right) \ , \qquad (B.4)$$

where a and b are real (a fact that is clear from the context of our work). This means that one may rewrite the result in Eq.(B.3) in a slightly diferent but equivalent way using the formula in Eq.(B.4).

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