Energy of the Bose Laughlin quantum Hall state of few electrons at half filling of the lowest Landau level

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Abstract

We report exact analytical results for the energy of the Bose Laughlin state of small systems of electrons at half filling factor of the lowest Landau level. The results apply to a disk geometry and the number of particles in the system consists of up to four electrons immersed in a neutralizing jellium background. Exact calculation of various forms of energy becomes feasible after one transforms all the quantities of interest in terms of Jacobi coordinates. Analytical results of this nature serve as excellent benchmarks to test the accuracy of various computational methods and approaches used in the field.

I. INTRODUCTION

The physics of quantum Hall effects is essentially the physics of two remarkable phenomena known as the integer quantum Hall effect (IQHE) [1–4] and the fractional quantum Hall effect (FQHE) [5–15]. Both phenomena occur in a two dimensional (2D) electronic system in a strong perpendicular magnetic field. When electrons in a 2D system are subject to a perpendicular magnetic field, their energy values are quantized to a set of discrete levels called Landau levels (LLs). Each LL can accommodate up to a certain number of electrons, namely, the number of available quantum states (the degeneracy) of each LL is some quantity, N_s. The actual number of electrons divided by this quantity gives the filling factor defined as $\nu = N/N_s$ where N is the number of electrons. The filling factor uniquely defines a quantum Hall many-body state. If $N \geq N_s$ the filling factor is $\nu \geq 1$ and the IQHE happens when ν is integer. The origin of the IQHE can be understood from the basic quantum mechanics [16] in terms of single particle states without inclusion of electron interactions [17]. On the other hand, when $N < N_s$, the filling factor is $\nu < 1$. For such a case, electrons partially fill the available quantum states in the lower spin sub-band of the lowest Landau level (LLL). FQHE occurs at fractional values $\nu = 1/3$, $1/5$, and so on.

In a strong magnetic field when all electrons are accommodated in the LLL, the kinetic energy of electrons is frozen to a constant value. As a result, the physics of the system is dominated by electron–electron interactions. FQHE represents a novel quantum many-body incompressible electronic liquid phase that arises due to strong electron-electron correlations. The most robust FQHE states occur in the LLL at filling factors, $\nu = 1/3$ and $1/5$. A good description of such states is provided by Laughlin's wave function [6]. Laughlin's approach when applied to a filling factor, $\nu = 1/2$ leads to a Bose Laughlin state for the electrons. The Bose Laughlin wave function is not directly related to the physics of the FQHE, but it has been speculated that such a wave function might be a good candidate to describe a rapidly rotating, dilute Bose-Einstein condensate (BEC) at very low temperature [18–27]. The effect of fast rotation in a BEC system of harmonically confined 2D bosons mimics a perpendicular magnetic field [28–30]. It has been predicted that, for fast rotation, the harmonically confined BEC system enters in the quantum Hall regime [31–39]. A strongly correlated state in this regime may be described by a Bose Laughlin wave function for the appropriate even-denominator filling factor. A weakly interacting BEC that rotates in either a harmonic trap or a weakly anharmonic trapping potential was recently studied. It was found that, in the case of a purely harmonic potential, the gas makes a transition from the mean-field regime to the correlated Laughlin regime [40]. Quantum Hall states of bosons in rotating anharmonic traps have also been considered in a model of bosons in the LLL in a rotating trap where the confinement potential is a sum of a quadratic and a quartic term [41]. In all these cases, the interactions between particles are modeled by a Dirac delta potential. Calculations show a transition from a pure Laughlin state to a state containing an additional giant vortex at the center of the trap (a Laughlin quasi-hole). Results seem to indicate that when a BEC rotates in a purely harmonic potential with an angular frequency which is close to the trap frequency, its many-body state becomes a Bose Laughlin state. However, it is remarked that achieving these states experimentally is very difficult since in a real experiment no trapping potential is ever exactly harmonic [42].

The renewal of interest on the Bose Laughlin state calls for the necessity of analytical results that would serve as undisputed benchmarks to test the accuracy of various complicated calculations [43–45]. The objective of this work is to address this need and report some exact analytical results for the energy of a quantum Hall system with $2 \leq N \leq 4$ electrons described by a Bose Laughlin wave function for filling factor $\nu = 1/2$ of the LLL.

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II. MODEL

In this work we adopt a disk geometry model [46–48]. In the disk geometry, the electrons are confined to a 2D plane. A positive uniformly charged disk represents the jellium background [49, 50] and ensures the overall charge neutrality of the system. We consider systems of $N \geq 2$ electrons each with charge $-e$ ($e > 0$) and mass m_e in presence of a positive uniformly charged finite disk with area, $\Omega_N = \pi R_N^2$ where R_N is the radius of the disk. The electrons are subject to a uniform perpendicular magnetic field of the form:

$$\vec{B} = (0, 0, -B) \ , \tag{1}$$

where B is the magnitude of the magnetic field. The choice of the negative sign of the z-component of $\vec{B}$ is a matter of convenience [51]. With this choice, one can express the resulting wave functions in terms of the complex variable, $z = x + i\,y$ rather than its complex conjugate. It is assumed that the electrons have their spins fully polarized in the direction of the magnetic field. The density of the system (number of electrons per unit area) or, equivalently, the uniform density of the background is constant:

$$\rho_0 = \frac{N}{\Omega_N} \ . \tag{2}$$

For a quantum Hall state with a given filling factor, ν the density of the system can also be written as:

$$\rho_0 = \frac{\nu}{2\,\pi\,l_0^2} \ , \tag{3}$$

where $l_0 = \sqrt{\hbar/(e\,B)}$ is the electron's magnetic length and $\hbar$ is the reduced Planck's constant. By using Eq.(2) in conjunction with Eq.(3) one can calculate that the radius of the positive neutralizing background disk varies with N as:

$$\frac{R_N}{l_0} = \sqrt{\frac{2\,N}{\nu}} \ . \tag{4}$$

The quantum Hamiltonian for the system is:

$$\hat{H} = \hat{K} + \hat{V} \ , \tag{5}$$

where $\hat{K}$ and $\hat{V}$ are, respectively, the kinetic and potential energy operators. The kinetic energy operator (in a magnetic field) is:

$$\hat{K} = \frac{1}{2\,m_e} \sum_{i=1}^{N} \left[\hat{\vec{p}}_i + e\,\vec{A}(\vec{r}_i)\right]^2 \ , \tag{6}$$

where $\hat{\vec{p}}$ is the usual 2D linear momentum operator, $\vec{A}(\vec{r})$ is the magnetic vector potential for a symmetric gauge and $\vec{r}$ is a 2D position vector. The total potential energy operator is written as:

$$\hat{V} = \hat{V}_{ee} + \hat{V}_{eb} + \hat{V}_{bb} \ , \tag{7}$$

and consists of electron-electron (ee), electron-background (eb) and background-background (bb) interaction potential energy terms:

$$\hat{V}_{ee} = \sum_{i<j}^{N} v(\vec{r}_i - \vec{r}_j) \ , \tag{8}$$

$$\hat{V}_{eb} = -\rho_0 \sum_{i=1}^{N} \int_{\Omega_N} d^2r \ v(\vec{r}_i - \vec{r}) \ , \tag{9}$$

and

$$\hat{V}_{bb} = \frac{\rho_0^2}{2} \int_{\Omega_N} d^2r \int_{\Omega_N} d^2r' \ v(\vec{r} - \vec{r}') \ . \tag{10}$$

The interaction potential appearing in the expressions above is a standard Coulomb interaction potential:

$$v(\vec{r}_i - \vec{r}_j) = \frac{k_e\,e^2}{|\vec{r}_i - \vec{r}_j|} \ , \tag{11}$$

where $r_{ij} = |\vec{r}_i - \vec{r}_j|$ is the separation distance between a pair of particles/elementary charges. It is customary in the literature not to include Coulomb's electric constant, k_e in expressions for the interaction potential and/or energy. In all expressions above, $\vec{r}_i$ (or $\vec{r}_j$) denote electronic 2D position vectors while $\vec{r}$ and $\vec{r}'$ are background coordinates that are confined within the disk.

III. ENERGY OF THE BOSE LAUGHLIN STATE AT HALF FILLING

We consider a quantum Hall state of electrons at filling factor $\nu = 1/2$ of the LLL. Electrons are fermions, therefore, the correct quantum state for the electrons should be described by an antisymmetric wave function with respect to the particle's coordinates given that the system is considered to be fully-spin polarized. However, Laughlin's approach for this specific filling factor leads to a symmetric wave function. We call such a symmetric wave function, a Bose Laughlin wave function [52] and its form is:

$$\Psi(z_1, \ldots, z_N) = \prod_{i<j}^{N} (z_i - z_j)^2 \ \exp\left(-\sum_{j=1}^{N} \frac{|z_j|^2}{4\,l_0^2}\right) \ , \tag{12}$$

where $z_j = x_j + i\,y_j$ are 2D position vectors in complex notation and $i = \sqrt{-1}$ is the imaginary number. Since the wave function in Eq.(12) is symmetric under the exchange of coordinates, it represents a Bose liquid state. In order to avoid any misunderstanding, we again point out that the wave function in Eq.(12) is not the true quantum ground state for a system of electrons at filling factor $\nu = 1/2$ of the LLL. The Bose Laughlin wave

function would had been the expected quantum ground state at filling factor $\nu = 1/2$ should the electrons had been bosons (but they are not).

Because the wave function lies entirely in the LLL, the expectation value of the kinetic energy operator is a constant:

$$\frac{\langle \hat{K} \rangle}{N} = \frac{\hbar\,\omega_c}{2} \ , \tag{13}$$

where ω_c is the cyclotron frequency. In order to calculate the total ground state interaction energy of the system, we must calculate each of the following terms: $\langle \hat{V}_{ee} \rangle$, $\langle \hat{V}_{eb} \rangle$ and $\langle \hat{V}_{bb} \rangle$, where in a short-hand notation $\langle \hat{O} \rangle$ denotes the standard quantum expectation value of a given operator, $\langle \Psi | \hat{O} | \Psi \rangle / \langle \Psi | \Psi \rangle$ with respect to the wave function used to describe the system. One can write $\langle \hat{V}_{ee} \rangle$ as:

$$\langle \hat{V}_{ee} \rangle = \frac{N(N-1)}{2} \left\langle v(\vec{r}_1 - \vec{r}_2) \right\rangle \ , \tag{14}$$

and this expression applies to systems with an arbitratry number of $N \geq 2$ electrons. The expression for $\langle \hat{V}_{eb} \rangle$ can be written as:

$$\langle \hat{V}_{eb} \rangle = -\rho_0 \int d^2 r_1 \, \rho_N(\vec{r}_1) \int_{\Omega_N} d^2 r \ v(\vec{r}_1 - \vec{r}) \ , \tag{15}$$

where $\rho_N(\vec{r}_1)$ is the one-particle density function:

$$\rho_N(\vec{r}_1) = N \, \frac{\int d^2 r_2 \ldots \int d^2 r_N \, |\Psi(z_1, \ldots, z_N)|^2}{\langle \Psi | \Psi \rangle} \ , \tag{16}$$

and

$$\langle \Psi | \Psi \rangle = \int d^2 r_1 \int d^2 r_2 \ldots \int d^2 r_N \, |\Psi(z_1, \ldots, z_N)|^2 \ , \tag{17}$$

represents the norm. The quantity $\langle \hat{V}_{bb} \rangle$ is:

$$\langle \hat{V}_{bb} \rangle = \frac{\rho_0^2}{2} \int_{\Omega_N} d^2 r \int_{\Omega_N} d^2 r' \, v(\vec{r} - \vec{r}\,') \ . \tag{18}$$

Note that $\int_{\Omega_N}$ means integration over a finite disk region while $\int$ means integration over the whole 2D space. The total ground state interaction energy per particle can be written as a sum of three terms:

$$\epsilon = \epsilon_{ee} + \epsilon_{eb} + \epsilon_{bb} \ , \tag{19}$$

where $\epsilon = \langle \hat{V} \rangle / N$, $\epsilon_{ee} = \langle \hat{V}_{ee} \rangle / N$, $\epsilon_{eb} = \langle \hat{V}_{eb} \rangle / N$ and $\epsilon_{bb} = \langle \hat{V}_{bb} \rangle / N$ are, respectively, the total, ee, eb and bb interaction energies per particle.

Among the quantities above, only ϵ_{bb} dos not depend on the nature of the wave function used. The calculation of $\langle \hat{V}_{bb} \rangle$ and consequently ϵ_{bb} is straightforward but not very easy to perform. Details of this calculation are provided in Appendix. A. The final ϵ_{bb} result for the Bose Laughlin state at $\nu = 1/2$ consisting of an arbitrary number N of particles is written as:

$$\epsilon_{bb} = \frac{8}{3\pi} \sqrt{\frac{N}{4}} \ \frac{k_e \, e^2}{l_0} \ . \tag{20}$$

The layout of the calculation of $\langle \hat{V}_{eb} \rangle$ and consequently ϵ_{eb} is given in Appendix. B.

The calculation of $\langle \hat{V}_{ee} \rangle$ and consequently ϵ_{ee} involves the magnitude squared of the Bose Laughlin wave function in Eq.(12) which can be written as:

$$|\Psi(z_1, \ldots, z_N)|^2 = \Big[F_N(\vec{r}_1, \ldots, \vec{r}_N) \Big]^2 \exp\left[-\frac{S_N(\vec{r}_1, \ldots, \vec{r}_N)}{2\, l_0^2} \right] \ , \tag{21}$$

where

$$F_N(\vec{r}_1, \ldots, \vec{r}_N) = \prod_{i<j}^{N} (\vec{r}_i - \vec{r}_j)^2 \ , \tag{22}$$

and

$$S_N(\vec{r}_1, \ldots, \vec{r}_N) = \sum_{j=1}^{N} r_j^2 \ . \tag{23}$$

In the expressions above, all coordinates, $\vec{r}_j$ are 2D vectors while $r_j = |\vec{r}_j| \geq 0$ are their corresponding magnitudes.

The Jacobi coordinates are described in various publications [53, 54]. For this reason, we highlight very briefly their main properties without going into much details. The Jacobi coordinates, $\vec{\xi}_1, \ldots, \vec{\xi}_N$ for a system of N identical particles with positions, $\vec{r}_1, \ldots, \vec{r}_N$ are written as:

$$\begin{cases} \vec{\xi}_1 = \vec{r}_1 - \vec{r}_2 \\ \vec{\xi}_2 = \frac{\vec{r}_1 + \vec{r}_2}{2} - \vec{r}_3 \\ \cdots \quad \cdots \quad \cdots \\ \vec{\xi}_j = \frac{\vec{r}_1 + \ldots + \vec{r}_j}{j} - \vec{r}_{j+1} \\ \cdots \quad \cdots \quad \cdots \\ \vec{\xi}_N = \vec{R} = \frac{\vec{r}_1 + \ldots + \vec{r}_N}{N} \ , \end{cases} \tag{24}$$

where $\vec{R}$ is the center-of-mass coordinate of the system. The Jacobi coordinates are orthogonal in any dimension:

$$\int \prod_{k=1}^{N} d\vec{r}_k = \int \prod_{k=1}^{N} d\vec{\xi}_k \quad ; \quad N \geq 2 \ . \tag{25}$$

The Jacobi coordinates satisfy the following relation:

$$\sum_{k=1}^{N} r_k^2 = \sum_{j=1}^{N-1} \frac{j}{j+1} \, \xi_j^2 + N R^2 \quad ; \quad N \geq 2 \ . \tag{26}$$

The usefulness of Jacobi coordinates stands on the fact that the two functions, $F_N(\vec{r}_1, \ldots, \vec{r}_N)$ and $S_N(\vec{r}_1, \ldots, \vec{r}_N)$ can be conveniently written in their

terms. As a result, the exact calculation of many quantities is facilitated when all the relevant expressions are rewritten in terms of the Jacobi coordinates. Explicit analytic expressions for $F_N(\vec{r}_1,\ldots,\vec{r}_N)$ and $S_N(\vec{r}_1,\ldots,\vec{r}_N)$ in terms of the Jacobi coordinates, $\vec{\xi}_1,\ldots,\vec{\xi}_N$ are provided in Ref.[55] where the Laughlin state at filling factor $\nu=1/3$ was studied. For example, for $N=2$ and 3 electrons one has:

$$\begin{cases} F_2(\vec{\xi_1}) = \xi_1^2 \\ \\ S_2(\vec{\xi_1},\vec{R}) = \frac{\xi_1^2}{2} + 2\,R^2 \end{cases} \tag{27}$$

and

$$\begin{cases} F_3(\vec{\xi_1},\vec{\xi_2}) = \frac{\xi_1^2}{16}\left[(4\;\xi_2^2+\xi_1^2)^2 - 16\;(\vec{\xi_1}\;\vec{\xi_2})^2\right] \\ \\ S_3(\vec{\xi_1},\vec{\xi_2},\vec{R}) = \frac{\xi_1^2}{2} + \frac{2\,\xi_2^2}{3} + 3\,R^2\;. \end{cases} \tag{28}$$

The expressions are a little bit more elaborate for $N=4$ electrons:

$$F_4(\vec{\xi_1},\vec{\xi_2},\vec{\xi_3}) = \frac{\xi_1^2}{16}\;\left[(4\;\xi_2^2+\xi_1^2)^2 - 16\;(\vec{\xi_1}\;\vec{\xi_2})^2\right]\;\left(\xi_3^2+\frac{4}{9}\xi_2^2-\frac{4}{3}\;\vec{\xi_2}\;\vec{\xi_3}\right)\;\left[\left(\frac{\xi_1^2}{4}+\xi_3^2+\frac{\xi_2^2}{9}+\frac{2}{3}\;\vec{\xi_2}\;\vec{\xi_3}\right)^2 - \frac{1}{9}\left(\vec{\xi_1}\;\vec{\xi_2}+3\;\vec{\xi_1}\;\vec{\xi_3}\right)^2\right] \tag{29}$$

and

$$S_4(\vec{\xi_1},\vec{\xi_2},\vec{\xi_3},\vec{R}) = \frac{\xi_1^2}{2} + \frac{2\;\xi_2^2}{3} + \frac{3\;\xi_3^2}{4} + 4\,R^2\;. \tag{30}$$

Since all quantities of interest can be written in terms of the Jacobi coordinates, the only steps left to finalize an analytic calculation involve lengthy but relatively straighforward calculations of integrals. The most difficult integrals originate from the $\langle\hat{V}_{ee}\rangle$ term because this quantity involves two-body operators. For such a case, one can write the interaction energy per particle using Jacobi coordinates as:

$$\epsilon_{ee} = \frac{\langle\hat{V}_{ee}\rangle}{N} = \frac{(N-1)}{2}\;\frac{\int d^2\xi_1\cdots\int d^2\xi_{N-1}\int d^2R\left[F_N(\vec{\xi_1},\cdots,\vec{\xi}_{N-1})\right]^2\;e^{-\frac{1}{2\,l_0^2}S_N(\vec{\xi_1},\cdots,\vec{\xi}_{N-1},\vec{R})}\;\frac{k_e\,e^2}{\xi_1}}{\int d^2\xi_1\cdots\int d^2\xi_{N-1}\int d^2R\left[F_N(\vec{\xi_1},\cdots,\vec{\xi}_{N-1})\right]^2\;e^{-\frac{1}{2\,l_0^2}S_N(\vec{\xi_1},\cdots,\vec{\xi}_{N-1},\vec{R})}}\;. \tag{31}$$

Calculations for $N=2,3$ and 4 particles may appear challenging, but, in our opinion, they are straightforward. Details of the energy calculations for a system of $N=2$ particles are provided in Appendix. C. We think that such mathematical details would be useful for the reader to understand the flow of the calculations not only for the specific $N=2$ case, but also for a more general situation of a system with an arbitrary number N of particles. For the sake of brevity, we skip the details of the calculations for $N\geq 3$ particles and go straight to the final analytic results for various energy terms (ϵ_{ee} can also be found in Ref. [29]) that correspond to a Bose Laughlin state of electrons at filling factor $\nu=1/2$ of the LLL. In the following expressions, $\exp(x)$ represents the usual exponential function while $I_n(x)$ are modified Bessel functions [56] of the first kind of order n.

A. N=2

$$\begin{cases} \epsilon_{ee} = \frac{3\sqrt{\pi}}{32}\;\frac{k_e\,e^2}{l_0} \\ \\ \epsilon_{eb} = -\sqrt{\frac{\pi}{2}}\;\frac{11\,I_0(2)+55\,I_1(2)}{16\,\exp(2)}\;\frac{k_e\,e^2}{l_0} \\ \\ \epsilon_{bb} = \frac{4\sqrt{2}}{3\,\pi}\;\frac{k_e\,e^2}{l_0} \end{cases} \tag{32}$$

B. N=3

$$\begin{cases} \epsilon_{ee} = \frac{1353\sqrt{\pi}}{8192}\;\frac{k_e e^2}{l_0} \\ \\ \epsilon_{eb} = -\sqrt{\frac{\pi}{2}}\;\frac{5659\,I_1(3)-2893\,I_0(3)}{352\,\exp(3)}\;\frac{k_e\,e^2}{l_0} \\ \\ \epsilon_{bb} = \frac{4}{\sqrt{3}\,\pi}\;\frac{k_e\,e^2}{l_0} \end{cases} \tag{33}$$

C. N=4

$$\begin{cases} \epsilon_{ee} = \frac{4200674715\,\sqrt{\pi}}{18555600896}\ \frac{k_e e^2}{l_0} \\ \\ \epsilon_{eb} = -\sqrt{\frac{\pi}{2}}\ \frac{177876789\,I_1(4) - 142531687\,I_0(4)}{1698816\,\exp(4)}\ \frac{k_e\,e^2}{l_0} \\ \\ \epsilon_{bb} = \frac{8}{3\,\pi}\ \frac{k_e\,e^2}{l_0} \end{cases} \quad (34)$$

The numerical values of the corresponding analytical energy results after being approximated with an accuracy of five digits after the decimal point are given in Table I. In Table I we also show specific numerical values of various forms of energy. The reported results are rounded at the fifth digit after the decimal point.

TABLE I: Interaction energies per particle (in units of $k_e\,e^2/l_0$) for the Bose Laughlin state at filling factor $\nu = 1/2$ for systems with $N = 2, 3$ and 4 electrons in a disk geometry. The energy estimates are rounded at the fifth digit after the decimal point. The various interaction energies, namely, ϵ_{ee}, ϵ_{eb}, ϵ_{bb} and $\epsilon = \epsilon_{ee} + \epsilon_{eb} + \epsilon_{ee}$ represent, respectively, the electron-electron (ee), electron-background (eb), background-background (bb), and total energy per particle.

N	ϵ_{ee}	ϵ_{eb}	ϵ_{bb}	ϵ
2	0.16617	-1.19326	0.60021	-0.42688
3	0.29274	-1.46283	0.73511	-0.43498
4	0.40125	-1.69042	0.84883	-0.44034

IV. CONCLUSIONS

In this work we report exact analytic results for the energy of Bose Laughlin states corresponding to small systems of $N = 2, 3$ and 4 electrons at half filling factor of the LLL. The results obtained for various forms of energy per particle apply to a quantum Hall system of electrons in a disk geometry. As seen from the results in Table I, one notices that $\epsilon_{eb} \approx -2\,\epsilon_{bb}$ even for small values of N as the ones considered in this work. By comparing the expression in Eq.(15) to that in Eq.(18) one expects such a conclusion to be valid when $\rho(\vec{r}_1) \approx \rho_0$ with the understanding that $\rho(\vec{r}_1)$ becomes negligible for $|\vec{r}_1| > R_N$. Our first guess would had been that this condition is satisified for relatively large values of N. However, we are somehow surprised to see that the condition holds reasonably well also for small systems with $2 \leq N \leq 4$ electrons. As already seen from Monte Carlo simulation studies, the overall energy per particle decreases monotonically as N increases. The thermodynamic ($N \to \infty$) value of the energy per particle for the Bose Laughlin state at half filling of the LLL was estimated in Ref. [52] to be $-0.48415\ k_e\,e^2/l_0$. This implies that the relative difference between the energy value for $N = 4$ and its $N \to \infty$ counterpart is only about 9 %.

The exact energy results obtained in this work can be immediately used to test the accuracy of numerical approaches or treatments used to study strongly correlated systems of electrons similar in nature to the quantum Hall states currently studied. From this perspective, one may rely on them to gauge the accuracy of Monte Carlo calculations for quantum Hall systems of electrons in a disk geometry. For example, Ref.[52] reports some Monte Carlo results for the energy of the Bose Laughlin state at even-denominator filling factors at various N including $N = 4$ electrons. It is reassuring to see that the Monte Carlo results for $N = 4$ electrons at filling factor $\nu = 1/2$ reported in Ref.[52] agree rather accurately with the exact analytical value calculated in this work (such analytical result were unavailable at that time). The small discrepancies noted may likely originate from the use of tabulated functions in Monte Carlo calculations to calculate the eb energy term. It is also important to note that, although we limited our calculations to systems with up to $N = 4$ electrons, the method employed can in principle be extended to larger systems. The only drawback in such a scenario is an undesirable increase in the mathematical complexity of various expressions.

For a given quantum Hall 2D system of N particles, any conventional calculation of the expectation value of a quantum operator with respect to a wave function will involve $2\,N$ dimensional integrals. The benefit of using the Jacobi transformations is that, in specific cases and depending on the nature of the wave function, one may reduce the number of variables. Such is the situation that arises for a Laughlin-like wave function. The polynomial part of a Laughlin-like wave function depends on 2D position vectors $\vec{r}_1, \ldots, \vec{r}_N$ for a system of N particles and is very difficult to deal with. However, rewriting this quantity in terms of Jacobi coordinates leads to expressions that depend on $N-1$ variables, $\vec{\xi}_1, \ldots, \vec{\xi}_{N-1}$ for the same system of N particles. This is an important simplification when it comes to small systems of $N = 2 - 4$ particles as can be illustrated by looking at the general expression of the interaction energy per particle in terms of Jacobi coordinates given in Eq.(31). By looking at the mathematical structure of the expression in Eq.(31) one notices that the two-body Coulomb interaction term under the numerator's integrand can be written as a function of only one variable (in this case ξ_1) and the N-th Jacobi variable, $\vec{\xi}_N = \vec{R}$ will eventually "disappear" from the calculations. As stated earlier, one can, in principle, extend such calculations to larger systems with $N \geq 5$ particles. However, as one may deduce by looking at the expression for $F_4(\vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_3)$ in Eq.(29), what happens is that the polynomial functions become very complicated for $N \geq 5$. In a nutshell, there are limitations to the method as N increases. One may find still useful to push the calculations beyond $N = 4$ particles which is the value where we stopped. However, we think that the method loses its elegance and simplicity if pushed to systems with $N \geq 5$ particles. Analytical results of this nature can be very useful but they cannot substitute

more powerful numerical methods such as Monte Carlo simulations which can be extended to systems with hundreds of particles. Therefore, Monte Carlo simulation methods or similarly powerful numerical methods should be chosen when dealing with any many-body system that is cumbersome to be treated analytically.

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Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

APPENDIX A: BACKGROUND-BACKGROUND ENERGY

We want to calculate the background-background energy of a uniformly charged disk:

$$\langle \hat{V}_{bb} \rangle = \frac{\rho_0^2}{2} \int_{\Omega_N} d^2 r \int_{\Omega_N} d^2 r' \frac{k_e\, e^2}{|\vec{r} - \vec{r}'|} \ , \tag{A1}$$

where $\vec{r}$ and $\vec{r}'$ are 2D vectors. This quantity represents the Coulomb repulsion self-energy of the charge contained on a uniformly charged disk with area, $\Omega_N = \pi\, R_N^2$ containing a total charge $Q_N = N\, e$. Note that, in the above expression, $\rho_0 = N/\Omega_N$ is uniform particle density (not uniform charge density). The quantity in Eq.(A1) can be written more explicitely in 2D polar coordinates as:

$$\langle \hat{V}_{bb} \rangle = \frac{\rho_0^2}{2} \int_0^{R_N} dr\, r \int_0^{2\,\pi} d\varphi \int_0^{R_N} dr'\, r' \int_0^{2\,\pi} d\varphi' \, \frac{k_e\, e^2}{|\vec{r} - \vec{r}'|} \ , \tag{A2}$$

where $r = |\vec{r}| \geq 0$ and $r' = |\vec{r}'| \geq 0$. The calculation of the integral in Eq.(A2) can be facilitated by writing:

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{m=-\infty}^{+\infty} \int_0^{\infty} dk\, e^{i\, m\, (\varphi - \varphi')}\, J_m(k\, r)\, J_m(k\, r') \ , \tag{A3}$$

where $J_m(x)$ are Bessel functions of the first kind of order m. Integration over the angular variables results in:

$$\int_0^{2\,\pi} d\varphi \int_0^{2\,\pi} d\varphi' \, \frac{1}{|\vec{r} - \vec{r}'|} = (2\,\pi)^2 \int_0^{\infty} dk\, J_0(k\, r)\, J_0(k\, r') \ . \tag{A4}$$

One uses the result from Eq.(A4) to express the quantity in Eq.(A2) as:

$$\langle \hat{V}_{bb} \rangle = \frac{\rho_0^2}{2}\, (k_e\, e^2)\, (2\,\pi)^2 \int_0^{\infty} dk \int_0^{R_N} dr\, r\, J_0(k\, r) \int_0^{R_N} dr'\, r'\, J_0(k\, r') \ . \tag{A5}$$

The following formula applies:

$$\int_0^a dx\, x\, J_0(x) = a\, J_1(a) \ . \tag{A6}$$

Based on such formula one calculates that:

$$\langle \hat{V}_{bb} \rangle = \frac{\rho_0^2}{2}\, (k_e\, e^2)\, (2\,\pi)^2 \int_0^{\infty} dk \left[R_N\, \frac{J_1(k\, R_N)}{k} \right]^2 \ . \tag{A7}$$

Another integral formula is worth mentioning:

$$\int_0^{\infty} dx \left[\frac{J_1(x)}{x} \right]^2 = \frac{4}{3\,\pi} \ . \tag{A8}$$

A straightforward application of the formula in Eq.(A8) leads to:

$$\langle \hat{V}_{bb} \rangle = \frac{\rho_0^2}{2}\, (k_e\, e^2)\, (2\,\pi)^2\, R_N^3\, \frac{4}{3\,\pi} \ . \tag{A9}$$

One may group the terms in Eq.(A9) the following way:

$$\langle \hat{V}_{bb} \rangle = \frac{8}{3\,\pi}\, (k_e\, e^2)\, \rho_0^2\, \frac{(\pi\, R_N^2)^2}{R_N} \ . \tag{A10}$$

The expression in Eq.(A10) allows one to obtain the following compact result:

$$\langle \hat{V}_{bb} \rangle = \frac{8}{3\,\pi}\, \frac{k_e\, (N\, e)^2}{R_N} \ . \tag{A11}$$

We know from Eq.(4) that $R_N/l_0 = \sqrt{2N/\nu}$. By substituting the expression for R_N from Eq.(4) in Eq.(A11) one obtains:

$$\langle \hat{V}_{bb} \rangle = \frac{8\, N}{3\,\pi} \sqrt{\frac{\nu\, N}{2}}\, \frac{k_e\, e^2}{l_0} \ . \tag{A12}$$

From here, the background-background energy per particle follows:

$$\epsilon_{bb} = \frac{\langle \hat{V}_{bb} \rangle}{N} = \frac{8}{3\pi} \sqrt{\frac{\nu\, N}{2}}\, \frac{k_e\, e^2}{l_0} \ . \tag{A13}$$

The filling factor is $\nu = 1/2$ in our case and as a result one has:

$$\epsilon_{bb} = \frac{8}{3\pi} \sqrt{\frac{N}{4}}\, \frac{k_e\, e^2}{l_0} \quad ; \quad \nu = 1/2 \ . \tag{A14}$$

It is obvious from start that the background-background energy term does not depend on the nature of the wave function used to describe the quantum system.

APPENDIX B: ELECTRON-BACKGROUND ENERGY

We want to calculate the following quantity:

$$\langle \hat{V}_{eb} \rangle = -\rho_0 \int d^2 r_1\, \rho_N(\vec{r}_1) \int_{\Omega_N} d^2 r\ v(\vec{r}_1 - \vec{r}) \ , \tag{B1}$$

where $v(\vec{r}_1 - \vec{r}) = k_e\, e^2/|\vec{r}_1 - \vec{r}|$ and

$$\rho_N(\vec{r}_1) = N\, \frac{\int d^2 r_2 \ldots \int d^2 r_N\, |\Psi(z_1, \ldots, z_N)|^2}{\langle \Psi | \Psi \rangle} \ , \tag{B2}$$

is the one-particle density function corresponding to a quantum state of N particles. The value of $\langle \hat{V}_{eb} \rangle$ depends on the nature of the wave function that is used to describe the system since the one-particle density function $\rho_N(\vec{r}_1)$ depends on the wave function. We can write the quantity in Eq.(B1) more explicitely as:

$$\langle \hat{V}_{eb} \rangle = -\rho_0 \int d^2 r_1\, \rho_N(\vec{r}_1) \int_0^{R_N} dr\, r \int_0^{2\,\pi} d\varphi\ \frac{k_e\, e^2}{|\vec{r}_1 - \vec{r}|} \ . \tag{B3}$$

One can expand $1/|\vec{r}_1 - \vec{r}|$ the same way as in Eq.(A3) to obtain: the result:

$$\int_0^{2\,\pi} d\varphi\, \frac{1}{|\vec{r}_1 - \vec{r}|} = 2\,\pi \int_0^{\infty} dk\, J_0(k\, r_1)\, J_0(k\, r) \ . \tag{B4}$$

This allows us to write:

$$\langle \hat{V}_{eb} \rangle = -\rho_0\, (k_e\, e^2)\, (2\,\pi) \int d^2 r_1\, \rho_N(\vec{r}_1) \int_0^{R_N} dr\, r \int_0^{\infty} dk\, J_0(k\, r_1)\, J_0(k\, r) \ . \tag{B5}$$

After rerranging the terms, one writes

$$\langle \hat{V}_{eb} \rangle = -\rho_0\, (k_e\, e^2)\, (2\,\pi) \int d^2 r_1\, \rho_N(\vec{r}_1) \int_0^{\infty} dk\, J_0(k\, r_1) \int_0^{R_N} dr\, r\, J_0(k\, r) \ . \tag{B6}$$

The formula in Eq.(A6) leads to the following result:

$$\int_0^{R_N} dr\, r\, J_0(k\, r) = R_N \frac{J_1(k\, R_N)}{k} \ . \tag{B7}$$

This means that we can write the expression in Eq.(B6) as:

$$\langle \hat{V}_{eb} \rangle = -\rho_0\, (k_e\, e^2)\, (2\,\pi)\, R_N \int_0^{\infty} \frac{dk}{k} J_1(kR_N) f_N(k) \ , \tag{B8}$$

where

$$f_N(k) = \int d^2 r_1\, \rho_N(\vec{r}_1)\, J_0(k\, r_1) \ , \tag{B9}$$

is an auxiliary function. For a quantum Hall state with filling factor ν, one has $\rho_0 = \nu/(2\,\pi\, l_0^2)$ which allows us to write:

$$\langle \hat{V}_{eb} \rangle = -\nu\, \frac{R_N}{l_0}\, \frac{k_e\, e^2}{l_0} \int_0^{\infty} \frac{dk}{k} J_1(kR_N) f_N(k) \ . \tag{B10}$$

Since $R_N/l_0 = \sqrt{2N/\nu}$ one can write the quantity in Eq.(B10) as:

$$\langle \hat{V}_{eb} \rangle = -\sqrt{2\,\nu\, N}\ \frac{k_e\, e^2}{l_0} \int_0^{\infty} \frac{dk}{k} J_1(kR_N) f_N(k) \ . \tag{B11}$$

The electron-background enegy per particle follows:

$$\epsilon_{eb} = \frac{\langle \hat{V}_{eb} \rangle}{N} = -\sqrt{\frac{2\,\nu}{N}}\, \frac{k_e\, e^2}{l_0} \int_0^{\infty} \frac{dk}{k} J_1(k\, R_N)\, f_N(k) \ . \tag{B12}$$

For the particular case of filling factor $\nu = 1/2$ one has:

$$\epsilon_{eb} = -\frac{1}{\sqrt{N}} \; \frac{k_e \, e^2}{l_0} \int_0^\infty \frac{dk}{k} J_1(k R_N) \, f_N(k) \;\; ; \;\; \nu = 1/2 \; . \tag{B13}$$

Note that the calculation of ϵ_{eb} involves a multi-step approach. One must, first, calculate $\rho_N(\vec{r}_1)$. From there one must obtain $f_N(k)$ defined in Eq.(B9) and, finally, one must calculate the integral in Eq.(B13).

APPENDIX C: ENERGY FOR $N = 2$

By using the Jacobi variables, one can write the interaction energy per particle for a system of $N = 2$ particles as:

$$\epsilon_{ee} = \frac{1}{2} \, \frac{\int d^2\xi_1 \int d^2R \left[F_2(\vec{\xi}_1) \right]^2 \, e^{-\frac{1}{2\,l_0^2} S_2(\vec{\xi}_1, \vec{R})} \; \frac{k_e \, e^2}{\xi_1}}{\int d^2\xi_1 \int d^2R \left[F_2(\vec{\xi}_1) \right]^2 \, e^{-\frac{1}{2\,l_0^2} S_2(\vec{\xi}_1, \vec{R})}} \; , \tag{C1}$$

where $F_2(\vec{\xi}_1) = \xi_1^2$ and $S_2(\vec{R}, \vec{\xi}_1) = \frac{\xi_1^2}{2} + 2\,R^2$. One can rewrite the quantity in Eq.(C1) as:

$$\epsilon_{ee} = \frac{1}{2} \, \frac{\int d^2\xi_1 \, \xi_1^4 \, e^{-\frac{\xi_1^2}{4\,l_0^2}} \; \frac{k_e \, e^2}{\xi_1}}{\int d^2\xi_1 \, \xi_1^4 \, e^{-\frac{\xi_1^2}{4\,l_0^2}}} \; . \tag{C2}$$

The integrals above are simple and the final result is:

$$\epsilon_{ee} = \frac{3\sqrt{\pi}}{32} \; \frac{k_e \, e^2}{l_0} \; . \tag{C3}$$

The one-particle density function for a system of $N = 2$ particles can be written as:

$$\rho_{N=2}(\vec{r}_1) = 2 \, \frac{\int d^2r_2 \, |\Psi(z_1, z_2)|^2}{\int d^2r_1 \int d^2r_2 \, |\Psi(z_1, z_2)|^2} \; , \tag{C4}$$

where

$$|\Psi(z_1, z_2)|^2 = (\vec{r}_1 - \vec{r}_2)^4 \, e^{-\frac{1}{2\,l_0^2}(r_1^2 + r_2^2)} \; . \tag{C5}$$

After completing the integrals one can express the final result as:

$$\rho_{N=2}(\vec{r}_1) = \rho_0 \; e^{-\frac{1}{2}\left(\frac{r_1}{l_0}\right)^2} \; \left[1 + \left(\frac{r_1}{l_0} \right)^2 + \frac{1}{8} \left(\frac{r_1}{l_0} \right)^4 \right] \; , \tag{C6}$$

where $\rho_0 = \nu/(2\,\pi\,l_0^2)$ is the uniform density at filling factor $\nu = 1/2$. The one-particle density function is isotropic. It depends only on variable $r_1 = |\vec{r}_1| \geq 0$ and does not depend on angle φ_1. The reader can verify, starting from the expression in Eq.(C6) that:

$$\int d^2r_1 \, \rho_{N=2}(\vec{r}_1) = 2 \; . \tag{C7}$$

Since $\rho_{N=2}(\vec{r}_1)$ is isotropic one has:

$$f_{N=2}(k) = 2\,\pi \int_0^\infty dr_1 \, r_1 \, \rho_{N=2}(\vec{r}_1) \, J_0(k\,r_1) \; . \tag{C8}$$

The final result after carrying out the integration is:

$$f_{N=2}(k) = e^{-\frac{1}{2}(k\,l_0)^2} \; \left[2 - (k\,l_0)^2 + \frac{1}{16} \, (k\,l_0)^4 \right] \; , \tag{C9}$$

Based on Eq.(B13) one has for $N = 2$:

$$\epsilon_{eb} = -\frac{1}{\sqrt{2}} \, \frac{k_e \, e^2}{l_0} \int_0^\infty \frac{dk}{k} \, J_1\left(2\sqrt{2}\,k\,l_0 \right) \, f_{N=2}(k) \; . \tag{C10}$$

The integrals are standard and the final result is:

$$\epsilon_{eb} = -\sqrt{\frac{\pi}{2}} \; \frac{11\,I_0(2) + 55\,I_1(2)}{16\,\exp(2)} \; \frac{k_e \, e^2}{l_0} \; , \tag{C11}$$

where $I_n(x)$ are modified Bessel functions of the first kind of order n. Note that in this case we denote the exponential function as $\exp(x)$ and not e^x so that there is no confusion with the symbol e of charge.

The result for the ϵ_{bb} for a system of $N = 2$ particles is easily obtained from the general formula in Eq.(A14) and is written as:

$$\epsilon_{bb} = \frac{4\sqrt{2}}{3\,\pi} \; \frac{k_e \, e^2}{l_0} \; . \tag{C12}$$

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