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Communications in Mathematical Physics

ISSN 0010-3616
Volume 375
Number 3

Commun. Math. Phys. (2020)
375:2281-2334
DOI 10.1007/s00220-019-03670-7

Springer

Commun. Math. Phys. 375, 2281–2334 (2020)
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-019-03670-7

Communications in
Mathematical
Physics

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Received: 19 February 2019 / Accepted: 28 October 2019
Published online: 9 January 2020 – © Springer-Verlag GmbH Germany, part of Springer Nature 2020

Abstract: We study limit theorems in the context of random perturbations of dispersing billiards in finite and infinite measure. In the context of a planar periodic Lorentz gas with finite horizon, we consider random perturbations in the form of movements and deformations of scatterers. We prove a central limit theorem for the cell index of planar motion, as well as a mixing local limit theorem for piecewise Hölder continuous observables. In the context of the infinite measure random system, we prove limit theorems regarding visits to new obstacles and self-intersections, as well as decorrelation estimates. The main tool we use is the adaptation of anisotropic Banach spaces to the random setting.

Contents

Introduction

The Lorentz process is a physically interesting mechanical system modeled by mathematical billiards with chaotic behavior. Introduced by Sinai in [37], it has been studied extensively by many authors, see [8,9,12] and other related references. It is the deterministic motion of a point particle starting from a random phase point and undergoing specular reflections on the boundaries of strictly convex scatterers. Throughout this paper we will consider a $\mathbb{Z}^2$-periodic random configuration of scatterers, with finite horizon. The diffusion limit of the planar Lorentz process can be described by a Wiener process [9], and is thus closely related to the central limit theorem (CLT) and local limit theorem (LLT).

The history of the LLT goes back to the historic De Moivre Laplace theorem for independent identically distributed (iid) Bernoulli random variables. It has then been generalized in many contexts. The CLT appears as a consequence of the LLT. In the context of dynamical systems, the first LLT was established by Guivarc'h and Hardy for subshifts of finite type [22]. The method they used, also used by Nagaev in [27], was based on perturbations of an associated transfer operator and has since been used for many expanding and hyperbolic dynamical systems. This method is now often called the Nagaev–Guivarc'h method. For the Sinai billiard (with fixed scatterers), the LLT was proved by Szász and Varjú in [35] using Young towers and the Nagaev-Guivarc'h method. Also using Young towers, Pène established and used in [29–31] some precise versions of the LLT to prove further limit theorems for the Sinai billiard (see also her works with Saussol [33] and with Thomine [34] for other applications of the LLT).

The goal of this article is to prove the LLT, as well as several of its applications, in the context of randomly deforming scatterers in a dispersing Lorentz gas with finite horizon. In this context the use of Young towers does not appear very adequate, since a different tower is associated to every different $\mathbb{Z}^2$-periodic configuration of scatterers. It is therefore much more natural to work directly with the billiard transformations since these transformations act on the same space $\bar{M}_0$ and preserve the same measure. To this end, we will work with the spaces considered in [15–17], which are spaces $\mathcal{B}$, $\mathcal{B}_w$ made of distributions instead of being spaces of functions contained in L^p for some $p > 1$ as in [22,35]. This will complicate our study. One advantage of the approach used by Demers and Zhang is that the Banach spaces they construct in [16] are the same for natural families of billiard transformations.

Since we are interested in random iterations of billiard transformations, we will consider the full random billiard system corresponding to the skew product transforma-

tion which takes into account both the billiard configuration (position and speed) and the randomness of the configuration of scatterers. Let us mention that Aimino, Nicol and Vaienti established in [2] an LLT (together with other limit theorems) for random iterations of expanding dynamical systems. Their approach was based on the Nagaev-Guivarc'h method applied to the restriction of the transfer operator of the full random system to functions depending only on the phase space coordinate (and not on the random coordinate). The advantage of their method is that they worked on a simple Banach space (in which the randomness of the transformations is not taken into account). But the disadvantage is that they had to reprove for this restricted operator theorems that were already known for transfer operators. In the present paper, we apply directly the Nagaev-Guivarc'h method to the transfer operator of the full random system acting on suitable Banach spaces $\widetilde{\mathcal{B}}$, $\widetilde{\mathcal{B}}_w$ which are easily defined using $\mathcal{B}$, $\mathcal{B}_w$. As a consequence, our results apply to observables that may depend on both the position and speed of the billiard, as well as the random coordinate.

This article is organized as follows. In Sect. 1, we specify our assumptions and notation. In Sect. 2, we state our main limit theorems: LLT, asymptotic estimate of the return time to the initial scatterer, asymptotic behavior of the number of self-intersections, annealed and quenched limit theorem for a random billiard in random scenery, limit theorems for some ergodic sums of the planar random billiard (in infinite measure), mixing and decorrelation for the planar random billiard (in infinite measure). In Sect. 3, we study the spectral properties of the transfer operator of the full random system. Section 4 is devoted to the proof of our main results under general spectral assumptions.

1. Notation and Assumptions

1.1. Deterministic billiard systems. Let $I \geq 1$ and let $O_1, \ldots, O_I$ be I convex open subsets of $\mathbb{R}^2$, having $\mathcal{C}^3$ boundary with strictly positive curvature, and such that the closure of the sets $(U_{i,\ell} := \ell + O_i)_{i=1,\ldots,I;\ \ell\in\mathbb{Z}^2}$ are pairwise disjoint. We consider the $\mathbb{Z}^2$-periodic billiard table $Q := \mathbb{R}^2 \setminus \bigcup_{\ell\in\mathbb{Z}^2} \bigcup_{i=1}^{I} (U_{i,\ell})$. We assume moreover that every line meets ∂Q (i.e. that the horizon is finite). We are interested in the behavior of a point particle moving in Q at unit speed, going straight inside Q, and reflecting elastically off ∂Q (the reflected direction being the symmetric of the incident one with respect to the normal line to Q at the reflection point).

We consider the **planar billiard system** (M_0, μ_0, T_0) modeling the behavior of the point particle at reflection times. A configuration is given by a pair $(q, \vec{v}) \in M_0$ representing position and velocity, and corresponding to a reflected vector off ∂Q, with

$$M_0 := \{(q, \vec{v}) \in \mathbb{R}^2 \times \mathbb{R}^2 :\ q \in \partial Q,\ \|\vec{v}\| = 1,\ \langle \vec{n}(q), \vec{v}\rangle \geq 0\},$$

where $\vec{n}(q)$ is the unit vector, normal to ∂Q at q and directed into Q. The transformation T_0 maps a reflected vector to the reflected vector at the next reflection time. This transformation preserves the measure μ_0 given by $d\mu_0 = \tilde{c} \cos\varphi \, dr \, d\varphi$ (where r is the parametrized arclength coordinate on ∂Q corresponding to q and φ is the algebraic measure of the angle $(\widehat{\vec{n}(q), \vec{v}})$ and where $\tilde{c} = 1/(2\sum_{i=1}^{I} |\partial O_i|)$, the reason for the choice of $\tilde{c}$ will be clear in a few lines).

For every $i \in \{1, \ldots, I\}$ and every $\ell \in \mathbb{Z}^2$, we define $M_{i,\ell} := \{(q, \vec{v}) \in M_0 \ :\ q \in \partial U_{i,\ell}\}$ for the set of reflected vectors based on the obstacle $U_{i,\ell}$. For every $\ell \in \mathbb{Z}^2$, we will call an **ℓ-cell** the set $\bar{M}_\ell := \bigcup_{i=1}^{I} M_{i,\ell}$.

Identifying the boundary of each scatterer ∂O_i with a circle $\mathbb{S}_i$ of length $|\partial O_i|$, we define $\bar{M}_0 := \cup_{i=1}^{I} \mathbb{S}_i \times [-\pi/2, \pi/2]$. Thus $\bar{M}_0$ is a parametrization of $\bar{M}_{(0,0)}$ in the coordinates (r, φ) introduced above. Note that many configurations of obstacles O_i result in the same parametrized space $\bar{M}_0$. We shall exploit this fact when defining the classes of random perturbations that we shall consider.

Because of its $\mathbb{Z}^2$ periodicity, the planar billiard system can be identified with a $\mathbb{Z}^2$-cylindrical extension over a dynamical system $(\bar{M}_0, \bar{\mu}_0, \bar{T}_0)$. Indeed, using the notation $x + \ell = (q + \ell, \vec{v})$ for every $x = (q, \vec{v}) \in M_0$ and every $\ell \in \mathbb{Z}^2$, we observe that there exists a transformation $\bar{T}_0 : \bar{M}_0 \to \bar{M}_0$ (corresponding to the billiard map modulo $\mathbb{Z}^2$) and a function $\Phi_0 : \bar{M}_0 \to \mathbb{Z}^2$ called a **cell-change** such that

$$T_0(x + \ell) = \bar{T}_0(x) + \ell + \Phi_0(x) .$$

This transformation $\bar{T}_0$ preserves the probability measure $\bar{\mu}_0 := \mu_0|_{\bar{M}_0}$ (the fact that $\bar{\mu}_0$ is a probability comes from our choice for the normalizing constant $\tilde{c}$).

In the following, identifying a couple $(x, \ell) \in \bar{M}_0 \times \mathbb{Z}^2$ with $x + \ell \in M_0$, we identify (M_0, μ_0, T_0) with the $\mathbb{Z}^2$-cylindrical extension of $(\bar{M}_0, \bar{\mu}_0, \bar{T}_0)$ by Φ_0, i.e. we identify M_0 with $\bar{M}_0 \times \mathbb{Z}^2$, μ_0 with $\bar{\mu}_0 \otimes \mathfrak{m}$, where $\mathfrak{m} := \sum_{k \in \mathbb{Z}^2} \delta_k$ is the counting measure on $\mathbb{Z}^2$.

1.2. Random perturbations of the initial billiard system. Before describing the random perturbations we shall consider, we describe a class of maps $\bar{\mathcal{F}}$ on $\bar{M}_0$ with uniform properties from which we will draw random sequences of maps. The class $\bar{\mathcal{F}}$ we will use is a slightly simplified version of the one introduced in [16]. The perturbations in [16] allowed billiards with infinite horizon, while for the present work we will assume a finite horizon condition and that the invariant measure is absolutely continuous with respect to the Lebesgue measure, which simplifies several of our assumptions.

We consider a probability space $(E, \mathfrak{T}, \eta)$ containing 0 and a family $(T_\omega)_{\omega \in E}$ of $\mathbb{Z}^2$-periodic planar Sinai billiard systems (with finite horizon) defined on M_0, the quotient billiard maps (modulo $\mathbb{Z}^2$ for the position) $\bar{T}_\omega$ of which are in $\bar{\mathcal{F}}$, and below we will choose $\bar{\mathcal{F}}_{\vartheta_0}(\bar{T}_0)$ as a small ϑ_0-neighbourhood of our original map $\bar{T}_0$, see (5).

For any $\underline{\omega} \in E^{\mathbb{N}}$, we will consider random iterations of the form $T^k_{\underline{\omega}} := T_{\omega_{k-1}} \circ \cdots \circ T_{\omega_0}$. Here $\underline{\omega} = (\omega_k)_{k \geq 0}$, and $T_{\omega_k} \in \mathcal{F}$, for any $k \geq 0$, where $\mathcal{F}$ is a collection of $\mathbb{Z}^2$ extensions of $\bar{\mathcal{F}}$. This will be formalized below. In our model, the modification of environment is applied during the reflection time of the particle; the particle stays on the obstacle and moves with it during the modification of the billiard system. At its k-th reflection time, the particle arrives on an obstacle in an environment parametrized by ω_{k-1}, but when it leaves it sees the environment ω_k.

We identify (M_0, μ_0, T_ω) with the $\mathbb{Z}^2$-extension of $(\bar{M}_0, \bar{\mu}_0, \bar{T}_\omega)$ by some function $\Phi_\omega : \bar{M}_0 \to \mathbb{Z}^2$ which is constant on each connected component of continuity of $\bar{T}_\omega$. We define **the random billiard system** $(\bar{M}, \bar{\mu}, \bar{T})$, corresponding to random iterations of maps in $\bar{\mathcal{F}}$, by setting:

$$\bar{M} := \bar{M}_0 \times E^{\mathbb{N}}, \qquad \bar{\mu} := \bar{\mu}_0 \otimes \eta^{\otimes \mathbb{N}}, \qquad \bar{T}(x, (\omega_k)_{k \geq 0}) := (\bar{T}_{\omega_0} x, (\omega_{k+1})_{k \geq 0}) .$$

We also define **the planar random billiard system** (M, μ, T) with:

$$M := M_0 \times E^{\mathbb{N}}, \qquad \mu := \mu_0 \otimes \eta^{\otimes \mathbb{N}}, \qquad T\left(x, \ell, (\omega_k)_{k \geq 0}\right) := (T_{\omega_0}(x, \ell), (\omega_{k+1})_{k \geq 0}) .$$

This dynamical system is a $\mathbb{Z}^2$-extension of $(\bar{M}, \bar{\mu}, \bar{T})$ by $\Phi : \bar{M} \to \mathbb{Z}^2$ given by:

$$\Phi(x, (\omega_k)_{k\geq 0}) = \Phi_{\omega_0}(x)) .$$

Observe that

$$\begin{aligned} T^n(x, \ell, (\omega_k)_k) &= (T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_0}(x, \ell), (\omega_{n+k})_k) \\ &= (\bar{T}_{\omega_{n-1}} \circ \cdots \circ \bar{T}_{\omega_0}(x), \ell + S_n(x, (\omega_k)_k), (\omega_{n+k})_k), \end{aligned}$$

with

$$S_n(x, (\omega_k)_k) := \sum_{k=0}^{n-1} \Phi \circ \bar{T}^k(x, (\omega_k)_k) = \sum_{k=0}^{n-1} \Phi_{\omega_k} \circ \bar{T}_{\omega_{k-1}} \circ \cdots \circ \bar{T}_{\omega_0}(x),$$

corresponding to the cell change, starting from x, after n iterations of maps labeled successively by $\omega_0, \ldots, \omega_{n-1}$.

Notation 1.1. *As exemplified by the definitions above, we will use overlines such as $\bar{\mu}$, $\bar{M}$, $\bar{T}$ to denote objects associated with the quotient random system, defined in finite measure. When we introduce a subscript such as $\bar{\mu}_0$, $\bar{M}_0$, $\bar{T}_\omega$, these denote objects which are not functions of the random coordinate, but are still defined on the quotient space.*

1.3. A uniform family of maps. We fix the phase space $\bar{M}_0 = \cup_{i=1}^{I} \mathbb{S}_i \times [-\pi/2, \pi/2]$ as described above. Define $\mathcal{S}_0 = \{\varphi = \pm\frac{\pi}{2}\}$ and for a fixed $k_0 \in \mathbb{N}$ with value to be chosen in (3), for $k \geq k_0$ we define the homogeneity strips,

$$\mathbb{H}_k = \left\{(r, \varphi) \in \bar{M}_0 : \tfrac{\pi}{2} - \tfrac{1}{k^2} < \varphi < \tfrac{\pi}{2} - \tfrac{1}{(k+1)^2}\right\}, \tag{1}$$

and the strips $\mathbb{H}_{-k}$ are defined similarly in a neighborhood of $\varphi = -\pi/2$. For the class of maps defined below, we will work with the extended singularity set $\mathcal{S}_{0,H} = \mathcal{S}_0 \cup (\cup_{k\geq k_0} \partial\mathbb{H}_{\pm k})$. Thus for any $F \in \bar{\mathcal{F}}$, the set $\mathcal{S}^F_{\pm n} := \cup_{i=0}^{n} F^{\mp i}\mathcal{S}_{0,H}$ represents the singularity set for $F^{\pm n}$.

We suppose $\bar{\mathcal{F}}$ is a class of maps $F : \bar{M}_0 \circlearrowleft$ such that each $F \in \bar{\mathcal{F}}$ is a C^2 diffeomorphism of $\bar{M}_0 \backslash \mathcal{S}^F_1$ onto $\bar{M}_0 \backslash \mathcal{S}^F_{-1}$ and satisfies the following properties.

(H1) Hyperbolicity and singularities. There exist continuous families of stable and unstable cones, $C^s(x)$ and $C^u(x)$ in the tangent space of $\bar{M}_0$ at $x \in \bar{M}_0 \backslash \mathcal{S}_{-1}$ and $x \in \bar{M}_0 \backslash \mathcal{S}_1$, respectively, which are strictly invariant in the following sense: $DF(x)C^u(x) \subsetneq C^u(Fx)$ and $DF^{-1}(x)C^s(x) \subsetneq C^s(F^{-1}x)$ for all $F \in \bar{\mathcal{F}}$ wherever DF and DF^{-1} are defined.

We assume the sets $\cup_{i=0}^{n} F^{\pm i}\mathcal{S}_0$ (without homogeneity strips) comprise finitely many smooth curves for each $n \in \mathbb{N}$, while the sets $\mathcal{S}^F_{\pm n}$ (with homogeneity strips) have countably many smooth curves. $\mathcal{S}^F_n$ is uniformly transverse[1] to $C^u(x)$ and $\mathcal{S}^F_{-n}$ is uniformly transverse to $C^s(x)$ for each $n \geq 0$. Moreover, $C^s(x)$ and $C^u(x)$ are uniformly transverse on $\bar{M}_0$ and $C^s(x)$ is uniformly transverse to the horizontal and vertical directions on all of $\bar{M}_0$.[2]

[1] The uniformity is assumed to be a lower bound on the angle between these curves and the relevant cone, which is indepedent of $x \in \bar{M}_0$, $n \in \mathbb{N}$ and $F \in \bar{\mathcal{F}}$.

[2] This is not a restrictive assumption for perturbations of the Lorentz gas since the standard cones for the associated billiard map satisfy this property [12, Section 4.5].

We assume there exist constants $C_e > 0$ and $\Lambda > 1$ such that for all $F \in \bar{\mathcal{F}}$ and $n \geq 0$,

$$\begin{aligned} \|DF^n(x)v\| &\geq C_e^{-1}\Lambda^n\|v\|, \forall v \in C^u(x), \text{ and } \|DF^{-n}(x)v\| \\ &\geq C_e^{-1}\Lambda^n\|v\|, \forall v \in C^s(x), \end{aligned} \tag{2}$$

where $\|\cdot\|$ is the Euclidean norm on the tangent space to $\bar{M}_0$.

Finally, near singularities, we assume the maps in $\bar{\mathcal{F}}$ behave like dispersing billiards: there exists $C_a > 0$ such that

$$C_a\|v\| \leq \|DF^{-1}(x)v\|\cos\varphi(F^{-1}x) \leq C_a^{-1}\|v\|, \quad \forall v \in C^s(x),$$

where $\varphi(z)$ denotes the angle φ at the point $z = (r, \varphi) \in \bar{M}_0$. We also require that the second derivative is bounded by,[3]

$$C_a \leq \|D^2F^{-1}(x)\|\cos^3\varphi(F^{-1}x) \leq C_a^{-1}.$$

(H2) Families of stable and unstable curves. We call a $\mathcal{C}^2$ curve $W \subset \bar{M}_0$ a *stable curve* with respect to the class $\bar{\mathcal{F}}$ if the unit tangent to W lies in $C^s(x)$ for all $x \in W$. We say W is *homogeneous* if it lies in a single homogeneity strip $\mathbb{H}_k$. We define homogeneous unstable curves analogously.

Let $\widehat{\mathcal{W}}^s$ denote the set of $\mathcal{C}^2$ homogeneous stable curves in $\bar{M}_0$ whose curvature is bounded above by a constant $B > 0$. We assume there exists B large enough that $F^{-1}W$ is a union of elements of $\widehat{\mathcal{W}}^s$ for all $W \in \widehat{\mathcal{W}}^s$ and $F \in \bar{\mathcal{F}}$. A family $\widehat{\mathcal{W}}^u$ of unstable curves is defined analogously.

(H3) One-step expansion. Assume there exists an adapted norm $\|\cdot\|_*$ on the tangent space to $\bar{M}_0$, equivalent to $\|\cdot\|$, in which the constant C_e in (2) can be taken to be 1. This yields a uniform expansion and contraction in one step for maps in the class $\bar{\mathcal{F}}$.

Let $W \in \widehat{\mathcal{W}}^s$. For $F \in \bar{\mathcal{F}}$, we subdivide $F^{-1}W$ into maximal homogeneous curves $V_i = V_i(F) \in \widehat{W}^s$. We denote by $|J_{V_i}F|_*$ the minimum contraction on V_i under F in the metric induced by the adapted norm $\|\cdot\|_*$. We assume that k_0 in (1) can be chosen sufficiently large that,

$$\lim_{\delta\to 0}\sup_{F\in\bar{\mathcal{F}}}\sup_{\substack{W\in\widehat{\mathcal{W}}^s \\ |W|<\delta}}\sum_i |J_{V_i}F|_* < 1, \tag{3}$$

where $|W|$ denotes the arclength of W.

In addition, if we weaken the power of the Jacobian slightly, we assume that the sum above still converges (although it need not be a contraction). Choosing δ_0 so that the expression in (3) is < 1 for $\delta < \delta_0$, we assume there exists $\zeta_0 \in (0, 1)$ and $C_1 > 0$ such that for all $\delta \in (0, \delta_0)$ and $\zeta \in [\zeta_0, 1]$,

$$\sup_{F\in\bar{\mathcal{F}}}\sup_{\substack{W\in\widehat{\mathcal{W}}^s \\ |W|<\delta}}\sum_i |J_{V_i}F|^{\zeta}_{\mathcal{C}^0(V_i)} \leq C_1.$$

(H4) Bounded distortion. There exists a constant $C_d > 0$ with the following properties. Let $W' \in \widehat{\mathcal{W}}^s$ and for $F \in \bar{\mathcal{F}}$, $n \in \mathbb{N}$, let $x, y \in W \subset F^{-n}W'$ such that F^iW is a homogeneous stable curve for each $0 \leq i \leq n$. Then,

$$\left|\frac{J_W F^n(x)}{J_W F^n(y)} - 1\right| \leq C_d d_W(x, y)^{1/3}, \tag{4}$$

[3] Since F^{-1} is $\mathcal{C}^2$ on $\bar{M}_0\backslash(\mathcal{S}_0 \cup F\mathcal{S}_0)$, setting $x = (r, \varphi)$ and $F^{-1}(x) = (r_{-1}, \varphi_{-1})$, we may define the norm $\|D^2F^{-1}(x)\|$ to be the maximum over all the second partials of (r_{-1}, φ_{-1}) with respect to (r, φ) at x.

where $J_W F^n$ denotes the (stable) Jacobian of F^n along W with respect to arclength.

(H5) Invariant measure. All the maps $F \in \bar{\mathcal{F}}$ have the same invariant measure $\bar{\mu}_0$.

Remark 1.2. Assumption (H5) can be replaced more generally with the requirement that all $F \in \bar{\mathcal{F}}$ preserve the same measure $\bar{\mu}$ which is absolutely continuous with respect to Lebesgue and mixing. In addition, $\bar{\mu}$ should satisfy the following technical assumptions: For $k \geq k_0$, $\bar{\mu}(\mathbb{H}_k) = O(k^{-q})$ for some $q > 4$; also, $\bar{\mu}$ can be disintegrated into measures μ_α along any measurable foliation of M_0 into stable manifolds $\{W_\alpha, \alpha \in \mathcal{A}\}$, with a factor measure λ, such that

$$\bar{\mu}_0(A) = \int_{\alpha\in\mathcal{A}} \int_{x\in W_\alpha} 1_A(x)\, d\mu_\alpha d\lambda(\alpha),$$

where $d\mu_\alpha = \rho_\alpha dm_\alpha$ satisfies a regularity condition: $|\ln \rho_\alpha(x) - \ln \rho_\alpha(y)| \leq C_F d_{W_\alpha}(x, y)^{1/3}$, for some constant $C_F \geq C_d$, $d_W(x, y)$ is the distance of x and y measured along the curve W, and m_α is arclength measure on W_α.

This generalization to other smooth invariant measures is of interest, for example, when considering perturbations in the form of certain soft potentials rather than hard scatterers, or the case of external forces due to gradient fields. See for instance [3,11] and their inclusion in a similar perturbative framework [16].

A crucial lemma, which will allow us to draw random sequences from the class $\bar{\mathcal{F}}$, is the following.

Lemma 1.3. *Fix a class $\bar{\mathcal{F}}$ satisfying* **(H1)**–**(H5)** *with uniform constants. Let $\underline{\omega} \in E^{\mathbb{N}}$, and suppose $\bar{T}_{\omega_k} \in \bar{\mathcal{F}}$ for all $k \geq 0$.*

Then for all $n \in \mathbb{N}$, the composition $\bar{T}^n_{\underline{\omega}} := \bar{T}_{\omega_{n-1}} \circ \cdots \circ \bar{T}_{\omega_0}$ satisfies assumptions **(H1)**–**(H5)***, with possibly larger constants (that are nonetheless independent of n and $\underline{\omega}$), and with respect to the singularity sets $\mathcal{S}_n^{\bar{T}_{\underline{\omega}}} = \cup_{k=0}^{n-1} \bar{T}^{-1}_{\omega_0} \circ \cdots \circ \bar{T}^{-1}_{\omega_k} \mathcal{S}_{0,H}$.*

Lemma 1.3 is proved in [16, Section 5.3].

1.4. Distance in the class $\bar{\mathcal{F}}$. To define a notion of distance $d_{\bar{\mathcal{F}}}(\cdot,\cdot)$ in the class of maps $\bar{\mathcal{F}}$, let $F_1, F_2 \in \bar{\mathcal{F}}$ and for $\epsilon > 0$, let $N_\epsilon(\mathcal{S}_{-1}^{F_i})$ denote the ϵ-neighborhood of the singularity set $\mathcal{S}_{-1}^{F_i}$. We say $d_{\bar{\mathcal{F}}}(F_1, F_2) \leq \epsilon$ if for all $x \notin N_\epsilon(\mathcal{S}_{-1}^{F_1} \cup \mathcal{S}_{-1}^{F_2})$:

(C1) $d((F_1)^{-1}(x), (F_2)^{-1}(x)) \leq \epsilon$;

(C2) $\left| \dfrac{J_W F_i(x)}{J_W F_j(x)} - 1 \right| \leq \epsilon$, for all $W \in \widehat{\mathcal{W}}^s$ and $x \in W$, $i, j = 1, 2$;

(C3) $\|D(F_1)^{-1}(x)v - D(F_2)^{-1}(x)v\| \leq \sqrt{\epsilon}$, for any unit vector v tangent to $W \in \widehat{\mathcal{W}}^s$ at x.

For $F_0 \in \bar{\mathcal{F}}$ and $\vartheta_0 > 0$, define

$$\bar{\mathcal{F}}_{\vartheta_0}(F_0) = \{F \in \bar{\mathcal{F}} : d_{\bar{\mathcal{F}}}(F, F_0) < \vartheta_0\}, \tag{5}$$

to be the ϑ_0 neighborhood of F_0 in $\bar{\mathcal{F}}$.

We remark that this definition of distance does not require the sets $\mathcal{S}_{-1}^{F_1}$ and $\mathcal{S}_{-1}^{F_2}$ to be close in any sense, only that the maps are $\mathcal{C}^1$-close outside an ϵ-neighborhood of the union of the two singularity sets. Next, we describe a perturbation family of billiards that satisfying assumptions (**H1**)–(**H5**), to illustrate that these assumptions are reasonable.

1.5. Applications: deterministic perturbations. Given I intervals $J_1, \dots J_I$, we fix the phase space $\bar{M}_0 = \bigcup_{i=1}^{I} J_i \times [-\pi/2, \pi/2]$ on which the maps in class $\bar{\mathcal{F}}$ are defined. We use the notation $\bar{Q} = (\{O_i\}_{i=1}^{I}; \{J_i\}_{i=1}^{I})$ to denote the configuration of scatterers $O_1, \dots, O_l$ placed on the billiard table such that $|\partial O_i| = |J_i|$, $i = 1, \dots, I$. We identify the endpoints of J_i so that each J_i can be identified with a circle and each component of $\bar{M}_0$ is a cylinder. Since we have fixed $J_1, \dots, J_I$, $\bar{M}_0$ remains the same for all configurations $\bar{Q}$ that we consider. For each such configuration, we define

$$\tau_{\min}(\bar{Q}) = \inf\{\tau(x) : \tau(x) \text{ is defined for the configuration } \bar{Q}\}.$$

Similarly, we define $\tau_{\max}$, as well as $\mathcal{K}_{\min}(\bar{Q})$ and $\mathcal{K}_{\max}(\bar{Q})$, which denote the minimum and maximum curvatures respectively of the ∂O_i in the configuration $\bar{Q}$. The constant $E_{\max}(\bar{Q})$ denotes the maximum C^3 norm of the ∂O_i in $\bar{Q}$.

For each fixed $\tau_*, \mathcal{K}_*, E_* > 0$, define $\mathcal{Q}_1(\tau_*, \mathcal{K}_*, E_*)$ to be the collection of all configurations $\bar{Q}$ such that:

$$\tau_* \le \tau_{\min}(\bar{Q}) \le \tau_{\max}(\bar{Q}) \le \tau_*^{-1}, \quad \mathcal{K}_* \le \mathcal{K}_{\min}(\bar{Q}) \le K_{\max}(\bar{Q}) \le \mathcal{K}_*^{-1}, \quad E_{\max}(\bar{Q}) \le E_*.$$

Let $\bar{\mathcal{F}}_1(\tau_*, \mathcal{K}_*, E_*)$ be the corresponding set of billiard maps induced by the configurations in $\mathcal{Q}_1$. The following lemma is proved in [16].

Lemma 1.4 ([16, Theorem 2.7]). *Fix intervals $J_1, \dots, J_I$ and let $\tau_*, \mathcal{K}_*, E_* > 0$. The family $\bar{\mathcal{F}}_1(\tau_*, \mathcal{K}_*, E_*)$ satisfies* **(H1)–(H5)** *with uniform constants depending only on τ_*, $\mathcal{K}_*$ and E_*.*

We fix an initial configuration of scatterers $\bar{Q}_0 \in \mathcal{Q}_1(\tau_*, \mathcal{K}_*, E_*)$ and consider configurations $\bar{Q}$ which alter each ∂O_i in $\bar{Q}_0$ to a curve $\partial \tilde{O}_i$ having the same arclength as ∂O_i. We consider each ∂O_i as a parametrized curve $u_i : J_i \to \mathbb{R}^2$ and each $\partial \tilde{O}_i$ as parametrized by $\tilde{u}_i$. Define

$$\Delta(\bar{Q}, \bar{Q}_0) = \sum_{i=1}^{l} |u_i - \tilde{u}_i|_{C^2(J_i, \mathbb{R}^2)}.$$

The following is proved in [16] .

Lemma 1.5 ([16, Theorem 2.8]). *Choose $\vartheta_0 \le \min\{\tau_*/2, \mathcal{K}_*/2\}$ and let $\bar{\mathcal{F}}_A(\bar{Q}_0, E_*; \vartheta_0)$ be the set of all billiard maps corresponding to configurations $\bar{Q}$ such that $\Delta(\bar{Q}, \bar{Q}_0) \le \vartheta_0$ and $E_{\max}(\bar{Q}) \le E_*$. Then $\bar{\mathcal{F}}_A(\bar{Q}_0, E_*; \vartheta_0) \subset \bar{\mathcal{F}}_1(\tau_*/2, \mathcal{K}_*/2, E_*)$ and $d_{\bar{\mathcal{F}}}(\bar{F}_1, \bar{F}_2) \le C|\vartheta_0|^{1/3}$ for any $\bar{F}_1, \bar{F}_2 \in \bar{\mathcal{F}}_A(Q_0, E_*; \vartheta_0)$.*

The importance of these results is that together, they will imply that the transfer operators associated to maps in the neighborhood $\bar{\mathcal{F}}_{\vartheta_0}(\bar{T}_0)$ have a uniform spectral gap if the transfer operator associated with $\bar{T}_0$ has a spectral gap. Moreover, small changes in the configuration of scatterers are seen to generate small differences in the distance $d_{\bar{\mathcal{F}}}(\cdot, \cdot)$.

Remark 1.6. The assumption in the discussion above and in Sect. 1.3 that the perturbed scatterers $\partial \tilde{O}_i$ have the same arclength as the original ∂O_i is made so that all maps in $\bar{\mathcal{F}}$ act on the same phase space $\bar{M}_0$, and so all the associated transfer operators act on the same Banach space. This can be relaxed slightly if all scatterers are scaled by the same constant. Then we can reparametrize each $\partial \tilde{O}_i$ (no longer according to arclength) using the same interval J_i as for ∂O_i. This will change the derivative of the maps acting

on this configuration of scatterers, but since the constants appearing in **(H1)**–**(H5)** have some leeway built into the inequalities, for small reparametrizations the same properties will continue to hold.

Unfortunately, to scale scatterers $\partial \tilde{O}_i$ by different constants as described in [16, Remark 2.9], one would need to eliminate assumption **(H5)** since then the measure $\bar{\mu}_0$ would not be preserved.

2. Main Results

In this section, we consider all $\bar{T}_\omega \in \bar{\mathcal{F}}_{\vartheta_0}(\bar{T}_0)$, for some $\vartheta_0 > 0$ small enough and a fixed map $\bar{T}_0 : \bar{M}_0 \circlearrowleft$.

2.1. Local limit theorem. Adapting the proof of [16, Corollary 2.4] (with the slight difference that, here, the observable $\Phi(x, \underline{\omega})$ we are interested in depends also on $\underline{\omega}$), we will prove the following central limit theorem.

Theorem 2.1 (Central limit theorem for the cell index). *With respect to $\bar{\mu}$, the covariance matrix of $(S_n/\sqrt{n})_n$ converges to a non-negative symmetric function*

$$\Sigma^2 := \left(\mathbb{E}_{\bar{\mu}}\left[\Phi^{(i)}.\Phi^{(j)}\right] + \sum_{k\geq 1} \mathbb{E}_{\bar{\mu}}\left[\Phi^{(i)}.\Phi^{(j)} \circ \bar{T}^k + \Phi^{(j)}.\Phi^{(i)} \circ \bar{T}^k\right] \right)_{i,j=1,2}, \quad (6)$$

where, for every $j = 1, 2$, $\Phi^{(j)}$ is the j-th coordinate of Φ, and using . to denote multiplication.

Moreover $(S_n/\sqrt{n})_n$ converges in distribution to a centered Gaussian distribution with covariance matrix Σ^2.

The fact that Σ^2 is positive if ϑ_0 is small enough will be proved in Lemma 3.18 (using a continuity argument). In Sect. 3.2, we will define a Banach space $\widetilde{\mathcal{B}}$, containing a class of distributions on $\bar{M}$, and its dual $\widetilde{\mathcal{B}}'$. For a function $g : \bar{M} \to \mathbb{R}$, define the functional H_g, by

$$H_g(\cdot) := \mathbb{E}_{\bar{\mu}}[g.\,\cdot\,]. \quad (7)$$

Remark 3.1 and Lemma 3.3 will give conditions on g that guarantee that $H_g \in \widetilde{\mathcal{B}}'$.

Theorem 2.2 (Local limit theorem). *For every $f, g : \bar{M} \to \mathbb{R}$ such that $H_g \in \widetilde{\mathcal{B}}'$ and such that $f \in \widetilde{\mathcal{B}}$,*

$$\mathbb{E}_{\bar{\mu}}\left[f.\mathbf{1}_{\{S_n=\ell\}}.g \circ \bar{T}^n\right] = \frac{\exp\left(-\frac{\Sigma^{-2}\ell.\ell}{2n}\right)}{2\pi n\sqrt{\det \Sigma^2}} \mathbb{E}_{\bar{\mu}}[f]\mathbb{E}_{\bar{\mu}}[g] + O\left(n^{-\frac{3}{2}} \|f\|_{\widetilde{\mathcal{B}}} \|H_g\|_{\widetilde{\mathcal{B}}'}\right). \quad (8)$$

Remark 2.3. Due to Lemma 3.3 and Remark A.1, it suffices for the conclusion of Theorem 2.2 that $f(\cdot, \underline{\omega})$ and $g(\cdot, \underline{\omega})$ be piecewise Hölder continuous on $\bar{M}_0$ (with Hölder bounds that are uniform in $\underline{\omega}$). For instance, the coordinates $\Phi^{(i)}$ of the displacement function Φ satisfy these conditions, as well as the free flight function for the billiard map $\bar{T}_\omega$, $\tau(\cdot, \omega)$.

2.2. Return time, visit to new obstacles and self intersections. We define $\mathcal{I}_0(x,\underline{\omega}) := i$ if $x \in \bigcup_{\ell\in\mathbb{Z}^2} M_{i,\ell}$ as the index in $\{1,\dots,I\}$ of the obstacle on which the particle is at time 0 and $\mathcal{I}_k := \mathcal{I}_0 \circ T^k$. Since the quantity $\mathcal{I}_0(x,\underline{\omega})$ does not depend on $\underline{\omega}$, we will also write $\mathcal{I}_0(x)$ for this quantity. Note that $\mathcal{I}_k(x,\underline{\omega})$ does not depend on the index ℓ of the cell containing x, this allows us to define also $\mathcal{I}_k$ on $\bar{M}$ (by projection).

Observe that the fact that the point particle is on the obstacle (i,ℓ) at the k-th reflection time (i.e. $T^k(x,\underline{\omega}) \in M_{i,\ell}$) can be rewritten:

$$(\ell_0 + S_k(\bar{x},\underline{\omega}), \mathcal{I}_k(\bar{x},\underline{\omega})) = (\ell, i),$$

if $x = (\bar{x},\ell_0) \in \bar{M}_0 \times \mathbb{Z}^2$. We are interested here in the study of the probability that a point particle starting[4] from $\bar{M}\times\{\mathbf{0}\}$ does not come back to its original obstacle until time n, that is in $\bar{\mu}(B_n)$ with

$$B_n := \{\forall k = 1,\dots,n \,:\, (\mathcal{I}_k, S_k) \neq (\mathcal{I}_0,(0,0))\} \subset \bar{M}\,.$$

We also study the probability that the obstacle visited at time n has not been visited before, that is $\bar{\mu}(B'_n)$ with

$$B'_n := \{\forall k = 0,\dots,n-1 \,:\, (\mathcal{I}_k, S_k) \neq (\mathcal{I}_n, S_n)\} \subset \bar{M}\,.$$

Observe that, because of the reversibility of our model, $\bar{\mu}(B_n) = \bar{\mu}(B'_n)$.

Theorem 2.4. *We have the following asymptotics*

$$\bar{\mu}(B_n) = \bar{\mu}(B'_n) = \frac{2I\pi\sqrt{\det\Sigma^2}}{\log n} + O\left((\log n)^{-\frac{4}{3}}\right), \quad as \quad n\to+\infty\,.$$

In Sect. 4.2, we give a proof of the above asymptotic estimates of $\bar{\mu}(B_n)$ and $\bar{\mu}(B'_n)$ in a more general context. This result will appear as an easy and direct consequence of the local limit theorem, Theorem 2.2. We now consider the number of couples of times at which the point particle hits the same obstacle:

$$\mathcal{V}_n := \sum_{i,j=1}^{n} \mathbf{1}_{\{S_j=S_i,\ \mathcal{I}_j=\mathcal{I}_i\}}.$$

Theorem 2.5. *$\bar{\mu}$-almost surely, we have:*

$$\lim_{n\to\infty} \frac{\mathcal{V}_n}{n\log n} = \frac{1}{\pi\sqrt{\det\Sigma^2}} \frac{\sum_{a=1}^{I}|\partial O_a|^2}{\left(\sum_{b=1}^{I}|\partial O_b|\right)^2}.$$

The proof of the previous result is delicate as it uses a precise estimate of the variance of $\mathcal{V}_n$. As can be seen from the works by Bolthausen [5] and by Deligiannidis and Utev [14], going from a rough to a precise estimate of the variance of the number of self intersections requires important additional work. In Sect. 4.3, we give a proof of this result under general spectral assumptions. Our argument provides, in the case of random walks, an alternative argument to the one given by Deligiannidis and Utev in [14]. Let us indicate that even if we use the general scheme of the previous unpublished paper [31]

[4] Throughout the paper, we shall use the notation $\mathbf{0} = (0,0)$ as an element of $\mathbb{Z}^2$.

(in which an analogous result is proved for a single billiard map), this general scheme being just the natural decomposition already used by Bolthausen in [5] to get a non-optimal estimate of the variance, the method we use in the present paper to establish our crucial estimates is different from [31]. In particular our method enables us to get rid of some assumptions (bounded cell change function, Banach spaces continuously injected in some L^p) that were satisfied and used in [31].

The two previous results (probability to visit a new site, precise asymptotics for the number of self-intersections), in addition to being interesting in their own right, will greatly help us to prove the result of the next section.

2.3. Billiard in random scenery. We consider the following billiard dynamics. We assume that the phase space for the initial configuration of the particle is $\bar{M}_0$, with initial distribution $\bar{\mu}_0$ and that the particle will experiment random iterations of billiard maps T_{ω_k}, with $(\omega_k)_{k\geq 0}$ a sequence of i.i.d. random variables with common distribution η, idependently of the initial configuration. To each obstacle (i,ℓ), we associate a random variable $\xi_{(i,\ell)}$ defined on some probability space $(\Omega,\mathbb{P})$. We assume that these random variables $\xi_{(i,\ell)}$ are i.i.d., centered, and square integrable. We assume that, each time the point particle hits the obstacle (i,ℓ), it wins the value $\xi_{(i,\ell)}$. Let $\mathcal{Z}_n$ be the total amount won by the particle up to the n-th reflection. For every n, we consider the linearized process $(\widetilde{\mathcal{Z}}_n(t))_{t\geq 0}$ defined by

$$\widetilde{\mathcal{Z}}_n(t) = \mathcal{Z}_{\lfloor nt\rfloor} + (nt - \lfloor nt\rfloor)(\mathcal{Z}_{\lfloor nt\rfloor+1} - \mathcal{Z}_{\lfloor nt\rfloor})\,.$$

Formally speaking $\mathcal{Z}_n$ and $\widetilde{\mathcal{Z}}_n$ are defined on the probability space $(\bar{M}\times\Omega, \bar{\mu}\otimes\mathbb{P})$.

Theorem 2.6. *For every $\mathcal{T}>0$, the sequence of processes $((\widetilde{\mathcal{Z}}_n(t)/\sqrt{n\log n})_{t\in[0,\mathcal{T}]})_n$ in $C([0,\mathcal{T}])$ converges in distribution with respect to $\bar{\mu}\otimes\mathbb{P}$ to $(B_t)_{t\in[0,\mathcal{T}]}$, where $B=(B_t)_{t\geq 0}$ is a Brownian motion such that*

$$\mathbb{E}[B_1^2] = \frac{\sigma_\xi^2}{\pi\sqrt{\det\Sigma^2}}\frac{\sum_{a=1}^I|\partial O_a|^2}{\left(\sum_{b=1}^I|\partial O_b|\right)^2}.$$

If, moreover, there exists $\chi>0$ such that $\mathbb{E}[|\xi_{(1,0)}|^2(\log^+|\xi_{(1,0)}|)^\chi|)]<\infty$, then, for $\mathbb{P}$-almost every realization of $(\xi_{i,\ell})_{i,\ell}$, $(\widetilde{\mathcal{Z}}_n)_n$ converges in distribution to the same Brownian motion B.

Let us indicate that it should be possible to remove the additional assumption $\mathbb{E}[|\xi_{(1,0)}|^2(\log^+|\xi_{(1,0)}|)^\chi|)]<\infty$ by using our estimates, combined with the very recent preprint [13] instead of [21].

Let us say a few words about the historical background of this result. Limit distributional theorems of analogous processes when S_n is replaced by a random walk on $\mathbb{Z}^d$ were first established at the end of the 70's by Borodin in [6,7] and by Kesten and Spitzer in [26], by Boltausen [5] in dimension 2 ten years later, and more recently by Deligiannidis and Utev in [14] and by Castell, Guillotin-Plantard and the second author in [10]. Let us also remark that when the random walk is the one dimensional simple symmetric random walk on $\mathbb{Z}$, the random walk in random scenery corresponds to an ergodic sum of a dynamical system, the so-called T,T^{-1}-transformation. This dynamical system has been introduced in a list of open problems by Weiss [39, problem 2, p. 682] in the early

1970s. This dynamical system is a famous natural example of a K-transformation which is not Bernoulli and even not loosely Bernoulli as has been shown by Kalikow in [24].

We prove Theorem 2.6 in a more general context in Sect. 4.4. As noticed by Deligiannidis and Utev in [14] in the context of random walks, the estimate provided by Theorem 2.5 simplifies greatly the proof of Theorem 2.6 compared to [5,30] ([30] contained a proof of this result for a single billiard map, with the use of the properties of Young towers). Furthermore, we simplify also the tightness argument used by Bolthausen in [5].

2.4. Limit theorems in infinite measure. The following results are consequences of our perturbation result (Proposition 3.17), combined with the general results of [34] and of [32].

Our next result deals with the asymptotic behavior of additive functionals of S_n, that is of quantities of the form $\sum_{k=0}^{n-1} g(S_k)$, for summable functions $g : \mathbb{Z}^2 \to \mathbb{R}$. This can be seen as the ergodic sum $\sum_{k=0}^{n-1} G \circ T^k$ with $G(x, \ell, \underline{\omega}) := g(\ell)$.

Theorem 2.7 (Additive functionals of S_n). *If g is summable (i.e. $\sum_{\ell\in\mathbb{Z}^2} |g(\ell)| < \infty$), then*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} g(S_k)}{\log n} = \frac{1}{2\pi\sqrt{\det \Sigma^2}} \sum_{\ell\in\mathbb{Z}^2} g(\ell)\, \mathcal{E}\,,$$

where $\mathcal{E}$ is an exponential random variable with expectation 1 and where the convergence is in the sense of distribution with respect to any probability measure absolutely continuous with respect to $\bar{\mu}$.

If moreover $\sum_{\ell\in\mathbb{Z}^2} g(\ell) = 0$ and $\sum_{\ell\in\mathbb{Z}^2} |\ell|^\varepsilon |g(\ell)| < \infty$, for some $\varepsilon > 0$, then

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} g(S_k)}{\sqrt{\log n}} = \frac{1}{\sqrt{2\pi}(\det \Sigma^2)^{\frac{1}{4}}} \sigma_g \sqrt{\mathcal{E}}\,\mathcal{N},$$

where the convergence is again in distribution, $\mathcal{E}$ is as above, $\mathcal{N}$ is a standard Gaussian random variable independent of $\mathcal{E}$ and

$$\sigma_g^2 := \sum_{\ell\in\mathbb{Z}^2} (g(\ell))^2 + 2\sum_{k\geq 1} \left(\sum_{\ell,\ell'\in\mathbb{Z}^2} g(\ell) g(\ell') \bar{\mu}_0(S_k = \ell - \ell') \right).$$

For $g : M_0 \to \mathbb{R}$, define $H_{g,\ell} : \mathcal{B} \to \mathbb{R}$ by $H_{g,\ell}(h) = \mathbb{E}_{\bar{\mu}_0}[g(\cdot, \ell)h]$. We also obtain the decay rates of correlations for the process generated by our random systems in infinite measure:

Theorem 2.8 (Mixing and decorrelation in infinite measure). *Let $K \geq 1$. Let $f, g : M_0 \to \mathbb{R}$ be two functions such that*

$$\sum_{\ell\in\mathbb{Z}^2} |\ell|^{2K} \left(\|f(\cdot, \ell)\|_{\mathcal{B}} + \|H_{g,\ell}\|_{\mathcal{B}'} \right) < \infty\,.$$

Then, there exist real numbers $C_0(f, g), \ldots, C_K(f, g)$ such that,

$$\mathbb{E}_\mu[\mathfrak{f}.\mathfrak{g}\circ T^n] = \int_{M_0\times E^{\mathbb{N}}} f.g\circ T_{\omega_n}\circ\cdots\circ T_{\omega_1}\, d\mu_0\, d\eta^{\otimes\mathbb{N}}((\omega_n)_n)$$
$$= \sum_{m=0}^{K}\frac{C_m(f,g)}{n^{m+1}} + o(n^{-K-1}),$$

with $C_0(f,g) = \frac{1}{2\pi\sqrt{\det\Sigma^2}}\int_{M_0} f\, d\mu_0 \int_{M_0} g\, d\mu_0$ *and setting* $\mathfrak{f}(x,\ell,\underline{\omega}) := f(x,\ell)$ *and* $\mathfrak{g}(x,\ell,\underline{\omega}) := g(x,\ell)$ *to be the extensions of* f *and* g *to* $M_0\times E^{\mathbb{N}}$.

3. Transfer Operators

In order to prove our main limit theorems, we will study the transfer operators associated with the random maps T and $\bar{T}$ as perturbations of the transfer operator associated with a fixed quotient billiard map $\bar{T}_0$.

In this section, we fix a class of maps $\bar{\mathcal{F}}$ satisfying **(H1)**–**(H5)** with uniform constants. $\bar{T}$ denotes the quotient of the full random map T, while $\bar{T}_\omega$, $\omega\in E$ denotes a quotient billiard map belonging to $\bar{\mathcal{F}}$, following the notation defined in Sect. 1.2.

Using **(H3)**, choose $\delta_0 > 0$ for which there exists $\theta < 1$ so that (3) gives,

$$\sup_{\bar{T}_\omega\in\bar{\mathcal{F}}_{\vartheta_0}}\sup_{\substack{W\in\widehat{\mathcal{W}}^s\\ |W|<\delta_0}}\sum_i |J_{V_i}\bar{T}_\omega|_* \le \theta. \tag{9}$$

We then define $\mathcal{W}^s\subset\widehat{\mathcal{W}}^s$ to be those stable curves in $\widehat{\mathcal{W}}^s$ whose length is at most δ_0.

Following [15], for any $\bar{T}_\omega\in\bar{\mathcal{F}}$ and $n\ge 0$, define $\bar{T}_\omega^{-n}\mathcal{W}^s\subset\mathcal{W}^s$ to be the set of homogeneous stable curves $W\in\mathcal{W}^s$ whose images $\bar{T}_\omega^i W\in\mathcal{W}^s$ for $0\le i\le n$. For $p\in[0,1]$ and letting $\mathcal{C}^p(\bar{T}_\omega^{-n}\mathcal{W}^s)$ denote those functions ψ which are p-Hölder continuous on elements of $\bar{T}_\omega^{-n}\mathcal{W}^s$, it follows from **(H1)** that $\psi\circ\bar{T}_\omega\in\mathcal{C}^p(\bar{T}^{-n-1}\mathcal{W}^s)$. Thus if $f\in(\mathcal{C}^p(\bar{T}_\omega^{-n-1}\mathcal{W}^s))'$ is an element of the dual of $\mathcal{C}^p(\bar{T}_\omega^{-n-1}\mathcal{W}^s)$, then $\mathcal{L}_{\bar{T}_\omega}: (\mathcal{C}^p(\bar{T}_\omega^{-n-1}\mathcal{W}^s))'\to(\mathcal{C}^p(\bar{T}_\omega^{-n}\mathcal{W}^s))'$ is defined by

$$\mathcal{L}_{\bar{T}_\omega} f(\psi) = f(\psi\circ\bar{T}_\omega),\qquad \forall\ \psi\in\mathcal{C}^p(\bar{T}_\omega^{-n}\mathcal{W}^s).$$

If in addition, f is a finite signed measure absolutely continuous with respect to $\bar{\mu}_0$, then we identify f with its density in $L^1(\bar{\mu}_0)$, which we shall also denote f, i.e. $f(\psi) = \int_{\bar{M}_0}\psi\, f\, d\bar{\mu}_0$. With this identification, we write $L^1(\bar{\mu}_0)\subset(\mathcal{C}^p(\bar{T}_\omega^{-n}\mathcal{W}^s))'$ for each $n\in\mathbb{N}$. Then acting on $L^1(\bar{\mu}_0)$, $\mathcal{L}_{\bar{T}_\omega}$ has the following familiar expression,

$$\mathcal{L}^n_{\bar{T}_\omega} f = f\circ\bar{T}_\omega^{-n},\qquad \text{for any } n\ge 0.$$

For brevity, sometimes we will denote $\mathcal{L}_{\bar{T}_\omega}$ by $\mathcal{L}_\omega$.

Let P be the transfer operator of $\bar{T}$ with respect to $\bar{\mu} := \bar{\mu}_0\otimes\eta^{\otimes\mathbb{N}}$. This operator is given by

$$Pf(y,(\omega_k)_{k\ge 0}) = \int_E \mathcal{L}_{\omega_{-1}} f(\cdot,(\omega_{k-1})_{k\ge 0})(y)\, d\eta(\omega_{-1}).$$

Let us write $\cdot$ for the usual scalar product on $\mathbb{R}^2$. We consider the family of operators $(P_u)_{u\in\mathbb{R}^2}$ given by

$$P_u f(y,(\omega_k)_k) := P\left(e^{iu\cdot\Phi} f\right)(y,(\omega_k)_k) = \int_E \mathcal{L}_{u,\omega_{-1}} f(\cdot,(\omega_{k-1})_{k\ge 0})(y)\, d\eta(\omega_{-1}),$$

where

$$\mathcal{L}_{u,\omega_{-1}} f = \mathcal{L}_{\omega_{-1}}(e^{iu\cdot\Phi_{\omega_{-1}}} f)\,.$$

Note that

$$P_u^n f = P^n(e^{iu\cdot S_n} f)\,.$$

Using results of [16], we will see that if we restrict $\bar{T}_\omega$ to a neighborhood $\bar{\mathcal{F}}_{\vartheta_0}(\bar{T}_0)$ according to (5), then P is a small (depending on ϑ_0) perturbation of the transfer operator $\mathcal{P}_0$ of the product system $(\bar{M}, \bar{\mu} := \bar{\mu}_0 \times \eta^{\otimes\mathbb{N}}, \bar{T}_0 \times \sigma)$, where σ is the shift over $E^{\mathbb{N}}$ (i.e. $\sigma((\omega_k)_{k\geq 0}) = (\omega_{k+1})_{k\geq 0}$) and where $(\bar{T}_0 \times \sigma)(x, \underline{\omega}) := (\bar{T}_0(x), \sigma(\underline{\omega}))$.

3.1. Banach spaces $\mathcal{B}$ and $\mathcal{B}_w$. We start by defining Banach spaces $\mathcal{B} \subset \mathcal{B}_w$ of distributions on $\bar{M}_0$, on which the transfer operators $\mathcal{L}_\omega$ associated to $\bar{T}_\omega \in \bar{\mathcal{F}}$ are well-behaved.

In order to define our norms, we first require a notion of distance $d_{\mathcal{W}^s}(\cdot,\cdot)$ between stable curves as well as a distance $d(\cdot,\cdot)$ defined among functions supported on these curves.

Due to the transversality condition on the stable cones $C^s(x)$ given by **(H1)**, each $W \in \mathcal{W}^s$ can be viewed as the graph of a function $\varphi_W(r)$ of the arc length parameter r. For each $W \in \mathcal{W}^s$, let J_W denote the interval on which φ_W is defined and set $G_W(r) = (r, \varphi_W(r))$ to be its graph so that $W = \{G_W(r) : r \in J_W\}$. We let m_W denote the unnormalized arclength measure on W, defined using the Euclidean metric.

Let $W_1, W_2 \in \mathcal{W}^s$ and let φ_{W_i}, G_{W_i} denote the corresponding functions defined above, for $i = 1, 2$. Denote by $\ell(J_{W_1} \triangle J_{W_2})$ the length of the symmetric difference between J_{W_1} and J_{W_2}. If W_1 and W_2 belong to the same homogeneity strip, we define the distance between them to be

$$d_{\mathcal{W}^s}(W_1, W_2) = \ell(J_{W_1} \triangle J_{W_2}) + |\varphi_{W_1} - \varphi_{W_2}|_{\mathcal{C}^1(J_{W_1}\cap J_{W_2})};$$

otherwise, we set $d_{\mathcal{W}^s}(W_1, W_2) = \infty$.

For $0 \leq p \leq 1$, let $\tilde{\mathcal{C}}^p(W)$ denote the set of continuous complex-valued functions on W with Hölder exponent p, measured in the Euclidean metric. Denote by $\mathcal{C}^p(W)$ the closure of $\mathcal{C}^\infty(W)$ in the $\tilde{\mathcal{C}}^p$-norm[5]:

$$|\psi|_{\mathcal{C}^p(W)} = |\psi|_{\mathcal{C}^0(W)} + C_W^{(p)}(\psi),$$

where $C_W^{(p)}(\psi)$ is the Hölder constant of ψ along W. It is remarkable to note that that with this definition,

$$|\psi_1\psi_2|_{\mathcal{C}^p(W)} \leq |\psi_1|_{\mathcal{C}^p(W)}|\psi_2|_{\mathcal{C}^p(W)}.$$

$\tilde{\mathcal{C}}^p(\bar{M}_0)$ and $\mathcal{C}^p(\bar{M}_0)$ can be defined similarly.

Given two curves $W_1, W_2 \in \mathcal{W}^s$ with $d_{\mathcal{W}^s}(W_1, W_2) < \infty$, and two test functions $\psi_i \in \mathcal{C}^p(W_i, \mathbb{C})$, *the distance between* ψ_1, ψ_2 is defined[6] as:

$$d(\psi_1, \psi_2) = |\psi_1 \circ G_{W_1} - \psi_2 \circ G_{W_2}|_{\mathcal{C}^0(J_{W_1}\cap J_{W_2})}.$$

[5] While $\mathcal{C}^p(W)$ is smaller than $\tilde{\mathcal{C}}^p(W)$, it does contain $\mathcal{C}^{p'}(W)$ for all $p' > p$.

[6] Note that $d(\psi_1, \psi_2)$ is only a pseudo-metric while $d_{\mathcal{W}^s}(\cdot,\cdot)$ does not satisfy the triangle inequality, yet they both serve as useful notions of distance when deriving the necessary Lasota-Yorke inequalities.

We will define the relevant Banach spaces by closing $\mathcal{C}^1(\bar{M}_0)$ with respect to the following set of norms. Fix $0 < p \le \frac{1}{3}$. Given a function $f \in \mathcal{C}^1(\bar{M}_0)$, define the *weak norm* of f by

$$|f|_w := \sup_{W\in\mathcal{W}^s} \sup_{\substack{\psi\in\mathcal{C}^p(W)\\ |\psi|_{\mathcal{C}^p(W)}\le 1}} \int_W f\psi \, dm_W. \tag{10}$$

Choose[7] $q, \gamma, \varsigma > 0$ such that $\varsigma \le 1 - \zeta_0$, $q < p$ and $\gamma \le \min\{\varsigma, p - q\}$. We define the *strong stable norm* of f as

$$\|f\|_s := \sup_{W\in\mathcal{W}^s} \sup_{\substack{\psi\in\mathcal{C}^q(W)\\ |\psi|_{\mathcal{C}^q(W)}\le |W|^{-\varsigma}}} \int_W f\psi \, dm_W \tag{11}$$

and the *strong unstable norm* as

$$\|f\|_u := \sup_{\varepsilon\le\varepsilon_0} \sup_{\substack{W_1,W_2\in\mathcal{W}^s\\ d_{\mathcal{W}^s}(W_1,W_2)\le\varepsilon}} \sup_{\substack{\psi_i\in\mathcal{C}^p(W_i)\\ |\psi_i|_{\mathcal{C}^p(W)}\le 1\\ d(\psi_1,\psi_2)=0}} \frac{1}{\varepsilon^\gamma} \left| \int_{W_1} f\psi_1 \, dm_W - \int_{W_2} f\psi_2 \, dm_W \right| \tag{12}$$

where $\varepsilon_0 > 0$ is chosen less than δ_0, the maximum length of $W \in \mathcal{W}^s$ which is determined by (9). The *strong norm* of f is defined by

$$\|f\|_{\mathcal{B}} = \|f\|_s + c_0\|f\|_u,$$

where c_0 is a small constant chosen so that the uniform Lasota-Yorke inequalities in [16, Theorem 2.2] hold.

We define $\mathcal{B}$ to be the completion of $\mathcal{C}^1(\bar{M}_0)$ in the strong norm[8] and $\mathcal{B}_w$ to be the completion of $\mathcal{C}^1(\bar{M}_0)$ in the weak norm.

Remark 3.1. Due to [16, Lemma 3.4], we have for $f \in \mathcal{B}_w$,

$$|f(\psi)| \le |f|_w \Big(|\psi|_\infty + \sup_{W\in\mathcal{W}^s} C_W^{(p)}(\psi) \Big), \qquad \text{for all } \psi \in \mathcal{C}^p(\mathcal{W}^s).$$

This permits us to extend $\mathbb{E}_{\bar{\mu}_0}[\cdot]$ to a linear continuous form on $\mathcal{B}_w$ (and so on $\mathcal{B}$) since

$$\forall f \in C^1(\bar{M}_0), \quad \mathbb{E}_{\bar{\mu}_0}[f] = \int_{\bar{M}_0} f \, d\bar{\mu}_0 = f(\mathbf{1}_{\bar{M}_0}).$$

We begin by recalling some properties of $\mathcal{B}$ and $\mathcal{B}_w$ proved in [15–17].

Lemma 3.2. a) *[15, Lemma 3.7] $\mathcal{B}$ contains piecewise Hölder continuous functions f with exponent $\zeta > \gamma/(1-\gamma)$ as described in Lemma 3.3 below.*

b) [16, Lemma 3.5] $(\cos\varphi)^{-1} \in \mathcal{B}$. *Thus, Lebesgue measure $m = (\cos\varphi)^{-1}\bar{\mu}_0 \in \mathcal{B}$ and so is fm for any f as in item (a) above.*

[7] The restrictions on the constants are placed according to the dynamical properties summarized in **(H1)**–**(H5)**. For example, $p \le 1/3$ due to the distortion bounds in **(H4)**, while $\varsigma \le 1 - \zeta_0$ due to **(H3)**, which is relevant for the uniform Lasota-Yorke inequalities (Lemma 3.14).

[8] As a measure, $f \in \mathcal{C}^1(\bar{M}_0)$ is identified with $f\,d\bar{\mu}_0$ according to our earlier convention. As a consequence, Lebesgue measure $dm = (\cos\varphi)^{-1}d\bar{\mu}_0$ is not automatically included in $\mathcal{B}$ since $(\cos\varphi)^{-1} \notin \mathcal{C}^1(\bar{M}_0)$. It follows from [16, Lemma 3.5] that in fact, $m \in \mathcal{B}$ (and $\mathcal{B}_w$).

c) [15, Lemma 2.1] *$\mathcal{L}_\omega$ is well-defined as a continuous linear operator on both $\mathcal{B}$ and $\mathcal{B}_w$ for any $\bar{T}_\omega \in \bar{\mathcal{F}}$. Moreover, there exists a sequence of continuous*[9] *inclusions $\mathcal{C}^\zeta(\bar{M}_0) \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{B}_w \hookrightarrow (\mathcal{C}^p(\bar{M}_0))'$, for all $\zeta > \gamma/(1-\gamma)$.*

d) [15, Lemma 3.10] *The unit ball of $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is compactly embedded in $(\mathcal{B}_w, |\cdot|_w)$.*

The following lemma is crucial for describing the types of discontinuities allowed in elements of $\mathcal{B}$ and for proving that the operator $\mathcal{L}_{u,\omega}$ is analytic in u.

Lemma 3.3. *Let $\mathfrak{P}$ be a (mod 0) countable partition of $\bar{M}_0$ into open, simply connected sets such that: (1) for each $k \in \mathbb{N}$, there is an $N_k < \infty$ such that at most N_k elements $Z \in \mathfrak{P}$ intersect $\mathbb{H}_k$; (2) there are constants $K, C_0 > 0$ such that for each $Z \in \mathfrak{P}$ and $W \in \mathcal{W}^s$, $Z \cap W$ comprises at most K connected components and for any*[10] *$\varepsilon > 0$, $m_W(N_\varepsilon(\partial Z) \cap W) \le C_0\varepsilon$.*

a) [17, Lemma 3.5] *Let $\zeta > \gamma/(1-\gamma)$. If $f \in \mathcal{C}^\zeta(Z)$ for each $Z \in \mathfrak{P}$ and $\sup_{Z\in\mathfrak{P}} |f|_{\mathcal{C}^\zeta(Z)} < \infty$, then $f \in \mathcal{B}$ and $\|f\|_{\mathcal{B}} \le C \sup_{Z\in\mathfrak{P}} |f|_{\mathcal{C}^\zeta(Z)}$, for some $C > 0$ independent of f. In particular, $\mathcal{C}^\zeta(\bar{M}_0) \subset \mathcal{B}$ for each $\zeta > \gamma/(1-\gamma)$.*

b) [17, Lemma 3.7] *Suppose in addition that $\zeta > \max\{p, \gamma/(1-\gamma)\}$ and there is a uniform bound on the N_k above. If g satisfies $\sup_{Z\in\mathfrak{P}} |g|_{\mathcal{C}^\zeta(Z)} < \infty$ and $f \in \mathcal{B}$, then $fg \in \mathcal{B}$ and $\|fg\|_{\mathcal{B}} \le C\|f\|_{\mathcal{B}} \sup_{Z\in\mathfrak{P}} |g|_{\mathcal{C}^\zeta(Z)}$ for some $C > 0$ independent of f and g.*

3.2. Banach spaces $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{B}}_w$. In this section, we introduce the associated Banach spaces $\widetilde{\mathcal{B}}_w$ and $\widetilde{\mathcal{B}}$ on $\bar{M}$ on which P acts suitably. $\widetilde{\mathcal{B}}$ will correspond to a set of Lipschitz functions from $E^{\mathbb{N}}$ to $\mathcal{B}$ and $\widetilde{\mathcal{B}}_w$ will correspond to the set of uniformly bounded functions from $E^{\mathbb{N}}$ to $\mathcal{B}_w$. For convenience, we will identify elements of $\mathcal{B}^{E^{\mathbb{N}}}$ with distributions f on $\bar{M}_0 \times E^{\mathbb{N}}$ such that $f(\cdot, \underline{\omega}) \in \mathcal{B}$ for all $\underline{\omega} \in E^{\mathbb{N}}$. Let $L(\mathcal{B}, \mathcal{B})$ denote the set of bounded linear operators on $\mathcal{B}$ and let $\|\cdot\|_{L(\mathcal{B},\mathcal{B})}$ denote the norm on $L(\mathcal{B}, \mathcal{B})$ induced by $\|\cdot\|_{\mathcal{B}}$.

Let $\varkappa > \sup_{\omega\in E} \|\mathcal{L}_\omega\|_{L(\mathcal{B},\mathcal{B})} \ge 1$. Let us define

$$\widetilde{\mathcal{B}} := \{f \in \mathcal{B}^{E^{\mathbb{N}}} \ : \ \|f\|_{\widetilde{\mathcal{B}}} < \infty\},$$

with

$$\|f\|_{\widetilde{\mathcal{B}}} := \sup_{\underline{\omega}\in E^{\mathbb{N}}} \|f(\cdot, \underline{\omega})\|_{\mathcal{B}} + \sup_{\underline{\omega}\neq\underline{\omega}'} \frac{\|f(\cdot, \underline{\omega}) - f(\cdot, \underline{\omega}')\|_{\mathcal{B}}}{d(\underline{\omega}, \underline{\omega}')},$$

and with

$$d((\omega_k)_k, (\omega'_k)_k) = \varkappa^{-\min\{k\ge 0:\omega_k\neq\omega_{k'}\}}.$$

It is immediate from this definition and the definition of $\mathcal{B}$, that $\widetilde{\mathcal{B}}$ is the completion in the $\|\cdot\|_{\widetilde{\mathcal{B}}}$ norm of the set of functions

$$(C^1(\bar{M}_0))^{E^{\mathbb{N}}} = \{f : \bar{M}_0 \times E^{\mathbb{N}} \to \mathbb{C} \ : \ f(\cdot, \underline{\omega}) \in C^1(\bar{M}_0) \ \forall \underline{\omega} \in E^{\mathbb{N}}\}.$$

[9] The first three of these are also injective. The fourth can be made injective by introducing a weight $|W|^{-\eta}$ for test functions ψ in the weak norm (as appears in the definition of $\|\cdot\|_s$) and requiring $\eta > p$ (see, for example, [17, Lemma 3.8]).

[10] In fact, Lemma 3.5 of [17] allows a nondegenerate tangency between $\partial\mathfrak{P}$ and the stable cone: $m_W(N_\varepsilon(\partial Z) \cap W) \le C_0\varepsilon^{t_0}$, for some $t_0 > 0$. But we will not need this weaker condition here so we assume $t_0 = 1$ in order to simplify the proofs and also the definition of the norms (which otherwise would depend on t_0).

In particular, $\widetilde{\mathcal{B}}$ is a Banach space.

Remark 3.4. It will be worthwhile to notice that, due to Lemma 3.3(a), for every $\omega \in E$, the coordinates of Φ_ω belong to $\mathcal{B}$, so that the coordinates of Φ are in $\widetilde{\mathcal{B}}$.

We also define

$$\widetilde{\mathcal{B}}_w := \{f \in (\mathcal{B}_w)^{E^{\mathbb{N}}} \ : \ |f|_{\widetilde{\mathcal{B}}_w} < \infty\},$$

with $|f|_{\widetilde{\mathcal{B}}_w} := \sup_{\underline{\omega}\in E^{\mathbb{N}}} |f(\cdot,\underline{\omega})|_w$. As with $\widetilde{\mathcal{B}}$, the space $\widetilde{\mathcal{B}}_w$ can also be realized as the completion of $(C^1(\bar{M}_0))^{E^{\mathbb{N}}}$ in the $|\cdot|_{\widetilde{\mathcal{B}}_w}$ norm.

Remark 3.5. Using Remark 3.1, we extend $\mathbb{E}_{\tilde{\mu}}[\cdot]$ to a continuous linear form on $\widetilde{\mathcal{B}}_w$ (and so on $\widetilde{\mathcal{B}}$) by setting

$$\forall f \in \widetilde{\mathcal{B}}_w, \quad \mathbb{E}_{\tilde{\mu}}[f] = \int_{E^{\mathbb{N}}} \mathbb{E}_{\tilde{\mu}_0}[f(\cdot,\underline{\omega})]\, d\eta^{\otimes\mathbb{N}}(\underline{\omega}).$$

It follows from Lemma 3.3(a) that for any obstacle O_a, $1_{O_a} \in \mathcal{B}$, and from Lemma 3.3(b) that $f(\cdot,\underline{\omega}) \mapsto 1_{O_a} f(\cdot,\underline{\omega})$ is a bounded linear operator on $\mathcal{B}$ for each $\underline{\omega} \in E^{\mathbb{N}}$ and $f \in \widetilde{\mathcal{B}}$. Thus $f \mapsto 1_{O_a} f$ is a bounded linear operator on $\widetilde{\mathcal{B}}$ as well.

We introduce the following notation for convenience.

Notation 3.6. *For any positive integer m, any $\tilde{\underline{\omega}}_m \in E^m$ and any $\underline{\omega} \in E^{\mathbb{N}}$, we will write $(\tilde{\underline{\omega}}_m, \underline{\omega})$ as the element of $E^{\mathbb{N}}$ obtained by concatenation; i.e. such that the first m terms correspond to those of $\tilde{\underline{\omega}}_m$ and that the term of order $m+k$ corresponds to the term of order k of $\underline{\omega}$.*

Lemma 3.7. (a) *Let n be a positive integer. Denote the norm $\|\cdot\|_\sigma$, for $\sigma \in \{w, s, u\}$. If $(f(\cdot,\tilde{\underline{\omega}}_n))_{\tilde{\underline{\omega}}_n \in E^n}$ is a measurable (in $\tilde{\underline{\omega}}_n$) family of elements of $\mathcal{B}_w$ such that*

$$\sup_{\tilde{\underline{\omega}}_n \in E^n} \|f(\cdot,\tilde{\underline{\omega}}_n)\|_\sigma < \infty,$$

then

$$\left\| \int_{E^n} f(\cdot,\tilde{\underline{\omega}}_n)\, d\eta^{\otimes n}(\tilde{\underline{\omega}}_n) \right\|_\sigma \le \int_{E^n} \|f(\cdot,\tilde{\underline{\omega}}_n)\|_\sigma \, d\eta^{\otimes n}(\tilde{\underline{\omega}}_n) \le \sup_{\tilde{\underline{\omega}}_n \in E^n} \|f(\cdot,\tilde{\underline{\omega}}_n)\|_\sigma .$$

(b) *If $(H_\omega)_{\omega\in E}$ is a measurable (in ω) family of uniformly bounded operators on $\mathcal{B}$ (resp. $\mathcal{B}_w$), then $H : f(x,\underline{\omega}) \mapsto \int_E H_{\tilde{\omega}}(f(x,(\tilde{\omega},\underline{\omega})))\, d\eta(\tilde{\omega})$ defines a continuous linear operator on $\widetilde{\mathcal{B}}$ (resp. $\widetilde{\mathcal{B}}_w$) with operator norm dominated by $\sup_{\omega\in E} \|H_\omega\|_{L(\mathcal{B},\mathcal{B})}$.*

Proof. (a) is just the triangle inequality. Let us prove Item (b). Let $f \in \widetilde{\mathcal{B}}$ or in $\widetilde{\mathcal{B}}_w$ and writing $\|\cdot\|_\sigma$ for the associated norm, due to (a), for every $\underline{\omega} \in E^{\mathbb{N}}$, we have

$$\begin{aligned} \|Hf(\cdot,\underline{\omega})\|_\sigma &\le \sup_{\tilde{\omega}\in E} \left\| H_{\tilde{\omega}} f(\cdot,(\tilde{\omega},\underline{\omega})) \right\|_\sigma \\ &\le \sup_{\tilde{\omega}} \|H_{\tilde{\omega}}\|_\sigma \, \|f(\cdot,(\tilde{\omega},\underline{\omega}))\|_\sigma \le \sup_{\tilde{\omega}} \|H_{\tilde{\omega}}\|_\sigma \sup_{\underline{\omega}'} \|f(\cdot,\underline{\omega}')\|_\sigma , \end{aligned}$$

which proves (b) if $f \in \widetilde{\mathcal{B}}_w$. If, in addition, $f \in \widetilde{\mathcal{B}}$, then for all $\underline{\omega}, \underline{\omega}' \in E^{\mathbb{N}}$,

$$
\begin{aligned}
&\|Hf(\cdot,\underline{\omega}) - Hf(\cdot,\underline{\omega}')\|_{\mathcal{B}} \\
&\quad\le \sup_{\tilde\omega} \|H_{\tilde\omega}\|_\sigma \sup_{\omega\in E} \|f(\cdot,(\omega,\underline{\omega})) - f(\cdot,(\omega,\underline{\omega}'))\|_\sigma \\
&\quad\le \sup_{\tilde\omega} \|H_{\tilde\omega}\|_\sigma \sup_{\underline{\omega},\underline{\omega}'\in E^{\mathbb{N}}} \frac{\|f(\cdot,(\omega,\underline{\omega})) - f(\cdot,(\omega,\underline{\omega}'))\|_\sigma}{d(\tilde{\underline{\omega}},\tilde{\underline{\omega}}')} \varkappa^{-1} d(\underline{\omega},\underline{\omega}')\,,
\end{aligned}
$$

where $\tilde{\underline{\omega}} = (\omega,\underline{\omega})$ and $\tilde{\underline{\omega}}' = (\omega,\underline{\omega}')$. □

Remark 3.8. The previous lemma ensures in particular that P acts continuously on both $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{B}}_w$ since $\mathcal{L}_\omega$ acts uniformly continuously on both $\mathcal{B}$ and $\mathcal{B}_w$.

A key step in our proof is the study of the spectral properties on $\widetilde{\mathcal{B}}$ of P and of the family of operators P_u defined by

$$
P_u := P(e^{iu\cdot\Phi}\cdot)\,.
$$

The next lemma ensures, in particular, that P_u is a linear operator on $\widetilde{\mathcal{B}}$. Denote by $\Phi^{(1)}$ and $\Phi^{(2)}$ the components of the vector Φ.

Lemma 3.9. *For every $u \in \mathbb{R}^2$, any positive integer m and any $i_1,\dots,i_m \in \{1,2\}$, $P(\Phi^{(i_1)}\dots\Phi^{(i_m)}e^{iu\cdot\Phi}\cdot)$ is a linear operator on $\widetilde{\mathcal{B}}$ and on $\widetilde{\mathcal{B}}_w$, with operator norms uniformly in $O(\sup_{\omega\in E}\|\Phi\|_\infty^m)$.*

Proof. This proof is a variation of the argument used in [16, Section 5.2]. Recall that $Pg(\cdot,\underline{\omega}) = \int_E g(\bar T_\omega^{-1}(\cdot),(\omega,\underline{\omega}))\,d\eta(\omega)$, so that

$$
\begin{aligned}
&P(\Phi^{(i_1)}\dots\Phi^{(i_m)}e^{iu\cdot\Phi}f)(\cdot,\underline{\omega}) \\
&\quad= \int_E \left(\Phi_\omega^{(i_1)}\dots\Phi_\omega^{(i_m)}e^{iu\cdot\Phi_\omega}\right)\circ\bar T_\omega^{-1}\mathcal{L}_\omega f(\cdot,(\omega,\underline{\omega}))\,d\eta(\omega)\,.
\end{aligned}
$$

Let $\omega \in E$. The singularity set for $\Phi_\omega\circ\bar T_\omega^{-1}$ is contained in $\mathcal{S}_0 \cup \bar T_\omega\mathcal{S}_0$, which by assumption **(H1)** is comprised of finitely many smooth curves that are uniformly transverse to $C^s(x)$. Let $\mathcal{Z}$ denote the (finite) partition of $\bar M_0\backslash(\mathcal{S}_0\cup\bar T_\omega\mathcal{S}_0)$ into its maximal connected components, and note that $\mathcal{Z}$ satisfies the hypotheses of Lemma 3.3. In particular, $\mathcal{Z}\cap\mathbb{H}_k$ consists of only a finite number of connected components, which is bounded independently of k for $|k|\ge k_0$. Note that $\Phi_\omega\circ\bar T_\omega^{-1}$ is constant on each element of $\mathcal{Z}$. We use Lemma 3.3(b) to estimate, for every $f\in\widetilde{\mathcal{B}}$,

$$
\begin{aligned}
&\|\mathcal{L}_\omega(\Phi_\omega^{(i_1)}\dots\Phi_\omega^{(i_m)}e^{iu\cdot\Phi}f)(\cdot,\underline{\omega})\|_{\mathcal{B}} \\
&\quad= \|(i\cdot\Phi_\omega)^n\circ\bar T_\omega^{-1}\,(\mathcal{L}_\omega f)(\cdot,\underline{\omega})\|_{\mathcal{B}} \\
&\quad\le C\sup_{Z\in\mathcal{Z}} |(\Phi_\omega^{(i_1)}\dots\Phi_\omega^{(i_m)}e^{iu\cdot\Phi_\omega})\circ\bar T_\omega^{-1}|_{\mathcal{C}^1(Z)}\|\mathcal{L}_\omega f(\cdot,(\omega,\underline{\omega}))\|_{\mathcal{B}} \\
&\quad\le C'\|\Phi_\omega\|_\infty^m\|f(\cdot,(\omega,\underline{\omega}))\|_{\mathcal{B}}\,.
\end{aligned}
\tag{13}
$$

Analogously, for every $f\in\widetilde{\mathcal{B}}_w$,

$$
|\mathcal{L}_\omega(\Phi_\omega^{(i_1)}\dots\Phi_\omega^{(i_m)}e^{iu\cdot\Phi}f)(\cdot,\underline{\omega})|_w \le C'\|\Phi\|_\infty^m|f(\cdot,(\omega,\underline{\omega}))|_w\,,
$$

and we conclude by Item (b) of Lemma 3.7. □

3.3. P_u as a perturbation of a quasicompact operator. For the remainder of Sect. 3, we fix a billiard map $\bar{T}_0$, and for $\vartheta_0 > 0$, define $\bar{\mathcal{F}}_{\vartheta_0}(\bar{T}_0)$ as in (5). Our main results in this setting will be that for ϑ_0 sufficiently small, both P and P_u are quasi-compact and have a spectral gap in $\widetilde{\mathcal{B}}$. These statements are contained in Proposition 3.15 and Theorem 3.17.

Recall $P_u := P(e^{iu\cdot\Phi}\cdot)$. Our next result states that P_u is a small perturbation (as $\vartheta_0 \to 0$) of $\mathcal{P}_u := \mathcal{P}(e^{iu\cdot\Phi_0}\cdot)$, where $\mathcal{P}$ is the transfer operator $\mathcal{P}_0$ of the direct product $(\bar{M}, \bar{\mu}, \bar{T}_0' := \bar{T}_0 \times \sigma)$, i.e.

$$\mathcal{P}(f)(y, (\omega_k)_{\geq 0}) = \int_E \mathcal{L}_0 f(\cdot, (\omega_{k-1})_k)(y)\, d\eta(\omega_{-1})\,,$$

and

$$\mathcal{P}_u(f)(y, (\omega_k)_{\geq 0}) = \int_E \mathcal{L}_{u,0} f(\cdot, (\omega_{k-1})_k)(y)\, d\eta(\omega_{-1})\,.$$

Here, $\mathcal{L}_0 = \mathcal{L}_{\bar{T}_0}$ and $\mathcal{L}_{u,0} = \mathcal{L}_{\bar{T}_0}(e^{iu\cdot\Phi_0}\cdot)$.

Proposition 3.10. *There exists $C > 0$ such that for every $u \in \mathbb{R}^2$ and every $f \in \widetilde{\mathcal{B}}$,*

$$|P_u f - \mathcal{P}_u f|_{\widetilde{\mathcal{B}}_w} \leq C \|f\|_{\widetilde{\mathcal{B}}} \vartheta_0^{\frac{\gamma}{2}}\,.$$

Before proving this proposition, we state the following lemma.

Lemma 3.11. *There exists $C > 0$ such that for all $\omega \in E$ and $u \in \mathbb{R}^2$,*

$$|\mathcal{L}_{u,\omega} f - \mathcal{L}_{u,0} f|_w \leq C \|f\|_{\mathcal{B}} d_{\bar{\mathcal{F}}}(\bar{T}_\omega, \bar{T}_0)^{\gamma/2}, \quad \forall\, f \in \mathcal{B}.$$

Proof. This lemma for $u = 0$ is proved in [16, Theorem 2.3]. We must show that the relevant estimates are independent of u. For the convenience of the reader, we reproduce the main points of the argument.

Let $\varepsilon = d_{\bar{\mathcal{F}}}(\bar{T}_\omega, \bar{T}_0)$ and let $W \in \mathcal{W}^s$, $f \in \mathcal{C}^1(\bar{M}_0)$ and $\psi \in \mathcal{C}^p(W)$ with $|\psi|_{\mathcal{C}^p(W)} \leq 1$. Following [16, Sect. 5] (also [16, Sect. 4.3]) we decompose $\bar{T}_0^{-1}W$ and $\bar{T}_\omega^{-1}W$ into matched and unmatched pieces on which $\bar{T}_0$ and $\bar{T}_\omega$ are continuous, respectively, $\bar{T}_0^{-1}W = (\cup_j U_j^0) \cup (\cup_k V_k^0)$ and $\bar{T}_\omega^{-1}W = (\cup_j U_j^\omega) \cup (\cup_k V_k^\omega)$. The matched pieces U_j^0 and U_j^ω can be connected by a foliation of vertical line segments defined on a common r-interval I_j as in [16, eq. (4.12)]. The unmatched pieces satisfy $|\bar{T}_0 V_j^0|, |\bar{T}_\omega V_j^\omega| \leq C\varepsilon$.

Thus following [16, eq. (5.2)] we write,

$$\begin{aligned}\int_W (\mathcal{L}_{u,\omega} f - \mathcal{L}_{u,0} f)\psi\, dm_W &\leq \sum_{\ell,k} \left| \int_{V_k^\ell} f\, e^{iu\cdot\Phi_\ell}\, \psi \circ \bar{T}_\ell\, J_{V_k^\ell} \bar{T}_\ell\, dm_W \right| \\ &+ \sum_j \left| \int_{U_j^\omega} f\, e^{iu\cdot\Phi_\omega}\, \psi \circ \bar{T}_\omega\, J_{U_j^\omega} \bar{T}_\omega\, dm_W - \int_{U_j^0} f\, e^{iu\cdot\Phi_0}\, \psi \circ \bar{T}_0\, J_{U_j^0} \bar{T}_0\, dm_W \right|,\end{aligned} \tag{14}$$

where $\ell \in \{0, \omega\}$ in the first sum. We estimate the integrals over the unmatched pieces using the strong stable norm,

$$\begin{aligned}&\int_{V_k^\ell} f\, e^{iu\cdot\Phi_\ell}\, \psi \circ \bar{T}_\ell\, J_{V_k^\ell} \bar{T}_\ell\, dm_W \\ &\leq \|f\|_s |V_k^\ell|^\varsigma |e^{iu\cdot\Phi_\ell}\, \psi \circ \bar{T}_\ell|_{\mathcal{C}^p(V_k^\ell)} |J_{V_k^\ell} \bar{T}_\ell|_{\mathcal{C}^p(V_k^\ell)} \\ &\leq C \|f\|_s |T_\ell V_k^\ell|^\varsigma |e^{iu\cdot\Phi_\ell}\, \psi \circ \bar{T}_\ell|_{\mathcal{C}^p(V_k^\ell)} |J_{V_k^\ell} \bar{T}_\ell|^{1-\varsigma}_{\mathcal{C}^0(V_k^\ell)}\,,\end{aligned}$$

where we have used bounded distortion **(H4)** to bound

$$|J_{V_k^\ell}\bar{T}_\ell|_{\mathcal{C}^p(V_k^\ell)} \le C|J_{V_k^\ell}\bar{T}_\ell|_{\mathcal{C}^0(V_k^\ell)} \quad \text{and} \quad |V_k^\ell||J_{V_k^\ell}\bar{T}_\ell|_{\mathcal{C}^0(V_k^\ell)} \le C|\bar{T}_\ell V_k^\ell|\,.$$

Next, since $e^{iu\cdot\Phi_\ell}$ is constant on each V_k^ℓ, we have

$$|e^{iu\cdot\Phi_\ell}\psi\circ\bar{T}_\ell|_{\mathcal{C}^p(V_k^\ell)} \le |e^{iu\cdot\Phi_\ell}|_\infty|\psi\circ\bar{T}_\ell|_{\mathcal{C}^p(V_k^\ell)} = |\psi\circ\bar{T}_\ell|_{\mathcal{C}^p(V_k^\ell)}\,. \tag{15}$$

Finally, since $|\psi\circ\bar{T}_\ell|_{\mathcal{C}^p(V_k^\ell)} \le C|\psi|_{\mathcal{C}^p(W)}$ by **(H1)** (see [16, eq. (4.6)]), we complete the estimate on unmatched pieces,

$$\sum_{\ell,k}\left|\int_{V_k^\ell} f\, e^{iu\cdot\Phi_\ell}\,\psi\circ\bar{T}_\ell\, J_{V_k^\ell}\bar{T}_\ell\, dm_W\right| \le C\|f\|_s\varepsilon^\varsigma\sum_{\ell,k}|J_{V_k^\ell}\bar{T}_\ell|^{1-\varsigma}_{\mathcal{C}^0(V_k^\ell)}\,, \tag{16}$$

and the sum is uniformly bounded by **(H3)** since $\varsigma > \xi_0$.

Next we perform the estimate on matched pieces in (14). Since matched pieces lie in the same connected component of $\bar{M}_0\backslash(\mathcal{S}_1^{\bar{T}_\omega}\cup\mathcal{S}_1^{\bar{T}_0})$, we have $\Phi_\omega = \Phi_0$ on such components. Thus,

$$\begin{aligned}&\sum_j\left|\int_{U_j^\omega} f\;\; e^{iu\cdot\Phi_\omega}\,\psi\circ\bar{T}_\omega\, J_{U_j^\omega}\bar{T}_\omega\, dm_W - \int_{U_j^0} f\, e^{iu\cdot\Phi_0}\,\psi\circ\bar{T}_0\, J_{U_j^0}\bar{T}_0\, dm_W\right|\\ &\le \sum_j|e^{iu\cdot\Phi_0}|_{\mathcal{C}^0(U_j^0)}\left|\int_{U_j^\omega} f\,\psi\circ\bar{T}_\omega\, J_{U_j^\omega}\bar{T}_\omega\, dm_W - \int_{U_j^0} f\,\psi\circ\bar{T}_0\, J_{U_j^0}\bar{T}_0\, dm_W\right|\,.\end{aligned} \tag{17}$$

Since $|e^{iu\cdot\Phi_0}|_{\mathcal{C}^0(U_j^0)} = 1$, this is precisely the same expression as in [16, eq. (5.4)]. Thus combining [16, eq. (5.9)] with (16) proves the lemma, with constant independent of $u\in\mathbb{R}^2$. □

Proof of Proposition 3.10. This comes directly from Lemmas 3.11 and 3.7. Indeed, for every $f\in\widetilde{\mathcal{B}}$, we have

$$\begin{aligned}&\sup_{\underline{\omega}\in E^{\mathbb{N}}}\left|(P_u-\mathcal{P}_u)f(\cdot,\underline{\omega})\right|_w\\ &= \sup_{\underline{\omega}\in E^{\mathbb{N}}}\left|\int_E\left(\mathcal{L}_{u,\omega_{-1}}-\mathcal{L}_{u,0}\right)f(\cdot,(\omega_{-1},\underline{\omega}))\,d\eta(\omega_{-1})\right|_w\\ &\le \sup_{\underline{\omega}\in E^{\mathbb{N}}}\int_E\left|\left(\mathcal{L}_{u,\omega_{-1}}-\mathcal{L}_{u,0}\right)f(\cdot,(\omega_{-1},\underline{\omega}))\right|_w\,d\eta(\omega_{-1})\\ &\le C\sup_{\underline{\omega}'\in E^{\mathbb{N}}}\|f(\cdot,\underline{\omega}')\|_{\mathcal{B}}\,\vartheta_0^{\frac{\gamma}{2}} = C\|f\|_{\widetilde{\mathcal{B}}}\,\vartheta_0^{\frac{\gamma}{2}}\,,\end{aligned}$$

since $\bar{T}_\omega\in\bar{\mathcal{F}}_{\theta_0}(\bar{T}_0)$. □

Lemma 3.12. *$\mathcal{P}$ is quasicompact, 1 is its only dominating eigenvalue and it is a simple eigenvalue (with eigenspace $\mathbb{C}.\bar{\mu}$). In particular, there exists $\tilde{C}>0$ and $\tilde{\alpha}\in(0,1)$ such that*

$$\forall f\in\widetilde{\mathcal{B}},\quad \|\mathcal{P}^n f - \mathbb{E}_{\bar{\mu}}[f]\mathbf{1}_{\bar{M}}\|_{\widetilde{\mathcal{B}}} \le \tilde{C}\tilde{\alpha}^n\|f\|_{\widetilde{\mathcal{B}}}.$$

Proof. Due to [16, Theorem 2.2 and Corollary 2.4]) $\mathcal{L}_0$ is quasicompact, 1 is its only dominating eigenvalue and it is a simple eigenvalue (with eigenspace $\mathbb{C}.\mathbf{1}_{\bar{M}_0}$). In particular, there exists $\tilde{C} > 0$, $\tilde{\alpha}_0 \in (0,1)$ such that

$$\forall h \in \mathcal{B}, \quad \|\mathcal{L}_0^n h - \mathbb{E}_{\bar{\mu}_0}[h]\mathbf{1}_{\bar{M}_0}\|_{\mathcal{B}} \le \tilde{C}\tilde{\alpha}_0^n \|h\|_{\mathcal{B}}\,.$$

Let $f \in \widetilde{\mathcal{B}}$. Observe that

$$\mathcal{P}^n(f)(y,(\omega_k)_{k\ge 0}) = \int_{E^n} \mathcal{L}_0^n f(\cdot,(\omega_{k-n})_k)(y)\, d\eta^{\otimes n}(\omega_{-n},\dots,\omega_{-1})$$

and that

$$\mathbb{E}_{\bar{\mu}}[f] = \int_{\mathbb{E}^{\mathbb{N}}} \mathbb{E}_{\bar{\mu}_0}[f(\cdot,\underline{\omega}')]\, d\eta^{\otimes\mathbb{N}}(\underline{\omega}')\,.$$

First, setting $\tilde{\omega}_n = (\omega_{-n},\dots,\omega_{-1})$, we have, using Lemma 3.3(a),

$$\begin{aligned}
&\sup_{\underline{\omega}} \|\mathcal{P}^n(f)(\cdot,\underline{\omega}) - \mathbb{E}_{\bar{\mu}}[f]\mathbf{1}_{\bar{M}}\|_{\mathcal{B}}\\
&\quad= \sup_{\underline{\omega}} \left\| \int_{E^n} (\mathcal{L}_0^n f(\cdot,(\omega_{k-n})_k) - \mathbb{E}_{\bar{\mu}}[f])\, d\eta(\omega_{-1})\dots d\eta(\omega_{-n}) \right\|_{\mathcal{B}}\\
&\quad\le \sup_{\underline{\omega}} \left\| \int_{E^n} \mathcal{L}_0^n f(\cdot,(\omega_{k-n})_k) - \mathbb{E}_{\bar{\mu}_0}\left[f(\cdot,(\omega_{k-n})_k)\right]\, d\eta(\omega_{-1})\dots d\eta(\omega_{-n}) \right\|_{\mathcal{B}}\\
&\qquad+ \|\mathbf{1}_{\bar{M}_0}\|_{\mathcal{B}} \sup_{\underline{\omega}} \left| \int_{E^n} \Big(\mathbb{E}_{\bar{\mu}_0}\left[f(\cdot,(\tilde{\omega}_n,\underline{\omega}))\right]\right.\\
&\qquad\left. - \int_{E^{\mathbb{N}}} \mathbb{E}_{\bar{\mu}_0}(f(\cdot,(\tilde{\omega}_n,\underline{\omega}')))d\eta^{\otimes\mathbb{N}}(\underline{\omega}') \Big)\, d\eta^{\otimes n}(\tilde{\omega}_n) \right|\\
&\quad\le \sup_{\underline{\omega}} \int_{E^n} \left\| \mathcal{L}_0^n f(\cdot,(\omega_{k-n})_k) - \mathbb{E}_{\bar{\mu}_0}[f(\cdot,(\omega_{k-n})_k)] \right\|_{\mathcal{B}}\, d\eta(\omega_{-1})\dots d\eta(\omega_{-n})\\
&\qquad+\|\mathbf{1}_{\bar{M}_0}\|_{\mathcal{B}} \sup_{\underline{\omega},\underline{\omega}' : d(\underline{\omega},\underline{\omega}')<\varkappa^{-n}} \left|\mathbb{E}_{\bar{\mu}_0}\left[f(\cdot,\underline{\omega}) - f(\cdot,\underline{\omega}')\right]\right|\\
&\quad\le \tilde{C}\tilde{\alpha}_0^n \int_{E^n} \|f(\cdot,(\omega_{k-n})_k)\|_{\mathcal{B}}\, d\eta(\omega_{-1})\dots d\eta(\omega_{-n})\\
&\qquad+\|\mathbf{1}_{\bar{M}_0}\|_{\mathcal{B}}\|\mathbb{E}_{\bar{\mu}_0}[\cdot]\|_{\mathcal{B}'} \sup_{\underline{\omega},\underline{\omega}' : d(\underline{\omega},\underline{\omega}')<\varkappa^{-n}} \left\| f(\cdot,\underline{\omega}) - f(\cdot,\underline{\omega}') \right\|_{\mathcal{B}}\\
&\quad\le (\tilde{C}\tilde{\alpha}_0^n + C_1\varkappa^{-n})\|f\|_{\widetilde{\mathcal{B}}},
\end{aligned}$$

since $\mathbf{1}_{\bar{M}_0}$ is in $\mathcal{B}$ and $\mathbb{E}_{\bar{\mu}_0}[\cdot]$ is in the dual of $\mathcal{B}$ by Remark 3.1.

Second, for every $\underline{\omega}$ and $\underline{\omega}'$ in $E^{\mathbb{N}}$, we have

$$\begin{aligned}
&\|\mathcal{P}^n(f)(\cdot,\underline{\omega}) - \mathcal{P}^n(f)(\cdot,\underline{\omega}')\|_{\mathcal{B}}\\
&\quad= \left\| \int_{E^n} \left(\mathcal{L}_0^n f(\cdot,(\tilde{\omega}_n,\underline{\omega})) - \mathcal{L}_0^n f(\cdot,(\tilde{\omega}_n,\underline{\omega}))\right)\, d\eta^{\otimes n}(\tilde{\omega}_n) \right\|_{\mathcal{B}}\\
&\quad\le \int_{E^n} \left\| \mathcal{L}_0^n \left(f(\cdot,(\tilde{\omega}_n,\underline{\omega})) - f(\cdot,(\tilde{\omega}_n,\underline{\omega}))\right) \right\|_{\mathcal{B}}\, d\eta^{\otimes n}(\tilde{\omega}_n)
\end{aligned}$$

$$
\begin{aligned}
&\le \sup_{\underline{\omega}^{(1)},\underline{\omega}^{(2)}\,:\, d(\underline{\omega}^{(1)},\underline{\omega}^{(2)})<d(\underline{\omega},\underline{\omega}')\varkappa^{-n}} \left|\mathbb{E}_{\bar{\mu}_0}\left[f(\cdot,\underline{\omega}^{(1)})-f(\cdot,\underline{\omega}^{(2)})\right]\right| \\
&\quad +\tilde{C}\tilde{\alpha}_0^n\|f\|_{\widetilde{\mathcal{B}}} d(\underline{\omega},\underline{\omega}')\varkappa^{-n} \\
&\le \left\|\mathbb{E}_{\bar{\mu}_0}[\cdot]\right\|_{\mathcal{B}'} \sup_{\underline{\omega}^{(1)},\underline{\omega}^{(2)}\,:\, d(\underline{\omega}^{(1)},\underline{\omega}^{(2)})<d(\underline{\omega},\underline{\omega}')\varkappa^{-n}} \left\|f(\cdot,\underline{\omega}^{(1)})-f(\cdot,\underline{\omega}^{(2)})\right\|_{\mathcal{B}} \\
&\quad +\tilde{C}\tilde{\alpha}_0^n\|f\|_{\widetilde{\mathcal{B}}} d(\underline{\omega},\underline{\omega}')\varkappa^{-n} \\
&\le \left\|\mathbb{E}_{\bar{\mu}_0}[\cdot]\right\|_{\mathcal{B}'}\|f\|_{\widetilde{\mathcal{B}}} d(\underline{\omega},\underline{\omega}')\varkappa^{-n} + \tilde{C}\tilde{\alpha}_0^n\|f\|_{\widetilde{\mathcal{B}}} d(\underline{\omega},\underline{\omega}')\varkappa^{-n}\,.
\end{aligned}
$$

This proves the lemma with $\tilde{\alpha} = \max\{\tilde{\alpha}_0, \varkappa^{-1}\}$. □

3.4. Doeblin–Fortet–Lasota–Yorke type inequality for P_u. We next establish the spectral properties of P and P_u on $\widetilde{\mathcal{B}}$.

Proposition 3.13. *There exist $\tilde{C} > 0$ and $\tilde{\tau} \in (0,1)$, such that for every $n \ge 1$, $f \in \widetilde{\mathcal{B}}$, $u \in \mathbb{R}^2$ and $n \ge 0$,*

$$
\begin{aligned}
&|P_u^n f|_{\widetilde{\mathcal{B}}_w} \le \tilde{C}|f|_{\widetilde{\mathcal{B}}_w}, \\
&\|P_u^n f\|_{\widetilde{\mathcal{B}}} \le \tilde{C}\left(\tilde{\tau}^n\|f\|_{\widetilde{\mathcal{B}}} + |f|_{\widetilde{\mathcal{B}}_w}\right).
\end{aligned} \tag{18}
$$

This result will follow directly from the next lemma.

Lemma 3.14. *There exist $C > 0$ and $\tau \in (0,1)$, such for every $n \ge 1$, $\omega_1, \dots, \omega_n \in E$, $f \in \mathcal{B}$, $u \in \mathbb{R}^2$ and $n \ge 0$,*

$$
\begin{aligned}
&|\mathcal{L}_{u,\omega_1}\cdots\mathcal{L}_{u,\omega_n} f|_w \le C|f|_w, \\
&\|\mathcal{L}_{u,\omega_1}\cdots\mathcal{L}_{u,\omega_n} f\|_{\mathcal{B}} \le C\left(\tau^n\|f\|_{\mathcal{B}} + |f|_w\right).
\end{aligned} \tag{19}
$$

Proof. Here we denote $\mathcal{L}_{u,\underline{\omega}}^n := \mathcal{L}_{u,\omega_1}\cdots\mathcal{L}_{u,\omega_n}$, and $\bar{T}_{\underline{\omega}}^n = \bar{T}_{\omega_n}\circ\cdots\circ\bar{T}_{\omega_1}$. The above Lasota–Yorke inequalities are proved[11] for $\mathcal{L}_{\underline{\omega}}^n$ as long as each $\bar{T}_{\omega_k} \in \bar{\mathcal{F}}$ by [16, Proposition 5.6], with $\underline{\omega} = (\omega_k)_{k\ge1}$. As in the proof of Lemma 3.11, we must show that the constants appearing in the inequalities are independent of $u \in \mathbb{R}^2$, and all $\underline{\omega} \in E^{\mathbb{N}}$. We will use the fact that $S_n\Phi_{\underline{\omega}}$ is constant on elements of $\bar{M}_0 \backslash \mathcal{S}_n^{\bar{T}_{\underline{\omega}}}$.

We perform the weak norm estimate first. For $f \in \mathcal{C}^1(\bar{M}_0)$, $W \in \mathcal{W}^s$ and $\psi \in \mathcal{C}^p(W)$ with $|\psi|_{\mathcal{C}^p(W)} \le 1$, we must estimate,

$$
\int_W \mathcal{L}_{u,\underline{\omega}}^n f\,\psi\,dm_W = \sum_{W_i\in\mathcal{G}_n(W)} \int_{W_i} f e^{iu\cdot S_n\Phi_{\underline{\omega}}} J_{W_i}\bar{T}_{\underline{\omega}}^n\,\psi\circ\bar{T}_{\underline{\omega}}^n\,dm_{W_i},
$$

[11] The estimates in [16, Proposition 5.6] include a factor $\eta \ge 1$, which comes from the Jacobian of $\bar{T}_\omega$ with respect to $\bar{\mu}_0$. Since we have assumed that $J_{\bar{\mu}_0}\bar{T}_\omega = 1$ in our simplified version of **(H5)**, we have $\eta = 1$ in the present setting. Also note that the density function g for the random perturbation in [16] is identically 1 in our setting as well.

where $\mathcal{G}_n(W)$ are the components of $\bar{T}_{\underline{\omega}}^{-n} W$, subdivided so that they each belong to $\mathcal{W}^s$. Thus,

$$\begin{aligned}\int_W \mathcal{L}_{u,\underline{\omega}}^n f\, \psi\, dm_W &\le \sum_{W_i \in \mathcal{G}_n(W)} |f|_w |e^{iu\cdot S_n \Phi_{\underline{\omega}}} \psi \circ \bar{T}_{\underline{\omega}}^n|_{\mathcal{C}^p(W_i)} |J_{W_i} \bar{T}_{\underline{\omega}}^n|_{\mathcal{C}^p(W_i)} \\ &\le C|f|_w \sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} \bar{T}_{\underline{\omega}}^n|_{\mathcal{C}^0(W_i)}\,,\end{aligned}$$

where as in (15), we have used that $e^{iu\cdot S_n \Phi_{\underline{\omega}}}$ is constant on each W_i, so that

$$|e^{iu\cdot S_n \Phi_{\underline{\omega}}} \psi \circ \bar{T}_{\underline{\omega}}^n|_{\mathcal{C}^p(W_i)} \le |e^{iu\cdot S_n \Phi_{\underline{\omega}}}|_\infty |\psi \circ \bar{T}_{\underline{\omega}}^n|_{\mathcal{C}^p(W_i)} \le |\psi|_{\mathcal{C}^p(W)}\,. \tag{20}$$

The sum over the Jacobians is uniformly bounded by [16, Lemma 5.5]. Note that due to (20), the bound is independent of u, and thus prescisely the same as in [16, eq. (5.21)].

For the strong stable norm estimate, the same observation holds, again since $e^{iu\cdot S_n \Phi_{\underline{\omega}}}$ is constant on each W_i. Thus by [16, eq. 5.22],

$$\|\mathcal{L}_{u,\underline{\omega}}^n f\|_s \le C(\theta^{(1-\varsigma)n} + \Lambda^{-qn})\|f\|_s + C|f|_w\,.$$

For the strong unstable norm estimate, we must compare values of test functions on two stable curves W^1, W^2 that lie close together. As in the proof of Lemma 3.11 (see also [16, Sect. 4.3]), we decompose $T_{\underline{\omega}}^{-n} W^1$ and $T_{\underline{\omega}}^{-n} W^2$ into matched and unmatched pieces on which $T_{\underline{\omega}}^n$ is continuous, $T_{\underline{\omega}}^{-n} W^\ell = (\cup_j U_j^\ell) \cup (\cup_k V_k^\ell)$, $\ell = 1, 2$. The matched pieces U_j^1 and U_j^2 can be connected by a transverse foliation of unstable curves and are defined over a common r-interval as in [16, eq. (4.3)].

Since for each j, U_j^1 and U_j^2 lie in the same component of $\bar{M}_0 \backslash \mathcal{S}_n^{\bar{T}_{\underline{\omega}}}$, it follows that $S_n \Phi_{\underline{\omega}}$ has the same constant value on both curves and so factors right out of the Lasota–Yorke inequalities, precisely as in (17). Since $|e^{iu\cdot S_n \Phi_{\underline{\omega}}}| = 1$, the estimate on unmatched pieces can be performed as in (16). Thus by [16, eq. (5.23)],

$$\|\mathcal{L}_{u,\underline{\omega}}^n f\|_u \le C\Lambda^{-\gamma n}\|f\|_u + C_1^n \|f\|_s\,.$$

Combining the inequalities for the stable and unstable components as in [16, Sect. 4] then completes the proof of the Lasota–Yorke inequality for the strong norm. □

Proof of Proposition 3.13. Observe that

$$(P_u^n f)(\cdot, \underline{\omega}) = \int_{E^n} \mathcal{L}_{u,\omega_{-n}} \cdots \mathcal{L}_{u,\omega_{-1}} f(\cdot, (\omega_{k-n})_{k\ge 0})(y)\, d\eta^{\otimes n}(\omega_{-n}, \ldots, \omega_{-1})\,.$$

Due to Lemma 3.7 and to the first inequality of Lemma 3.14, for any $f \in \widetilde{\mathcal{B}}_w$ and $n \ge 1$,

$$\begin{aligned}\left|P_u^n f\right|_{\widetilde{\mathcal{B}}_w} &= \sup_{\underline{\omega} \in E^{\mathbb{N}}} \left|(P_u^n f)(\cdot, \underline{\omega})\right|_w \\ &\le \sup_{\underline{\omega} \in E^{\mathbb{N}}} \int_{E^n} \left|\mathcal{L}_{u,\omega_{-n}} \cdots \mathcal{L}_{u,\omega_{-1}} f(\cdot, (\omega_{k-n})_{k\ge 0})\right|_w d\eta^{\otimes n}(\omega_{-n}, \ldots, \omega_{-1}) \\ &\le C \sup_{\underline{\omega}' \in E^{\mathbb{N}}} \left|f(\cdot, \underline{\omega}')\right|_w = C|f|_{\widetilde{\mathcal{B}}_w}\,.\end{aligned}$$

Analogously, using again Lemma 3.7 and, this time, the second inequality of Lemma 3.14, we obtain, for any $f \in \widetilde{\mathcal{B}}$ and $n \geq 1$,

$$\sup_{\underline{\omega} \in E^{\mathbb{N}}} \left\| (P_u^n f)(\cdot, \underline{\omega}) \right\|_{\mathcal{B}} \leq C \left(\tau^n \sup_{\underline{\omega} \in E^{\mathbb{N}}} \left\| f(\cdot, \underline{\omega}) \right\|_{\mathcal{B}} + \sup_{\underline{\omega} \in E^{\mathbb{N}}} \left| f(\cdot, \underline{\omega}) \right|_w \right).$$

Finally, using Lemma 3.7,

$$\begin{aligned}
&\sup_{\underline{\omega} \neq \underline{\omega}'} \frac{\| P_u^n f(\cdot, \omega) - P_u^n f(\cdot, \underline{\omega}') \|_{\mathcal{B}}}{d(\underline{\omega}, \underline{\omega}')} \\
&= \sup_{\underline{\omega} \neq \underline{\omega}'} \frac{\| \int_{E^n} \mathcal{L}_{u,\omega_{-n}} \cdots \mathcal{L}_{u,\omega_{-1}} \left(f(\cdot, (\tilde{\omega}, \underline{\omega})) - f(\cdot, (\tilde{\omega}, \underline{\omega}')) \right) d\eta^{\otimes n}(\tilde{\omega}) \|_{\mathcal{B}}}{d(\underline{\omega}, \underline{\omega}')} \\
&\leq \sup_{\underline{\omega} \neq \underline{\omega}'} \frac{\int_{E^n} \| \mathcal{L}_{u,\omega_{-n}} \cdots \mathcal{L}_{u,\omega_{-1}} \left(f(\cdot, (\tilde{\omega}, \underline{\omega})) - f(\cdot, (\tilde{\omega}, \underline{\omega}')) \right) \|_{\mathcal{B}} \, d\eta^{\otimes n}(\tilde{\omega})}{d(\underline{\omega}, \underline{\omega}')} \\
&\leq \int_{E^n} \sup_{\underline{\omega} \neq \underline{\omega}'} \frac{\| \mathcal{L}_{u,\omega_{-n}} \cdots \mathcal{L}_{u,\omega_{-1}} \left(f(\cdot, (\tilde{\omega}, \underline{\omega})) - f(\cdot, (\tilde{\omega}, \underline{\omega}')) \right) \|_{\mathcal{B}}}{d(\underline{\omega}, \underline{\omega}')} \, d\eta^{\otimes n}(\tilde{\omega}) \\
&\leq \int_{E^n} \sup_{\underline{\omega}_0 \neq \underline{\omega}_0'} \frac{\| \mathcal{L}_{u,\omega_{-n}} \cdots \mathcal{L}_{u,\omega_{-1}} \left(f(\cdot, \underline{\omega}_0) - f(\cdot, \underline{\omega}_0') \right) \|_{\mathcal{B}}}{\varkappa^n d(\underline{\omega}_0, \underline{\omega}_0')} \, d\eta^{\otimes n}(\tilde{\omega}) \\
&\leq \varkappa^{-n} C \sup_{\underline{\omega} \neq \underline{\omega}'} \frac{\| \mathcal{L}_{\omega_n} \cdots \mathcal{L}_{\omega_1} (h(\cdot, \underline{\omega}) - h(\cdot, \underline{\omega}')) \|_{\mathcal{B}}}{d(\underline{\omega}, \underline{\omega}')} \\
&\leq \varkappa^{-n} \sup_{\omega \in E} \| \mathcal{L}_\omega \|_{L(\mathcal{B},\mathcal{B})}^n \sup_{\underline{\omega} \neq \underline{\omega}'} \frac{\| h(\cdot, \underline{\omega}) - h(\cdot, \underline{\omega}') \|_{\mathcal{B}}}{d(\underline{\omega}, \underline{\omega}')}
\end{aligned}$$

since $\varkappa > \sup_{\omega \in E} \| \mathcal{L}_\omega \|_{L(\mathcal{B},\mathcal{B})}$, we obtain that P_u satisfies Doeblin–Fortet–Lasota–Yorke conditions for ($\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{B}}_w$). □

3.5. Quasicompactness of P and of P_u.

Proposition 3.15. *If ϑ_0 is small enough, P is quasicompact on $\widetilde{\mathcal{B}}$, 1 is its only dominating eigenvalue and it is a simple eigenvalue (with eigenspace $\mathbb{C}.\mathbf{1}_{\bar{M}}$). In particular, there exist $\tilde{C} > 0$ and $\tilde{\alpha} \in (0, 1)$, such that*

$$\forall f \in \widetilde{\mathcal{B}}, \quad \| P^n f - \mathbb{E}_{\bar{\mu}}[f] \mathbf{1}_{\bar{M}} \|_{\widetilde{\mathcal{B}}} \leq \tilde{C} \tilde{\alpha}^n \| f \|_{\widetilde{\mathcal{B}}}.$$

Proof. For ϑ_0 sufficiently small, P satisfies the Lasota–Yorke inequalities of Proposition 3.13 uniformly in ϑ_0. Thus by Proposition 3.10 and the Keller–Liverani perturbation theorem [25, Corollary 1], the spectra and spectral projectors of P and $\mathcal{P}$ on $\widetilde{\mathcal{B}}$ are close for ϑ_0 small. Since the spectral gap for $\mathcal{P}$ on $\widetilde{\mathcal{B}}$ is uniform in ϑ_0 by Lemma 3.12, it follows that P has a spectral gap on $\widetilde{\mathcal{B}}$ with a single and simple peripherical eigenvalue, provided ϑ_0 is sufficiently small. Since P is the dual operator of $f \mapsto f \circ \bar{T}$, the spectral radius of P is 1 and 1 is an eigenvalue of P. We conclude that 1 is the dominating eigenvalue and that it is simple. □

Proposition 3.16. *P_u, as an operator acting on $\widetilde{\mathcal{B}}$, is an analytic perturbation of P.*

Proof. Observe that the n-th derivative of $u \mapsto P_u$ is the operator defined by

$$f \mapsto i^n P\left(\Phi^{(i_1)} \dots \Phi^{(i_n)} e^{iu\cdot\Phi} f\right).$$

Due to Lemma 3.9 and to classical results on analytic functions, we conclude that, in $L(\widetilde{\mathcal{B}}, \widetilde{\mathcal{B}})$, $u \mapsto P_u$ is analytic on $\mathbb{R}^2$ and that

$$P_u = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{A}_{n,\omega}, \quad \text{with } \mathcal{A}_n f(u) = P((iu\cdot\Phi)^n f),$$

where $\mathcal{A}_n f(u)$ is n-linear in u. □

Our main results will follow from the following technical result.

Theorem 3.17. *The function $\mathbf{1}_{\tilde{M}}$ is in $\widetilde{\mathcal{B}}$ and $\mathbb{E}_{\tilde{\mu}}[\cdot]$ is a continuous linear form on $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{B}}_w$.*
If ϑ_0 is small enough, there exist $\beta \in (0,\pi)$, $C > 0$ and $\alpha \in (0,1)$, three analytic maps $u \mapsto \lambda_u$ from $[-\beta,\beta]^2$ to $\mathbb{C}$, $u \mapsto N_u$ and $u \mapsto \Pi_u$ from $[-\beta,\beta]^2$ to $L(\widetilde{\mathcal{B}}, \widetilde{\mathcal{B}})$ such that

a) $\lambda_0 = 1$, $\Pi_0 := \mathbb{E}_{\tilde{\mu}}[\cdot]\mathbf{1}_{\tilde{M}}$,
b) *for every $u \in [-\beta,\beta]^2$ and every integer $n \geq 1$, $P_u^n = \lambda_u^n \Pi_u + N_u^n$, $\Pi_u N_u = N_u \Pi_u = 0$, $\Pi_u^2 = \Pi_u$, and $\|N_u^n\|_{L(\widetilde{\mathcal{B}},\widetilde{\mathcal{B}})} \leq C\alpha^n$.*
Moreover, for every integer $k \geq 0$, $\|(N_u^n)^{(k)}\|_{L(\widetilde{\mathcal{B}},\widetilde{\mathcal{B}})} = O(\alpha^n)$, where $(N_u^n)^{(k)}$ means the k-th derivative of N_u^n.
c) *for every $u \in [-\pi,\pi]^2 \backslash [-\beta,\beta]^2$ and every integer $n \geq 1$, we have $\|P_u^n\|_{L(\widetilde{\mathcal{B}},\widetilde{\mathcal{B}})} \leq C\alpha^n$.*
d) *The positive symmetric matrix Σ^2 given by (6) satisfies $\lambda_u = 1 - \frac{1}{2}(\Sigma^2 u\cdot u) + O(|u|^3)$.*

Proof of Theorem 3.17. The fact that $\mathbf{1}_{\tilde{M}}$ is in $\widetilde{\mathcal{B}}$ comes from the fact that $\mathbf{1}_{\tilde{M}_0}$ is in $\mathcal{B}$. As seen in Remark 3.5, $\mathbb{E}_{\tilde{\mu}}[\cdot]$ is a continuous form on $\widetilde{\mathcal{B}}$. The proof of the remaining part of the theorem relies on Propositions 3.10, 3.13, 3.15 and 3.16.

Propositions 3.13, 3.15 and 3.16 immediately imply the existence of a spectral gap for P_u for $|u|$ sufficiently small, using standard perturbation theory [19, VII.6 Theorem 9]. This yields the analyticity and items (a) and (b) of the proposition with β depending on ϑ_0 and the uniform constants depending on the family $\mathcal{F}_{\vartheta_0}$, but not on the probability measure η.

For item (c), due to [1, Lemma 4.3], it is enough to prove that, if ϑ_0 is small enough, then for every $u \in [-\pi,\pi]^2 \backslash [-\beta,\beta]^2$, P_u admits no eigenvalue of modulus 1. Assume the contrary. There would exist a sequence of operators $(P_{u_k}^{(k)})_k$ corresponding to a sequence of vanishing neighbourhoods $(E_k)_k$ of $\bar{T}_0$ in $\mathcal{F}$ and with $\beta \leq |u_k| \leq \pi$ and $\rho(P_{u_k}^{(k)}) = 1$, where $\rho(\cdot)$ denotes the spectral radius. Up to extracting a subsequence, we also have $\lim_{k\to+\infty} u_k = u_\infty$. But, due to Proposition 3.10 and since $u \mapsto \mathcal{L}_{u,0}$ is continuous from $\mathbb{R}^2$ to $L(\mathcal{B},\mathcal{B})$, we would deduce that

$$\lim_{k\to+\infty} \|P_{u_k}^{(k)} - \mathcal{P}_{u_\infty}\|_{L(\mathcal{B},\mathcal{B}_w)} = 0.$$

Combining this with Proposition 3.13 and with the perturbation theorem of [25], this would imply that $\rho(\mathcal{P}_{u_\infty}) = 1$, which would contradict Proposition C.2. We conclude that, as soon as ϑ_0 is sufficiently small, $\sup_{\beta\leq|u|\leq\pi} \rho(P_u) < 1$ as claimed.

It remains to prove item (d). Due to [16, Corollary 2.4], for any initial probability measure $\nu \in \mathcal{B}$, $(S_n/\sqrt{n})_n$ converges in distribution to a (possibly generalized) centered Gaussian random variable with variance Σ^2. As in [15, Proof of Theorem 2.6], Σ^2 is the variance of $(S_n/\sqrt{n})_n$, as $n \to \infty$. Thus Σ^2 is given by the Green-Kubo formula (6) as long as the correlations $\mathbb{E}_{\bar{\mu}}[\Phi^{(i)}.\Phi^{(j)} \circ \bar{T}^k]$ are summable. Indeed, the spectral gap for P (Proposition 3.15) implies that the correlations decay exponentially in k since P is the transfer operator for $\bar{T}$ with respect to the measure $\bar{\mu}$.

Moreover, due to item (b) of the present theorem,

$$\sup_{t\in[-\beta,\beta]^2} |\mathbb{E}_{\bar{\mu}}[e^{itS_n}] - \lambda_t^n \mathbb{E}_{\bar{\mu}}[\Pi_t(\mathbf{1})]| = O(\alpha^n)$$

and so

$$\lim_{n\to+\infty} \lambda^n_{t/\sqrt{n}} = e^{-\frac{1}{2}(\Sigma^2 t\cdot t)}$$

with uniform convergence on any compact set of $\mathbb{R}^2$. This implies that

$$\lim_{n\to+\infty} n \log\left(\lambda_{t/\sqrt{n}}\right) = -\frac{1}{2}(\Sigma^2 t\cdot t)\,.$$

On the other hand, $\log(\lambda_{t/\sqrt{n}}) \sim (\lambda_{t/\sqrt{n}} - 1)$ as $n \to +\infty$. Hence

$$\lim_{n\to+\infty} n(\lambda_{t/\sqrt{n}} - 1) = -\frac{1}{2}(\Sigma^2 t\cdot t).$$

Setting $u = t/\sqrt{n}$, we can then deduce the stated Taylor expansion since $u \mapsto \lambda_u$ is analytic. The positivity of Σ^2 follows from the next lemma. □

Lemma 3.18. *If ϑ_0 is small enough, Σ^2 is positive.*

Proof. Recall that Σ^2 has been defined in (6). We consider Σ_0^2 being defined by

$$\Sigma_0^2 := \left(\mathbb{E}_{\bar{\mu}}\left[\Phi_0^{(i)}.\Phi_0^{(j)}\right] + \sum_{k\geq 1} \mathbb{E}_{\bar{\mu}_0}\left[\Phi_0^{(i)}.\Phi_0^{(j)} \circ \bar{T}_0^k + \Phi_0^{(j)}.\Phi_0^{(i)} \circ \bar{T}_0^k\right]\right)_{i,j=1,2}. \tag{21}$$

It is enough to prove that Σ^2 converges to Σ_0^2 as ϑ_0 goes to 0. We use (6) together with the fact that Σ_0^2 satisfies an analogous formula (with $\Phi(x,\underline{\omega})$ replaced by $\Phi_0(x)$ and with $\bar{T}(x,\underline{\omega})$ replaced by $\bar{T}_0(x)$). Therefore

$$\Sigma^2 - \Sigma_0^2 = A_0 + 2\sum_{k\geq 1} A_k\,,$$

with $A_k := \mathbb{E}_{\bar{\mu}}\left[\Phi.\Phi \circ \bar{T}^k\right] - \mathbb{E}_{\bar{\mu}_0}[\Phi_0.\Phi_0 \circ \bar{T}_0^k]$. Extending the definition of Φ_0 on $\bar{M}$ by setting $\Phi_0(x,\underline{\omega}) := \Phi_0(x)$, we obtain

$$\begin{aligned} A_k &:= \mathbb{E}_{\bar{\mu}}\left[\Phi.\Phi \circ \bar{T}^k - \Phi_0.\Phi_0 \circ (\bar{T}_0')^k\right] \\ &= \mathbb{E}_{\bar{\mu}}\left[P^k\Phi.\Phi - \mathcal{P}^k\Phi_0.\Phi_0\right] \\ &= \mathbb{E}_{\bar{\mu}}\left[(\Phi-\Phi_0).P^k\Phi\right] + \mathbb{E}_{\bar{\mu}}\left[\Phi_0.P^k(\Phi-\Phi_0)\right] + \mathbb{E}_{\bar{\mu}}\left[\Phi_0.(P^k\Phi_0 - \mathcal{P}^k\Phi_0)\right]. \end{aligned}$$

The two first terms of the right hand side of this formula are less than

$$4\|\Phi\|_\infty^2 \sup_{\omega\in E} \bar\mu_0(\Phi_\omega - \Phi_0 \neq \mathbf{0}),$$

which goes to 0 as $\vartheta_0 \to 0$. The third term is dominated by

$$k \max\left(\|P\|_{L(\widetilde{\mathcal{B}},\widetilde{\mathcal{B}})}, \|\mathcal{P}\|_{L(\widetilde{\mathcal{B}}_w,\widetilde{\mathcal{B}}_w)}\right)^{k-1} \|P-\mathcal{P}\|_{\mathcal{L}(\widetilde{\mathcal{B}},\widetilde{\mathcal{B}}_w)} \|\Phi_0\|_{\widetilde{\mathcal{B}}} \|\mathbb{E}_{\bar\mu}[\Phi_0\cdot]\|_{\widetilde{\mathcal{B}}'_w}.$$

We deduce that this quantity goes to 0 using Remark 3.4, Lemma 3.9, and Proposition 3.10, and since $\mathbb{E}_{\bar\mu}[\Phi_0\cdot] = \mathbb{E}_{\bar\mu}[\mathcal{P}(\Phi_0\cdot)]$ (applying Lemma 3.9 with $E = \{0\}$).

We conclude with the use of the dominated convergence theorem, since

$$\begin{aligned}\left|\mathbb{E}_{\bar\mu}\left[\Phi.\Phi\circ\bar T^k\right]\right| &= \left|\mathbb{E}_{\bar\mu}\left[P^k\Phi.\Phi\right]\right| \\ &\le \|\Phi\|_{\widetilde{\mathcal{B}}}\widetilde{C}\widetilde{\alpha}^k\|\mathbb{E}_{\bar\mu}[\Phi\cdot]\|_{\widetilde{\mathcal{B}}'} = \|\Phi\|_{\widetilde{\mathcal{B}}}\widetilde{C}\widetilde{\alpha}^k\|\mathbb{E}_{\bar\mu}[P(\Phi\cdot)]\|_{\widetilde{\mathcal{B}}'},\end{aligned}$$

where we used Proposition 3.15 since $\mathbb{E}_{\bar\mu}[\Phi] = \mathbf{0}$, and a similar bound holds for $\mathbb{E}_{\bar\mu_0}[\Phi_0.\Phi_0\circ\bar T_0^k]$. □

4. Limit Theorems Under General Assumptions and Proofs of Our Results for Billiards

We start with the proof of our results which are direct consequences of Theorem 3.17 and of general results existing in the literature.

Proofs of Theorems 2.1, 2.7 and 2.8. The convergence in distribution of Theorem 2.1 is a direct corollary of Theorem 3.17 by Lévy's continuity theorem (as in [22,23,27]) since, for every $u \in \mathbb{R}^2$, $\mathbb{E}_{\bar\mu}\left[e^{i\frac{u}{\sqrt{n}}\cdot S_n}\right] = \mathbb{E}_{\bar\mu}\left[P^n_{u/\sqrt{n}}\mathbf{1}_{\bar M}\right] \sim \lambda^n_{u/\sqrt{n}} \sim e^{-\frac{1}{2}(\Sigma^2 u\cdot u)}$ as n goes to infinity. Theorem 3.17 provides the announced expression for Σ^2.

Theorem 3.17 gives exactly [34, Hypothesis 3.1] (with $(A,\mu,T) = (\bar M,\bar\mu,\bar T)$, $F=\Phi$, $U=[-\beta,\beta]^2$, $\mathcal{B}=\widetilde{\mathcal{B}}$, $M=1$, $d=2$, $R_u = N_u$, $r=\alpha$ and $L\equiv 1$). Therefore applying [34, Theorem 2.4] (with $(\widetilde A,\widetilde\mu,\widetilde T) = (M,\mu,T)$, $(A,\mu,T) = (\bar M,\bar\mu,\bar T)$, $F=\Phi$, $\mathfrak{a}_n=\sqrt{n}$, $\alpha=2$, $d=2$), we obtain Theorem 2.7.

Observe now that Theorem 3.17 implies that $(P_s)_s$ satisfies Condition (H_2) of [32, Definition 3.1] with respect to $(\widetilde{\mathcal{B}},\infty,\infty,3,\Sigma^2)$ (using Condition (H_1) of [32, Definition 2.1]). Thus, applying [32, Theorem 3.2] and using the formulas given in [32, Remark 3.3], we get Theorem 2.8. □

We will prove the other results in a general context. About these results, let us mention that Theorem 2.4 and the first part of Theorem 2.6 have been proved in [18,29] and in [30] for a single billiard map. We give here the proof in a more general context with a significant simplification in the proof of Theorem 2.6 due to the better estimate of the variance of the auto-intersection and to some simplification in the Bolthausen tightness argument. The second part of Theorem 2.6 uses a general argument from [21]. Theorem 2.5 exists for a single billiard map, but only in an unpublished paper by the second author [31]. Let us indicate that the generality of the proof we give in the present paper is possible due to important modifications of the proof. Indeed we state general results enabling the study of $\mathbb{Z}^2$-extension with unbounded (square integrable) step function and we do not use the fact that the Banach space we consider is continuously

injected in L^p for a suitable $p > 1$ (this property was true for Young Banach spaces on towers constructed in [38] for a single billiard map but not for the spaces we consider here); both of these conditions were used in the proof of [31].

We will prove the limit theorems we are interested in under the following general hypothesis.

Assumption 4.1. Let (M, μ, T) be a $\mathbb{Z}^2$-extension of a probability preserving dynamical system $(\bar{M}, \bar{\mu}, \bar{T})$ by a function $\Phi : \bar{M} \to \mathbb{C}$. Let P be the transfer operator associated with $\bar{T}$ with respect to $\bar{\mu}$ and let $(P_u := P(e^{iu\cdot\Phi}\cdot))_{u\in\mathbb{R}^2}$. We assume that these operators act on two Banach spaces $\widetilde{\mathcal{B}}_1$ and $\widetilde{\mathcal{B}}_2$ such that $\mathbf{1}_{\bar{M}} \in \widetilde{\mathcal{B}}_1 \hookrightarrow \widetilde{\mathcal{B}}_2$ (continuous inclusion) and that $\mathbb{E}_{\bar{\mu}}[\cdot]$ is a continuous linear form[12] on $\widetilde{\mathcal{B}}_2$.
Assume that there exist $\beta \in (0, \pi)$, $C > 0$ and $\alpha \in (0, 1)$, three continuous maps $u \mapsto \lambda_u$ from $[-\beta, \beta]^2$ to $\mathbb{C}$, $u \mapsto N_u$ and $u \mapsto \Pi_u$ from $[-\beta, \beta]^2$ to $L(\widetilde{\mathcal{B}}_1, \widetilde{\mathcal{B}}_2)$ such that

(A1) for every $u \in [-\beta, \beta]^2$ and every integer $n \geq 1$,

$$P_u^n = \lambda_u^n \Pi_u + N_u^n, \quad \Pi_u N_u = N_u \Pi_u = 0, \quad \Pi_u^2 = \Pi_u$$

and $\|N_u^n\|_{L(\widetilde{\mathcal{B}}_1, \widetilde{\mathcal{B}}_1)} \leq C\alpha^n$.

(A2) for every $u \in [-\pi, \pi]^2 \backslash [-\beta, \beta]^2$ and every integer $n \geq 1$, we have $\|P_u^n\|_{L(\widetilde{\mathcal{B}}_1, \widetilde{\mathcal{B}}_1)} \leq C\alpha^n$.

(A3) $u \mapsto \Pi_u$, seen as a $L(\widetilde{\mathcal{B}}_1, \widetilde{\mathcal{B}}_2)$-valued function, is differentiable at 0 and $\Pi_0 := \mathbb{E}_{\bar{\mu}}[\cdot]\mathbf{1}_{\bar{M}}$,

(A4) There exists a positive symmetric matrix Σ^2 such that $\lambda_u = 1 - \frac{1}{2}(\Sigma^2 u\cdot u) + O(|u|^3)$.

In this general context, we will also use the following notation and considerations. We write S_n for the ergodic sum $S_n := \sum_{k=0}^{n-1} \Phi \circ \bar{T}^k$. It will be crucial to notice that $P_u^n = P^n(e^{iu\cdot S_n}\cdot)$.

We consider a partition of $\bar{M}$ in I subsets $\bar{O}_1, \dots, \bar{O}_I$ of $\bar{\mu}$ positive measure (corresponding to $(\partial O_i \times S^1) \times E^{\mathbb{N}}$ in our example). We consider the function $\mathcal{I}_0$ which, at every $x \in \bar{M}$, associates the index $\mathcal{I}_0(x)$ of the atom $\bar{O}_{\mathcal{I}_0(x)}$ of the partition containing x. We also define $\mathcal{I}_k := \mathcal{I}_0 \circ \bar{T}^k$.

We remark that our random map $\bar{T}$ with $\bar{T}_\omega \in \bar{\mathcal{F}}_{\vartheta_0}(\bar{T}_0)$ for all $\omega \in E$ satisfies all the items of Assumption 4.1 due to Theorem 3.17.

4.1. Local limit theorem: general result and proof of Theorem 2.2. For every $n \in \mathbb{N}^*$, $\ell \in \mathbb{Z}^2$ and $h \in \widetilde{\mathcal{B}}_1$, we set:

$$\mathcal{H}_{\ell,n} h := P^n \left(\mathbf{1}_{\{S_n=\ell\}} h\right). \tag{22}$$

Recall that

$$\mathbf{1}_{\{k=\ell\}} = \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^2} e^{i(k-\ell)\cdot u}\, du \tag{23}$$

where du is understood as $du_1 du_2$ for $u = (u_1, u_2) \in \mathbb{R}^2$ (integral with respect to the Lebesgue measure), which leads us to the following formula

$$\mathcal{H}_{\ell,n} h = \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^2} e^{-i\ell\cdot u} P_u^n h\, du\,. \tag{24}$$

[12] Up to extending by continuity the definition of $\mathbb{E}_{\bar{\mu}}[\cdot]$.

Theorem 4.2. *Assume general Assumption 4.1. Then*

$$\sup_{\ell\in\mathbb{Z}^2}\left\|\mathcal{H}_{\ell,n}-\frac{e^{-\frac{1}{2n}\Sigma^{-2}\ell\cdot\ell}}{2\pi n\sqrt{\det\Sigma^2}}\Pi_0\right\|_{L(\widetilde{\mathcal{B}}_1,\widetilde{\mathcal{B}}_2)}=O(n^{-\frac{3}{2}})\,.$$

Moreover, there exists $K_0\geq 1$ such that for every integer $n\geq 0$ and every $\ell\in\mathbb{Z}^2$,

$$\|\mathcal{H}_{\ell,n}\|_{L(\widetilde{\mathcal{B}}_1,\widetilde{\mathcal{B}}_1)}\leq\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2}\|P_u^n\|_{L(\widetilde{\mathcal{B}}_1,\widetilde{\mathcal{B}}_1)}\,du\leq\frac{K_0}{n+1}\,,\tag{25}$$

Proof. Up to a change of β, there exists $a>0$ such that, for every $u\in[-\beta,\beta]^2$, $|\lambda_u|\leq\exp(-a|u|^2)$. Hence, using Assumption 4.1, we have the following equalities in $L(\widetilde{\mathcal{B}}_1,\widetilde{\mathcal{B}}_2)$:

$$\begin{aligned}
&\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2}e^{-i\ell\cdot u}P_u^n\,du=\frac{1}{(2\pi)^2}\int_{[-\beta,\beta]^2}e^{-i\ell\cdot u}P_u^n\,du+O(\alpha^n)\\
&=\frac{1}{(2\pi)^2}\int_{[-\beta,\beta]^2}e^{-i\ell\cdot u}\lambda_u^n\Pi_u\,du+O(\alpha^n)\\
&=\frac{1}{(2\pi)^2}\int_{[-\beta,\beta]^2}e^{-i\ell\cdot u}\lambda_u^n(\Pi_0+O(u))\,du+O(\alpha^n)\\
&=\frac{1}{(2\pi)^2}\int_{[-\beta,\beta]^2}e^{-i\ell\cdot u}\left(e^{-\frac{1}{2}(\Sigma^2u.u)}+O(|u|^3)\right)^n\Pi_0+O(e^{-an|u|^2}|u|)\,du+O(\alpha^n)\,.
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2}e^{-i\ell\cdot u}P_u^n\,du\\
&=\frac{1}{(2\pi)^2n}\int_{[-\beta\sqrt{n},\beta\sqrt{n}]^2}e^{-i\ell\cdot\frac{v}{\sqrt{n}}}e^{-\frac{1}{2}(\Sigma^2v.v)}\Pi_0\\
&\quad+O\left(ne^{-a(n-1)\frac{|v|^2}{n}}\frac{|v|^3}{n^{\frac{3}{2}}}+e^{-a|v|^2}\frac{v}{\sqrt{n}}\right)dv+O(\alpha^n)\\
&=\frac{1}{(2\pi)^2n}\int_{[-\beta\sqrt{n},\beta\sqrt{n}]^2}e^{-i\ell\cdot\frac{v}{\sqrt{n}}}e^{-\frac{1}{2}(\Sigma^2v.v)}\Pi_0\\
&\quad+O\left(e^{-\frac{a}{2}|v|^2}\frac{|v|^3}{\sqrt{n}}+e^{-a|v|^2}\frac{v}{\sqrt{n}}\right)dv+O(\alpha^n)\\
&=\frac{1}{(2\pi)^2n}\int_{[-\beta\sqrt{n},\beta\sqrt{n}]^2}e^{-i\ell\cdot\frac{v}{\sqrt{n}}}e^{-\frac{1}{2}(\Sigma^2v.v)}\Pi_0\,dv+O(n^{-\frac{3}{2}})\\
&=\frac{1}{(2\pi)^2n}\int_{\mathbb{R}^2}e^{-i\frac{\ell}{\sqrt{n}}\cdot v}e^{-\frac{1}{2}(\Sigma^2v.v)}\Pi_0\,dv+O(n^{-\frac{3}{2}})\\
&=\frac{e^{-\frac{1}{2n}\Sigma^{-2}\ell\cdot\ell}}{2\pi n\sqrt{\det\Sigma^2}}\Pi_0+O(n^{-\frac{3}{2}})\,,
\end{aligned}$$

where we have changed variables, $v=u\sqrt{n}$, and the O are in $L(\widetilde{\mathcal{B}}_1,\widetilde{\mathcal{B}}_2)$ with uniform bound. This bound is in $L(\widetilde{\mathcal{B}}_1,\widetilde{\mathcal{B}}_2)$ and not in $L(\widetilde{\mathcal{B}}_1,\widetilde{\mathcal{B}}_1)$ because according to Assumption (A3), the map $u\mapsto\Pi_u$ is differentiable from $[-\beta,\beta]$ to $L(\widetilde{\mathcal{B}}_1,\widetilde{\mathcal{B}}_2)$ and a priori not from $[-\beta,\beta]$ to $L(\widetilde{\mathcal{B}}_1,\widetilde{\mathcal{B}}_1)$.

For the second estimate, we write

$$\begin{aligned}
&\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} \|P_u^n\|_{\mathcal{L}(\widetilde{\mathcal{B}}_1,\widetilde{\mathcal{B}}_1)}\,du\\
&=\frac{1}{(2\pi)^2}\int_{[-\beta,\beta]^2} |\lambda_u|^n \|\Pi_u\|_{\mathcal{L}(\widetilde{\mathcal{B}}_1,\widetilde{\mathcal{B}}_1)}\,du + O(\alpha^n)\\
&\le \frac{1}{(2\pi)^2}\int_{[-\beta,\beta]^2} e^{-a|u|^2 n} \sup_{u\in[-\beta,\beta]} \|\Pi_u\|_{\mathcal{L}(\widetilde{\mathcal{B}}_1,\widetilde{\mathcal{B}}_1)}\,du + O(\alpha^n)\\
&\le O(n^{-1})\,,
\end{aligned}$$

using again the change of variable $v = u\sqrt{n}$. □

Due to Theorem 3.17, Theorem 2.2 is a direct consequence of the following result.

Corollary 4.3. *Assume general Assumption 4.1. Let $f, g : \bar{M} \to \mathbb{R}$ such that $H_g(\cdot) := \mathbb{E}_{\bar{\mu}}[g\cdot] \in \widetilde{\mathcal{B}}_2'$ and such that $f \in \widetilde{\mathcal{B}}_1$. Then*

$$\mathbb{E}_{\bar{\mu}}\left[f.\mathbf{1}_{\{S_n=\ell\}}.g\circ\bar{T}^n\right] = \frac{\exp\left(-\frac{\Sigma^{-2}\ell\cdot\ell}{2n}\right)}{2\pi n\sqrt{\det\Sigma^2}}\mathbb{E}_{\bar{\mu}}[f]\mathbb{E}_{\bar{\mu}}[g] + O\left(n^{-\frac{3}{2}}\|f\|_{\widetilde{\mathcal{B}}_1}\|H_g\|_{\widetilde{\mathcal{B}}_2'}\right). \tag{26}$$

Proof. Observe that we have

$$\mathbb{E}_{\bar{\mu}}\left[f.\mathbf{1}_{\{S_n=\ell\}}.g\circ\bar{T}^n\right] = \mathbb{E}_{\bar{\mu}}\left[P^n(f.\mathbf{1}_{\{S_n=\ell\}}).g\right] = H_g\left(P^n(\mathbf{1}_{\{S_n=\ell\}}f)\right),$$

recalling (7). We conclude due to Theorem 4.2. □

4.2. Return time to the original obstacle: general result and proof of Theorem 2.4. Recall that $\mathcal{I}_k(x)$ corresponds to the index of the atom $\bar{O}_{\mathcal{I}_k(x)}$ containing $\bar{T}^k x$ and that $S_n(x)$ corresponds to the label of the copy of $\bar{M}$ in M containing $T^k(x,\mathbf{0})$. We also define $\mathcal{I}_k$ on $\bar{M}$ by canonical projection. We consider the set B_n of $x \in \bar{M}$ such that the orbit $(T^n(x,\mathbf{0}))_{n\ge 0}$ won't return to the initial atom $\bar{O}_{\mathcal{I}_0(x)} \times \{\mathbf{0}\}$ until time n:

$$B_n := \{\forall k = 1,\dots,n \,:\, (\mathcal{I}_k, S_k) \neq (\mathcal{I}_0,(0,0))\} \subset \bar{M}\,.$$

Analogously we define B_n' the set of points $x \in \bar{M}$ for which the atom visited at time n has not been visited before:

$$B_n' := \{\forall k = 0,\dots,n-1 \,:\, (\mathcal{I}_k, S_k) \neq (\mathcal{I}_n, S_n)\} \subset \bar{M}\,. \tag{27}$$

We set $B_n(a) := \bar{O}_a \cap B_n$ and $B_n'(a) := \bar{T}^{-n}(\bar{O}_a) \cap B_n$. We prove the following result on the probability of these sets.

Proposition 4.4. *Assume general Assumption 4.1.*

If $\mathbf{1}_{\bar{O}_a} \in \widetilde{\mathcal{B}}_1$ and if $f \mapsto \mathbb{E}_{\bar{\mu}}[f\mathbf{1}_{B_k(a)}]$ are uniformly bounded (uniformly in k) in $\widetilde{\mathcal{B}}_2'$, then

$$\bar{\mu}(B_n(a)) = \frac{2\pi\sqrt{\det\Sigma^2}}{\log n} + O\left((\log n)^{-\frac{4}{3}}\right). \tag{28}$$

If $f \mapsto \mathbb{E}_{\bar{\mu}}[f\mathbf{1}_{\bar{O}_a}]$ is in $\widetilde{\mathcal{B}}_2'$ and if $P^k\mathbf{1}_{B'_k(a)}$ are uniformly bounded (uniformly in k) in $\widetilde{\mathcal{B}}_1$, then

$$\bar{\mu}(B'_n(a)) = \frac{2\pi\sqrt{\det\Sigma^2}}{\log n} + O\left((\log n)^{-\frac{4}{3}}\right). \tag{29}$$

Proof. As in [29], we follow the idea of the proof of Dvoretzky and Erdös [20] and adapt it to our context. Considering the last visit time to $\bar{O}_a \times \{\mathbf{0}\}$ of $(T^k(x,\mathbf{0}))$ until time n, we write

$$\bar{\mu}(\bar{O}_a) = \sum_{k=0}^{n} \bar{\mu}\left(\bar{O}_a \cap \{S_k = \mathbf{0}\} \cap \bar{T}^{-k}(B_{n-k}(a))\right) \tag{30}$$

and, analogously,

$$\bar{\mu}(\bar{O}_a) = \bar{\mu}\left(\bar{T}^{-n}\bar{O}_a\right) = \sum_{k=0}^{n} \bar{\mu}\left((\bar{T}^{-n}\bar{O}_a) \cap \{S_n - S_{n-k} = \mathbf{0}\} \cap B'_{n-k}(a)\right) \tag{31}$$

considering the first visit time to $\bar{O}_a \times \{S_n\}$ before time n. Moreover, due to Corollary 4.3 and to our assumptions on $\bar{O}_a$ and on $B_n(a)$, there exists $C'' > 0$ such that

$$\forall k \in \mathbb{N}^*, \quad \left|\bar{\mu}\left(\bar{O}_a \cap \{S_k = \mathbf{0}\} \cap \bar{T}^{-k}(B_{n-k}(a))\right) - \frac{\bar{\mu}(\bar{O}_a)\bar{\mu}(B_{n-k}(a))}{2k\pi\sqrt{\det\Sigma^2}}\right| \le \frac{C''}{k^{\frac{3}{2}}}. \tag{32}$$

Since $\bar{\mu}\left((\bar{T}^{-n}\bar{O}_a) \cap \{S_n - S_{n-k} = \mathbf{0}\} \cap B'_{n-k}(a)\right) = \mathbb{E}_{\bar{\mu}}\left[\mathbf{1}_{\bar{O}_a} P^k\left(\mathbf{1}_{\{S_k=\mathbf{0}\}} P^{n-k}\mathbf{1}_{B'_{n-k}(a)}\right)\right]$, and using Theorem 4.2, we also have

$$\forall k \in \mathbb{N}^*, \quad \left|\bar{\mu}\left((\bar{T}^{-n}\bar{O}_a) \cap \{S_n - S_{n-k} = \mathbf{0}\} \cap B'_{n-k}(a)\right) - \frac{\bar{\mu}(\bar{O}_a)\bar{\mu}(B'_{n-k}(a))}{2k\pi\sqrt{\det\Sigma^2}}\right| \le \frac{C''}{k^{\frac{3}{2}}}. \tag{33}$$

We will prove (28) using (30) and (32). The proof of (29) using (31) and (33) follows the same scheme, and we omit it.

$$\begin{aligned}
\bar{\mu}(\bar{O}_a) &\ge \sum_{k=\lceil m_n\rceil}^{n-1} \frac{\bar{\mu}(\bar{O}_a)\bar{\mu}(B_{n-1-k}(a))}{2k\pi\sqrt{\det\Sigma^2}} + \sum_{k=m_n}^{n} \frac{C''}{k^{\frac{3}{2}}} \\
&\ge \bar{\mu}(B_n(a))\,(\log(n) - \log(m_n))\,\frac{\bar{\mu}(\bar{O}_a)}{2\pi\sqrt{\det\Sigma^2}} + \sum_{k=m_n}^{n} \frac{C''}{k^{\frac{3}{2}}} \\
&\ge \log(n)\,\bar{\mu}(B_n(a))\left(1 - \frac{\log(m_n)}{\log n}\right)\frac{\bar{\mu}(\bar{O}_a)}{2\pi\sqrt{\det\Sigma^2}} + O(m_n^{-\frac{1}{2}}),
\end{aligned}$$

with $m_n = \lceil(\log n)^2\rceil$, which leads to

$$\log(n)\,\bar{\mu}(B_n(a)) \le 2\pi\sqrt{\det\Sigma^2} + O\left(\frac{\log\log n}{\log n}\right). \tag{34}$$

Moreover

$$\begin{aligned}\bar\mu(\bar O_a) &\le \sum_{k=0}^{m_n'-1}\bar\mu(B_{\lceil n\log n\rceil-k}(a)) + \sum_{k=m_n'}^{\lceil n\log n\rceil-n}\frac{\bar\mu(\bar O_a)\bar\mu(B_{\lceil n\log n\rceil-k}(a))}{2k\pi\sqrt{\det\Sigma^2}}\\ &\quad+\sum_{k=\lceil n\log n\rceil-n+1}^{\lceil n\log n\rceil}\frac{\bar\mu(\bar O_a)\bar\mu(B_{\lceil n\log n\rceil}(a))}{2k\pi\sqrt{\det\Sigma^2}} + \sum_{k=m_n'}^{\lceil n\log n\rceil}\frac{C''}{k^{\frac32}}\\ &\le \frac{m_n'}{\log n}+\bar\mu(B_n(a))\left(\left(\log(n\log n-n+1)-\log(m_n'-1)\right)\frac{\bar\mu(\bar O_a)}{2\pi\sqrt{\det\Sigma^2}}\right)\\ &\quad+\frac{\log(n\log n)-\log(n\log n-n)}{2\pi\sqrt{\det\Sigma^2}}+C''(m_n')^{-\frac12},\end{aligned}$$

where we used the facts that $\bar\mu(B_{\lceil n\log n\rceil-k}(a))\le\bar\mu(B_n(a))=O((\log n)^{-1})$ for every $k\le\lceil n\log n\rceil-n$ and that $\bar\mu(B_m(a))\le 1$ for $k>\lceil n\log n\rceil-n$. This leads us to

$$\begin{aligned}\bar\mu(\bar O_a) &\le \log n\frac{\bar\mu(\bar O_a)}{2\pi\sqrt{\det\Sigma^2}}\bar\mu(B_n(a))\left(1+O\left(\frac{\log\log n+\log m_n'}{\log n}\right)\right)\\ &\quad+O\left(\frac{m_n'}{\log n}+(m_n')^{-\frac12}\right),\\ &\le \log n\frac{\bar\mu(\bar O_a)}{2\pi\sqrt{\det\Sigma^2}}\bar\mu(B_n(a))\left(1+O\left(\frac{\log\log n}{\log n}\right)\right)\\ &\quad+O\left((\log n)^{-\frac13}\right),\end{aligned}$$

by taking $m_n':=\lfloor(\log n)^{\frac23}\rfloor$ and so

$$\log(n)\,\bar\mu(B_n(a))\ \ge\ 2\pi\sqrt{\det\Sigma^2}+O\left((\log n)^{-\frac13}\right). \tag{35}$$

The proposition follows from (34) and (35). □

In view of applying Proposition 4.4 in our context of random iterations of billiards, we will use the following result.

Lemma 4.5 (Estimate for random iterations of billiard maps). *Assume we are in the particular case of billiards, with assumptions and notations of Sects. 1–3. There exists $K_1>0$ such that, for every positive integer ℓ, for every $(\omega_1,\dots,\omega_\ell)\in E^\ell$, for every uniformly bounded function $g:\bar M_0\to\mathbb{R}$ which is uniformly p-Hölder continuous on connected components of $\bar M_0\setminus\left(\cup_{k=1}^{\ell}\bar T_{\omega_1}^{-1}\circ\cdots\circ\bar T_{\omega_k}^{-1}(\mathcal{S}_{0,H})\right)$, and for all $f\in\mathcal{B}_w$,*

$$|\mathbb{E}_{\bar\mu_0}[f\,g]|\le K_1|f|_w\left(|g|_\infty+\sup_{C\in\mathcal{C}_{\omega_1,\dots,\omega_\ell}}C_{g|C}^{(p)}\right). \tag{36}$$

Moreover, for every $f\in\mathcal{B}$,

$$\|\mathcal{L}_{u,\omega_\ell}\dots\mathcal{L}_{u,\omega_1}(gf)\|_{\mathcal{B}}\le K_1\|f\|_{\mathcal{B}}\left(|g|_\infty+\sup_{C\in\mathcal{C}_{\omega_1,\dots,\omega_\ell}}C_{g|C}^{(p)}\right), \tag{37}$$

where $\mathcal{C}_{\omega_1,\dots,\omega_\ell}$ *is the set of connected components of* $\bar{M}_0 \setminus \left(\cup_{k=1}^{\ell} \bar{T}_{\omega_1}^{-1} \circ \cdots \circ \bar{T}_{\omega_k}^{-1}(\mathcal{S}_{0,H})\right)$ *and where* $C_{g|C}^{(p)}$ *is the Hölder constant of* g *restricted to* C.

The proof of Lemma 4.5 can be found in "Appendix A".

Remark 4.6. The purpose of Lemma 4.5 in our billiard context is to show that K_1 can be chosen independently of ℓ. If one wishes similar bounds on piecewise Hölder continuous functions on M_0 with respect to a fixed partition, then Remark 3.1 and Lemma 3.3 provide such estimates under general conditions on the boundaries of partition elements.

Indeed, we will apply the lemma to the function $g = \mathbf{1}_{B_n(a)}$, where $B_n(a)$ is defined in Sect. 2.2 (see also Sect. 4.2).

Next we are ready to prove the main Theorem 2.4.

Proof of Theorem2.4. Assumption 4.1 follows from Theorem 3.17. The other assumptions of Proposition 4.4 follow from Lemma 4.5 since $\mathbf{1}_{B_n(a)}$ satisfies the assumptions on g in that lemma (uniformly in n). □

4.3. Number of self-intersections: general result and Proof of Theorem 2.5. We consider the number of self-intersections $\mathcal{V}_n$ of the process $(\mathcal{I}_k, S_k)_k$ defined by

$$\mathcal{V}_n := \sum_{k,\ell=1}^{n} \mathbf{1}_{\{S_\ell = S_k,\ \mathcal{I}_\ell = \mathcal{I}_k\}}. \tag{38}$$

Theorem 4.7. *Assume general Assumption 4.1 with* $\widetilde{\mathcal{B}}_2 = \widetilde{\mathcal{B}}_1$. *Assume moreover:*

(A5) the operator $f \mapsto f\mathbf{1}_{\bar{O}_a}$ *is a linear operator on* $\widetilde{\mathcal{B}}_1$ *for every* $a \in \{1, \dots, I\}$.

Then $(\mathcal{V}_n/(n \log n))_n$ *converges* $\bar{\mu}$*-almost surely to* $\frac{1}{\pi\sqrt{\det \Sigma^2}} \sum_{a=1}^{I} \bar{\mu}(\mathcal{I}_0 = a)^2$.

The proof of Theorem 2.5 will follow from the following lemmas. Recalling (38), let us write $E_{k,\ell} := \{S_k = S_\ell, \mathcal{I}_k = \mathcal{I}_\ell\}$ and $E_\ell := E_{0,\ell}$.

Lemma 4.8. *Assume general assumptions of Theorem 4.7. For* $\ell > k$, *we have*

$$\bar{\mu}\left(E_{k,\ell} \cap \bar{T}^{-k}\bar{O}_a\right) = \frac{\left(\bar{\mu}(\bar{O}_a)\right)^2}{2\pi\sqrt{\det \Sigma^2}(\ell - k)} + O((\ell - k)^{-\frac{3}{2}}),$$

and so

$$\bar{\mu}(E_{k,\ell}) = \frac{c_1}{\ell - k} + O((\ell - k)^{-\frac{3}{2}}) \quad \text{and} \quad \mathbb{E}_{\bar{\mu}}[\mathcal{V}_n] = 2c_1 n \log n + O(n),$$

with $c_1 := \frac{1}{2\pi\sqrt{\det \Sigma^2}} \sum_{a=1}^{I} \bar{\mu}(\mathcal{I}_0 = a)^2$.

Proof. Since $\bar{\mu}$ is $\bar{T}$-invariant, for $k < \ell$, recalling (22) we have

$$\begin{aligned}
\bar{\mu}\left(E_{k,\ell} \cap \bar{T}^{-k}\bar{O}_a\right) &= \bar{\mu}\left(E_{\ell-k} \cap \bar{O}_a\right) = \bar{\mu}(\mathcal{I}_0 = a,\ S_{\ell-k} = 0, \mathcal{I}_{\ell-k} = a) \\
&= \mathbb{E}_{\bar{\mu}}\left[\mathbf{1}_{\bar{O}_a} \mathcal{H}_{0,\ell-k}(\mathbf{1}_{\bar{O}_a})\right] \\
&= \frac{\bar{\mu}(\bar{O}_a)^2}{2\pi\sqrt{\det \Sigma^2}(\ell - k)} + O((\ell - k)^{-\frac{3}{2}}),
\end{aligned}$$

due to Theorem 4.2 since $\mathbf{1}_{\bar{O}_a} \in \widetilde{\mathcal{B}}_1$ and since $\mathbb{E}_{\bar{\mu}}[\mathbf{1}_{\bar{O}_a} \cdot] \in \widetilde{\mathcal{B}}_1'$. Hence

$$\bar{\mu}(E_{k,\ell}) = \sum_{a=1}^{I} \bar{\mu}\left(E_{k,\ell} \cap \bar{T}^{-k}\bar{O}_a\right) = \frac{\sum_{a=1}^{I}\left(\bar{\mu}(\bar{O}_a)\right)^2}{2\pi\sqrt{\det \Sigma^2}(\ell-k)} + O((\ell-k)^{-\frac{3}{2}}),$$

and

$$\begin{aligned}\mathbb{E}_{\bar{\mu}}[\mathcal{V}_n] &= \sum_{k,\ell=1}^{n} \bar{\mu}(E_{k,\ell}) = n + 2\sum_{1\le k<\ell\le n} \bar{\mu}(E_{\ell-k}) \\ &= n + 2\sum_{m=1}^{n-1}(n-m)\bar{\mu}(E_m) = O(n) + 2c_1 n\log n\,.\end{aligned}$$

□

Lemma 4.9. *Assume general assumptions of Theorem 4.7. There exists $C_1 > 0$ such that for all non-negative integers n, m, k, for all $i, j, i', j' \in \{1, \dots, I\}$, and for all $N_1, N_2 \in \mathbb{Z}^2$, we have*

$$|\operatorname{Cov}_{\bar{\mu}}(\mathbf{1}_{\{\mathcal{I}_0=i, S_n=N_1, \mathcal{I}_n=i'\}}, \mathbf{1}_{\{\mathcal{I}_{n+m}=j, S_{n+m+k}-S_{n+m}=N_2, \mathcal{I}_{n+m+k}=i'\}})| \le \frac{C_1\alpha^m}{(n+1)(k+1)}.$$

In particular

$$|\operatorname{Cov}_{\bar{\mu}}(\mathbf{1}_{E_{0,n}}, \mathbf{1}_{E_{n+m,n+m+k}})| \le \frac{I^2\, C_1\alpha^m}{(n+1)(k+1)}.$$

Proof. The covariance we are interested in can be rewritten

$$\begin{aligned}&\operatorname{Cov}_{\bar{\mu}}\left(\mathbf{1}_{\bar{O}_i}\mathbf{1}_{\{S_n=N_1\}}\mathbf{1}_{\bar{O}_{i'}} \circ \bar{T}^n, (\mathbf{1}_{\bar{O}_j}\mathbf{1}_{\{S_k=N_2\}}\mathbf{1}_{\bar{O}_{j'}} \circ \bar{T}^k) \circ \bar{T}^{n+m}\right) \\ &= \mathbb{E}_{\bar{\mu}}\Big[P^{n+m+k}\Big(\Big(\mathbf{1}_{\bar{O}_i}\mathbf{1}_{\{S_n=N_1\}}\mathbf{1}_{\bar{O}_{i'}} \circ \bar{T}^n - \mathbb{E}_{\bar{\mu}}[\mathbf{1}_{\bar{O}_i}\mathbf{1}_{\{S_n=N_1\}}\mathbf{1}_{\bar{O}_{i'}} \circ \bar{T}^n]\Big) \\ &\quad (\mathbf{1}_{\bar{O}_j}\mathbf{1}_{\{S_k=N_2\}}\mathbf{1}_{\bar{O}_{j'}} \circ \bar{T}^k) \circ \bar{T}^{n+m}\Big)\Big]\,.\end{aligned}$$

Moreover, using several times $P^m(f\, g\circ \bar{T}^m) = g\, P^m(f)$ and the definition of $\mathcal{H}_{\ell,n}$, we obtain that this quantity is equal to

$$\mathbb{E}_{\bar{\mu}}\left[\mathbf{1}_{\bar{O}_{j'}}\,\mathcal{H}_{N_2,k}\Big(\mathbf{1}_{\bar{O}_j}(P^m - \mathbb{E}_{\bar{\mu}})\left(\mathbf{1}_{\bar{O}_{i'}}\mathcal{H}_{N_1,n}\left(\mathbf{1}_{\bar{O}_i}\right)\right)\Big)\right]$$

and so is bounded by

$$\begin{aligned}&a_{j'}\cdot\|\mathcal{H}_{N_2,k}\|_{L(\widetilde{\mathcal{B}}_1,\widetilde{\mathcal{B}}_1)}\cdot a_j\cdot\|P^m - \mathbb{E}_{\bar{\mu}}\|_{L(\widetilde{\mathcal{B}}_1,\widetilde{\mathcal{B}}_1)}\cdot a_{i'}\cdot\|\mathcal{H}_{N_1,n}\|_{L(\widetilde{\mathcal{B}}_1,\widetilde{\mathcal{B}}_1)}\|\mathbf{1}_{\bar{O}_i}\|_{\widetilde{\mathcal{B}}_1} \\ &\le \frac{K_0^2}{(n+1)(k+1)} C\alpha^m a_{j'}a_j a_i \|\mathbf{1}_{\bar{O}_i}\|_{\widetilde{\mathcal{B}}_1}\,,\end{aligned}$$

due to (25) and assumption (A5) of Theorem 4.7, together with (A1) of Assumptions 4.1 applied to $u = 0$. Here $a_i := \|\mathbf{1}_{\bar{O}_i} \times \cdot\|_{L(\widetilde{\mathcal{B}}_1,\widetilde{\mathcal{B}}_1)}$.

This gives the first estimate of the lemma. To get the second one from the first one, we just observe that

$$\mathbf{1}_{E_{k,\ell}} := \sum_{i=1}^{I} \mathbf{1}_{\bar{O}_i \cap \{S_{\ell-k}=0\} \cap \bar{T}^{-(\ell-k)}\bar{O}_i} \circ \bar{T}^k .$$

□

We will use the notation $A_n \sim B_n$ for two positive quantities whenever $\lim_{n\to\infty} \frac{A_n}{B_n} = 1$.

Lemma 4.10. *Assume general assumptions of Theorem 4.7. We have* $Var_{\bar{\mu}}(\mathcal{V}_n) \sim cn^2$, *with*

$$c := \frac{\left(\sum_{a=1}^{I} (\bar{\mu}(\bar{O}_a))^2\right)^2}{\det \Sigma^2} \left(\frac{1+2J}{\pi^2} - \frac{1}{6}\right),$$
$$J := \int_{y_1,y_2,y_3>0: y_1+y_2+y_3<1} \frac{1-y_1-y_2-y_3}{y_1y_2+y_2y_3+y_1y_3}\, dy_1 dy_2 dy_3.$$

The proof of Lemma 4.10 is rather technical and involved, so we move it to the "Appendix B".

Proof of Theorem 4.7. Set $n_k := \exp(\sqrt{k}\log k)$. For every $\varepsilon > 0$, due to the Bienaymé-Chebychev inequality and using Lemmas 4.8 and 4.10,

$$\sum_{k\geq 1} \bar{\mu}\left(|\mathcal{V}_{n_k} - \mathbb{E}_{\bar{\mu}}[\mathcal{V}_{n_k}]| > \varepsilon \mathbb{E}_{\bar{\mu}}[\mathcal{V}_{n_k}]\right) \leq \sum_{k\geq 1} \frac{Var_{\bar{\mu}}(\mathcal{V}_{n_k})}{\varepsilon^2 (\mathbb{E}_{\bar{\mu}}[\mathcal{V}_{n_k}])^2}$$
$$= \sum_{k\geq 1} O((\log n_k)^{-2}) = \sum_{k\geq 1} O(k^{-1}(\log k)^{-2}) < \infty .$$

Hence $(\mathcal{V}_{n_k}/\mathbb{E}_{\bar{\mu}}[\mathcal{V}_{n_k}])_k$ converges $\bar{\mu}$-almost surely to 1. Due to Lemma 4.8, $(\mathcal{V}_{n_k}/(n_k \log n_k))_k$ converges almost surely to $2c_1$. Since $n_k \log n_k \sim n_{k+1}\log n_{k+1}$ and since $(\mathcal{V}_n)_n$ is increasing, if $n \in \{n_k, \ldots, n_{k+1}\}$, then

$$\mathcal{V}_{n_k}/(n_{k+1}\log n_{k+1}) \leq \mathcal{V}_n/(n\log n) \leq \mathcal{V}_{n_{k+1}}/(n_k \log n_k),$$

and so $(\mathcal{V}_n/(n\log n))_n$ converges $\bar{\mu}$-almost surely to $2c_1$. □

Proof of Theorem 2.5. Due to Remark 3.5, Theorem 3.17 and to Lemma 4.5, the assumptions of Theorem 4.7 are satisfied. Therefore $(\mathcal{V}_n/(n\log n))_n$ converges $\bar{\mu}$-almost surely to

$$\frac{1}{\pi\sqrt{\det\Sigma^2}} \sum_{a=1}^{I} \bar{\mu}(\mathcal{I}_0 = a)^2 = \frac{1}{\pi\sqrt{\det\Sigma^2}} \sum_{a=1}^{I} \left(\frac{2|\partial O_a|}{2\sum_{b=1}^{I}|\partial O_b|}\right)^2$$
$$= \frac{1}{\pi\sqrt{\det\Sigma^2}} \frac{\sum_{a=1}^{I}|\partial O_a|^2}{\left(\sum_{b=1}^{I}|\partial O_b|\right)^2} . \tag{39}$$

□

4.4. Random scenery: general result and proof of Theorem 2.6. Assume that to each atom $\bar{O}_i \times \{\ell\}$ is associated a random variable $\xi_{i,\ell}$, independent and identically distributed across $i \in [1, \dots I]$ and $\ell \in \mathbb{Z}^2$, centered with variance σ_ξ^2 and defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We define the random variable (defined on $\bar{M} \times \Omega$):

$$\mathcal{Z}_n := \sum_{k=0}^{n-1} \xi_{\mathcal{I}_k, S_k}.$$

We also define a linearly interpolated version of $\mathcal{Z}_n$ by,

$$\widetilde{\mathcal{Z}}_n(t) := \mathcal{Z}_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor)\xi_{(\mathcal{I}_{\lfloor nt \rfloor + 1}, S_{\lfloor nt \rfloor + 1})}.$$

Theorem 4.11 *(*Annealed and ξ-quenched CLT for $\mathcal{Z}$*). Assume general Assumption 4.1 and that,*

i) *for every $a \in \{1, \dots, I\}$, $f \mapsto \mathbf{1}_{\bar{O}_a} f$ is a continuous linear operator on $\widetilde{\mathcal{B}}_1$;*
ii) *and $\sup_{k \in \mathbb{N}} \|P^k \mathbf{1}_{B'_k(a)}\|_{\widetilde{\mathcal{B}}_1} < \infty$ (recalling (27));*
iii) *there exists $c' > 0$ such that $\mathbb{E}_{\bar{\mu}}[|S_n|^2] \sim c' n$.*

Then, $(\widetilde{\mathcal{Z}}_n := (\widetilde{\mathcal{Z}}_n(t)/\sqrt{n \log n})_{t>0})_n$ converges in distribution, with respect to $\bar{\mu} \otimes \mathbb{P}$ (and to the uniform norm on $C([0, \mathcal{T}])$ for every $\mathcal{T} > 0$), to a Brownian motion $B = (B_t)_{t \geq 0}$ such that $\mathbb{E}[B_1^2] = \frac{\sigma_\xi^2}{\pi \sqrt{\det \Sigma^2}} \sum_{a=1}^I \bar{\mu}(\mathcal{I}_0 = a)^2$.

If, moreover, there exists $\chi > 0$ such that $\mathbb{E}[|\xi_{(1,0)}|^2 (\log^+ |\xi_{(1,0)}|)^\chi|)] < \infty$, then, for $\mathbb{P}$-a.e. realization of $(\xi_{i,\ell})_{i,\ell}$, $(\widetilde{\mathcal{Z}}_n)_n$ converges also in distribution, with respect to $\bar{\mu}$, to the same Brownian motion B.

As said before, it should be possible to remove the additional assumption $\mathbb{E}[|\xi_{(1,0)}|^2 (\log^+ |\xi_{(1,0)}|)^\chi|)] < \infty$ by using our estimates, combined with the very recent preprint [13] instead of [21].

Proof of Theorem 2.6. Using Theorem 4.11, we prove Theorem 2.6. Assumption 4.1 holds in the setting of Theorem 2.6 due to Theorem 3.17. Moreover, assumption (i) of Theorem 4.11 follows from Remark 3.5, while assumption (ii) follows from Lemma 4.5 and (iii) comes from Theorem 2.1. With the hypotheses of Theorem 4.11 verified, Theorem 2.6 follows using the same calculation as in (39). □

We proceed to prove Theorem 4.11.

For the annealed central limit theorem, we mostly follow the proof by Bolthausen for random walks in random scenery in dimension 2 [5]. In comparison with [30], the fact that the almost sure convergence of $\mathcal{V}_n$ has been proved greatly simplifies the proof.

Lemma 4.12. *Assume the general assumptions of Theorem 4.11. Fix $\vartheta > 0$. For $\bar{\mu}$-almost every $x \in \bar{M}$, $\sup_\ell \sum_{k=1}^n \mathbf{1}_{\{S_k = \ell\}} = o(n^\vartheta)$.*

Proof. For every $\ell \in \mathbb{Z}^2$ and every $N \in \mathbb{N}^*$,

$$\mathbb{E}_{\bar{\mu}}\left[\left(\sum_{k=1}^n \mathbf{1}_{\{S_k = \ell\}}\right)^N\right] \leq N! \sum_{1 \leq k_1 \leq \dots \leq k_N \leq n} \bar{\mu}\left(S_{k_1} = S_{k_2} = \dots = S_{k_N} = \ell\right)$$
$$= N! \sum_{1 \leq k_1 \leq \dots \leq k_N \leq n} \mathbb{E}_{\bar{\mu}}\left[\mathcal{H}_{0,k_N - k_{N-1}} \cdots \mathcal{H}_{0,k_2 - k_1} \mathcal{H}_{\ell,k_1}(\mathbf{1})\right]$$

$$\leq N! \sum_{1\leq k_1\leq\cdots\leq k_N\leq n} \frac{(K_0)^N}{(k_1+1)(k_2-k_1+1)\cdots(k_N-k_{N-1}+1)}$$
$$= O(K_0^N N!\,(\log n)^N)\,, \tag{40}$$

due to Theorem 4.2. Moreover, due to (iii) combined with a result by Billingsley (see [4] and [36])

$$\mathbb{E}_{\bar\mu}\left[\max_{k=1,\dots,n}|S_k|^2\right] = O(n(\log n)^2)$$

and so due to the Markov inequality, for every $s>0$, $\bar\mu\left(\max_{k=1,\dots,n}|S_k|>n^{1+s}\right)\leq \frac{\mathbb{E}_{\bar\mu}\left[\max_{k=1,\dots,n}|S_k|^2\right]}{n^{2+2s}} = O(n^{-1-s})$. Now fix $\vartheta>0$. Then,

$$\begin{aligned}
&\bar\mu\left(\sup_{\ell}\sum_{k=1}^{n}\mathbf{1}_{\{S_k=\ell\}}>n^{\vartheta}\right)\\
&\quad\leq \bar\mu\left(\max_{k=1,\dots,n}|S_k|>n^{1+\vartheta}\right)+\bar\mu\left(\sup_{|\ell|\leq n^{1+\vartheta}}\sum_{k=1}^{n}\mathbf{1}_{\{S_k=\ell\}}>n^{\vartheta}\right)\\
&\quad\leq O(n^{-1-\vartheta})+(2n^{1+\vartheta}+1)^2\sup_{|\ell|\leq n^{1+\vartheta}}\bar\mu\left(\sum_{k=1}^{n}\mathbf{1}_{\{S_k=\ell\}}>n^{\vartheta}\right)\\
&\quad\leq O(n^{-1-\vartheta}+(\log n)^N n^{2+2\vartheta-\vartheta N})\,,
\end{aligned}$$

where we used the inequality $\mathbb{E}[X>n^{\vartheta}]\leq\mathbb{E}[X^N]n^{-\vartheta N}$ for any $N\in\mathbb{N}^*$ combined with (40). Now choosing $N>(3+3\vartheta)/\vartheta$, we conclude the proof of the lemma by the Borel-Cantelli lemma. □

Recall that, for $x\in\bar M$, the random variable $\mathcal{Z}_n(x)$ can be rewritten: $\mathcal{Z}_n(x)=\sum_{k=1}^n \xi_{\mathcal{I}_k,S_k} = \sum_{i=1}^I\sum_{\ell\in\mathbb{Z}^2}\xi_{i,\ell}\mathcal{N}_n(i,\ell)(x)$, where $\mathcal{N}_n(i,\ell)(x):=\sum_{k=1}^n\mathbf{1}_{\{S_k=\ell,\mathcal{I}_k=i\}}(x)$ is the number of visits to the obstacle of index (i,ℓ) up to time n and where $(\xi_{i,\ell})_{i,\ell}$ is a sequence of i.i.d. centered square integrable random variables defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$.

Note that the variance of $\mathcal{Z}_n(x)$ (with respect to $\mathbb{P}$) is $\sigma_\xi^2\mathcal{V}_n(x)$, where $\sigma_\xi^2:=\mathbb{E}[\xi^2_{(1,\mathbf{0})}]$, since, under $\mathbb{P}$, $\mathcal{Z}_n(x)$ is a sum of independent random variables of respective variances $\sigma_\xi^2(\mathcal{N}_n(i,\ell)(x))^2$.

Lemma 4.13 (Convergence of finite-dimensional distributions). *Assume the assumptions of Theorem 4.11. For every $m\geq 1$, every $0<t_1<t_2<\cdots<t_m$, For $\bar\mu$-almost every $x\in\bar M$, $\left(\sum_{j=1}^m a_j\left(\mathcal{Z}_{\lfloor nt_j\rfloor}-\mathcal{Z}_{\lfloor nt_{j-1}\rfloor}\right)(x)/\sqrt{n\log n}\right)_n$ converges in distribution (with respect to $\mathbb{P}$) to a centered Gaussian random variable with variance $2c_1\sigma_\xi^2\sum_{j=1}^m a_j^2(t_j-t_{j-1})$.*

Proof. We fix $x \in \bar{M}$. The variance of $\sum_{j=1}^m a_j \left(\mathcal{Z}_{\lfloor nt_j \rfloor} - \mathcal{Z}_{\lfloor nt_{j-1} \rfloor}\right)(x)$ (with respect to $\mathbb{P}$) is equal to, recalling (38),

$$\begin{aligned}
&\sigma_\xi^2 \sum_{i=1}^I \sum_{\ell \in \mathbb{Z}^2} \left(\sum_{j=1}^m a_j \left(\mathcal{N}_{\lfloor nt_j \rfloor}(i,\ell)(x) - \mathcal{N}_{\lfloor nt_{j-1} \rfloor}(i,\ell)(x) \right) \right)^2 \\
&= \sigma_\xi^2 \sum_{i=1}^I \sum_{\ell \in \mathbb{Z}^2} \sum_{j,j'=1}^m a_j a_{j'} \sum_{k=\lfloor nt_{j-1} \rfloor + 1}^{\lfloor nt_j \rfloor} \sum_{k'=\lfloor nt_{j'-1} \rfloor + 1}^{\lfloor nt_{j'} \rfloor} \mathbf{1}_{\{S_k = \ell, \mathcal{I}_k = i, S_{k'} = \ell, \mathcal{I}_{k'} = i\}}(x) \\
&= \sigma_\xi^2 \sum_{j,j'=1}^m a_j a_{j'} \sum_{k=\lfloor nt_{j-1} \rfloor + 1}^{\lfloor nt_j \rfloor} \sum_{k'=\lfloor nt_{j'-1} \rfloor + 1}^{\lfloor nt_{j'} \rfloor} \mathbf{1}_{\{S_k = S_{k'}, \mathcal{I}_k = \mathcal{I}_{k'}\}}(x) \\
&= \sigma_\xi^2 \left(\sum_{j=1}^m a_j^2 \mathcal{V}_{\lfloor nt_j \rfloor - \lfloor nt_{j-1} \rfloor} \circ \bar{T}^{\lfloor nt_{j-1} \rfloor} \right. \\
&\quad + \sum_{1 \le j < j' \le m} a_j a_{j'} \left(\left(\mathcal{V}_{\lfloor nt_{j'} \rfloor - \lfloor nt_{j-1} \rfloor} - \mathcal{V}_{\lfloor nt_{j'-1} \rfloor - \lfloor nt_{j-1} \rfloor} \right) \circ \bar{T}^{\lfloor nt_{j-1} \rfloor} \right. \\
&\quad \left. \left. + \left(\mathcal{V}_{\lfloor nt_{j'-1} \rfloor - \lfloor nt_j \rfloor} - \mathcal{V}_{\lfloor nt_{j'} \rfloor - \lfloor nt_j \rfloor} \right) \circ \bar{T}^{\lfloor nt_j \rfloor} \right) \right) \\
&\sim 2c_1 \sigma_\xi^2 \sum_{j=1}^m a_j^2 (t_j - t_{j-1}) n \log n \,,
\end{aligned} \tag{41}$$

for $\bar{\mu}$-a.e. $x \in \bar{M}$, due to the proof of Theorem 4.7 (since $(\mathcal{V}_n/(n \log n))_n$ converges $\bar{\mu}$-almost surely to $2c_1$, as well as any sequence of random variables with the same marginal distributions).

Note that, with respect to $\mathbb{P}$, $\sum_{j=1}^m a_j \left(\mathcal{Z}_{\lfloor nt_j \rfloor} - \mathcal{Z}_{\lfloor nt_{j-1} \rfloor}\right)(x)$ is a sum of independent centered random variables with variances

$$\left(\sigma_{n,i,\ell}^2(x) := \sigma_\xi^2 \left(\sum_{j=1}^m a_j (\mathcal{N}_{\lfloor nt_j \rfloor}(i,\ell)(x) - \mathcal{N}_{\lfloor nt_{j-1} \rfloor}(i,\ell)(x)) \right)^2 \right)_{i,\ell} .$$

Hence, due to Lemma 4.12 and to the Lindeberg Theorem, for $\bar{\mu}$-almost every $x \in \bar{M}$, the sequence of random variables

$$\left(\frac{\sum_{j=1}^m a_j \left(\mathcal{Z}_{\lfloor nt_j \rfloor} - \mathcal{Z}_{\lfloor nt_{j-1} \rfloor}\right)(x)}{Var(\sum_{j=1}^m a_j (\mathcal{Z}_{\lfloor nt_j \rfloor} - \mathcal{Z}_{\lfloor nt_{j-1} \rfloor})(x))} \right)_n$$

converges in distribution (with respect to $\mathbb{P}$) to a standard Gaussian random variable. The conclusion then follows from (41). □

Lemma 4.14. *Under the assumptions of Theorem 4.11, the sequence of random variables* $\left(\widetilde{\mathcal{Z}}_n(t)/\sqrt{n \log n}\right)_n$ *is tight (with respect to* $\bar{\mu} \otimes \mathbb{P}$*) in* $C([0, \mathcal{T}])$ *for every* $\mathcal{T} > 0$.

Proof. Due to Theorem 4.7, it is enough to prove the tightness of $\left(\widetilde{\mathcal{Z}}_n(t)/\sqrt{\sigma_\xi^2 \mathcal{V}_n}\right)_n$. Due to [4, Lemma p. 88], it is enough to prove that

$$\lim_{\lambda\to+\infty} \limsup_{n\to+\infty} \lambda^2 (\bar{\mu}\otimes\mathbb{P})\left(\max_{k=1,\dots,n} |\mathcal{Z}_k| \geq \lambda\sigma_\xi\sqrt{\mathcal{V}_n}\right) = 0. \tag{42}$$

We modify the proof of tightness of Bolthausen in [5]. For completeness, we explain the adaptations to make. Following [5] (see also [30, bottom of page 824], using the fact that $(\mathcal{Z}_n)_n$ has positively associated increments knowing $(S_n)_n$, we obtain that, for any $\lambda > \sqrt{2}$,

$$(\bar{\mu}\otimes\mathbb{P})\left(\max_{j\leq n} |\mathcal{Z}_j| \geq \lambda\sigma_\xi\sqrt{\mathcal{V}_n}\right) \leq 2(\bar{\mu}\otimes\mathbb{P})\left(|\mathcal{Z}_n| > (\lambda-\sqrt{2})\sigma_\xi\sqrt{\mathcal{V}_n}\right).$$

Now we simplify the conclusion of [5]. Since we know that $(\mathcal{Z}_n/\sqrt{\mathcal{V}_n})_n$ converges in distribution to a Gaussian random variable Y, so

$$\limsup_{n\to+\infty} (\bar{\mu}\otimes\mathbb{P})\left(\max_{j\leq n} |\mathcal{Z}_j| \geq \lambda\sigma_\xi\sqrt{\mathcal{V}_n}\right) \leq 2\mathbb{P}\left(|Y| > (\lambda-\sqrt{2})\sigma_\xi\right)$$

and $\mathbb{P}(|Y| > x) = O(e^{-c_Y x^2})$ for some $c_Y > 0$, which proves (42) and so the tightness. □

Proof of Theorem 4.11. The first result of Theorem 4.11 is a direct consequence of Lemmas 4.13 and 4.14.

Now let us prove the last point. For this, we use the general argument developed by Guillotin-Plantard, Dos Santos and Poisat in [21]. Indeed the proof of [21] only uses the following assumptions:

- Γ is a denumerable set,
- $\widetilde{S} := (\widetilde{S}_n)_{n\geq 0}$ is a sequence of Γ-valued random variables,
- $\xi := (\xi)_{y\in\Gamma}$ is a sequence of independent identically distributed real valued random variables, which are centered and such that $\mathbb{E}[|\xi_y|^2(\log^+|\xi_y|)^\chi|)] < \infty$ for some $\chi > 0$,
- the sequences of random variables ξ and $\widetilde{S}$ are independent,
- $\left(\frac{1}{\sqrt{n\log n}}(\sum_{k=0}^{\lfloor nt\rfloor-1}\xi_{\widetilde{S}_k} + (nt-\lfloor nt\rfloor)\xi_{\widetilde{S}_{\lfloor nt\rfloor}})\right)_{t\in[0,1]}$ converges in distribution in $C(0,T)$ to the Brownian motion B,
- $\sup_{y\in\Gamma}\mathbb{E}[\widetilde{N}_n(y)] = O(\log n)$ with $\widetilde{N}_n(y) := \#\{k=0,\dots,n-1 : \widetilde{S}_k = y\} = \sum_{k=0}^{n-1}\mathbf{1}_{\{\widetilde{S}_k=y\}}$ being the local time of $\widetilde{S}$.
- $\sum_{y\in\Gamma}(\mathbb{E}[(\widetilde{N}_n(y))])^2 = O(n)$, with the same notation.
- $\mathbb{P}(\widetilde{S}_n \notin \{\widetilde{S}_0,\dots,\widetilde{S}_{n-1}\}) = O((\log n)^{-1})$.

We apply this to $\Gamma = \{1,\dots,I\}\times\mathbb{Z}^2$ and $\widetilde{S}_n = (\mathcal{I}_n, S_n)$. For the antepenultimate condition, observe that, due to Corollary 4.3,

$$\mathbb{E}[\widetilde{N}_n(a,\ell)] = \sum_{k=0}^{n-1}\mathbb{E}_{\bar{\mu}}\left[\mathbf{1}_{\{S_k=\ell\}}.\mathbf{1}_{\bar{O}_a}\circ\bar{T}^n\right] = \sum_{k=0}^{n-1}\mathbb{E}_{\bar{\mu}}\left[\mathbf{1}_{\bar{O}_a}\mathcal{H}_{\ell,k}(\mathbf{1})\right] = O(\log n).$$

For the penultimate condition,

$$\sum_{y\in\Gamma}(\mathbb{E}[(\widetilde{N}_n(y))])^2 = \sum_{y\in\Gamma}\sum_{k,j=0}^{n-1}\mathbb{E}_{\bar\mu\times\bar\mu}[\mathbf{1}_{\widetilde{S}_k=\widetilde{S}'_j=y}] = \sum_{i,j=0}^{n-1}\mathbb{E}_{\bar\mu\times\bar\mu}[\mathbf{1}_{\widetilde{S}_k=\widetilde{S}'_j}],$$

considering an independent copy $\widetilde{S}' = (\widetilde{S}'_n = (\mathcal{I}'_n, S'_n))_n$ of $\widetilde{S}$. Now, using again (23) combined with Assumption 4.1 with β and $a > 0$ as in the proof of Theorem 4.2, we obtain

$$\begin{aligned}
\mathbb{E}_{\bar\mu\times\bar\mu}\left[\mathbf{1}_{\widetilde{S}_k=\widetilde{S}'_j}\right] &\le \mathbb{E}_{\bar\mu\times\bar\mu}\left[\mathbf{1}_{S_k=S'_j}\right] \\
&= \int_{[-\pi,\pi]^2}\mathbb{E}_{\bar\mu\times\bar\mu}\left[e^{iu\cdot S_k}e^{-iu\cdot S'_j}\right]du \\
&= \int_{[-\pi,\pi]^2}\mathbb{E}_{\bar\mu}\left[e^{iu\cdot S_i}\right]\mathbb{E}_{\bar\mu}\left[e^{-iu\cdot S'_j}\right]du \\
&= \int_{[-\pi,\pi]^2}\mathbb{E}_{\bar\mu}\left[P_u^k\mathbf{1}\right]\mathbb{E}_{\bar\mu}\left[P_{-u}^j\mathbf{1}\right]du \\
&\le \int_{[-\beta,\beta]^2} e^{-ak|u|^2}\left|\mathbb{E}_{\bar\mu}\left[\Pi_u\mathbf{1}\right]\right| e^{-aj|u|^2}\left|\mathbb{E}_{\bar\mu}\left[\Pi_u\mathbf{1}\right]\right| du + O(\alpha^{k+j}) \\
&\le \int_{\mathbb{R}^2} e^{-ak|u|^2}\left|\mathbb{E}_{\bar\mu}\left[\Pi_u\mathbf{1}\right]\right| e^{-aj|u|^2}\left|\mathbb{E}_{\bar\mu}\left[\Pi_u\mathbf{1}\right]\right| du + O(\alpha^{k+j}) \\
&= O(|1+k+j|^{-1})\,.
\end{aligned}$$

Therefore

$$\sum_{y\in\Gamma}(\mathbb{E}[(\widetilde{N}_n(y))])^2 = O\left(\sum_{0\le j,k\le n-1}\frac{1}{1+k+j}\right) = O(n)\,.$$

The last condition comes from the second part of Proposition 4.4. Note that in order to invoke Proposition 4.4, we need that the operator $f \mapsto \mathbb{E}_{\bar\mu}[f\,\mathbf{1}_{\bar O_a}]$ is continuous on $\widetilde{\mathcal{B}}'_2$. This follows from the fact that we have assumed (i) in the statement of the theorem, that $f \mapsto f\mathbf{1}_{\bar O_a}$ is a continuous operator on $\widetilde{\mathcal{B}}_1$, and that by Assumption 4.1, $\mathbb{E}_{\bar\mu}[\cdot]$ acts continuously on $\widetilde{\mathcal{B}}_2$. The second condition needed to conclude (29) from Proposition 4.4 is precisely assumption (ii) in the statement of the theorem. □

Acknowledgements. This work was begun at the AIM workshop *Stochastic Methods for Non-Equilibrium Dynamical Systems*, in June 2015. Part of this work was carried out during visits by the authors to ESI, Vienna in 2016, to CIRM, Luminy in 2017 and 2018, and to BIRS, Canada in 2018, and by a visit of FP to the University of Massachusetts at Amherst in 2018. MD was supported in part by NSF Grant DMS 1800321. FP is grateful to the IUF for its important support.

Appendix A. Proof of Lemma 4.5

Here we prove the Lemma 4.5, which was used in Sect. 4.2, especially used in the proof of Theorem 2.4.

Let us prove that (36) holds true. By density, it suffices to perform the estimate for $f \in \mathcal{C}^1(\bar{M}_0)$. In the proof below, we use the fact that the invariant measure $\bar{\mu}_0$ is absolutely continuous with respect to the Lebesgure measure.

Choose $\ell \geq 1$ and fix $\underline{\omega}_\ell := (\omega_1, \ldots, \omega_\ell)$. Let g be as in the statement of the lemma. For brevity, denote by $\bar{T}^\ell_{\underline{\omega}_\ell} = \bar{T}_{\omega_\ell} \circ \cdots \circ \bar{T}_{\omega_1}$ the composition of random maps and by $\mathcal{L}^\ell_{\underline{\omega}_\ell}$ its associated transfer operator. Also, set $H^p_\ell(g) = |g|_\infty + \sup_{C \in \mathcal{C}_{\omega_1,\ldots,\omega_\ell}} C^{(p)}_{g|C}$. We must estimate

$$\mathbb{E}_{\bar{\mu}_0}[f\, g] = \int_{\bar{M}_0} f\, g\, d\bar{\mu}_0 = \int_{\bar{M}_0} \mathcal{L}^\ell_{\underline{\omega}_\ell} f \cdot g \circ (\bar{T}^\ell_{\underline{\omega}_\ell})^{-1}\, d\bar{\mu}_0.$$

To do this, we decompose $\bar{M}_0$ into a countable collection of local rectangles, each foliated by a smooth collection of stable curves on which we may apply our norms. This technique follows closely the decomposition used in [16, Lemma 3.4].

We partition each connected component of $\bar{M}_0 \setminus (\cup_{|k|\geq k_0} \mathbb{H}_k)$, into finitely many boxes B_j whose boundary curves are elements of $\mathcal{W}^s$ and $\mathcal{W}^u$, as well as the horizontal boundaries of $\mathbb{H}_{\pm k_0}$. We construct the boxes B_j so that each has diameter in $(\delta/2, \delta)$, for some $\delta > 0$, and is foliated by a smooth foliation of stable curves $\{W_\xi\}_{\xi \in \Xi_j}$, such that each curve W_ξ is stretched completely between the two unstable boundaries of B_j. Indeed, due to the continuity of the cones $C^s(x)$ from **(H1)**, we can choose δ sufficiently small that the family $\{W_\xi\}_{\xi \in \Xi_j}$ is a family of parallel line segments.

We disintegrate the measure $\bar{\mu}_0$ on B_j into a family of conditional probability measures $d\mu_\xi = c_\xi \cos\varphi\, dm_{W_\xi}, \xi \in \Xi_j$, where c_ξ is a normalizing constant, and a factor measure $\lambda_j(\xi)$ on the index set Ξ_j. Since $\bar{\mu}_0$ is absolutely continuous with respect to Lebesgue measure on $\bar{M}_0$, we have $\lambda_j(\Xi_j) = \bar{\mu}_0(B_j) = \mathcal{O}(\delta^2)$.

Similarly, on each homogeneity strip $\mathbb{H}_t$, $t \geq k_0$, we choose a smooth foliation of parallel line segments $\{W_\xi\}_{\xi \in \Xi_t} \subset \mathbb{H}_t$ which completely cross $\mathbb{H}_t$. Due to the uniform transversality of the stable cone with $\partial\mathbb{H}_t$, we may choose a single index set Ξ_t for each homogeneity strip. We again disintegrate $\bar{\mu}_0$ into a family of conditional probability measures $d\mu_\xi = c_\xi \cos\varphi\, dm_{W_\xi}, \xi \in \Xi_t$, and a transverse measure $\lambda_t(\xi)$ on the index set Ξ_t. This implies that $\lambda_t(\Xi_t) = \bar{\mu}_0(\mathbb{H}_t) = \mathcal{O}(|t|^{-5})$ for each $|t| \geq k_0$.

Notice that on each homogeneity strip $\mathbb{H}_k$, the function $\cos\varphi$ satisfies,

$$|\log\cos\varphi(x) - \log\cos\varphi(y)| \leq C d(x, y)^{1/3} \tag{43}$$

for some uniform constant $C > 0$ (uniform in k).

We are ready to estimate the required integral. Let $\mathcal{G}_\ell(W_\xi)$ denote the components of $(\bar{T}^\ell_{\underline{\omega}_\ell})^{-1} W_\xi$, with long pieces subdivided to have length between $\delta_0/2$ and δ_0, as in the proof of Lemma 3.14.

$$
\begin{aligned}
&\int \mathcal{L}^\ell_{\underline{\omega}_\ell} f\cdot g\circ(\bar T^\ell_{\underline{\omega}_\ell})^{-1}\,d\bar\mu_0\\
&\quad=\sum_j\int_{B_j}\mathcal{L}^\ell_{\underline{\omega}_\ell} f\cdot g\circ(\bar T^\ell_{\underline{\omega}_\ell})^{-1}d\bar\mu_0+\sum_{|t|\ge k_0}\int_{\mathbb{H}_t}\mathcal{L}^\ell_{\underline{\omega}_\ell} f\cdot g\circ(\bar T^\ell_{\underline{\omega}_\ell})^{-1}d\bar\mu_0\\
&\quad=\sum_j\int_{\Xi_j}\int_{W_\xi}\mathcal{L}^\ell_{\underline{\omega}_\ell} f\cdot g\circ(\bar T^\ell_{\underline{\omega}_\ell})^{-1}\,d\mu_\xi d\lambda_j(\xi)\\
&\qquad+\sum_{|t|\ge k_0}\int_{\Xi_t}\int_{W_\xi}\mathcal{L}^\ell_{\underline{\omega}_\ell} f\cdot g\circ(\bar T^\ell_{\underline{\omega}_\ell})^{-1}\,d\mu_\xi d\lambda_t(\xi)\\
&\quad=\sum_j\int_{\Xi_j}\sum_{W_{\xi,i}\in\mathcal{G}_\ell(W_\xi)}\int_{W_{\xi,i}} f\,g\,c_\xi\cos\varphi\circ\bar T^\ell_{\underline{\omega}_\ell}\,J_{W_{\xi,i}}\bar T^\ell_{\underline{\omega}_\ell}\,dm_{W_{\xi,i}}d\lambda_j(\xi)\\
&\qquad+\sum_{|t|\ge k_0}\int_{\Xi_t}\sum_{W_{\xi,i}\in\mathcal{G}_\ell(W_\xi)}\int_{W_{\xi,i}} f\,g\,c_\xi\cos\varphi\circ\bar T^\ell_{\underline{\omega}_\ell}\,J_{W_{\xi,i}}\bar T^\ell_{\underline{\omega}_\ell}\,dm_{W_{\xi,i}}d\lambda_t(\xi)\,.
\end{aligned}
$$

Next we use the assumption that g is Hölder continuous on connected components of $\bar M_0\setminus(\cup_{k=1}^\ell \bar T^{-1}_{\omega_1}\circ\cdots\circ\bar T^{-1}_{\omega_k}(\mathcal{S}_{0,H}))$. Since elements of $\mathcal{G}_\ell(W_\xi)$ are also subdivided according to these singularity sets, we have that g is Hölder continuous on each $W_{\xi,i}\in\mathcal{G}_\ell(W_\xi)$. Thus,

$$
\begin{aligned}
&\int_{W_{\xi,i}} f\,g\,c_\xi\cos\varphi\circ\bar T^\ell_{\underline{\omega}_\ell}\,J_{W_{\xi,i}}\bar T^\ell_{\underline{\omega}_\ell}\,dm_{W_{\xi,i}}\\
&\quad\le|f|_w|g|_{\mathcal{C}^p(W_{\xi,i})}c_\xi|\cos\varphi\circ\bar T^\ell_{\underline{\omega}_\ell}|_{\mathcal{C}^p(W_{\xi,i})}|J_{W_{\xi,i}}\bar T^\ell_{\underline{\omega}_\ell}|_{\mathcal{C}^p(W_{\xi,i})}\\
&\quad\le|f|_w H^p_\ell(g)|J_{W_{\xi,i}}\bar T^\ell_{\underline{\omega}_\ell}|_{\mathcal{C}^0(W_{\xi,i})}\frac{C}{|W_\xi|},
\end{aligned}
$$

where we used (43) in the last estimate, as well as the fact that the normalizing constant c_ξ is proportional to $|W_\xi|^{-1}$. This implies that

$$
\begin{aligned}
\mathbb{E}_{\bar\mu_0}[f\,g]\le C|f|_w H^p_\ell(g)\Big(&\sum_j\int_{\Xi_j}\sum_{W_{\xi,i}\in\mathcal{G}_\ell(W_\xi)}|J_{W_{\xi,i}}\bar T^\ell_{\underline{\omega}_\ell}|_{\mathcal{C}^0(W_{\xi,i})}|W_\xi|^{-1}\,d\lambda_j(\xi)\\
&+\sum_{|t|\ge k_0}\int_{\Xi_t}\sum_{W_{\xi,i}\in\mathcal{G}_\ell(W_\xi)}|J_{W_{\xi,i}}\bar T^\ell_{\underline{\omega}_\ell}|_{\mathcal{C}^0(W_{\xi,i})}|W_\xi|^{-1}\,d\lambda_t(\xi)\Big).
\end{aligned}
$$

Now $\sum_{W_{\xi,i}\in\mathcal{G}_\ell(W_\xi)}|J_{W_{\xi,i}}\bar T^\ell_{\underline{\omega}_\ell}|_{\mathcal{C}^0(W_{\xi,i})}$ is bounded by a uniform constant independent of ξ and $\underline{\omega}_\ell$ by [16, Lemma 5.5(b)]. Moreover, $\int_{\Xi_j}|W_\xi|^{-1}d\lambda_j(\xi)\le C\delta_0$ for some constant $C>0$ since we chose our foliation to be comprised of long cone-stable curves. We conclude that the first term to the right hand side of the last inequality is uniformly bounded by $C_1|f|_w H^p_\ell(g)$ since the sum over j is finite.

For the second term on the right hand side of the last inequality, we again use [16, Lemma 5.5(b)] as well as the fact that $|W_\xi|^{-1}=\mathcal{O}(t^3)$ for $\xi\in\Xi_t$, while $\lambda_t(\Xi_t)=\mathcal{O}(t^{-5})$. Thus

$$
\sum_{|t|\ge k_0}\int_{\Xi_t}|W_\xi|^{-1}d\lambda_t(\xi)\le\sum_{|t|\ge k_0}Ct^{-2}\le Ck_0^{-1}.
$$

We conclude that

$$\left|\mathbb{E}_{\bar\mu_0}[f\,g]\right| \le K_1 |f|_w H_\ell^p(g),$$

for some uniform constant K_1 depending on $\bar{\mathcal{F}}_{\vartheta_0}$, but not on f, ℓ or $\underline{\omega}_\ell$. This completes the proof of (36).
To prove (37), we follow the proof of Lemma 3.14. Note that for $f \in \mathcal{C}^1(\bar M_0)$, $W \in \mathcal{W}^s$, and a test function ψ, we have

$$\int_W \mathcal{L}_{u,\omega_\ell} \dots \mathcal{L}_{u,\omega_1}(fg)\,\psi\, dm_W = \sum_{W_i} \int_{W_i} fg\, e^{iu\cdot S_\ell}\, \psi\circ \bar T^\ell_{\underline{\omega}_\ell}\, J_{W_i}\bar T^\ell_{\underline{\omega}_\ell}\, dm_{W_i},$$

where the sum is taken over $W_i \in \mathcal{G}_\ell(W)$, the components of $(\bar T^\ell_{\underline{\omega}_\ell})^{-1}W$, subdivided as before. This is the same type of expression as in [16, eq. (5.24)] or [16, eq. (4.4)], but now the test function is

$$g\, e^{iu\cdot S_\ell}\, \psi\circ \bar T^\ell_{\underline{\omega}_\ell}\, J_{W_i}\bar T^\ell_{\underline{\omega}_\ell}$$

rather than simply $\psi\circ \bar T^\ell_{\underline{\omega}_\ell}\, J_{W_i}\bar T^\ell_{\underline{\omega}_\ell}$. Since S_ℓ is constant on each $W_i \in \mathcal{G}_\ell(W)$, and we have assumed that g is (uniformly in ℓ) Hölder continuous on each $W_i \in \mathcal{G}_\ell(W)$, the proof of the Lasota–Yorke inequalities follows as in the proof of [16, Proposition 5.6]. The bound (37) then follows as in the proof of Lemma 3.14.

Remark A.1. As a consequence of this lemma, if $g : \bar M \to \mathbb{R}$ is a bounded measurable function such that, for every $\underline{\omega} = (\omega_k)_{k\ge 0} \in E^{\mathbb{N}}$, there exists positive integer $\ell_{\underline{\omega}}$ such that $g(\cdot,\underline{\omega})$ is p-Hölder on every connected component (uniformly on $\underline{\omega}$) of $\bar M_0 \setminus \left(\cup_{k=0}^{\ell_{\underline{\omega}}-1} \bar T^{-1}_{\omega_0}\circ\dots\circ \bar T^{-1}_{\omega_{\ell(\underline{\omega})-1}}(\mathcal{S}_{0,H})\right)$. Then, for every $f \in \widetilde{\mathcal{B}}_w$, we have

$$\begin{aligned}\left|\mathbb{E}_{\bar\mu}[gf]\right| &= \left|\int_E \mathbb{E}_{\bar\mu_0}[g(\cdot,\underline{\omega})f(x,\underline{\omega})]\,d\eta(\underline{\omega})\right| \\ &= K_1\|f\|_{\widetilde{\mathcal{B}}_w}\left(\|g\|_\infty + \sup_{\underline{\omega}\in E^{\mathbb{N}}}\sup_{C\in\mathcal{C}_{\omega_1,\dots,\omega_\ell(\omega)}} C^{(p)}_{(g(\cdot,\underline{\omega}))_{|C}}\right),\end{aligned}$$

with the same notations as in the previous lemma. Therefore, $\mathbb{E}_{\bar\mu}[g\cdot]$ is in $\widetilde{\mathcal{B}}'_w$.

Appendix B. Proof of Lemma 4.10

Note that $\mathcal{V}_n = n + 2\sum_{1\le k<\ell\le n} \mathbf{1}_{\{S_\ell = S_k, \mathcal{I}_\ell = \mathcal{I}_k\}}$. Hence

$$Var_{\bar\mu}(\mathcal{V}_n) = 4\sum_{1\le k_1<\ell_1\le n}\ \sum_{1\le k_2<\ell_2\le n} D_{k_1,\ell_1,k_2,\ell_2},$$

with $D_{k_1,\ell_1,k_2,\ell_2} := \bar\mu(E_{k_1,\ell_1}\cap E_{k_2,\ell_2}) - \bar\mu(E_{k_1,\ell_1})\bar\mu(E_{k_2,\ell_2})$. It follows that

$$\left|Var_{\bar\mu}(\mathcal{V}_n) - 8(A_2+A_3)\right| \le 8(A_1+A_4), \tag{44}$$

with

$$A_1 := \sum_{1\le k_1<\ell_1\le k_2<\ell_2\le n} \left|D_{k_1,\ell_1,k_2,\ell_2}\right|, \quad A_2 := \sum_{1\le k_1<k_2<\ell_1<\ell_2\le n} D_{k_1,\ell_1,k_2,\ell_2},$$

$$A_3 := \sum_{1<k_1<k_2<\ell_2<\ell_1\le n} D_{k_1,\ell_1,k_2,\ell_2}, \quad A_4 := \sum_{(k_1,k_2,\ell_1,\ell_2)\in E_n\cup F_n} \left|D_{k_1,\ell_1,k_2,\ell_2}\right|,$$

with

$$E_n := \{(k_1,k_2,\ell_1,\ell_2)\in\{1,\dots,n\} \,:\, k_1=k_2<\min(\ell_1,\ell_2)\},$$
$$F_n := \{(k_1,k_2,\ell_1,\ell_2)\in\{1,\dots,n\} \,:\, \max(k_1,k_2)<\ell_1=\ell_2\}.$$

We will start with the two easiest estimates: the estimates of the error terms A_1 and A_4. The method we will use to estimate the main terms A_2 and A_3 differs from [31]. Due to Lemma 4.9,

$$A_1 \le I^2 \sum_{1\le k_1<\ell_1\le k_2<\ell_2\le n} \frac{C_1\alpha^{k_2-\ell_1}}{(\ell_1-k_1)(\ell_2-k_2)} = O(n(\log n)^2) = o(n^2).$$

Let us now prove that $A_4 = o(n^2)$ by writing

$$\begin{aligned}
&\sum_{(k_1,k_2,\ell_1,\ell_2)\in E_n} \left|D_{k_1,\ell_1,k_2,\ell_2}\right| \\
&\le 2 \sum_{1\le k<\ell_1\le\ell_2\le n} \left(\bar\mu(E_{k,\ell_1}\cap E_{k,\ell_2}) + \bar\mu(E_{k,\ell_1})\bar\mu(E_{k,\ell_2})\right) \\
&\le 2 \sum_{1\le k<\ell_1\le\ell_2\le n} \left(\bar\mu(S_{\ell_1}=S_{\ell_2}=S_k) + \bar\mu(S_{\ell_1}=S_k)\bar\mu(S_{\ell_2}=S_k)\right) \\
&\le 2 \sum_{1\le k<\ell_1\le\ell_2\le n} \left(\mathbb{E}_{\bar\mu}\left[\mathcal{H}_{0,\ell_2-\ell_1}\mathcal{H}_{0,\ell_1-k}(\mathbf{1})\right] + \mathbb{E}_{\bar\mu}\left[\mathcal{H}_{0,\ell_1-k}(\mathbf{1})\right]\mathbb{E}_{\bar\mu}\left[\mathcal{H}_{0,\ell_2-k}(\mathbf{1})\right]\right) \\
&\le K_0' \sum_{1\le k<\ell_1\le\ell_2\le n} \left(\frac{1}{(\ell_1-k)(\ell_2-\ell_1+1)} + \frac{1}{(\ell_1-k)(\ell_2-k)}\right)
\end{aligned}$$

for some $K_0' > 0$ due to Theorem 4.2, since $\mathbb{E}_{\bar\mu}[\cdot]$ is a continuous linear operator on $\widetilde{\mathcal{B}}_1$ and since $\mathbf{1}\in\widetilde{\mathcal{B}}_1$. This leads to $\sum_{(k_1,k_2,\ell_1,\ell_2)\in E_n}\left|D_{k_1,\ell_1,k_2,\ell_2}\right| = O(n(\log n)^2)$. Analogously, we obtain $\sum_{(k_1,k_2,\ell_1,\ell_2)\in F_n}\left|D_{k_1,\ell_1,k_2,\ell_2}\right| = O(n(\log n)^2)$. Hence $A_4 = o(n^2)$.

For A_2, we study separately the terms $\bar\mu(E_{k_1,\ell_1}\cap E_{k_2,\ell_2})$ and the terms $\bar\mu(E_{k_1,\ell_1})\bar\mu(E_{k_2,\ell_2})$. First by Lemma 4.8,

$$\begin{aligned}
&\sum_{1\le k_1<k_2<\ell_1<\ell_2\le n} \bar\mu(E_{k_1,\ell_1})\bar\mu(E_{k_2,\ell_2}) \\
&= c_1^2 \sum_{1\le k_1<k_2<\ell_1<\ell_2\le n} \left((\ell_1-k_1)^{-1} + O((\ell_1-k_1)^{-3/2})\right) \\
&\quad \left((\ell_2-k_2)^{-1} + O((\ell_2-k_2)^{-3/2})\right) \\
&= o(n^2) + c_1^2 \sum_{1\le k_1<k_2<\ell_1<\ell_2\le n} \frac{1}{(\ell_1-k_1)(\ell_2-k_2)}, \qquad (45)
\end{aligned}$$

where we used the fact that

$$\begin{aligned}
&\sum_{1\le k_1<k_2<\ell_1<\ell_2\le n} \frac{1}{\ell_1-k_1}\frac{1}{(\ell_2-k_2)^{\frac{3}{2}}}\\
&\le \sum_{m_1,m_2,m_3,m_4=1}^{n} \frac{1}{m_2+m_3}\frac{1}{(m_3+m_4)^{\frac{3}{2}}}\\
&\le n\sum_{m_3=1}^{n}\sum_{m_2=1}^{n}\frac{1}{m_2+m_3}\sum_{m_4=1}^{n}\frac{1}{(m_3+m_4)^{\frac{3}{2}}}\\
&= O\left(n\sum_{m_3=1}^{n}\log n\, m_3^{-\frac{1}{2}}\right)\\
&= O(n^{\frac{3}{2}}\log n) = o(n^2)\,.
\end{aligned}$$

Therefore, due to the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
&\sum_{1\le k_1<k_2<\ell_1<\ell_2\le n} \bar\mu(E_{k_1,\ell_1})\bar\mu(E_{k_2,\ell_2})\\
&= o(n^2) + c_1^2 n^2 \int_{\frac{1}{n}\le\frac{\lfloor nx\rfloor}{n}<\frac{\lfloor ny\rfloor}{n}<\frac{\lceil nz\rceil}{n}<\frac{\lceil nt\rceil}{n}\le 1} \frac{dxdydzdt}{\left(\frac{\lceil nz\rceil}{n}-\frac{\lfloor nx\rfloor}{n}\right)\left(\frac{\lceil nt\rceil}{n}-\frac{\lfloor ny\rfloor}{n}\right)}\\
&\sim c_1^2 n^2 \int_{0<x<y<z<t<1} \frac{dxdydzdt}{(z-x)(t-y)}\\
&= c_1^2\frac{\pi^2}{12}n^2 = \frac{n^2}{48\det\Sigma^2}\left(\sum_{a=1}^{I}\bar\mu(\mathcal{I}_0=a)^2\right)^2. \qquad (46)
\end{aligned}$$

The rest of the estimate of A_2 is new (it is different from [31]). Fix for the moment $1\le k_1<k_2<\ell_1<\ell_2\le n$. Note that

$$\begin{aligned}
&\bar\mu(E_{k_1,\ell_1}\cap E_{k_2,\ell_2})\\
&= \sum_{a,b=1}^{I}\bar\mu\Big(\bar T^{-k_1}\bar O_a\cap\bar T^{-k_2}\bar O_b\cap\bar T^{-\ell_1}(\bar O_a)\\
&\quad\cap\bar T^{-\ell_2}\bar O_b\cap\{S_{k_2}-S_{k_1}=-(S_{\ell_1}-S_{k_2})=S_{\ell_2}-S_{\ell_1}\}\Big)\,.
\end{aligned}$$

Using now (23) as for (24), we observe that $\mathbf{1}_{\{S_{k_2}-S_{k_1}=-(S_{\ell_1}-S_{k_2})=S_{\ell_2}-S_{\ell_1}\}}$ is equal to the following quantity

$$\frac{1}{(2\pi)^4}\int_{([-\pi,\pi]^2)^2} e^{iu\cdot((S_{k_2}-S_{k_1})+(S_{\ell_1}-S_{k_2}))}e^{iv\cdot((S_{\ell_2}-S_{\ell_1})+(S_{\ell_1}-S_{k_2}))}\,du\,dv\,,$$

which is also equal to

$$\begin{aligned}
&\frac{1}{(2\pi)^4}\int_{([-\pi,\pi]^2)^2} e^{iu\cdot(S_{k_2}-S_{k_1})}e^{i(u+v)\cdot(S_{\ell_1}-S_{k_2})}e^{iv\cdot(S_{\ell_2}-S_{\ell_1})}\,du\,dv\\
&= \frac{1}{(2\pi)^4}\int_{([-\pi,\pi]^2)^2} e^{iu\cdot S_{k_2-k_1}\circ\bar T^{k_1}}e^{i(u+v)\cdot S_{\ell_1-k_2}\circ\bar T^{k_2}}e^{iv\cdot S_{\ell_2-\ell_1}\circ\bar T^{\ell_1}}\,du\,dv\,.
\end{aligned}$$

Now using the P-invariance and $\bar{T}$-invariance of $\bar{\mu}$ and several times the formula $P^m(f.g\circ\bar{T}^m) = gP^m(f)$, we obtain

$$\bar{\mu}(E_{k_1,\ell_1}\cap E_{k_2,\ell_2}) = \sum_{a,b=1}^{I}\frac{1}{(2\pi)^4}\int_{([-\pi,\pi]^2)^2}\mathbb{E}_{\bar{\mu}}\left[\mathbf{1}_{\bar{O}_b}P_v^{\ell_2-\ell_1}\left(\mathbf{1}_{\bar{O}_a}P_{u+v}^{\ell_1-k_2}\left(\mathbf{1}_{\bar{O}_b}P_u^{k_2-k_1}(\mathbf{1}_{\bar{O}_a})\right)\right)\right]du\,dv\,.$$

Due to our spectral assumptions, we observe that

$$P_u^n = \lambda_u^n\Pi_u + O(\alpha^n)\,,$$

up to defining $\lambda_u = e^{-\frac{1}{2}\Sigma^2u\cdot u}$ for u outside $[-\beta,\beta]^2$ and so, proceding as in the proof of Theorem 4.2, we obtain that, for every $n\geq 2$ and every $u,v\in[-\pi,\pi]^2$,

$$\begin{aligned}P_u^n &= e^{-\frac{n}{2}\Sigma^2u\cdot u}\mathbb{E}_{\tilde{\mu}}[\cdot]\mathbf{1} + O(\alpha^n) + O(e^{-2na|u|^2}(|u|+n|u|^3))\\ &= e^{-\frac{n}{2}\Sigma^2u\cdot u}\mathbb{E}_{\tilde{\mu}}[\cdot]\mathbf{1} + O(e^{-na|u|^2}|u|)\,,\end{aligned}$$

and $|\lambda_u^n|\leq e^{-2a|u|^2}$ for some $a>0$ (such that $e^{-2a|\pi|^2}>\alpha^n$, $\max(\lambda_u^{n-1}, e^{-\frac{n-1}{2}\Sigma^2u\cdot u})\leq e^{-2an|u|^2}$) since $n|u|^2e^{-2na|u|^2} = O(e^{-na|u|^2})$. Therefore, we obtain

$$\begin{aligned}&\mathbb{E}_{\bar{\mu}}\left[\mathbf{1}_{\bar{O}_b}P_v^{\ell_2-\ell_1}\left(\mathbf{1}_{\bar{O}_a}P_{u+v}^{\ell_1-k_2}\left(\mathbf{1}_{\bar{O}_b}P_u^{k_2-k_1}(\mathbf{1}_{\bar{O}_a})\right)\right)\right]\\ &\quad= (\bar{\mu}(\bar{O}_a)\bar{\mu}(\bar{O}_b))^2e^{-\frac{1}{2}Q(\Sigma u,\Sigma v)} + O\left((|u|+|v|)e^{-naQ(u,v)}\right)\,,\end{aligned}\tag{47}$$

where we have set

$$\begin{aligned}Q(u,v) &:= (\ell_2-\ell_1)|v|^2 + (\ell_1-k_2)|u+v|^2 + (k_2-k_1)|u|^2\\ &= (\ell_2-k_2)|v|^2 + 2(\ell_1-k_2)u\cdot v + (\ell_1-k_1)|u|^2\\ &= (A_Q(u,v))\cdot(A_Q(u,v)) = |A_Q(u,v)|^2\,,\end{aligned}$$

with $A_Q^2 := \begin{pmatrix}\ell_1-k_1 & 0 & \ell_1-k_2 & 0\\ 0 & \ell_1-k_1 & 0 & \ell_1-k_2\\ \ell_1-k_2 & 0 & \ell_2-k_2 & 0\\ 0 & \ell_1-k_2 & 0 & \ell_2-k_2\end{pmatrix}$ which is symmetric with determinant

$$\begin{aligned}\det A_Q^2 &= (\ell_1-k_1)^2(\ell_2-k_2)^2 + (\ell_1-k_2)^4 - 2(\ell_1-k_2)^2(\ell_1-k_1)(\ell_2-k_2)\\ &= ((k_2-k_1)(\ell_1-k_2) + (k_2-k_1)(\ell_2-\ell_1) + (\ell_1-k_2)(\ell_2-\ell_1))^2.\end{aligned}\tag{48}$$

Due to the form of A_Q^2, we observe that A_Q^2 has eigenvectors of the forms $(*,0,*,0)$ and $(0,*,0,*)$, that it has two double eigenvalues of sum (without multiplicity) $\ell_1-k_1+\ell_2-k_2$ and of product (without multiplicity) $\sqrt{\det A_Q^2}$. Therefore its dominating eigenvalue is smaller than the sum and so is less than $4\max(k_2-k_1,\ell_1-k_2,\ell_2-\ell_1)$ and so (using the fact that the product of the two eigenvalues is larger than the maximum times the median of these three values) the smallest eigenvalue of A_Q^2 cannot be smaller than a quarter of

the median of $k_2-k_1, \ell_1-k_2, \ell_2-\ell_1$, that we denote by $med(k_2-k_1, \ell_1-k_2, \ell_2-\ell_1)$. So

$$\begin{aligned}\int_{([-\pi,\pi]^2)^2} e^{-nQ(\Sigma u,\Sigma v)}\,dudv &= (\det\Sigma)^{-2}\int_{(\Sigma[-\pi,\pi]^2)^2} e^{-nQ(u,v)}\,dudv\\ &= (\det A_Q)^{-1}(\det\Sigma)^{-2}\int_{A_Q(\Sigma([-\pi,\pi]^2)^2)} e^{-|(x,y)|^2}\,dxdy\\ &= (\det A_Q)^{-1}(\det\Sigma)^{-2}\left(\int_{(\mathbb{R}^2)^2} e^{-|(x,y)|^2}\,dxdy + O(e^{-a_1 med(k_2-k_1,\ell_1-k_2,\ell_2-\ell_1)^2})\right)\\ &= (2\pi)^2(\det A_Q)^{-1}(\det\Sigma)^{-2}\left(1+O(e^{-a_1 med(k_2-k_1,\ell_1-k_2,\ell_2-\ell_1)^2})\right),\end{aligned}$$

for some $a_1>0$. Moreover

$$\begin{aligned}\int_{(\mathbb{R}^2)^2}|(u,v)|e^{-naQ(u,v)}\,dudv &= (\det A_Q)^{-1}\int_{(\mathbb{R}^2)^2}|A_Q^{-1}(u,v)|e^{-a|(x,y)|^2}\,dxdy\\ &= O\left((\det A_Q)^{-1}\,med(k_2-k_1,\ell_1-k_2,\ell_2-\ell_1)^{-\frac12}\right).\end{aligned}$$

Therefore

$$\begin{aligned}&\bar\mu(E_{k_1,\ell_1}\cap E_{k_2,\ell_2})\\ &= \frac{\left(\sum_{a=1}^I \bar\mu(\bar O_a)^2\right)^2}{(2\pi)^2\,\det A_Q\,\det\Sigma^2}\left(1+O\left(med(k_2-k_1,\ell_1-k_2,\ell_2-\ell_1)^{-\frac12}\right)\right). \quad (49)\end{aligned}$$

But using (48),

$$\begin{aligned}&\sum_{1\le k_1<k_2<\ell_1<\ell_2\le n}(\det A_Q)^{-1}\\ &= \sum_{1\le k_1<k_2<\ell_1<\ell_2\le n}\frac{1}{(k_2-k_1)(\ell_1-k_2)+(k_2-k_1)(\ell_2-\ell_1)+(\ell_1-k_2)(\ell_2-\ell_1)}\\ &= \sum_{m_1,m_2,m_3,m_4\ge1\,:\,m_1+m_2+m_3+m_4\le n}\frac{1}{m_2m_3+m_2m_4+m_3m_4}\\ &= n^2\int_{(0,+\infty)^4}\frac{\mathbf{1}_{\left\{\frac{\lceil ny_1\rceil}{n}+\frac{\lceil ny_2\rceil}{n}+\frac{\lceil ny_3\rceil}{n}+\frac{\lceil ny_4\rceil}{n}\le1\right\}}}{\frac{\lceil ny_2\rceil}{n}\frac{\lceil ny_3\rceil}{n}+\frac{\lceil ny_2\rceil}{n}\frac{\lceil ny_4\rceil}{n}+\frac{\lceil ny_3\rceil}{n}\frac{\lceil ny_4\rceil}{n}}\,dy_1\,dy_2\,dy_3\,dy_4\\ &\sim n^2\int_{(0,+\infty)^4}\frac{\mathbf{1}_{\{y_1+y_2+y_3+y_4\le1\}}}{y_2y_3+y_2y_4+y_3y_4}\,dy_1\,dy_2\,dy_3\,dy_4,\end{aligned}$$

due to the dominated convergence theorem. Therefore

$$\begin{aligned}&\sum_{1\le k_1<k_2<\ell_1<\ell_2\le n}(\det A_Q)^{-1}\sim n^2\\ &\int_{(0,+\infty)^3}\frac{(1-y_2-y_3-y_4)\mathbf{1}_{\{y_2+y_3+y_4\le1\}}}{y_2y_3+y_2y_4+y_3y_4}\,dy_2\,dy_3\,dy_4 = n^2J. \quad (50)\end{aligned}$$

Analogously

$$\begin{aligned}
&\sum_{1\le k_1<k_2<\ell_1<\ell_2\le n} (\det A_Q)^{-1}\,(med(k_2-k_1,\ell_1-k_2,\ell_2-\ell_1))^{-\frac{1}{2}}\\
&= \sum_{m_1,m_2,m_3,m_4\ge 1\,:\,m_1+m_2+m_3+m_4\le n} \frac{1}{(m_2m_3+m_2m_4+m_3m_4)\,med(m_2,m_3,m_4)^{\frac{1}{2}}}\\
&\le n \sum_{1\le m_2\le m_3\le m_4\le n} \frac{1}{(m_2m_3+m_2m_4+m_3m_4)\,m_3^{\frac{1}{2}}} \qquad (51)\\
&\le n \sum_{1\le m_2\le m_3\le m_4\le n} \frac{1}{m_3^{\frac{3}{2}}m_4} \;\le\; n\log n \sum_{m_2=1}^{n}\sum_{m_3=m_2}^{n} m_3^{-\frac{3}{2}}\\
&\le n\log n \sum_{m_2=1}^{n} O(m_2^{-\frac{1}{2}}) = O(n^{\frac{3}{2}}\log n) = o(n^2)\,.
\end{aligned}$$

Equations (49), (50) and (51) lead to

$$\sum_{1\le k_1<k_2<\ell_1<\ell_2\le n} \bar\mu(E_{k_1,\ell_1}\cap E_{k_2,\ell_2}) = \frac{\left(\sum_{a=1}^{I}\bar\mu(\bar O_a)^2\right)^2}{((2\pi)^2\det\Sigma^2)}J + o(n^2)\,.$$

Combining this with (46), we conclude that

$$A_2 \sim \frac{n^2}{\det\Sigma^2}\left(\sum_{a=1}^{I}\bar\mu(\mathcal{I}_0=a)^2\right)^2\left(\frac{-1}{48}+\frac{J}{4\pi^2}\right). \qquad (52)$$

The study of A_3 is the most delicate. We can observe that both sums $\sum_{1\le k_1<k_2<\ell_2<\ell_1\le n}\bar\mu(E_{k_1,\ell_1}\cap E_{k_2,\ell_2})$ and $\sum_{1\le k_1<k_2<\ell_2<\ell_1\le n}\bar\mu(E_{k_1,\ell_1})\bar\mu(E_{k_2,\ell_2})$ are in $O(n^2\log n)$. However, we will see that their difference is in n^2. Once again our proof differs from the one in [31] and is based on the same idea as the one used to prove A_2. We set $E_{k,\ell}(b) := E_{k,\ell}\cap\{\mathcal{I}_k = b\}$. Due to the first part of Lemma 4.8,

$$\begin{aligned}
A_3 &= \sum_{1\le k_1<k_2<\ell_2<\ell_1\le n} \bar\mu(E_{k_1,\ell_1}\cap E_{k_2,\ell_2}) - \bar\mu(E_{k_1,\ell_1})\bar\mu(E_{k_2,\ell_2})\\
&= o(n^2) + \sum_{1\le k_1<k_2<\ell_2<\ell_1\le n}\sum_{a,b=1}^{I}\left(-I_{k_1,k_1,l_1,l_2} + \bar\mu(O_{k_1,k_2,l_1,l_2}\cap S_{k_1,k_2,l_1,l_2})\right)\\
&= o(n^2) + \sum_{1\le k_1<k_2<\ell_2<\ell_1\le n}\sum_{a,b=1}^{I}\left(-I_{k_1,k_1,l_1,l_2}\right) \qquad (53)\\
&\quad + \sum_{1\le k_1<k_2<\ell_2<\ell_1\le n}\sum_{a,b=1}^{I}\left(\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2}\mathbb{E}_{\bar\mu}\left[\mathbf{1}_{\bar O_a}P_u^{\ell_1-\ell_2}\left(\mathbf{1}_{\bar O_b}\mathcal{H}_{0,\ell_2-k_2}\right.\right.\right.\\
&\quad \left.\left.\left.\left(\mathbf{1}_{\bar O_b}P_u^{k_2-k_1}\left(\mathbf{1}_{\bar O_a}\right)\right)\right)\right]du\right), \qquad (54)
\end{aligned}$$

where

$$I_1(k_1,k_1,l_1,l_2)=\frac{(\bar\mu(\bar O_a))^2\bar\mu(E_{k_2,\ell_2}(b))}{2\pi\sqrt{\det\Sigma^2}(\ell_1-k_1)},$$
$$O_{k_1,k_2,l_1,l_2}=\bar O_a\cap\bar T^{-(k_2-k_1)}\bar O_b\cap\bar T^{-(\ell_2-k_1)}\bar O_b\cap\bar T^{-(\ell_1-k_1)}\bar O_a,$$
$$S_{k_1,k_2,l_1,l_2}=\{S_{\ell_2-k_2}\circ\bar T^{k_2-k_1}=0\}\cap\{S_{\ell_1-\ell_2}\circ\bar T^{\ell_2-k_1}=-S_{k_2-k_1}\}.$$

Now, as we did for (47) (and using Theorem 4.2), we get that

$$\mathbb{E}_{\bar\mu}\left[\mathbf{1}_{\bar O_a}P_u^{\ell_1-\ell_2}\left(\mathbf{1}_{\bar O_b}\mathcal{H}_{0,\ell_2-k_2}\left(\mathbf{1}_{\bar O_b}P_u^{k_2-k_1}\left(\mathbf{1}_{\bar O_a}\right)\right)\right)\right]$$
$$=(\bar\mu(\bar O_a))^2e^{-\frac{(\ell_1-\ell_2)+(k_2-k_1)}{2}|\Sigma u|^2}\mathbb{E}_{\bar\mu}\left[\mathbf{1}_{\bar O_b}\mathcal{H}_{0,\ell_2-k_2}\mathbf{1}_{\bar O_b}\right]+O\left(\frac{|u|}{\ell_2-k_2}e^{-na|u|^2}\right).$$

Therefore

$$\frac{1}{(2\pi)^2}\int_{(-\pi,\pi)^2}\mathbb{E}_{\bar\mu}\left[\mathbf{1}_{\bar O_a}P_u^{\ell_1-\ell_2}\left(\mathbf{1}_{\bar O_b}\mathcal{H}_{0,\ell_2-k_2}\left(\mathbf{1}_{\bar O_b}P_u^{k_2-k_1}\left(\mathbf{1}_{\bar O_a}\right)\right)\right)\right]du$$
$$=\frac{(\bar\mu(\bar O_a))^2\bar\mu\left(E_{k_2,\ell_2}(b)\right)}{2\pi(\ell_1-\ell_2+k_2-k_1)\sqrt{\det\Sigma^2}}+O\left(\frac{1}{(\ell_2-k_2)(\ell_1-\ell_2+k_2-k_1)^{\frac{3}{2}}}\right).\quad(55)$$

We will now prove that the term in O in this last formula is negligable. Indeed its sum over $\{1\le k_1\le k_2\le\ell_2\le\ell_1\le n\}$ is in O of the following quantity:

$$\sum_{m_1+m_2+m_3+m_4\le n}\left(\frac{1}{m_3(m_4+m_2)^{\frac{3}{2}}}\right)\le n\log n\sum_{m_2=1}^{n}\sum_{m_4=1}^{n}(m_4+m_2)^{-\frac{3}{2}}$$
$$\le O\left(n\log n\sum_{m_2=1}^{n}m_2^{-\frac{1}{2}}\right)=O(n^{\frac{3}{2}}\log n)=o(n^2).$$

This combined with (54) and (55) leads to

$$A_3=o(n^2)+\sum_{1\le k_1<k_2<\ell_2\le\ell_1\le n}\sum_{a,b=1}^{I}\frac{(\bar\mu(\bar O_a))^2\bar\mu\left(E_{k_2,\ell_2}(b)\right)}{2\pi\sqrt{\det\Sigma^2}}$$
$$\left(\frac{1}{\ell_1-\ell_2+k_2-k_1}-\frac{1}{\ell_1-k_1}\right),$$

i.e.

$$A_3=o(n^2)+\frac{\sum_a^I(\bar\mu(\mathcal{I}_0=a))^2}{2\pi\sqrt{\det\Sigma^2}}\sum_{m_1+m_2+m_3+m_4\le n}\left(\frac{c_1}{m_3}+O(m_3^{-\frac{3}{2}})\right)$$
$$\frac{m_3}{(m_2+m_4)(m_2+m_3+m_4)}$$
$$=o(n^2)+c_1^2\sum_{m_1+m_2+m_3+m_4\le n}\frac{1}{(m_2+m_4)(m_2+m_3+m_4)},$$

since

$$\sum_{m_1+m_2+m_3+m_4\le n} \frac{1}{m_3^{\frac{1}{2}}(m_2+m_4)(m_2+m_3+m_4)} = O\left(n \sum_{m_2,m_3,m_4=1}^{n} m_3^{-\frac{1}{2}}(m_2m_4)^{-1}\right) = o(n^2)\,.$$

Therefore, due to the Lebesgue dominated convergence theorem,

$$A_3 \sim n^2c_1^2 \int_{y_1,y_2,y_3,y_4>0:y_1+y_2+y_3+y_4<1} \frac{1}{(y_2+y_4)(y_2+y_3+y_4)}\,dy_1dy_2dy_3dy_4 \sim \frac{c_1^2}{2}n^2.$$

To conclude the proof of the lemma, we use the estimate for A_3 together with (44) and (52) to obtain,

$$\begin{aligned} 8A_2+8A_3 &= 4c_1^2n^2 + \frac{8n^2}{\det\Sigma^2}\left(\sum_{a=1}^{I}\bar\mu(\bar O_a)^2\right)^2\left(\frac{-1}{48}+\frac{J}{4\pi^2}\right) \\ &= \frac{n^2}{\det\Sigma^2}\left(\sum_{a=1}^{I}\bar\mu(\bar O_a)^2\right)^2\left[\frac{2J+1}{\pi^2}-\frac{1}{6}\right]. \end{aligned}$$

This finished the proof.

Appendix C. Spectrum of $\mathcal{P}_u$

In this appendix, we are interested in the spectrum of the family of operators $\mathcal{P}_u$. We start by stating a result for the unperturbed operators $\mathcal{L}_{u,0}$.

Lemma C.1. *Let $u \in \mathbb{R}^2$, $h \in \mathcal{B}$ and $\lambda \in \mathbb{C}$ be such that $\mathcal{L}_{u,0}h = \lambda h$ in $\mathcal{B}$ and $|\lambda| \ge 1$. Then either $h \equiv 0$ or $u \in 2\pi\mathbb{Z}^2$, $\lambda = 1$ and h is $\bar\mu_0$-almost surely constant.*

Proof. Recall that for $\psi \in \mathcal{C}^p(\bar M_0)$, we have $\psi \circ \bar T_0^n \in \mathcal{C}^p(\bar T^{-n}\mathcal{W}^s)$. Note that

$$\mathcal{L}_{u,0}h(\psi) = h(e^{iu\cdot\Phi_0}\psi\circ\bar T_0).$$

Thus for $n \ge 1$,

$$\mathcal{L}_{u,0}^n h(\psi) = h(e^{iu\cdot S_n\Phi_0}\psi\circ\bar T_0^n),$$

where $S_n\Phi_0 = \Phi_0+\Phi_0\circ\bar T_0+\cdots+\Phi_0\circ\bar T_0^{n-1}$ denotes the partial sum. By [16, Lemma 3.4], using the invariance of h,

$$\begin{aligned} |h(\psi)| &= |\lambda|^{-n}|h(e^{iu\cdot S_n\Phi_0}\psi\circ\bar T_0^n)| \\ &\le C|\lambda|^{-n}|h|_w\big(|e^{iu\cdot S_n\Phi_0}\psi\circ\bar T_0^n|_\infty + C^{(p)}_{\bar T_0^{-n}\mathcal{W}^s}(e^{iu\cdot S_n\Phi_0}\cdot\psi\circ\bar T_0^n)\big), \end{aligned} \tag{56}$$

where $C^{(p)}_{\bar{T}_0^{-n}\mathcal{W}^s}(\cdot)$ denotes the Hölder constant of exponent p measured along elements of $\bar{T}_0^{-n}\mathcal{W}^s$. Since $|e^{iu\cdot S_n\Phi_0}| = 1$ and $S_n\Phi_0$ is constant on each element of $\bar{T}_0^{-n}\mathcal{W}^s$, we have

$$\begin{aligned} &C^{(p)}_{\bar{T}_0^{-n}\mathcal{W}^s}(e^{iu\cdot S_n\Phi_0}\cdot\psi\circ\bar{T}_0^n)\\ &\quad\le |e^{iu\cdot S_n\Phi_0}|_\infty C^{(p)}_{\bar{T}_0^{-n}\mathcal{W}^s}(\psi\circ\bar{T}_0^n) + |\psi\circ\bar{T}_0^n|_\infty C^{(p)}_{\bar{T}_0^{-n}\mathcal{W}^s}(e^{iu\cdot S_n\Phi_0})\\ &\quad\le C\Lambda^{-pn}C^{(p)}_{\mathcal{W}^s}(\psi). \end{aligned}$$

Using this estimate in (56) and taking the limit as $n\to\infty$ yields $|h(\psi)| = 0$ if $|\lambda| > 1$ and $|h(\psi)| \le C|h|_w|\psi|_\infty$ for all $\psi\in\mathcal{C}^p(\mathcal{W}^s)$ if $|\lambda| = 1$. From this we conclude that the spectrum of $\mathcal{L}_{u,0}$ is always contained in the unit disk. Furthermore, when $|\lambda| = 1$, then h is a signed measure. For the remainder of the proof, we assume $|\lambda| = 1$.

Let $\mathbb{V}_{u,0}$ be the eigenspace of $\mathcal{L}_{u,0}$ corresponding to eigenvalue $\lambda_{u,0}$, and $\Pi_{u,0}$ the eigenprojection operator. Since we are assuming $\mathbb{V}_{u,0}$ is non-empty, Lemma 3.14 implies that $\mathcal{L}_{u,0}$ is quasi-compact with essential spectral radius bounded by $\tau < 1$. Moreover, Lemma 3.14 implies that $\|\mathcal{L}^n_{u,0}\|_{L(\mathcal{B},\mathcal{B})}$ remains bounded for all $n\ge 0$, so using [15, Lemma 5.1], we conclude that $\mathcal{L}_{u,0}$ has no Jordan blocks corresponding to its peripheral spectrum.

Using these facts, $\Pi_{u,0}$ has the representation

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^n\lambda^{-j}\mathcal{L}^j_{u,0} = \Pi_{u,0}.$$

In addition, for $f\in\mathcal{C}^1(\bar{M}_0)$, $\psi\in\mathcal{C}^p(\mathcal{W}^s)$,

$$\left|\Pi_{u,0}f(\psi)\right| = \left|\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^n\lambda^{-j}f((e^{iu\cdot S_j\Phi_0}\psi\circ\bar{T}_0^j)\right| \le |f|_\infty|\psi|_\infty.$$

Since $\Pi_{u,0}\mathcal{C}^1(\bar{M}_0)$ is dense in the finite dimensional space $\Pi_{u,0}\mathcal{B}$, therefore $\Pi_{u,0}\mathcal{C}^1(\bar{M}_0) = \Pi_{u,0}\mathcal{B} = \mathbb{V}_{u,0}$. So for $h\in\mathbb{V}_{u,0}$, there exists $f\in\mathcal{C}^1(\bar{M}_0)$ such that $\Pi_{u,0}f = h$. Now for each $\psi\in\mathcal{C}^p(\bar{M}_0)$,

$$|h(\psi)| = |\Pi_{u,0}f(\psi)| \le |f|_\infty\Pi_0 1(|\psi|) = |f|_\infty\bar{\mu}_0(|\psi|).$$

Thus h is absolutely continuous with respect to $\bar{\mu}_0$. For simplicity, we identify h and its density with respect to $\bar{\mu}_0$; then $h\in L^\infty(\bar{M}_0,\bar{\mu}_0)$. Now for any $\psi\in\mathcal{C}^p(\mathcal{W}^s)$, we have

$$\begin{aligned}\lambda\int_{\bar{M}_0}h\psi\,d\mu_0 &= \int_{\bar{M}_0}\mathcal{L}_0(e^{iu\cdot\Phi_0}h)\cdot\psi\,d\bar{\mu}_0\\ &= \int_{\bar{M}_0}(e^{iu\cdot\Phi_0}h)\circ\bar{T}_0^{-1}\cdot\psi\,d\bar{\mu}_0.\end{aligned}$$

Accordingly, $\lambda h = (e^{iu\cdot\Phi_0}h)\circ\bar{T}_0^{-1}$, $\bar{\mu}_0$-a.e. Or equivalently, we have $\lambda h\circ\bar{T}_0 = e^{iu\cdot\Phi_0}h$. Hence $\lambda^n h\circ\bar{T}_0^n = e^{iu\cdot S_n\Phi_0}h$.

Let G_λ be the closed multiplicative group generated by λ and let m_λ be the normalized Haar measure on G_λ. (G_λ is finite if λ is a root of unity; it is $\{z\in\mathbb{C} : |z| = 1\}$ otherwise.)

The dynamical system $(G_\lambda, m_\lambda, T_\lambda)$ is ergodic, where T_λ denotes multiplication by λ in G_λ. Due to [28], the dynamical system $(M_0 \times G_\lambda, \mu_0 \otimes m_\lambda, T_0 \times T_\lambda)$ in infinite measure is conservative and ergodic. But the function $H : M_0 \times G_\lambda \to \mathbb{C}$ defined as follows is $(T_0 \times T_\lambda)$-invariant:

$$\forall (\bar{x}, \ell, y) \in \bar{M}_0 \times \mathbb{Z}^2 \times G_\lambda, \quad H(\bar{x} + \ell, y) := yh(\bar{x})e^{-iu\cdot\ell}.$$

Indeed, for $\mu_0 \otimes m_\lambda$-a.e. $(\bar{x} + \ell, y) \in M_0 \times G_\lambda$,

$$\begin{aligned} H((T_0 \times T_\lambda)(\bar{x} + \ell, y)) &= H(\bar{T}_0(\bar{x}) + \ell + \Phi_0(\bar{x}), \lambda y) = \lambda y h(\bar{T}_0(\bar{x}))e^{-iu\cdot(\ell+\Phi_0(\bar{x}))} \\ &= ye^{-iu\cdot\ell}(\lambda h(\bar{T}_0(\bar{x}))e^{-iu\cdot\Phi_0(\bar{x})}) \\ &= ye^{-iu\cdot\ell}h(\bar{x}), \end{aligned}$$

due to our assumption on h. We conclude that H is a.e. equal to a constant, which implies that $u \in 2\pi\mathbb{Z}^2$, $\lambda = 1$, and h is $\bar{\mu}_0$-a.s. constant. □

Proposition C.2. *Given $\beta > 0$, there exists $C > 1$ and $\alpha \in (0, 1)$ such that*

$$\forall n \in \mathbb{N}^*, \quad \sup_{\beta \le |u| \le \pi} \|\mathcal{P}_u^n\|_{L(\widetilde{\mathcal{B}}, \widetilde{\mathcal{B}})} \le C\alpha^n .$$

Proof. Fix $\beta > 0$. Due to [1, Lemma 4.3], Lemma C.1, and the continuity in u provided by [17, Lemma 5.4] (see also Lemma 3.16 applied to $\mathcal{L}_{u,0}$ rather than P_u), we know that there exists $C > 1$ and $\alpha \in (0, 1)$ such that

$$\forall n \in \mathbb{N}^*, \quad \sup_{\beta \le |u| \le \pi} \|\mathcal{L}_{u,0}^n\|_{L(\mathcal{B}, \mathcal{B})} \le C\alpha^n .$$

Therefore, for every $f \in \widetilde{\mathcal{B}}$, we have

$$\begin{aligned} &\sup_{\underline{\omega} \in E^{\mathbb{N}}} \left\| \mathcal{P}_u^n f(x, \underline{\omega}) \right\|_{\mathcal{B}} \\ &= \sup_{\underline{\omega} \in E^{\mathbb{N}}} \left\| \int_{E^n} \mathcal{L}_{u,0}^n f(\cdot, (\tilde{\omega}, \underline{\omega}))\, d\eta^{\otimes n}(\tilde{\omega}) \right\|_{\mathcal{B}} \\ &\le \sup_{\underline{\omega} \in E^{\mathbb{N}}} \int_{E^n} \left\| \mathcal{L}_{u,0}^n f(\cdot, (\tilde{\omega}, \underline{\omega})) \right\|_{\mathcal{B}} d\eta^{\otimes n}(\tilde{\omega}) \\ &\le \sup_{\underline{\omega} \in E^{\mathbb{N}}} C\alpha^n \sup_{\underline{\omega}'} \left\| f(\cdot, \underline{\omega}') \right\|_{\mathcal{B}} \end{aligned}$$

where we used Lemma 3.7 to obtain the second line. Analogously,

$$\begin{aligned} &\sup_{\underline{\omega} \neq \underline{\omega}'} \frac{\left\| \mathcal{P}_u^n f(x, \underline{\omega}) - \mathcal{P}_u^n f(x, \underline{\omega}') \right\|_{\mathcal{B}}}{d(\underline{\omega}, \underline{\omega}')} \\ &= \sup_{\underline{\omega} \neq \underline{\omega}'} \frac{\left\| \int_{E^n} \mathcal{L}_{u,0}^n \left(f(\cdot, (\tilde{\omega}, \underline{\omega})) - f(\cdot, (\tilde{\omega}, \underline{\omega}')) \right) d\eta^{\otimes n}(\tilde{\omega}) \right\|_{\mathcal{B}}}{d(\underline{\omega}, \underline{\omega}')} \\ &\le \sup_{\underline{\omega} \neq \underline{\omega}'} \int_{E^n} \frac{\left\| \mathcal{L}_{u,0}^n \left(f(\cdot, (\tilde{\omega}, \underline{\omega})) - f(\cdot, (\tilde{\omega}, \underline{\omega}')) \right) \right\|_{\mathcal{B}}}{d(\underline{\omega}, \underline{\omega}')} d\eta^{\otimes n}(\tilde{\omega}) \end{aligned}$$

$$\leq C\alpha^n \varkappa^n \sup_{\underline{\omega}\neq\underline{\omega}'} \frac{\left\| f(\cdot,\underline{\omega}') - f(\cdot,\underline{\omega}') \right\|_{\mathcal{B}}}{d(\underline{\omega},\underline{\omega}')}.$$

We conclude by putting these two estimates together. □

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Communicated by C. Liverani