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ON THE MEASURE OF MAXIMAL ENTROPY FOR FINITE HORIZON SINAI BILLIARD MAPS

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The Sinai billiard map T on the two-torus, i.e., the periodic Lorentz gas, is a discontinuous map. Assuming finite horizon, we propose a definition h_* for the topological entropy of T. We prove that h_* is not smaller than the value given by the variational principle, and that it is equal to the definitions of Bowen using spanning or separating sets. Under a mild condition of sparse recurrence to the singularities, we get more: First, using a transfer operator acting on a space of anisotropic distributions, we construct an invariant probability measure μ_* of maximal entropy for T (i.e., $h_{\mu_*}(T) = h_*$), we show that μ_* has full support and is Bernoulli, and we prove that μ_* is the unique measure of maximal entropy and that it is different from the smooth invariant measure except if all nongrazing periodic orbits have multiplier equal to h_*. Second, h_* is equal to the Bowen–Pesin–Pitskel topological entropy of the restriction of T to a noncompact domain of continuity. Last, applying results of Lima and Matheus, as upgraded by Buzzi, the map T has at least Ce^{nh_*} periodic points of period n for all $n \in \mathbb{N}$.

1. Introduction

1.1. **Bowen–Margulis measures and measures of maximal entropy.** Half a century ago,[1] Margulis [Ma1] proved in his dissertation the following analogue of the prime number theorem for the closed geodesics Γ of a compact manifold of strictly negative (not necessarily constant) curvature: Let $h > 0$ be the topological entropy of the geodesic flow; then,

$$\#\{\Gamma \text{ such that } |\Gamma| \le L\} \sim_{L\to\infty} \frac{e^{hL}}{hL}\,. \tag{1.1}$$

(I.e., $\lim_{L\to\infty}(hLe^{-hL}\#\{\Gamma \text{ such that } |\Gamma| \le L\}) = 1$.) The main ingredient in the proof is an invariant probability measure for the flow, the Margulis (or Bowen–Margulis [Bo3]) measure μ_{top}. This measure—which coincides with volume in constant curvature, but not in general—is mixing (thus ergodic), and it can be written as a local product of its stable and unstable conditionals, where these conditional measures scale by $e^{\pm ht}$ under the action of the flow. These properties were essential to establish (1.1). The measure μ_{top} enjoys other remarkable properties, such as

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[1]See [Ma2] for the full english text.

equidistribution of closed geodesics. Finally, the measure $\mu_{\rm top}$ is the unique measure of maximal entropy of the flow, that is, the unique invariant measure with Kolmogorov entropy equal to the topological entropy of the flow.

These results were extended to more general smooth uniformly hyperbolic flows and diffeomorphisms, using the thermodynamic formalism of Bowen, Ruelle, and Sinai. In particular Parry–Pollicott [PaP] obtained a different proof of (1.1) using a dynamical zeta function. Later, based on Dolgopyat's [Do1] groundbreaking thesis (proving exponential mixing for the measure and giving a pole-free vertical strip for a zeta function), exponential error terms were obtained [PS1] for the counting asymptotics (1.1) in the case of surfaces or 1/4-pinched manifolds. Using [Do1, PS1], Stoyanov [St2] obtained exponential error terms for the closed orbits of a class of open planar convex billiards, which are smooth hyperbolic flows on their nonwandering set, a compact (fractal) invariant set. We refer to Sharp's survey in [Ma2] for more counting results in uniformly hyperbolic dynamics. We just mention here that, for some Axiom A flows with slower (nonexponential) mixing rates, it is possible [PS2] to get (weaker) error terms, of the form $\frac{e^{hL}}{hL}(1 + O(L^{-\delta}))$, for the asymptotics (1.1), by exploiting relevant operator bounds from [Do2] (corresponding to a resonance free domain for the transfer operator). This may be relevant for the Sinai billiards considered in the present work, as we do not expect them to mix exponentially fast for the measure of maximal entropy without additional assumptions.

Entropy is a fundamental invariant in dynamics and the study of measures of maximal entropy is a topic in its own right [Ka2]. Let us just mention here the discrete-time analogue of the counting theorem (1.1) which has been established in several situations (see also [Ka1] for more general results): Let $h > 0$ be the topological entropy of uniformly hyperbolic (Axiom A) diffeomorphism T, set $\mathrm{Fix}\, T^m = \{x : T^m(x) = x\}$; then Bowen showed [Bo1] that $\lim_{m\to\infty} \frac{1}{m} \log \#\mathrm{Fix}\, T^m = h$. In fact [Bo4], there is a constant $C > 0$ so that

$$Ce^{hm} \le \#\mathrm{Fix}\, T^m \le C^{-1}e^{hm} \qquad \forall m \ge 1 \,. \tag{1.2}$$

Uniqueness of the measure of maximal entropy has been extended to some geodesic flows in nonpositive curvature (i.e., weakening the hyperbolicity requirement). The breakthrough result of Knieper [Kn] for compact rank 1 manifolds has been recently given a new dynamical proof [B-T] (using Bowen's ideas as revisited by Climenhaga and Thompson). This is currently a very active topic; see, e.g., [CKW].

The present paper is devoted to the study of the measure of maximal entropy in a situation where uniform hyperbolicity holds, but the dynamics is not smooth: The singular set $\mathcal{S}_{\pm 1}$, i.e., those points where the map T (or the flow Φ) or its inverse are not C^1, is not empty. In this setting, the following integrability condition is crucial:

$$\int |\log d(x, \mathcal{S}_{\pm 1})| \, d\mu_{\rm top} < \infty \,. \tag{1.3}$$

Following Lima–Matheus [LM], we shall say that a measure μ satisfying the above integrability condition for a map T is T-adapted.

Condition (1.3) is prevalent in the rich literature about measures of maximal entropy for meromorphic maps of a compact Kähler manifold (see the survey [Fr], and, e.g., [DDG2] and the references therein) such as birational mappings. In this work, we are concerned with a different class of dynamics with singularities: the dispersing billiards introduced by Sinai [S] on the two-torus. A Sinai billiard on the torus is the periodic case of the planar Lorentz gas (1905) model for the motion of a single dilute electron in a metal. The scatterers (corresponding to the atoms of the metal) are assumed to be strictly convex, but they are not necessarily perfect discs. Such billiards have become foundational models in mathematical physics.

The Sinai billiard flow is continuous, but[2] not differentiable: the "grazing" orbits (those which are tangent to a scatterer) lead to singularities. Nevertheless, existence of a measure of maximal entropy for the billiard flow is granted, thanks to hyperbolicity. The topological entropy has been studied for the billiard flow [BFK]. However, uniqueness of the measure of maximal entropy, as well as mixing and the adapted condition (1.3) are not known. Since the transfer operator techniques we use are simpler to implement in the discrete-time case, we study in this paper the Sinai billiard map, which is the return map of the single point particle to the scatterers.

Sinai billiard maps preserve a smooth invariant measure μ_{SRB} which has been studied extensively: With respect to μ_{SRB}, the billiard is uniformly hyperbolic, ergodic, K-mixing, and Bernoulli [S, GO, SC, ChH]. The measure μ_{SRB} is T-adapted [KS]. Moreover, this measure enjoys exponential decay of correlations [Y] and a host of other limit theorems (see, e.g., [CM, Chapter 7] or [DZ1]). The billiard has many periodic orbits and thus many other ergodic invariant measures μ, but there are very few results regarding other invariant measures and they apply only to perturbations of μ_{SRB} [CWZ, DRZ]. Since the billiard map is discontinuous, the standard results [W] guaranteeing that the supremum of Kolmogorov entropy is attained and coincides with the topological entropy do not hold. It is natural to ask whether a measure of maximal entropy exists, and, in the affirmative, whether it is unique, ergodic, and mixing.

Another natural goal is to establish (1.2). Chernov asked (see [Gu, Problems 5 and 6]) whether a slightly weaker property than (1.2), namely

$$\lim_{m\to\infty} \frac{1}{m} \log \#\mathrm{Fix}\, T^m = h_{\text{top}}\,,$$

holds. (Chernov [Ch1] showed that $\liminf_{m\to\infty} \frac{1}{m} \log \#\mathrm{Fix}\, T^m \ge h_{\mu_{\text{SRB}}}$. For a related class of billiards, Stoyanov [St1] found finite constants C and H so that $\#\mathrm{Fix}\, T^m \le Ce^{Hm}$ for all $m \ge 1$.)

A detailed knowledge of the measure of maximal entropy, and the techniques developed to obtain this information, could potentially allow us not only to establish (1.2) for the billiard map, but also eventually to prove a prime number asymptotic of the form (1.1) for the billiard flow. Although lifting a measure of maximal entropy for the map should not directly give a measure of maximal entropy for the flow, we believe that the techniques and results of the present paper will be instrumental in understanding the measure of maximal entropy of the billiard flow.

We list our results in Section 1.2. In a nutshell, for all finite horizon planar Sinai billiards T satisfying a (mild) condition of "sparse recurrence" to the singular set, we construct a measure of maximal entropy, we show that it is unique, mixing (even Bernoulli), that it has full support, and that it is T-adapted. Our results combined with those of Lima–Matheus [LM] and a very recent preprint of Buzzi [Bu] give $C > 0$ such that the lower bound in (1.2) holds.

[2] In contrast, open billiards in the plane which satisfy a noneclipsing condition do not have any singularities on their nonwandering set, so that they fit in the Axiom A category [St2].

Finally, we mention that our technique for constructing and studying the invariant measure, which uses transfer operators but avoids coding, is reminiscent both of the construction of Margulis [Ma2] and the techniques of "laminar currents" introduced by Dujardin for birational mappings [Du] (see also [DDG2]).

1.2. **Summary of main results.** A Sinai billiard table Q on the two-torus $\mathbb{T}^2$ is a set $Q = \mathbb{T}^2 \setminus B$, with $B = \bigcup_{i=1}^{D} B_i$ for some finite number $D \geq 1$ of pairwise disjoint closed domains B_i with C^3 boundaries having strictly positive curvature (in particular, the domains are strictly convex). The sets B_i are called scatterers; see Figure 2 for some common examples. The billiard flow is the motion of a point particle traveling in Q at unit speed and undergoing elastic (i.e., specular) reflections at the boundary of the scatterers. (By definition, at a tangential—also called grazing—collision, the reflection does not change the direction of the particle.) This is also called a periodic Lorentz gas. As mentioned above, a key feature is that, although the billiard flow is continuous if one identifies outgoing and incoming angles, the tangential collisions give rise to singularities in the derivative [CM].

We shall be concerned with the associated billiard map T, defined to be the first collision map on the boundary of Q. Grazing collisions cause discontinuities in the billiard map $T : M \to M$. We assume, as in [Y], that the billiard table Q has *finite horizon* in the sense that the billiard flow on Q does not have any trajectories making only tangential collisions.

The first step is to find a suitable notion of topological entropy h_* for the discontinuous map T.

Let $M' \subset M$ be the (T-invariant but not compact) set of points whose future and past orbits are never grazing. By definition, T is continuous on M'. The (Bowen–Pesin–Pitskel) topological entropy $h_{\mathrm{top}}(F|_Z)$ can be defined for a map F on a noncompact set of continuity Z (see, e.g., [Bo2] and [Pes, §11 and App. II]). Chernov [Ch1] studied the topological entropy for a class of billiard maps including those of the present paper. In particular, he gave [Ch1, Thm 2.2] a countable symbolic dynamics description of two T-invariant subsets of M' of full Lebesgue measure in M', expressing their topological entropy in terms of those of the associated Markov chains. The entropies found there are both bounded above by $h_{\mathrm{top}}(T|_{M'})$, although Chernov does not prove their equality.

These existing results are not convenient for our purposes, however, since we have no control a priori on the measure of $M \setminus M'$. This is why we introduce (Definition 2.1) an ad hoc definition h_* of the topological entropy for the billiard map T on the compact set M.

Our first main result (Theorem 2.3) says that the topological entropies of T defined by spanning sets and separating sets coincide with the topological entropy h_*, that h_* can also be obtained by using the refinements of partitions of M into maximal connected components on which T and T^{-1} are continuous, and that $h_* \geq \sup\{h_\mu(T) : \mu \text{ is a } T\text{-invariant Borel probability measure on } M\}$.

To state our other main results, we need to quantify the recurrence to the singular set: Fix an angle φ_0 close to $\pi/2$ and $n_0 \in \mathbb{N}$. We say that a collision is φ_0-grazing if its angle with the normal is larger than φ_0 in absolute value. Let $s_0 \in (0, 1]$ be the smallest number such that

(1.4) any orbit of length n_0 has at most $s_0 n_0$ collisions which are φ_0-grazing.

Our sparse recurrence condition is

$$\text{there exist } n_0 \text{ and } \varphi_0 \text{ such that } h_* > s_0 \log 2 \,. \tag{1.5}$$

(Due to the finite horizon condition, we can choose φ_0 and n_0 such that $s_0 < 1$. We refer to Section 2.4 for further discussion of the condition.)

Assuming (1.5), our second main result (Theorem 2.4) is that T admits a unique invariant Borel probability measure μ_* of maximal entropy $h_* = h_{\mu_*}(T)$. In addition, $\mu_*(O) > 0$ for any open set and μ_* is[3] Bernoulli. Finally, the absolutely continuous invariant measure μ_{SRB} may coincide with μ_* *only* if all nongrazing periodic orbits have the same Lyapunov exponent, equal to h_*. (No dispersing billiards which satisfy this condition are known. See also Remark 1.2.)

Our third result is (Theorem 2.5) that h_* coincides with the Bowen–Pesin–Pitskel entropy $h_{\mathrm{top}}(T|_{M'})$ (still assuming (1.5)).

Next, Theorem 2.6 contains a key technical[4] estimate on the measures of neighbourhoods of singularity sets, (2.2), used to prove Theorems 2.4 and 2.5 under the assumption (1.5). Theorem 2.6 also states that μ_* has no atoms, that it gives zero mass to any stable or unstable manifold and any singularity set, that μ_* is T-adapted (in the sense of (1.3)), and that μ_*-almost every $x \in M$ has stable and unstable manifolds of positive length.

Finally, we obtain a lower bound $\#\mathrm{Fix}\, T^m \geq Ce^{h_* m}$ on the cardinality of the set of periodic orbits (Corollary 2.7 and the comments thereafter) whenever (1.5) holds.

1.3. **The transfer operator—organisation of the paper.** Our tool to construct the measure of maximal entropy is a transfer operator $\mathcal{L} = \mathcal{L}_{\mathrm{top}}$ with $\mathcal{L}f = \frac{f\circ T^{-1}}{J^sT\circ T^{-1}}$ analogous to the transfer operator $\mathcal{L}_{\mathrm{SRB}}f = (f/|\mathrm{Det}\, DT|)\circ T^{-1}$ which has proved very successful [DZ1] to study the measure μ_{SRB}. An important difference is that our transfer operator, $\mathcal{L}f$, is weighted by an unbounded[5] function ($1/J^sT$, where the stable Jacobian J^sT may tend to zero near grazing orbits). Using "exact" stable leaves instead of admissible approximate stable leaves will allow us to get rid of the Jacobian after a leafwise change of variables—the same change of variables in [DZ1] for the transfer operator $\mathcal{L}_{\mathrm{SRB}}$ associated with μ_{SRB} left them with J^sT, allowing countable sums over homogeneity layers to control distortion, and thus working with a Banach space giving a spectral gap and exponential mixing. In the present work, we relinquish the homogeneity layers to avoid unbounded sums (see, e.g., the logarithm needed to obtain the growth Lemma 5.1) and obtain a bounded operator, with spectral radius e^{h_*}. The price to pay is that we do not have the distortion control needed for Hölder-type moduli of continuity in the Banach norms of our weak and strong spaces $\mathcal{B} \subset \mathcal{B}_w$. The weaker modulus of continuity than in [DZ1] does not yield a spectral gap. We thus do not claim exponential mixing properties for the measure of maximal entropy μ_* constructed (in the spirit of the work of Gouëzel–Liverani [GL] for Axiom A diffeomorphisms) by combining right and left maximal eigenvectors $\mathcal{L}\nu = e^{h_*}\nu$ and $\mathcal{L}^*\tilde{\nu} = e^{h_*}\tilde{\nu}$ of the transfer operator.

[3]Recall that Bernoulli implies K-mixing, which implies strong mixing, which implies ergodic. In practice, we first show K-mixing and then bootstrap to Bernoulli.

[4]This estimate implies that almost every point approaches the singularity sets more slowly than any exponential rate (7.9); see, e.g., [LM] for an application of such rates of approach.

[5]The naive idea to introduce a bounded cutoff in the weight does not seem to work.

The paper is organised as follows: In Section 2, we give formal statements of our main results. Section 3 contains the proof of Theorem 2.3 about equivalent formulations of h_*. In Section 4, we define our Banach spaces $\mathcal{B}$ and $\mathcal{B}_w$ of anisotropic distributions, and we state the "Lasota–Yorke"-type estimates on our transfer operator $\mathcal{L}$. Section 5 contains key combinatorial growth lemmas, controlling the growth in complexity of the iterates of a stable curve. It also contains the definition of Cantor rectangles (Section 5.3). We next prove the "Lasota–Yorke" Proposition 4.7, the compact embedding of $\mathcal{B}$ in $\mathcal{B}_w$, and show that the spectral radius of $\mathcal{L}$ is equal to e^{h_*} in Section 6. The invariant probability measure μ_* is constructed in Section 7.1 by combining a right and left eigenvector (ν and $\tilde{\nu}$) of $\mathcal{L}$. Section 7.1 contains the proof of Theorem 2.6 about the measure of singular sets. Section 7.3 contains a key result of absolute continuity of the unstable foliation with respect to μ_* as well as the proof that μ_* has full support, exploiting ν-almost everywhere positive length of unstable manifolds from Section 7.2. We establish upper and lower bounds on the μ_*-measure of dynamical Bowen balls in Section 7.4, deducing from them a necessary condition for μ_{SRB} and μ_* to coincide. Using the absolute continuity from Section 7.3, we show in Section 7.5 that μ_* is K-mixing. In this section we also use the upper bounds on Bowen balls to see that μ_* is a measure of maximal entropy and prove the Bowen–Pesin–Pitskel Theorem 2.5. We deduce the Bernoulli property from K-mixing and hyperbolicity in Section 7.6, adapting[6] [ChH]. Finally, we show uniqueness in Section 7.7.

Our Hopf-argument proof of K-mixing requires showing absolute continuity of the unstable foliation for μ_*, a new result of independent interest, which is the content of Corollary 7.9. The "fragmentation" lemmas from Section 5, needed to get the lower bound on the spectral radius of the transfer operator, are also new. They imply, in particular, that the length $|T^{-n}W|$ of every local stable manifold W grows at the same exponential rate e^{nh_*} (Corollary 5.10).

We conclude this introduction with two remarks on the finite horizon condition.

Remark 1.1 (Finite horizon and collision time τ). For $x \in M$, let $\tau(x)$ denote the distance from x to $T(x)$. If τ is unbounded, i.e., if there is a collision-free trajectory for the flow, then there must be a flow trajectory making only tangential collisions. The reverse implication, however, is not true. Our[7] finite horizon assumption therefore implies that τ is bounded on M. Assuming only that τ is bounded is sometimes also called finite horizon [CM]. (If the scatterers B_i are viewed as open, then tangential collisions simply do not occur and the two definitions of finite horizon are reconciled.)

Remark 1.2 (Billiard with infinite horizon). Chernov [Ch1, §3.4] proved that the topological entropy of the Sinai billiard map T restricted to the noncompact set M' is infinite if the horizon is *not* finite, and together with Troubetskoy [CT] constructed invariant measures with infinite metric entropy for this map. Since the entropy of the smooth measure μ_{SRB} is finite, the measure μ_{SRB} does not maximise entropy for infinite horizon billiards. Chernov conjectured [Ch1, Remark 3.3] that this property holds for more general billiards, in particular for Sinai billiards with finite horizon.

[6] As pointed out to us by Y. Lima, we could instead apply [Sa1, Thm 3.1] to the lift of μ_* to the symbolic space constructed in [LM].

[7] We shall need the slightly stronger version, e.g., in Lemmas 3.4 and 3.5.

2. Full statement of main results

In this section, we formulate definitions of topological entropy for the billiard map that we shall prove are equivalent before stating formally all main results of this paper.

2.1. Definitions of topological entropy h_* of T on M.

We first introduce notation: Adopting the standard coordinates $x = (r, \varphi)$, for T, where r denotes arclength along ∂B_i and φ is the angle the post-collision trajectory makes with the normal to ∂B_i, the phase space of the map is the compact metric space M given by the disjoint union of cylinders,

$$M := \partial Q \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] = \bigcup_{i=1}^{D} \partial B_i \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] .$$

We denote each connected component of M by $M_i = \partial B_i \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. In the coordinates (r, φ), the billiard map $T : M \to M$ preserves [CM, §2.12] the smooth invariant measure[8] defined by $\mu_{\mathrm{SRB}} = (2|\partial Q|)^{-1} \cos\varphi \, dr d\varphi$.

We discuss next the discontinuity set of T: Letting $\mathcal{S}_0 = \{(r, \varphi) \in M : \varphi = \pm\pi/2\}$ denote the set of tangential collisions, then for each nonzero $n \in \mathbb{N}$, the set

$$\mathcal{S}_{\pm n} = \bigcup_{i=0}^{n} T^{\mp i} \mathcal{S}_0$$

is the singularity set for $T^{\pm n}$. In this notation, the T-invariant (noncompact) set M' of continuity of T is $M' = M \setminus \bigcup_{n \in \mathbb{Z}} \mathcal{S}_n$.

For $k, n \geq 0$, let $\mathcal{M}_{-k}^n$ denote the partition of $M \setminus (\mathcal{S}_{-k} \cup \mathcal{S}_n)$ into its maximal connected components. Note that all elements of $\mathcal{M}_{-k}^n$ are open sets. The cardinality of the sets $\mathcal{M}_0^n$ will play a key role in the estimates on the transfer operator in Section 4. We formulate the following definition with the idea that the growth rate of elements in $\mathcal{M}_{-k}^n$ should define the topological entropy of T, by analogy with the definition using a generating open cover (for continuous maps on compact spaces).

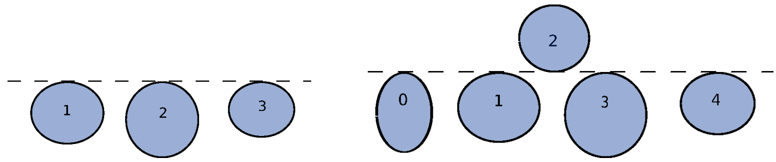
Definition 2.1. $h_* = h_*(T) := \limsup_{n\to\infty} \frac{1}{n} \log \#\mathcal{M}_0^n$.

The fact that the limsup defining h_* is a limit, as well as several equivalent characterizations involving the cardinality of related dynamical partitions or a variational principle, are proved in Theorem 2.3 (see Lemma 3.3).

Remark 2.2 ($h_*(T) = h_*(T^{-1})$). If $A \in \mathcal{M}_0^n$, then $T^n A \in \mathcal{M}_{-n}^0$ since $T^n \mathcal{S}_n = \mathcal{S}_{-n}$. Thus $\#\mathcal{M}_0^n = \#\mathcal{M}_{-n}^0$, and so $h_*(T) = h_*(T^{-1})$.

It will be convenient to express h_* in terms of the rate of growth of the cardinality of the refinements of a fixed partition, i.e., $\bigvee_0^n T^{-i}\mathcal{P}$, for some fixed $\mathcal{P}$. Although $\mathcal{M}_0^n$ is not immediately of this form, we will show that in fact h_* can be expressed in this fashion, obtaining along the way subadditivity of $\log \#\mathcal{M}_0^n$. For this, we introduce two sequences of partitions. Let $\mathcal{P}$ denote the partition of M into maximal connected sets on which T and T^{-1} are continuous. Define $\mathcal{P}_{-k}^n = \bigvee_{i=-k}^n T^{-i}\mathcal{P}$. Then, $n \mapsto \log \#\mathcal{P}_{-k}^n$ is subadditive for any fixed k, in particular the limit $\lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{P}_0^n$ exists.

[8] All measures in this work are finite Borel measures.



(a) (b)

FIGURE 1. (a) The billiard trajectory corresponding to the dotted line has symbolic itinerary 123, but is an isolated point in $\mathcal{P}_0^1$. Any open set with symbolic itinerary 12 cannot land on scatterer 3 (unless it first wraps around the torus). (b) The billiard trajectory corresponding to the dotted line and having symbolic trajectory 1234 is not isolated since it belongs to the boundary of an open set with the same symbolic sequence; however, the addition of scatterer 0 on the common tangency forces the point with symbolic trajectory 01234 to be isolated.

The interior of each element of $\mathcal{P}$ corresponds to precisely one element of $\mathcal{M}_{-1}^1$; however, its refinements $\mathcal{P}_{-k}^n$ may also contain some isolated points if three or more scatterers have a common tangential trajectory. Figure 1 displays two such examples (the pictures are local: we have not represented all discs needed to ensure finite horizon).

Let now $\mathring{\mathcal{P}}_{-k}^n$ denote the collection of interiors of elements of $\mathcal{P}_{-k}^n$. Then $\mathcal{P}_{-k}^n$ forms a finite partition of M, while $\mathring{\mathcal{P}}_{-k}^n$ forms a partition of $M \setminus (\mathcal{S}_{-k-1} \cup \mathcal{S}_{n+1})$ into open, connected sets. (We will show in Lemma 3.3 that $\mathring{\mathcal{P}}_{-k}^n = \mathcal{M}_{-k-1}^{n+1}$.)

Finally, we recall the classical Bowen [W] definitions of topological entropy for continuous maps using ε-separated and ε-spanning sets. Define the dynamical distance

$$d_n(x,y) := \max_{0\le i\le n} d(T^i x, T^i y)\,, \tag{2.1}$$

where $d(x,y)$ is the Euclidean metric on each M_i, and $d(x,y) = 10D\cdot\max_i \operatorname{diam}(M_i)$ if x and y belong to different M_i (this definition ensures we get a compact set), where D is the number of scatterers.

As usual, given $\varepsilon > 0$, $n \in \mathbb{N}$, we call E an (n,ε)-*separated set* if for all $x, y \in E$ such that $x \neq y$, we have $d_n(x,y) > \varepsilon$. We call F an (n,ε)-*spanning set* if for all $x \in M$, there exists $y \in F$ such that $d_n(x,y) \le \varepsilon$.

Let $r_n(\varepsilon)$ denote the maximal cardinality of any (n,ε)-separated set, and let $s_n(\varepsilon)$ denote the minimal cardinality of any (n,ε)-spanning set. We recall two related quantities:

$$h_{\rm sep} = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log r_n(\varepsilon)\,, \qquad h_{\rm span} = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log s_n(\varepsilon)\,.$$

Although $\lim_{n\to\infty}\frac{1}{n}\log\#\mathcal{P}_0^n$, $h_{\rm sep}$, and $h_{\rm span}$ are typically used for continuous maps, our first main result is that these naively defined quantities for the discontinuous billiard map T all agree with h_*, and they give an upper bound for the Kolmogorov entropy as follows.

Theorem 2.3 (Topological entropy of the billiard)**.** *The* $\limsup$ *in Definition* 2.1 *is a limit, and in fact the sequence* $\log\#\mathcal{M}_0^n$ *is subadditive. In addition, we have:*

(1) $h_* = \lim_{n\to\infty}\frac{1}{n}\log\#\mathcal{P}_0^n$;

(2) *the sequence* $\frac{1}{n}\log \#\mathring{\mathcal{P}}_0^n$ *also converges to* h_* *as* $n \to \infty$*;*
(3) $h_* = h_{\text{sep}}$ *and* $h_* = h_{\text{span}}$*;*
(4) $h_* \geq \sup\{h_\mu(T) : \mu$ *is a* T*-invariant Borel probability measure on* $M\}$*.*

The above theorem will follow from Lemmas 3.3, 3.4, 3.5, and 3.6.
(We shall obtain in Lemma 5.6 a superadditive property for $\log \#\mathcal{M}_0^n$.)

2.2. **The measure μ_* of maximal entropy.** Our next main result, existence and the Bernoulli property of a unique measure of maximal entropy, will be proved in Section 7, using the transfer operator $\mathcal{L}$ studied in Section 4.

Theorem 2.4 (Measure of maximal entropy for the billiard)**.** *If* $h_* > s_0 \log 2$*, then*

$$h_* = \max\{h_\mu(T) : \mu \text{ is a } T\text{-invariant Borel probability measure on } M\}\,.$$

Moreover, there exists a unique T*-invariant Borel probability measure* μ_* *such that* $h_* = h_{\mu_*}(T)$*. In addition,* μ_* *is Bernoulli and* $\mu_*(O) > 0$ *for all open sets* O*. Finally, if there exists a nongrazing periodic point* x *of period* p *such that* $\frac{1}{p}\log|\det(DT^{-p}|_{E^s}(x))| \neq h_*$ *then* $\mu_* \neq \mu_{\text{SRB}}$*.*

The above theorem follows from Propositions 7.11, 7.13, and 7.19, Corollary 7.17, and Proposition 7.21. (J. De Simoi has told us that [DKL, §4.4] the (possibly empty) set of planar billiard tables satisfying a noneclipsing condition (i.e., open billiards) for which $\frac{1}{p}\log|\det(DT^{-p}|_{E^s}(x))| = h_*$ for all p and all nongrazing p-periodic points x has infinite codimension.)

The existence of μ_* with $h_{\mu_*}(T) = h_*$, together with item (1) of Theorem 2.3 expressing h_* as a limit involving the refinements of a single partition, will allow us to interpret h_* as the Bowen–Pesin–Pitskel topological entropy of $T|_{M'}$ in Section 7.5:

Theorem 2.5 (h_* and Bowen–Pesin–Pitskel entropy)**.** *If* $h_* > s_0 \log 2$*, then* $h_* = h_{\text{top}}(T|_{M'})$*.*

2.3. **A key estimate on neighbourhood of singularities.** We call a smooth curve in M a *stable curve* if its tangent vector at each point lies in the stable cone, and define an *unstable curve* similarly. As mentioned in Section 1, the sets $\mathcal{S}_n$ are the singularity sets for T^n, $n \in \mathbb{Z} \setminus \{0\}$. The set $\mathcal{S}_n \setminus \mathcal{S}_0$ comprises [CM] a finite union of stable curves for $n > 0$ and a finite union of unstable curves for $n < 0$. For any $\epsilon > 0$ and any set $A \subset M$, we denote by $\mathcal{N}_\epsilon(A) = \{x \in M \mid d(x, A) < \epsilon\}$ the ϵ-neighbourhood of A.

The following key result gives information on the measure of neighbourhoods of the singularity sets (it is used in the proofs of Theorem 2.4 and, indirectly, Theorem 2.5).

Theorem 2.6 (Measure of neighbourhoods of singularity sets)**.** *Assume that* $h_* > s_0 \log 2$ *and let* μ_* *be the ergodic measure of maximal entropy constructed in* (7.1)*. The measure* μ_* *has no atoms, and for any local stable or unstable manifold* W *we have* $\mu_*(W) = 0$*. In addition,* $\mu_*(\mathcal{S}_n) = 0$ *for any* $n \in \mathbb{Z}$*.*

More precisely, for any $\gamma > 0$ *so that* $2^{s_0\gamma} < e^{h_*}$ *and* $n \in \mathbb{Z}$*, there exist* C *and* $\hat{C}_n < \infty$ *such that for all* $\varepsilon > 0$ *and any smooth curve* S *uniformly transverse to the stable cone,*

$$\mu_*(\mathcal{N}_\epsilon(S)) < \frac{C}{|\log \epsilon|^\gamma}\,, \quad \mu_*(\mathcal{N}_\epsilon(\mathcal{S}_n)) < \frac{\hat{C}_n}{|\log \epsilon|^\gamma}\,. \tag{2.2}$$

Since $h_* > s_0 \log 2$ we may take $\gamma > 1$, and we have

$$\int |\log d(x, \mathcal{S}_{\pm 1})| \, d\mu_* < \infty \,,$$

(i.e., μ_* is T-adapted [LM]*), and μ_*-almost every $x \in M$ has stable and unstable manifolds of positive length.*

Theorem 2.6 follows from Lemma 7.3 and Corollary 7.4.

This theorem is especially of interest for $\gamma > 1$, since in this case it implies that μ_*-almost every point does not approach the singularity sets faster than some exponential; see (7.9). In addition, it allows us to give a lower bound on the number of periodic orbits: For $m \geq 1$, let $\mathrm{Fix}\, T^m$ denote the set $\{x \in M \mid T^m(x) = x\}$. By [BSC] and [Ch1, Cor 2.4], there exist $h_C \geq h_{\mu_{\mathrm{SRB}}}(T) > 0$ and $C > 0$ with $\#\mathrm{Fix}\, T^m \geq C e^{h_C m}$ for all m. Our result is that (possibly up to a period p) we can take $h_C = h_*$ if $h_* > s_0 \log 2$.

Corollary 2.7 (Counting periodic orbits)**.** *If $h_* > s_0 \log 2$, then there exist $C > 0$ and $p \geq 1$ such that $\#\mathrm{Fix}\, T^{pm} \geq C e^{h_* pm}$ for all $m \geq 1$.*

Proof. The corollary follows from the work of Lima–Matheus [LM], which in turn relies on work of Gurevič [G1, G2] (see the proof of [Sa2, Thm 1.1]). We recall briefly the setup of [LM, Theorem 1.3]: Under assumptions (A1)-(A6), the authors construct for any T-adapted measure μ with positive Lyapunov exponent, a countable Markov partition that allows them to code a full μ-measure set of points. Once this partition has been constructed, [LM, Corollary 1.2] implies the above lower bound on periodic orbits for T with rate given by $h_\mu(T)$.

[LM, Theorem 1.3] applies to our measure of maximal entropy μ_* since it is T-adapted with positive Lyapunov exponent. In addition, conditions (A1)-(A4) of [LM] are requirements on the smoothness of the exponential map on the manifold, which are trivially satisfied in our setting since M is a finite union of cylinders and $\mathcal{S}_{\pm 1}$ is a finite union of curves. Finally, conditions (A5) and (A6) are requirements on the rate at which $\|DT\|$ and $\|D^2 T\|$ grow as one approaches $\mathcal{S}_1$. These are standard estimates for billiards and in the notation of [LM], if we choose $a = 2$, then conditions (A5) and (A6) hold, choosing there $\beta = 1/4$ and any $b > 1$. □

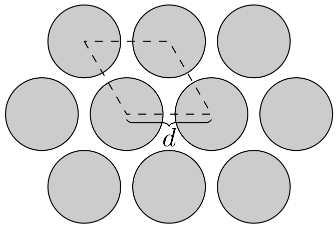
After the first version of our paper was submitted, J. Buzzi [Bu, v2] obtained results allowing one to bootstrap from Corollary 2.7 by exploiting the fact that T is topologically mixing, to show that if $h_* > s_0 \log 2$, then there exists $C > 0$ so that $\#\mathrm{Fix}\, T^m \geq C e^{h_* m}$ for all $m \geq 1$ [Bu, Theorem 1.5].

2.4. **On condition** (1.5) **of sparse recurrence to singularities.** We are not aware of any dispersing billiard on the torus for which the bound $h_* > s_0 \log 2$ from (1.5) fails. Let us start by mentioning that if there are no triple tangencies on the table—a generic condition—then $s_0 \leq 2/3$. To discuss this condition further, our starting point is claim (4) of Theorem 2.3, which implies by the Pesin entropy formula [KS],

$$h_* \geq h_{\mu_{\mathrm{SRB}}}(T) = \int \log J^u T \, d\mu_{\mathrm{SRB}} \,. \tag{2.3}$$

Thus it suffices to check $\chi^+_{\mu_{\mathrm{SRB}}} > s_0 \log 2$ in order to verify (1.5), where $\chi^+_{\mu_{\mathrm{SRB}}} = \int \log J^u T \, d\mu_{\mathrm{SRB}}$ is the positive Lyapunov exponent of μ_{SRB}.

First, we mention two numerical case studies from the literature.



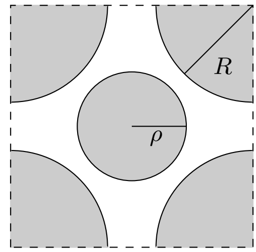


FIGURE 2. (a) The Sinai billiard on a triangular lattice studied in [BG] with angle $\pi/3$, scatterer of radius 1, and distance d between the centers of adjacent scatterers. (b) The Sinai billiard on a square lattice with scatterers of radii $\rho < R$ studied in [Ga]. The boundary of a single cell is indicated by dashed lines in both tables.

Baras and Gaspard [BG] studied the Sinai billiard corresponding to the periodic Lorentz gas with discs of radius 1 centered in a triangular lattice (Figure 2(a)). The distance d between points on the lattice is varied from $d = 2$ (when the scatterers touch) to $d = 4/\sqrt{3}$ (when the horizon becomes infinite). All computed values of the Lyapunov exponent[9] are greater than $\frac{2}{3}\log 2$ [BG, Table 1]. (Notably $\chi^+_{\mu_{\mathrm{SRB}}}$ does not decay as the minimum free flight-time $\tau_{\min}$ tends to zero.) For these billiard tables, since every segment with a double tangency is followed by *two* nontangential collisions, one can choose φ_0 and n_0 so that (1.4) is satisfied with $s_0 = 1/2$. Thus (1.5) holds for all computed values in this family of tables.

Garrido [Ga] studied the Sinai billiard corresponding to the periodic Lorentz gas with two scatterers of radii $\rho < R$ on the unit square lattice (Figure 2(b)). Setting $R = 0.4$, [Ga, Figure 6] computed $\chi^+_{\mu_{\mathrm{SRB}}}$ numerically for about 20 values of ρ ranging from $\rho = 0.1$ (when the scatterers touch) to $\rho = \frac{\sqrt{2}}{2} - 0.4$ (when the horizon becomes infinite). All computed values of $\chi^+_{\mu_{\mathrm{SRB}}}$ are greater than $0.8 > \log 2$ so that (1.5) holds for all such tables. (For these tables as well, one can in fact choose $s_0 = 1/2$.)

Secondly, for the family of tables studied by Garrido, we obtain an open set of pairs of parameters (ρ, R) satisfying (1.5) as follows. To ensure finite horizon and disjoint scatterers, the constraints are

$$\frac{1}{2} < \rho + R < \frac{\sqrt{2}}{2}, \quad \rho < R < \frac{1}{2}, \quad \text{and} \quad R > \frac{\sqrt{2}}{4}.$$

Since μ_{SRB} is a probability measure, denoting by $\mathcal{K}_{\min} > 0$ the minimum curvature and using a well-known [CM, eqs. (4.10) and (4.15)] bound for the unstable hyperbolicity exponent (see also [CM, Remark 3.47]) for the relation to entropy), we have,

$$\chi^+_{\mu_{\mathrm{SRB}}} \geq \log(1 + 2\tau_{\min}\mathcal{K}_{\min}).$$

[9]The reported values in [BG] are for the billiard flow. These can be converted to Lyapunov exponents for the map via the well-known formula $\chi^+_{\mathrm{map}} = \bar{\tau}\chi^+_{\mathrm{flow}}$, where $\bar{\tau}$ is the average free flight-time. For this billiard table, $\bar{\tau} = \frac{d^2\sqrt{3}}{4} - \frac{\pi}{2}$, using [CM, eq. (2.32)].

We find that this is greater than $(1/2)\log 2$ whenever $\tau_{\min}\mathcal{K}_{\min} > \frac{\sqrt{2}-1}{2}$. If $R > 1 - \frac{\sqrt{2}}{2} + \rho$, then $\tau_{\min} = 1 - 2R$, and $\mathcal{K}_{\min} = R^{-1}$, so that $\tau_{\min}\mathcal{K}_{\min} = R^{-1} - 2$. Thus if $R < \frac{2}{3+\sqrt{2}}$, then (1.5) holds. On the other hand if $R < 1 - \frac{\sqrt{2}}{2} + \rho$, then $\tau_{\min} = \frac{\sqrt{2}}{2} - R - \rho$ so that $\tau_{\min}\mathcal{K}_{\min} = \frac{\sqrt{2}}{2R} - 1 - \frac{\rho}{R}$. Thus (1.5) holds whenever $R < \frac{\sqrt{2}-2\rho}{1+\sqrt{2}}$. The union of these two sets is defined by the inequalities

$$\frac{\sqrt{2}}{4} < R < \frac{2}{3+\sqrt{2}}, \quad R < \frac{\sqrt{2}-2\rho}{1+\sqrt{2}}, \quad \text{and} \quad \rho + R > \frac{1}{2}.$$

We remark that this region intersects the line $R + \sqrt{2}\rho = \frac{\sqrt{2}}{2}$. This line corresponds to the set of tables which admit a period 8 orbit making 4 grazing collisions around the disc of radius ρ and 4 collisions at angle $\pi/4$ with the disc of radius R. For these tables, $s_0 = 1/2$, and we see that (1.5) admits tables with grazing periodic orbits.

Thirdly, it seems true that if there are no periodic orbits making at least one grazing collision, then for any $\epsilon > 0$ the constants n_0 and φ_0 can be chosen to ensure $s_0 < \epsilon$. This has led P.-A. Guihéneuf to conjecture that there exists a natural topology[10] on the set of billiard tables so that for any $\epsilon > 0$ the set of tables for which $s_0 < \epsilon$ is generic (that is, open and dense). This would immediately imply that our condition (1.5) is generically satisfied.

Finally, we mention that Diller, Dujardin, and Guedj [DDG1, Example 4.6] construct a birational map F having a measure of maximal entropy which is mixing but not F-adapted, by showing that F violates the Bedford–Diller [BD] recurrence condition. The Bedford–Diller condition does not have a natural analogue in our setting since double tangencies always occur. One could interpret our sparse recurrence condition $h_* > s_0 \log 2$ as its replacement. It would be interesting to find billiards for which $h_* \le s_0 \log 2$ and which admit a non-T-adapted measure of maximal entropy.

3. Proof of Theorem 2.3 (equivalent formulations of h_*)

In this section, we shall prove Theorem 2.3 through Lemmas 3.3, 3.4, 3.5, and 3.6.

We first recall some facts about the uniform hyperbolicity of T to introduce notation which will be used throughout. It is well known [CM] that T is uniformly hyperbolic in the following sense: First, the cones $C^u = \{(dr, d\varphi) \in \mathbb{R}^2 : \mathcal{K}_{\min} \le d\varphi/dr \le \mathcal{K}_{\max} + 1/\tau_{\min}\}$ and $C^s = \{(dr, d\varphi) \in \mathbb{R}^2 : -\mathcal{K}_{\min} \ge d\varphi/dr \ge -\mathcal{K}_{\max} - 1/\tau_{\min}\}$, are strictly invariant under DT and DT^{-1}, respectively, whenever these derivatives exist. Here, $\mathcal{K}_{\max}$ represent the maximum curvature of the scatterer boundaries and $\tau_{\max} < \infty$ is the largest free flight-time between collisions. Second, recalling that $\mathcal{K}_{\min} > 0$, $\tau_{\min} > 0$ denote the minimum curvature and the minimum free flight-time, and setting

$$\Lambda := 1 + 2\mathcal{K}_{\min}\tau_{\min}\,,$$

there exists $C_1 > 0$ such that for all $n \ge 0$,

$$\|DT^n(x)v\| \ge C_1\Lambda^n\|v\| \ \forall v \in C^u\,, \quad \|DT^{-n}(x)v\| \ge C_1\Lambda^n\|v\| \ \forall v \in C^s \tag{3.1}$$

[10] For a fixed number of scatterers, a candidate is given by the distance defined in [DZ2, §2.2, §3.4, Remark 2.9(b)].

for all x for which $DT^n(x)$, or, respectively, $DT^{-n}(x)$, is defined, so that Λ is a lower bound[11] on the hyperbolicity constant of the map T.

3.1. **Preliminaries.** The following lemma provides important information regarding the structure of the partitions $\mathcal{P}_{-k}^{n}$, which we will use to make an explicit connection between $\mathcal{M}_{-k}^{n}$ and $\mathring{\mathcal{P}}_{-k}^{n}$ in Lemma 3.3.

Lemma 3.1. *The elements of $\mathcal{P}_{-k}^{n}$ are connected sets for all $k \geq 0$ and $n \geq 0$.*

Proof. The statement is true by definition for $\mathcal{P} = \mathcal{P}_0^0$. We will prove the general statement by induction on k and n using the fact that $\mathcal{P}_{-k}^{n+1} = \mathcal{P}_{-k}^{n} \bigvee T^{-1}\mathcal{P}_{-k}^{n}$, and $\mathcal{P}_{-k-1}^{n} = \mathcal{P}_{-k}^{n} \bigvee T\mathcal{P}_{-k}^{n}$.

Fix $k, n \geq 0$, and assume the elements of $\mathcal{P}_{-k}^{n}$ are connected sets. Let $A_1, A_2 \in \mathcal{P}_{-k}^{n}$. If $T^{-1}A_1 \cap A_2$ is empty or is an isolated point, then it is connected. So suppose $T^{-1}A_1 \cap A_2$ has nonempty interior.

Clearly, $T^{-1}A_1$ is connected since T^{-1} is continuous on elements of $\mathcal{P}_{-k}^{n}$ for all $k, n \geq 0$. Notice that the boundary of A_1 is comprised of finitely many smooth stable and unstable curves in $\mathcal{S}_{-k} \cup \mathcal{S}_n$, as well as possibly a subset of $\mathcal{S}_0$ ([CM, Prop 4.45 and Exercise 4.46]; see also [CM, Fig 4.17]). We shall refer to these as the *stable* and *unstable parts of the boundary* of A_1. Similar facts apply to the boundaries of A_2 and TA_1.

We consider whether a stable part of the boundary of $T^{-1}A_1$ can cross a stable part of the boundary of A_2, and create two or more connected components of $T^{-1}A_1 \cap A_2$. Call these two boundary components γ_1 and γ_2 and notice that such an occurrence would force γ_1 and γ_2 to intersect in at least two points.

We claim the following fact: If a stable curve $S_i \subset T^{-i}\mathcal{S}_0$ intersects $S_j \subset T^{-j}\mathcal{S}_0$ for $i < j$, then S_j must terminate on S_i. This is because $T^iS_i \subset \mathcal{S}_0$, while $T^iS_j \subset T^{i-j}\mathcal{S}_0$ is still a stable curve, terminating on $\mathcal{S}_0$. A similar property holds for unstable curves in $\mathcal{S}_{-i}$. and $\mathcal{S}_{-j}$.

The claim implies that γ_1 and γ_2 both belong to $T^{-j}\mathcal{S}_0$ for some $1 \leq j \leq n$. But when such curves intersect, again, one must terminate on the other (crossing would violate injectivity of T^{-1}).

A similar argument precludes the possibility that unstable parts of the boundary cross one another multiple times. It follows that the only intersections allowed are stable/unstable boundaries of $T^{-1}A_1$ terminating on corresponding stable/unstable boundaries of A_2, or transverse intersections between stable components of $\partial(T^{-1}A_1)$ and unstable components of ∂A_2, and vice versa. This last type of intersection cannot produce multiple connected components due to the *continuation of singularities*, which states that every stable curve in $\mathcal{S}_{-n} \setminus \mathcal{S}_0$ is part of a monotonic and piecewise smooth decreasing curve which terminates on $\mathcal{S}_0$ (see [CM, Prop 4.47]). A similar fact holds for unstable curves in $\mathcal{S}_n \setminus \mathcal{S}_0$. This implies that $T^{-1}A_1 \cap A_2$ is a connected set, and since A_1 and A_2 were arbitrary, that $\mathcal{P}_{-k}^{n+1}$ is comprised entirely of connected sets.

Similarly, considering $TA_1 \cap A_2$ proves that all elements of $\mathcal{P}_{-k-1}^{n}$ are connected. $\square$

[11]Therefore, $h_{\mu_{\text{SRB}}}(T) = \int \log J^uT \, d\mu_{\text{SRB}} > \log \Lambda$ and the bound $\log(1 + 2\mathcal{K}_{\min}\tau_{\min}) > s_0 \log 2$ implies (1.5), as in Section 2.4.

From the proof of Lemma 3.1, we can see that, aside from isolated points, elements of $\mathcal{P}_{-k}^n$ consist of connected cells which are roughly "convex" and have boundaries comprised of stable and unstable curves.

Lemma 3.2. *There exists $C > 0$, depending on the table Q, such that for any $k, n \in \mathbb{N}$, $\#\mathring{\mathcal{P}}_{-k}^n \le \#\mathcal{P}_{-k}^n \le \#\mathring{\mathcal{P}}_{-k}^n + C(n + k + 1)$.*

Proof. It is clear from the definition of $\mathring{\mathcal{P}}_{-k}^n$ and $\mathcal{P}_{-k}^n$ that

$$\#\mathcal{P}_{-k}^n = \#\mathring{\mathcal{P}}_{-k}^n + \#\{\text{isolated points}\}\,,$$

where the isolated points in $\mathcal{P}_k^n$ can be created by multiple tangencies aligning in a particular manner, as described above (see Figure 1). Thus the first inequality is trivial.

The set of isolated points created at each forward iterate is contained in $\mathcal{S}_0 \cap T^{-1}\mathcal{S}_0$, while the set of isolated points created at each backward iterate is contained in $\mathcal{S}_0 \cap T\mathcal{S}_0$. We proceed to estimate the cardinality of these sets.

Let r_0 be sufficiently small such that for any segment $S \subset \mathcal{S}_0$ of length r_0, the image TS comprises at most $\tau_{\max}/\tau_{\min}$ connected curves on which T^{-1} is smooth [CM, Sect. 5.10]. For each i, the number of points in $\partial B_i \cap \mathcal{S}_0 \cap T^{-1}\mathcal{S}_0$ is thus bounded by $2|\partial B_i|\tau_{\max}/(\tau_{\min} r_0)$, where the factor 2 comes from the top and bottom boundary of the cylinder. Summing over i, we have $\#(\mathcal{S}_0 \cap T^{-1}\mathcal{S}_0) \le 2|\partial Q|\tau_{\max}/(\tau_{\min} r_0)$. Due to reversibility, a similar estimate holds for $\#(\mathcal{S}_0 \cap T\mathcal{S}_0)$. Since this bound holds at each iterate, the second inequality holds with $C = \frac{2|\partial Q|\tau_{\max}}{\tau_{\min} r_0}$. $\square$

3.2. **Formulations of h_* involving $\mathcal{P}$ and $\mathring{\mathcal{P}}$.** The following lemma gives claims (1) and (2) of Theorem 2.3.

Lemma 3.3. *The following holds for every $k \ge 0$. We have $\mathring{\mathcal{P}}_{-k}^n = \mathcal{M}_{-k-1}^{n+1}$ for every $n \ge 0$. Moreover, the following limits exist and are equal to h_*:*

$$h_* = \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{M}_{-k}^n = \lim_{n\to\infty} \frac{1}{n} \log \#\mathring{\mathcal{P}}_{-k}^n = \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{P}_{-k}^n\,.$$

Finally, the sequence $n \mapsto \log \#\mathcal{M}_{-k}^n$ is subadditive.

Proof. First notice that by Lemma 3.1, the elements of $\mathring{\mathcal{P}}_{-k}^n$ are open, connected sets whose boundaries are curves in $\mathcal{S}_{-k-1} \cup \mathcal{S}_{n+1}$. Since the elements of $\mathcal{M}_{-k-1}^{n+1}$ are the maximal open, connected sets with this property, it must be that $\mathring{\mathcal{P}}_{-k}^n$ is a refinement of $\mathcal{M}_{-k-1}^{n+1}$. Now suppose that the union of $O_1, O_2 \in \mathring{\mathcal{P}}_{-k}^n$ is contained in a single element $A \in \mathcal{M}_{-k-1}^{n+1}$. This is impossible since $\partial O_1, \partial O_2 \subset \mathcal{S}_{-k-1} \cup \mathcal{S}_{n+1}$, and at least part of these boundaries must lie inside A, contradicting the definition of A. So in fact, $\mathring{\mathcal{P}}_{-k}^n = \mathcal{M}_{-k-1}^{n+1}$.

We next show that the limit in terms of $\#\mathcal{P}_{-k}^n$ exists and is independent of k. It will follow that the limits in terms of $\#\mathcal{M}_{-k}^n$ and $\#\mathring{\mathcal{P}}_{-k}^n$ exist and coincide using the relation $\mathring{\mathcal{P}}_{-k}^n = \mathcal{M}_{-k-1}^{n+1}$ and Lemma 3.2.

Note that $\#\mathcal{P}_{-j}^n \le \#\mathcal{P}_{-k}^n$ whenever $0 \le j \le k$. For fixed k, we have $\#\mathcal{P}_{-k}^{n+m} \le \#\mathcal{P}_{-k}^n \cdot \#\left(\bigvee_{i=1}^m T^{-n-i}\mathcal{P}\right)$, and since $\#(\bigvee_{i=1}^m T^{-n-i}\mathcal{P}) = \#(\bigvee_{i=1}^m T^{-i}\mathcal{P})$ because T is invertible, it follows that $\#\mathcal{P}_{-k}^{n+m} \le \#\mathcal{P}_{-k}^n \cdot \#\mathcal{P}_{-k}^m$. Thus $\log \#\mathcal{P}_{-k}^n$ is subadditive as a function of n, and the limit in n converges for each k. Applying this to $k = 0$ implies that the limit defining h_* in Definition 2.1 exists.

Similar considerations show that $\#\mathcal{P}^n_{-k} \le \#\mathcal{P}^0_{-k} \cdot \#\mathcal{P}^n_0$, and so

$$h_* = \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{P}^n_0 \le \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{P}^n_{-k} \le \lim_{n\to\infty} \frac{1}{n} (\log \#\mathcal{P}^0_{-k} + \log \#\mathcal{P}^n_0) = h_* ,$$

so that the limit exists and is independent of k.

For the final claim, we shall see that $\log \#\mathring{\mathcal{P}}^n_{-k}$ is subadditive for essentially the same reason as $\log \#\mathcal{P}^n_{-k}$: Take a (nonempty) element P of $\mathring{\mathcal{P}}^{n+m}_1$. It is the interior of an intersection of elements of the form $T^{-j}A_j$ for some A_j in $\mathcal{P}$, for $j = 1$ to $n + m$. This is equal to the intersection of the interiors of $T^{-j}A_j$. But, since P is nonempty, none of the $T^{-j}A_j$ can have empty interior and so none of the A_j can have empty interior. Thus the interiors of A_j are in $\mathring{\mathcal{P}}$ as well. Now, splitting the intersection of the first n sets from the last m, we see that the intersection of the first n sets form an element of $\mathring{\mathcal{P}}^n_1$. For the last m sets, we can factor out T^{-n} at the price of making the set a bit bigger:

$$\mathrm{int}\,(T^{-n-j}(A_{-n-j})) \subseteq T^{-n}(\mathrm{int}\,(T^{-j}(A_{-n-j}))) ,$$

where $\mathrm{int}(\cdot)$ denotes the interior of a set. Doing this for $j = 1$ to m, we see that this intersection is contained in T^{-n} of an element of $\mathring{\mathcal{P}}^m_1$. It follows that $\#\mathring{\mathcal{P}}^{n+m}_1 \le \#\mathring{\mathcal{P}}^n_1 \cdot \#\mathring{\mathcal{P}}^m_1$, so taking logs, the sequence is subadditive. And then so is the sequence with $\mathcal{M}^n_0$ in place of $\mathring{\mathcal{P}}^{n-1}_1$. □

3.3. **Comparing h_* with the Bowen definitions.** We set $\mathrm{diam}^s(\mathcal{M}^n_{-k})$ equal to the maximum length of a stable curve in any element of $\mathcal{M}^n_{-k}$. Similarly, $\mathrm{diam}^u(\mathcal{M}^n_{-k})$ denotes the maximum length of an unstable curve in any element of $\mathcal{M}^n_{-k}$ while $\mathrm{diam}(\mathcal{M}^n_{-k})$ denotes the maximum diameter of any element of $\mathcal{M}^n_{-k}$.

The following lemma gives the first claim of (3) in Theorem 2.3.

Lemma 3.4. $h_* = h_{\mathrm{sep}}$.

Proof. Fix $\varepsilon > 0$. Let $\Lambda = 1 + 2\mathcal{K}_{\min}\tau_{\min}$ denote the lower bound on the hyperbolicity constant for T as in (3.1). Choose k_ε large enough that $\mathrm{diam}^s(\mathcal{M}^0_{-k_\varepsilon-1}) \le C_1^{-1}\Lambda^{-k_\varepsilon} < c_1\varepsilon$, for some $c_1 > 0$ to be chosen below. It follows that

$$\mathrm{diam}^u(\mathcal{M}^{n+1}_{-k_\varepsilon-1}) \le C_1^{-1}\Lambda^{-n} < c_1\varepsilon$$

for each $n \ge k_\varepsilon$. Using the uniform transversality of stable and unstable cones, we may choose $c_1 > 0$ such that $\mathrm{diam}(\mathcal{M}^{n+1}_{-k_\varepsilon-1}) < \varepsilon$ for all $n \ge k_\varepsilon$.

Now for $n \ge k_\varepsilon$, let E be an (n, ε)-separated set. Given $x, y \in E$, we will show that x and y cannot belong to the same set $A \in \mathring{\mathcal{P}}^{k_\varepsilon+n}_{-k_\varepsilon}$.

Since $x, y \in E$, there exists $j \in [0, n]$ such that $d(T^j(x), T^j(y)) > \varepsilon$. If $x \in A \in \mathring{\mathcal{P}}^{k_\varepsilon+n}_{-k_\varepsilon}$, then $x \in \bigcap_{i=-k_\varepsilon}^{k_\varepsilon+n} \mathrm{int}(T^{-i}P_i)$ for some choice of $P_i \in \mathcal{P}$. Then

$$T^j x \in \bigcap_{i=-k_\varepsilon-j}^{k_\varepsilon+n-j} T^{-i}P_{i+j} \subset \bigcap_{-k_\varepsilon}^{k_\varepsilon} T^{-i}P_{i+j} \in \mathcal{P}^{k_\varepsilon}_{-k_\varepsilon} . \tag{3.2}$$

Note that the element of $\mathcal{P}^{k_\varepsilon}_{-k_\varepsilon}$ to which $T^j(x)$ belongs must have nonempty interior since $T^{-i}P_i$ has nonempty interior for each $i \in [-k_\varepsilon, k_\varepsilon + n]$. If $y \in A$, then $T^j y$ would belong to the same element of $\mathcal{P}^{k_\varepsilon}_{-k_\varepsilon}$, which is impossible since $\mathrm{diam}(\mathring{\mathcal{P}}^{k_\varepsilon}_{-k_\varepsilon}) < \varepsilon$ and taking the closure of such sets does not change the diameter.

Thus $x, y \in E$ implies that x and y cannot belong to the same element of $\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon+n}$ with nonempty interior. On the other hand, if x belongs to an element of $\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon+n}$ with empty interior, then indeed the element containing x is an isolated point, and y cannot belong to the same element. Thus $\#E \le \#\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon+n}$.

Since this bound holds for every (n,ε)-separated set, we have $r_n(\varepsilon) \le \#\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon+n}$. Thus,

$$\lim_{n\to\infty} \frac{1}{n} \log r_n(\varepsilon) \le \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon+n} = h_* \,.$$

Since this bound holds for every $\varepsilon > 0$, we conclude $h_{\mathrm{sep}} \le h_*$.

To prove the reverse inequality, we claim that there exists $\varepsilon_0 > 0$, independent of $n \ge 1$ and depending only on the table Q, such that

$$\text{if } x, y \text{ lie in different elements of } \mathcal{M}_0^n, \text{ then } d_n(x,y) \ge \varepsilon_0. \tag{3.3}$$

To each point x in an element of $\mathcal{M}_0^n$, we can associate an itinerary $(i_0, i_1, \dots, i_n)$ such that $T^{i_j}(x) \in M_{i_j}$. If x, y have different itineraries, then for some $0 \le j \le n$, the points $T^j(x)$ and $T^j(y)$ lie in different components M_i, and so by definition (2.1) we have, $d_n(x,y) = 10D \cdot \max_i \operatorname{diam}(M_i)$.

Now suppose x, y lie in different elements of $\mathcal{M}_0^n$, but have the same itinerary. By definition of $\mathcal{M}_0^n$, the elements containing x and y are separated by curves in $\mathcal{S}_n$. Let j be the minimum index of such a curve. Then $T^{j-1}(x)$ and $T^{j-1}(y)$ lie on different sides of a curve in $\mathcal{S}_1 \setminus \mathcal{S}_0$. Due to the finite horizon condition (our slightly stronger version is needed here), there exists $\varepsilon_0 > 0$, depending only on the structure of $\mathcal{S}_1$, such that the two one-sided ε_0-neighbourhoods of each curve in $\mathcal{S}_1 \setminus \mathcal{S}_0$ are mapped at least ε_0 apart. Thus either $d(T^{j-1}(x), T^{j-1}(y)) \ge \varepsilon_0$ or $d(T^j(x), T^j(y)) \ge \varepsilon_0$.

With the claim proved, fix $n \in \mathbb{N}$ and $\varepsilon \le \varepsilon_0$, and define E to be a set comprising exactly one point from each element of $\mathcal{M}_0^n$. Then by the claim, E is (n,ε)-separated, so that $\#\mathcal{M}_0^n \le r_n(\varepsilon)$ for each $\varepsilon \le \varepsilon_0$. Taking $n \to \infty$ and $\varepsilon \to 0$ yields $h_* \le h_{\mathrm{sep}}$. □

The following lemma gives the second claim of (3) in Theorem 2.3.

Lemma 3.5. $h_* = h_{\mathrm{span}}$.

Proof. Fix $\varepsilon > 0$ and choose k_ε as in the proof of Lemma 3.4 so that

$$\operatorname{diam}(\mathcal{M}_{-k_\varepsilon-1}^{n+1}) < \varepsilon$$

for all $n \ge k_\varepsilon$. Choose one point x in each element of $\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon+n}$, and let F denote the collection of these points. We will show that F is an (n,ε)-spanning set for T.

Let $y \in M$ and let B_y be the element of $\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon+n}$ containing y. If B_y is an isolated point, then $y \in F$ and there is nothing to prove. Otherwise, let $x_y = F \cap B_y$. For each $j \in [0,n]$, using the analogous calculation as in (3.2), we must have $T^j(y), T^j(x_y) \in B_j \in \mathcal{P}_{-k_\varepsilon}^{k_\varepsilon}$. Since $\operatorname{diam}(\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon}) < \varepsilon$, this implies $d(T^j(y), T^j(x_y)) < \varepsilon$ for all $j \in [0,n]$. Thus F is an (n,ε)-spanning set. We have

$$\lim_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon) \le \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon+n} = h_* \,.$$

Since this is true for each $\varepsilon > 0$, it follows that $h_{\mathrm{span}} \le h_*$.

To prove the reverse inequality, recall ε_0 from the proof of Lemma 3.4. For $\varepsilon < \varepsilon_0$ and $n \in \mathbb{N}$, let F be an (n,ε)-spanning set. We claim $\#F \ge \#\mathcal{M}_0^n$. Suppose not. Then there exists $A \in \mathcal{M}_0^n$ which contains no elements of F. Let $y \in A$ and let

$x \in F$. By the claim in the proof of Lemma 3.4, $d_n(x,y) \geq \varepsilon_0$ since x and y lie in different elements of $\mathcal{M}_0^n$. Since this holds for all $x \in F$, it contradicts the fact that F is an (n, ε)-spanning set.

Since this is true for each (n, ε)-spanning set for $\varepsilon < \varepsilon_0$, we conclude that $s_n(\varepsilon) \geq \#\mathcal{M}_0^n$, and taking appropriate limits, $h_{\text{span}} \geq h_*$. □

3.4. **Easy direction of the variational principle for h_*.** Recall that given a T-invariant probability measure μ and a finite measurable partition $\mathcal{A}$ of M, the entropy of $\mathcal{A}$ with respect to μ is defined by $H_\mu(\mathcal{A}) = -\sum_{A \in \mathcal{A}} \mu(A) \log \mu(A)$, and the entropy of T with respect to $\mathcal{A}$ is $h_\mu(T, \mathcal{A}) = \lim_{n \to \infty} \frac{1}{n} H_\mu \left(\bigvee_{i=0}^{n-1} T^{-i} \mathcal{A} \right)$.

The following lemma gives the bound (4) in Theorem 2.3.

Lemma 3.6. *$h_* \geq \sup\{h_\mu(T) : \mu$ is a T-invariant Borel probability measure$\}$.*

Proof. Let μ be a T-invariant probability measure on M. We note that $\mathcal{P}$ is a generator for T since $\bigvee_{i=-\infty}^{\infty} T^{-i}\mathcal{P}$ separates points in M. Thus $h_\mu(T) = h_\mu(T, \mathcal{P})$ (see for example [W, Thm 4.17]). Then,

$$h_\mu(T, \mathcal{P}) = \lim_{n \to \infty} \frac{1}{n} H_\mu \left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{P} \right) = \lim_{n \to \infty} \frac{1}{n} H_\mu(\mathcal{P}_0^{n-1}) \leq \lim_{n \to \infty} \frac{1}{n} \log(\#\mathcal{P}_0^{n-1}) = h_* \, .$$

Thus $h_\mu(T) \leq h_*$ for every T-invariant probability measure μ. □

4. The Banach spaces $\mathcal{B}$ and $\mathcal{B}_w$ and the transfer operator $\mathcal{L}$

The measure of maximal entropy for the billiard map T will be constructed out of left and right eigenvectors of a transfer operator $\mathcal{L}$ associated with the billiard map and acting on suitable spaces $\mathcal{B}$ and $\mathcal{B}_w$ of anisotropic distributions. In this section we define these objects, state and prove the main bound, Proposition 4.7, on the transfer operator, and deduce from it Theorem 4.10, showing that the spectral radius of $\mathcal{L}$ on $\mathcal{B}$ is e^{h_*}.

Recalling that the stable Jacobian of T satisfies $J^sT \approx \cos\varphi$ [CM, eq. (4.20)], the relevant transfer operator is defined on measurable functions f by

$$\mathcal{L}f = \frac{f \circ T^{-1}}{J^sT \circ T^{-1}} \, . \tag{4.1}$$

In order to define the Banach spaces of distributions on which the operator $\mathcal{L}$ will act, we need preliminary notation: Let $\mathcal{W}^s$ denote the set of all nontrivial connected subsets W of stable manifolds for T so that W has length at most $\delta_0 > 0$, where $\delta_0 < 1$ will be chosen after (5.4), using the growth Lemma 5.1. Such curves have curvature bounded above by a fixed constant [CM, Prop 4.29]. Thus, $T^{-1}\mathcal{W}^s = \mathcal{W}^s$, up to subdivision of curves.

For every $W \in \mathcal{W}^s$, let $C^1(W)$ denote the space of C^1 functions on W and for every $\alpha \in (0,1)$ we let $C^\alpha(W)$ denote the closure[12] of $C^1(W)$ for the α-Hölder norm $|\psi|_{C^\alpha(W)} = \sup_W |\psi| + H_W^\alpha(\psi)$, where

$$H_W^\alpha(\psi) = \sup_{\substack{x,y \in W \\ x \neq y}} \frac{|\psi(x) - \psi(y)|}{d(x,y)^\alpha} \, . \tag{4.2}$$

[12] Working with the closure of C^1 will give injectivity of the inclusion of the strong space in the weak.

We write $\psi \in C^\alpha(\mathcal{W}^s)$ if $\psi \in C^\alpha(W)$ for all $W \in \mathcal{W}^s$, with uniformly bounded Hölder norm.

4.1. **Definition of norms and of the spaces $\mathcal{B}$ and $\mathcal{B}_w$.** Since the stable cone C^s is bounded away from the vertical, we may view each stable curve $W \in \mathcal{W}^s$ as the graph of a function $\varphi_W(r)$ of the arclength coordinate r ranging over some interval I_W, i.e.,

$$W = \{G_W(r) := (r, \varphi_W(r)) \in M : r \in I_W\}\,. \tag{4.3}$$

Given two curves $W_1, W_2 \in \mathcal{W}^s$, we may use this representation to define a distance[13] between them: Define

$$d_{\mathcal{W}^s}(W_1, W_2) = |I_{W_1} \triangle I_{W_2}| + |\varphi_{W_1} - \varphi_{W_2}|_{C^1(I_{W_1} \cap I_{W_2})}$$

if $I_{W_1} \cap I_{W_2} \neq \emptyset$. Otherwise, set $d_{\mathcal{W}^s}(W_1, W_2) = \infty$.

Similarly, given two test functions ψ_1 and ψ_2 on W_1 and W_2, respectively, we define a distance between them by

$$d(\psi_1, \psi_2) = |\psi_1 \circ G_{W_1} - \psi_2 \circ G_{W_2}|_{C^0(I_{W_1} \cap I_{W_2})}\,,$$

whenever $d_{\mathcal{W}^s}(W_1, W_2) < \infty$. Otherwise, set $d(\psi_1, \psi_2) = \infty$.

We are now ready to introduce the norms used to define the spaces $\mathcal{B}$ and $\mathcal{B}_w$. Besides $\delta_0 \in (0,1)$, and a constant $\varepsilon_0 > 0$ to appear below, they will depend on positive real numbers α, β, γ, and ς so that, recalling $s_0 \in (0,1)$ from[14] (1.4),

$$0 < \beta < \alpha \le 1/3\,, \qquad 1 < 2^{s_0\gamma} < e^{h_*}\,, \qquad 0 < \varsigma < \gamma\,. \tag{4.4}$$

(The condition $\alpha \le 1/3$ is used in Lemma 4.4 which is used to prove embedding into distributions. The number $1/3$ comes from the $1/k^2$ decay in the width of homogeneity strips (4.5). The upper bound on γ arises from use of the growth lemma from Section 5.1. See (5.4).)

For $f \in C^1(M)$, define the weak norm of f by

$$|f|_w = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in C^\alpha(W) \\ |\psi|_{C^\alpha(W)} \le 1}} \int_W f\, \psi\, dm_W\,.$$

Here, dm_W denotes unnormalized Lebesgue (arclength) measure on W.

Define the strong stable norm of f by[15]

$$\|f\|_s = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in C^\beta(W) \\ |\psi|_{C^\beta(W)} \le |\log|W||^\gamma}} \int_W f\, \psi\, dm_W\,,$$

[13] $d_{\mathcal{W}^s}$ is not a metric since it does not satisfy the triangle inequality; however, it is sufficient for our purposes to produce a usable notion of distance between stable manifolds. See [DRZ, Footnote 4] for a modification of $d_{\mathcal{W}^s}$ which does satisfy the triangle inequality.

[14] If $\gamma > 1$, we can get good bounds in Theorem 2.6. This is only possible if $h_* > s_0 \log 2$.

[15] The logarithmic modulus of continuity in $\|f\|_s$ is used to obtain a finite spectral radius.

(note that $|f|_w \le \max\{1, |\log \delta_0|^{-\gamma}\} \|f\|_s$). Finally, for $\varsigma \in (0, \gamma)$, define the strong unstable norm[16] of f by

$$\|f\|_u = \sup_{\varepsilon \le \varepsilon_0} \sup_{\substack{W_1, W_2 \in \mathcal{W}^s \\ d_{\mathcal{W}^s}(W_1, W_2) \le \varepsilon}} \sup_{\substack{\psi_i \in C^\alpha(W_i) \\ |\psi_i|_{C^\alpha(W_i)} \le 1 \\ d(\psi_1, \psi_2) = 0}} |\log \varepsilon|^\varsigma \left| \int_{W_1} f\, \psi_1 \, dm_{W_1} - \int_{W_2} f\, \psi_2 \, dm_{W_2} \right| .$$

Definition 4.1 (The Banach spaces)**.** The space $\mathcal{B}_w$ is the completion of $C^1(M)$ with respect to the weak norm $|\cdot|_w$, while $\mathcal{B}$ is the completion of $C^1(M)$ with respect to the strong norm, $\|\cdot\|_{\mathcal{B}} = \|\cdot\|_s + \|\cdot\|_u$.

In the next subsection, we shall prove the continuous embeddings $\mathcal{B} \subset \mathcal{B}_w \subset (C^1(M))^*$, i.e., elements of our Banach spaces are distributions of order at most one (see Proposition 4.2). Proposition 6.1 in Section 6.4 gives the compact embedding of the unit ball of $\mathcal{B}$ in $\mathcal{B}_w$.

4.2. **Embeddings into distributions on M.** In this section we describe elements of our Banach spaces $\mathcal{B} \subset \mathcal{B}_w$ as distributions of order at most one on M. (This does not follow from the corresponding result in [DZ1], in particular since we use exact stable leaves to define our norms.) We will actually show that they belong to the dual of a space $C^\alpha(\mathcal{W}^s_{\mathbb{H}})$ containing $C^1(M)$ that we define next: We did not require elements of $\mathcal{W}^s$ to be homogeneous. Now, defining the usual homogeneity strips

$$\mathbb{H}_k = \left\{(r, \varphi) \in M_i : \tfrac{\pi}{2} - \tfrac{1}{k^2} \le \varphi \le \tfrac{\pi}{2} - \tfrac{1}{(k+1)^2}\right\}, \quad k \ge k_0\,, \tag{4.5}$$

and analogously for $k \le -k_0$, we define $\mathcal{W}^s_{\mathbb{H}} \subset \mathcal{W}^s$ to denote those stable manifolds $W \in \mathcal{W}^s$ such that $T^n W$ lies in a single homogeneity strip for all $n \ge 0$. We write $\psi \in C^\alpha(\mathcal{W}^s_{\mathbb{H}})$ if $\psi \in C^\alpha(W)$ for all $W \in \mathcal{W}^s_{\mathbb{H}}$ with uniformly bounded Hölder norm. Similarly, we define $C^\alpha_{\cos}(\mathcal{W}^s_{\mathbb{H}})$ to comprise the set of functions ψ such that $\psi \cos \varphi \in C^\alpha(\mathcal{W}^s_{\mathbb{H}})$. Clearly $C^\alpha(\mathcal{W}^s_{\mathbb{H}}) \subset C^\alpha_{\cos}(\mathcal{W}^s_{\mathbb{H}})$.

Due to the uniform hyperbolicity (3.1) of T and the invariance of $\mathcal{W}^s$ and $\mathcal{W}^s_{\mathbb{H}}$, if $\psi \in C^\alpha(\mathcal{W}^s)$ (resp., $C^\alpha(\mathcal{W}^s_{\mathbb{H}})$), then $\psi \circ T \in C^\alpha(\mathcal{W}^s)$ (resp., $C^\alpha(\mathcal{W}^s_{\mathbb{H}})$). Also, since the stable Jacobian of T satisfies $J^s T \approx \cos \varphi$ [CM, eq. (4.20)] and is 1/3 log-Hölder continuous on elements of $\mathcal{W}^s_{\mathbb{H}}$ [CM, Lemma 5.27], then $\frac{\psi \circ T}{J^s T} \in C^\alpha_{\cos}(\mathcal{W}^s_{\mathbb{H}})$ for any $\alpha \le 1/3$.

We can now state our first embedding result. An embedding $\mathcal{B}_w \subset (\mathcal{F})^*$ (for $\mathcal{F} = C^1(M)$ or $\mathcal{F} = C^\alpha(\mathcal{W}^s_{\mathbb{H}})$) is understood in the following sense: for $f \in \mathcal{B}_w$ there exists $C_f < \infty$ such that, letting $f_n \in C^1(M)$ be a sequence converging to f in the $\mathcal{B}_w$ norm, for every $\psi \in \mathcal{F}$ the following limit exists:

$$f(\psi) = \lim_{n \to \infty} \int f_n \psi \, d\mu_{\mathrm{SRB}} \tag{4.6}$$

and satisfies $|f(\psi)| \le C_f \|\psi\|_{\mathcal{F}}$.

Proposition 4.2 (Embedding into distributions)**.** *The continuous embeddings*

$$C^1(M) \subset \mathcal{B} \subset \mathcal{B}_w \subset (C^\alpha(\mathcal{W}^s_{\mathbb{H}}))^* \subset (C^1(M))^*$$

[16] The logarithmic modulus of continuity appears in $\|f\|_u$ because of the logarithmic modulus of continuity in $\|f\|_s$. Its presence in $\|f\|_u$ causes the loss of the spectral gap.

hold, the first two embeddings[17] *being injective. Therefore, since* $C^1(M) \subset \mathcal{B} \subset \mathcal{B}_w$ *injectively and continuously, we have*

$$(\mathcal{B}_w)^* \subset \mathcal{B}^* \subset (C^1(M))^* .$$

Remark 4.3 (Radon measures). Proposition 4.2 has the following important consequence: If $f \in \mathcal{B}_w$ is such that $f(\psi)$ defined by (4.6) is nonnegative for all nonnegative $\psi \in \mathcal{F} = C^1(M)$, then, by Schwartz's [Sch, §I.4] generalisation of the Riesz representation theorem, it defines an element of the dual of $C^0(M)$, i.e., a Radon measure on M. If, in addition, $f(\psi) = 1$ for ψ the constant function 1, then this measure is a probability measure.

The following lemma is important for the third inclusion in Proposition 4.2. Recalling (4.2), we define $H^\alpha_{\mathcal{W}^s_{\mathbb{H}}}(\psi) = \sup_{W\in\mathcal{W}^s_{\mathbb{H}}} H^\alpha_W(\psi)$.

Lemma 4.4. *There exists* $C > 0$ *such that for any* $f \in \mathcal{B}_w$ *and* $\psi \in C^\alpha(\mathcal{W}^s_{\mathbb{H}})$, *recalling* (4.6),

$$|f(\psi)| \le C|f|_w\big(|\psi|_\infty + H^\alpha_{\mathcal{W}^s_{\mathbb{H}}}(\psi)\big) .$$

Proof. By density it suffices to prove the inequality for $f \in C^1(M)$. Let $\psi \in C^\alpha(\mathcal{W}^s_{\mathbb{H}})$. Since by our convention, we identify f with the measure $f d\mu_{\mathrm{SRB}}$, we must estimate,

$$f(\psi) = \int f\,\psi\, d\mu_{\mathrm{SRB}} .$$

In order to bound this integral, we disintegrate the measure μ_{SRB} into conditional probability measures $\mu^{W_\xi}_{\mathrm{SRB}}$ on maximal homogeneous stable manifolds $W_\xi \in \mathcal{W}^s_{\mathbb{H}}$ and a factor measure $d\hat\mu_{\mathrm{SRB}}(\xi)$ on the index set Ξ of homogeneous stable manifolds; thus $\mathcal{W}^s_{\mathbb{H}} = \{W_\xi\}_{\xi\in\Xi}$. According to the time reversal counterpart of [CM, Cor 5.30], the conditional measures $\mu^{W_\xi}_{\mathrm{SRB}}$ have smooth densities with respect to the arclength measure on W_ξ, i.e., $d\mu^{W_\xi}_{\mathrm{SRB}} = |W_\xi|^{-1}\rho_\xi dm_{W_\xi}$, where ρ_ξ is log-Hölder continuous with exponent 1/3. Moreover, $\sup_{\xi\in\Xi}|\rho_\xi|_{C^\alpha(W_\xi)} =: \bar C < \infty$ since $\alpha \le 1/3$.

Using this disintegration, we estimate[18] the required integral:

$$\begin{aligned}
(4.7)\qquad |f(\psi)| &= \left|\int_{\xi\in\Xi}\int_{W_\xi} f\,\psi\,\rho_\xi\,|W_\xi|^{-1} dm_{W_\xi}\, d\hat\mu_{\mathrm{SRB}}(\xi)\right| \\
&\le \int_{\xi\in\Xi} |f|_w |\psi|_{C^\alpha(W_\xi)}|\rho_\xi|_{C^\alpha(W_\xi)}|W_\xi|^{-1} d\hat\mu_{\mathrm{SRB}}(\xi) \\
&\le \bar C|f|_w\big(|\psi|_\infty + H^\alpha_{\mathcal{W}^s_{\mathbb{H}}}(\psi)\big)\int_{\xi\in\Xi}|W_\xi|^{-1} d\hat\mu_{\mathrm{SRB}}(\xi) .
\end{aligned}$$

This last integral is precisely that in [CM, Exercise 7.15] which measures the relative frequency of short curves in a standard family. Due to [CM, Exercise 7.22], the SRB measure decomposes into a proper family, and so this integral is finite. □

[17] We do not expect the third embedding to be injective, due to the logarithmic weight in the norm.

[18] This is where we use $f\mu_{\mathrm{SRB}}$: Replacing $\hat\mu_{\mathrm{SRB}}$ by the factor measure with respect to Lebesgue, this integral would be infinite. Using $\mathcal{W}^s$ rather than $\mathcal{W}^s_{\mathbb{H}}$ may produce a finite integral with respect to Lebesgue, but the ρ_ξ may not be uniformly Hölder continuous on the longer curves.

Proof of Proposition 4.2. The continuity and injectivity of the embedding of $C^1(M)$ into $\mathcal{B}$ are clear from the definition. The inequality $|\cdot|_w \le \|\cdot\|_s$ implies the continuity of $\mathcal{B} \hookrightarrow \mathcal{B}_w$, while the injectivity follows from the definition of $C^\beta(W)$ as the closure of $C^1(W)$ in the C^β norm, as described at the beginning of Section 4, so that $C^\alpha(W)$ is dense in $C^\beta(W)$.

Finally, since $C^1(M) \subset C^\alpha(\mathcal{W}^s_{\mathbb{H}})$, the continuity of the third and fourth inclusions follow from Lemma 4.4. □

4.3. **The transfer operator.** We now move to the key bounds on the transfer operator. First, we revisit the definition (4.1) in order to let $\mathcal{L}$ act on $\mathcal{B}$ and $\mathcal{B}_w$: We may define the transfer operator $\mathcal{L} : (C^\alpha_{\cos}(\mathcal{W}^s_{\mathbb{H}}))^* \to (C^\alpha(\mathcal{W}^s))^*$ by

$$\mathcal{L}f(\psi) = f\big(\tfrac{\psi \circ T}{J^sT}\big), \quad \psi \in C^\alpha(\mathcal{W}^s)\,.$$

When $f \in C^1(M)$, we identify f with the measure[19]

$$f d\mu_{\mathrm{SRB}} \in (C^\alpha_{\cos}(\mathcal{W}^s_{\mathbb{H}}))^*\,. \tag{4.8}$$

The measure above is (abusively) still denoted by f. For $f \in C^1(M)$ the transfer operator then indeed takes the form $\mathcal{L}f = (f/J^sT) \circ T^{-1}$ announced in (4.1) since, due to our identification (4.8), we have $\mathcal{L}f(\psi) = \int \mathcal{L}f\,\psi\, d\mu_{\mathrm{SRB}} = \int f\, \frac{\psi \circ T}{J^sT}\, d\mu_{\mathrm{SRB}}$.

Remark 4.5 (Viewing $f \in C^1$ as a measure). If we viewed instead f as the measure fdm, it is not clear whether the embedding Lemma 4.4 would still hold since the weight $\cos W$ (crucial to [DZ1, Lemma 3.9]) is absent from the norms. Along these lines, we do not claim that Lebesgue measure belongs to our Banach spaces.

Slightly modifying [DZ1] due to the lack of homogeneity strips, we could replace $|\psi|_{C^\alpha(W)} \le 1$ by $|\psi \cos\varphi|_{C^\alpha(W)} \le 1$ in our norms. Then it would be natural to view f as fdm, and the embedding Lemma 4.4 would hold, but the transfer operator would have the form

$$\mathcal{L}_{\cos} f = \frac{f \circ T^{-1}}{(J^sT \circ T^{-1})(JT \circ T^{-1})}\,,$$

where JT is the full Jacobian of the map (the ratio of cosines). We do not make such a change since it would only complicate our estimates unnecessarily. Note that the potentials of the operators $\mathcal{L}$ and $\mathcal{L}_{cos}$ differ by a coboundary, giving the same spectral radius.

It follows from submultiplicativity of $\#\mathcal{M}_0^n$ that $e^{nh_*} \le \#\mathcal{M}_0^n$ for all n. In Section 5.3, we shall prove the supermultiplicativity statement Lemma 5.6 from which we deduce the following upper bound for $\#\mathcal{M}_0^n$.

Proposition 4.6 (Exact exponential growth)**.** *Let $c_1 > 0$ be given by Lemma* 5.6. *Then for all $n \in \mathbb{N}$, we have $e^{nh_*} \le \#\mathcal{M}_0^n \le \frac{2}{c_1} e^{nh_*}$.*

The following proposition (proved in Section 6) gives the key norm estimates.

[19] To show the claimed inclusion just use that $d\mu_{\mathrm{SRB}} = (2|\partial Q|)^{-1} \cos\varphi\, dr d\varphi$.

Proposition 4.7. *Let c_1 be as in Proposition 4.6. There exist δ_0, $C > 0$, and $\varpi \in (0,1)$ such that*[20] *for all $f \in \mathcal{B}$,*

$$|\mathcal{L}^n f|_w \le \frac{C}{c_1\delta_0} e^{nh_*} |f|_w \quad \forall n \ge 0\,; \tag{4.9}$$

$$\|\mathcal{L}^n f\|_s \le \frac{C}{c_1\delta_0} e^{nh_*} \|f\|_s \quad \forall n \ge 0\,; \tag{4.10}$$

$$\|\mathcal{L}^n f\|_u \le \frac{C}{c_1\delta_0} (\|f\|_u + n^{\varpi}\|f\|_s) e^{nh_*} \quad \forall n \ge 0\,. \tag{4.11}$$

If $h_* > s_0 \log 2$ (where $s_0 < 1$ is defined by (1.4)), then in addition there exist $\varsigma > 0$ and $C > 0$ such that for all $f \in \mathcal{B}$

$$\|\mathcal{L}^n f\|_u \le \frac{C}{c_1\delta_0} (\|f\|_u + \|f\|_s) e^{nh_*} \quad \forall n \ge 0\,. \tag{4.12}$$

Remark 4.8. Replacing $|\log \epsilon|$ by $\log|\log \epsilon|$ in the definition of $\|f\|_u$, we can replace n^{ϖ} by a logarithm in (4.11).

In spite of compactness of the embedding $\mathcal{B} \subset \mathcal{B}_w$ (Proposition 6.1), the above bounds do *not* deserve to be called Lasota–Yorke estimates since (even replacing $\|\cdot\|_s + \|\cdot\|_u$ by $\|\cdot\|_s + c_u\|\cdot\|_u$ for small c_u and using footnote 20) they do not lead to bounds of the type $\|(e^{-h_*}\mathcal{L})^n f\|_{\mathcal{B}} \le \sigma^n \|f\|_{\mathcal{B}} + K_n |f|_w$ for some $\sigma < 1$ and finite constants K_n. We will nevertheless sometimes refer to them as "Lasota–Yorke" estimates, in quotation marks.

Proposition 4.7 combined with the following lemma imply that $\mathcal{L}$ is a bounded operator on both $\mathcal{B}$ and $\mathcal{B}_w$.

Lemma 4.9 (Image of a C^1 function)**.** *For any $f \in C^1(M)$ the image $\mathcal{L}f \in (C^\alpha(\mathcal{W}^s))^*$ is the limit of a sequence of C^1 functions in the strong norm $\|\cdot\|_{\mathcal{B}}$.*

Proof. Since our norms are weaker than the norms of [DZ1] (modulo the use of homogeneity layers there), the statement follows from replacing $\mathcal{L}_{\mathrm{SRB}}$ by $\mathcal{L}$ in the proofs of Lemmas 3.7 and 3.8 in [DZ1], and checking that the absence of homogeneity layers does not affect the computations. □

Proposition 4.7 gives the upper bounds in the following result (the bounds (4.14) and (4.15) are needed to construct a nontrivial maximal eigenvector in Proposition 7.1).

Theorem 4.10 (Spectral radius of $\mathcal{L}$ on $\mathcal{B}$)**.** *There exist $\varpi \in (0,1)$, $C < \infty$ such that,*

$$\|\mathcal{L}^n\|_{\mathcal{B}} \le C n^{\varpi} e^{nh_*} \quad \forall n \ge 0\,. \tag{4.13}$$

There exists $C > 0$ such that, letting 1 be the function $f \equiv 1$, we have,

$$\|\mathcal{L}^n 1\|_s \ge |\mathcal{L}^n 1|_w \ge C e^{nh_*} \quad \forall n \ge 0\,. \tag{4.14}$$

Recalling (4.9), the spectral radius of $\mathcal{L}$ on $\mathcal{B}$ and $\mathcal{B}_w$ is thus equal to $\exp(h_*) > 1$.

If $h_* > s_0 \log 2$ (with $s_0 < 1$ defined by (1.4)), then, if $\varsigma > 0$ and $\delta_0 > 0$ are small enough, there exists $\widetilde{C} < \infty$ such that,

$$\|\mathcal{L}^n\|_{\mathcal{B}} \le \widetilde{C} e^{nh_*} \quad \forall n \ge 0\,. \tag{4.15}$$

The above theorem is proved in Subection 6.3.

---

[20] In fact the strong stable norm satisfies a stronger inequality: $\|\mathcal{L}^n f\|_s \le \frac{C}{c_1\delta_0}(\sigma^n \|f\|_s + |f|_w) e^{nh_*}$ for some $\sigma < 1$. We omit the proof since we do not use this.

5. Growth lemma and fragmentation lemmas

This section contains combinatorial growth lemmas, controlling the growth in complexity of the iterates of a stable curve. They will be used to prove the "Lasota–Yorke" Proposition 4.7, to show Lemma 5.2, used in Section 6.3 to get the lower bound (4.14) on the spectral radius, and to show absolute continuity in Section 7.3.

In view of the compact embedding Proposition 6.1, and also to get Lemma 5.4 from Lemma 5.2, we must work with a more general class of stable curves: We define a set of cone-stable curves $\widehat{\mathcal{W}}^s$ whose tangent vectors all lie in the stable cone for the map, with length at most δ_0 and curvature bounded above so that $T^{-1}\widehat{\mathcal{W}}^s \subset \widehat{\mathcal{W}}^s$, up to subdivision of curves. Obviously, $\mathcal{W}^s \subset \widehat{\mathcal{W}}^s$. We define a set of cone-unstable curves $\widehat{\mathcal{W}}^u$ similarly.

For $W \in \widehat{\mathcal{W}}^s$, let $\mathcal{G}_0(W) = W$. For $n \geq 1$, define $\mathcal{G}_n(W) = \mathcal{G}_n^{\delta_0}(W)$ inductively as the smooth components of $T^{-1}(W')$ for $W' \in \mathcal{G}_{n-1}(W)$, where elements longer than δ_0 are subdivided to have length between $\delta_0/2$ and δ_0. Thus $\mathcal{G}_n(W) \subset \widehat{\mathcal{W}}^s$ for each n and $\bigcup_{U \in \mathcal{G}_n(W)} U = T^{-n}W$. Moreover, if $W \in \mathcal{W}^s$, then $\mathcal{G}_n(W) \subset \mathcal{W}^s$.

Denote by $L_n(W)$ those elements of $\mathcal{G}_n(W)$ having length at least $\delta_0/3$, and define $\mathcal{I}_n(W)$ to comprise those elements $U \in \mathcal{G}_n(W)$ for which T^iU is not contained in an element of $L_{n-i}(W)$ for $0 \leq i \leq n-1$.

A fundamental fact [Ch2, Lemma 5.2] we will use is that the growth in complexity for the billiard is at most linear:

$$\begin{aligned} &\exists\, K > 0 \text{ such that } \forall\, n \geq 0, \text{ the number of curves in } \mathcal{S}_{\pm n} \text{ that intersect} \\ &\qquad\qquad \text{at a single point is at most } Kn. \end{aligned} \tag{5.1}$$

5.1. Growth lemma.

Recall $s_0 \in (0,1)$ from (1.4). We shall prove the following.

Lemma 5.1 (Growth lemma). *For any $m \in \mathbb{N}$, there exists $\delta_0 = \delta_0(m) \in (0,1)$ such that for all $n \geq 1$, all $\bar\gamma \in [0,\infty)$, and all $W \in \widehat{\mathcal{W}}^s$, we have*

a) $$\sum_{W_i \in \mathcal{I}_n(W)} \left(\frac{\log|W|}{\log|W_i|}\right)^{\bar\gamma} \leq 2^{(ns_0+1)\bar\gamma}(Km+1)^{n/m}\ ;$$

b) $$\sum_{W_i \in \mathcal{G}_n(W)} \left(\frac{\log|W|}{\log|W_i|}\right)^{\bar\gamma}$$
$$\leq \min\Big\{2\delta_0^{-1}2^{(ns_0+1)\bar\gamma}\#\mathcal{M}_0^n,\ 2^{2\bar\gamma+1}\delta_0^{-1}\sum_{j=1}^{n}2^{js_0\bar\gamma}(Km+1)^{j/m}\#\mathcal{M}_0^{n-j}\Big\}.$$

Moreover, if $|W| \geq \delta_0/2$, then both factors $2^{(ns_0+1)\bar\gamma}$ can be replaced by $2^{\bar\gamma}$.

Proof. First recall that if $W \in \widehat{\mathcal{W}}^s$ is short, then

$$|T^{-1}W| \leq C|W|^{1/2} \quad \text{for some constant } C \geq 1, \text{ independent of } W \in \widehat{\mathcal{W}}^s, \tag{5.2}$$

[CM, Exercise 4.50]. The above bound can be iterated, giving $|T^{-\ell}W| \leq C'|W|^{2^{-\ell}}$, where $C' \leq C^2$, for any number of consecutive "nearly tangential" collisions (collisions with angle $|\varphi| > \varphi_0$). Since in every n_0 iterates, we have at most s_0n_0 nearly tangential collisions and $(1-s_0)n_0$ iterates that expand at most by a constant factor $\Lambda_1 > 1$ depending only on φ_0, we see that

$$\begin{aligned} &|T^{-n_0}W| \leq C|W|^{2^{-s_0n_0}}\Lambda_1^{(1-s_0)n_0} \\ &\implies |T^{-2n_0}W| \leq C^{1+2^{-s_0n_0}}|W|^{2^{-2s_0n_0}}\Lambda_1^{(1-s_0)n_0 2^{-s_0n_0}}\Lambda_1^{(1-s_0)n_0}\,. \end{aligned}$$

Iterating this inductively, we conclude

$$|T^{-j}W| \le C''|W|^{2^{-s_0 j}} \quad \text{for all } j \ge 1, \tag{5.3}$$

where $C'' \ge 1$ depends only on n_0 and Λ_1. Therefore, if δ_0 is smaller than $1/C''$, we have

$$\left(\frac{\log|W|}{\log|W_i|}\right)^{\bar\gamma} \le \left(2^{s_0 n}\left(1 - \frac{\log C''}{\log|W_i|}\right)\right)^{\bar\gamma} \le 2^{(ns_0+1)\bar\gamma} \ \forall\, W_i \in \mathcal{G}_n(W)\,,$$

since $|W_i| \le \delta_0$. Note that if $|W_i| \le |W|$, then $\frac{\log|W|}{\log|W_i|} \le 1$, so that such curves do not contribute large terms to the sums in parts (a) and (b) of the lemma.

(a) Using the above argument, for any $W \in \widehat{\mathcal{W}}^s$, we may bound the ratio of logs by $2^{(n+1)s_0\bar\gamma}$. Moreover, if $|W| \ge \delta_0/2$, then since $|W_i| \le \delta_0 < 2$, we have

$$\frac{\log|W|}{\log|W_i|} \le \frac{\log(\delta_0/2)}{\log\delta_0} = 1 - \frac{\log 2}{\log\delta_0} \le 2\,.$$

Now, fixing m and using the linear bound on complexity, choose $\delta_0 = \delta_0(m) > 0$ such that if $|W| \le \delta_0$, then $T^{-\ell}W$ comprises at most $K\ell+1$ connected components for $0 \le \ell \le 2m$. Such a choice is always possible by (5.2). Then for $n = mj + \ell$, we split up the orbit into $j-1$ increments of length m and the last increment of length $m+\ell$. Part (a) then follows by a simple induction, since elements of $\mathcal{I}_{mj}(W)$ must be formed from elements of $\mathcal{I}_{m(j-1)}(W)$ which have been cut by singularity curves in $\mathcal{S}_{-m}$. At the last step, this estimate also holds for elements of which have been cut by singularity curves in $\mathcal{S}_{-m-\ell}$ by choice of δ_0.

(b) The bound on the ratio of logs is the same as in part (a). The first bound on the cardinality of the sum follows by noting that each element of $\mathcal{G}_n(W)$ is contained in one element of $\mathcal{M}_0^n$. Moreover, due to subdivision of long pieces, there can be no more than $2\delta_0^{-1}$ elements of $\mathcal{G}_n(W)$ in a single element of $\mathcal{M}_0^n$.

For the second bound in part (b), we may assume that $|W| < \delta_0/2$; otherwise, we may bound the sum by $2^{\bar\gamma+1}\delta_0^{-1}\#\mathcal{M}_0^n$, which is optimal for what we need. For $|W| < \delta_0/2$, let $F_1(W)$ denote those $V \in \mathcal{G}_1(W)$ whose length is at least $\delta_0/2$. Inductively, define $F_j(W)$, for $2 \le j \le n-1$, to contain those $V \in \mathcal{G}_j(W)$ whose length is at least $\delta_0/2$, and such that T^kV is not contained in an element of $F_{j-k}(W)$ for any $1 \le k \le j-1$. Thus $F_j(W)$ contains elements of $\mathcal{G}_j(W)$ that are "long for the first time" at time j.

We group $W_i \in \mathcal{G}_n(W)$ by its "first long ancestor" as follows. We say W_i has *first long ancestor*[21] $V \in F_j(W)$ for $1 \le j \le n-1$ if $T^{n-j}W_i \subseteq V$. Note that such a j and V are unique for each W_i if they exist. If no such j and V exist, then W_i has been forever short and so must belong to $\mathcal{I}_n(W)$. Denote by $A_{n-j}(V)$ the set

[21] Note that "ancestor" refers to the backwards dynamics mapping W to W_i.

of $W_i \in \mathcal{G}_n(W)$ corresponding to one $V \in F_j(W)$. Now

$$
\begin{aligned}
&\sum_{W_i \in \mathcal{G}_n(W)} \left(\frac{\log |W|}{\log |W_i|} \right)^{\bar\gamma} \\
&= \sum_{j=1}^{n-1} \sum_{V_\ell \in F_j(W)} \sum_{W_i \in A_{n-j}(V_\ell)} \left(\frac{\log |W|}{\log |W_i|} \right)^{\bar\gamma} + \sum_{W_i \in \mathcal{I}_n(W)} \left(\frac{\log |W|}{\log |W_i|} \right)^{\bar\gamma} \\
&\le \sum_{j=1}^{n-1} \sum_{V_\ell \in F_j(W)} \left(\frac{\log |W|}{\log |V_\ell|} \right)^{\bar\gamma} \sum_{W_i \in A_{n-j}(V_\ell)} \left(\frac{\log |V_\ell|}{\log |W_i|} \right)^{\bar\gamma} + 2^{(ns_0+1)\bar\gamma}(Km+1)^{n/m} \\
&\le \sum_{j=1}^{n-1} \sum_{V_\ell \in F_j(W)} \left(\frac{\log |W|}{\log |V_\ell|} \right)^{\bar\gamma} 2^{\bar\gamma+1}\delta_0^{-1} \# \mathcal{M}_0^{n-j} + 2^{(ns_0+1)\bar\gamma}(Km+1)^{n/m} \\
&\le \sum_{j=1}^{n-1} 2^{(js_0+1)\bar\gamma}(Km+1)^{j/m} 2^{\bar\gamma+1}\delta_0^{-1} \# \mathcal{M}_0^{n-j} + 2^{(ns_0+1)\bar\gamma}(Km+1)^{n/m} \\
&\le 2^{2\bar\gamma+1}\delta_0^{-1} \sum_{j=1}^{n} 2^{js_0\bar\gamma}(Km+1)^{j/m} \# \mathcal{M}_0^{n-j} \, ,
\end{aligned}
$$

where we have applied part (a) from time 1 to time j and the first estimate in part (b) from time j to time n, since each $|V_\ell| \ge \delta_0/2$. □

With the growth lemma proved, we can choose m and the length scale δ_0 of curves in $\mathcal{W}^s$. Recalling K from (5.1) and the condition on γ from (4.4), we fix m so large that

$$
\frac{1}{m}\log(Km+1) < h_* - \gamma s_0 \log 2 \, , \tag{5.4}
$$

and we choose $\delta_0 = \delta_0(m)$ to be the corresponding length scale from Lemma 5.1. If $h_* > s_0 \log 2$, then we take $\gamma > 1$, so that in fact $\frac{1}{m}\log(Km+1) < h_* - s_0 \log 2$.

5.2. **Fragmentation lemmas.** The results in this subsection will be used in Sections 5.3 and 7.3. For $\delta \in (0, \delta_0)$ and $W \in \widehat{\mathcal{W}}^s$, define $\mathcal{G}_n^\delta(W)$ to be the smooth components of $T^{-n}W$, with long pieces subdivided to have length between $\delta/2$ and δ. (So $\mathcal{G}_n^\delta(W)$ is defined exactly like $\mathcal{G}_n(W)$, but with δ_0 replaced by δ.) Let $L_n^\delta(W)$ denote the set of curves in $\mathcal{G}_n^\delta(W)$ that have length at least $\delta/3$ and let $S_n^\delta(W) = \mathcal{G}_n^\delta(W) \setminus L_n^\delta(W)$. Define $\mathcal{I}_n^\delta(W)$ to be those curves in $\mathcal{G}_n^\delta(W)$ that have no ancestors[22] of length at least $\delta/3$, as in the definition of $\mathcal{I}_n(W)$ above. The following lemma and its corollary bootstrap from Lemma 5.1a) and will be crucial to get the lower bound on the spectral radius.

Lemma 5.2. *For each $\varepsilon > 0$, there exist $\delta \in (0, \delta_0]$ and $n_1 \in \mathbb{N}$, such that for $n \ge n_1$,*

$$
\frac{\# L_n^\delta(W)}{\# \mathcal{G}_n^\delta(W)} \ge \frac{1-2\varepsilon}{1-\varepsilon} \quad \textit{for all } W \in \widehat{\mathcal{W}}^s \textit{ with } |W| \ge \delta/3.
$$

Proof. Fix $\varepsilon > 0$ and choose n_1 so large that $3C_1^{-1}(Kn_1+1)\Lambda^{-n_1} < \varepsilon$ and $\Lambda^{n_1} > e$. Next, choose $\delta > 0$ sufficiently small that if $W \in \widehat{\mathcal{W}}^s$ with $|W| < \delta$, then $T^{-n}W$ comprises at most $Kn+1$ smooth pieces of length at most δ_0 for all $n \le 2n_1$.

[22]For $k < n$, we say that $U \in \mathcal{G}_k^\delta(W)$ is an *ancestor* of $V \in \mathcal{G}_n^\delta(W)$ if $T^{n-k}V \subseteq U$.

Let $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta/3$. We shall prove the following equivalent inequality for $n \geq n_1$:

$$\frac{\# S_n^\delta(W)}{\# \mathcal{G}_n^\delta(W)} \leq \frac{\varepsilon}{1-\varepsilon} .$$

For $n \geq n_1$, write $n = kn_1 + \ell$ for some $0 \leq \ell < n_1$. If $k = 1$, the above inequality is clear since $S^\delta_{n_1+\ell}(W)$ contains at most $K(n_1 + \ell) + 1$ components by assumption on δ and n_1, while $|T^{-(n_1+\ell)}W| \geq C_1\Lambda^{n_1+\ell}|W| \geq C_1\Lambda^{n_1+\ell}\delta/3$. Thus $\mathcal{G}_n^\delta(W)$ must contain at least $C_1\Lambda^{n_1+\ell}/3$ curves since each has length at most δ. Thus,

$$\frac{\# S^\delta_{n_1+\ell}(W)}{\# \mathcal{G}^\delta_{n_1+\ell}(W)} \leq 3C_1^{-1}\frac{K(n_1+\ell)+1}{\Lambda^{n_1+\ell}} \leq 3C_1^{-1}\frac{Kn_1+1}{\Lambda^{n_1}} < \varepsilon ,$$

where the second inequality holds for all $\ell \geq 0$ as long as $\frac{1}{n_1} \leq \log \Lambda$, which is true by choice of n_1.

For $k > 1$, we split n into $k - 1$ blocks of length n_1 and the last block of length $n_1 + \ell$. We group elements $W_i \in S^\delta_{kn_1+\ell}(W)$ by most recent[23] long ancestor $V_j \in L^\delta_{qn_1}(W)$: q is the greatest index in $[0, k-1]$ such that $T^{(k-q)n_1+\ell}W_i \subseteq V_j$ and $V_j \in L^\delta_{qn_1}(W)$. Note that since $|V_j| \geq \delta/3$, then $\mathcal{G}^\delta_{(k-q)n_1+\ell}(V_j)$ must contain at least $C_1\Lambda^{(k-q)n_1}/3$ curves since each has length at most δ. Thus using Lemma 5.1a) with $\bar\gamma = 0$, we estimate

(5.5)

$$\begin{aligned}
\frac{\# S^\delta_{kn_1+\ell}(W)}{\# \mathcal{G}^\delta_{kn_1+\ell}(W)} &= \frac{\sum_{W_i \in \mathcal{I}^\delta_{kn_1+\ell}(W)} 1}{\# \mathcal{G}^\delta_{kn_1+\ell}(W)} + \frac{\sum_{q=1}^{k-1} \sum_{V_j \in L^\delta_{qn_1}(W)} \sum_{W_i \in \mathcal{I}^\delta_{(k-q)n_1+\ell}(V_j)} 1}{\# \mathcal{G}^\delta_{kn_1+\ell}(W)} \\
&\leq \frac{(Kn_1+1)^k}{C_1\Lambda^{kn_1}/3} + \sum_{q=1}^{k-1} \frac{\sum_{V_j \in L^\delta_{qn_1}(W)} (Kn_1+1)^{k-q}}{\sum_{V_j \in L^\delta_{qn_1}(W)} C_1\Lambda^{(k-q)n_1}/3} \\
&\leq 3C_1^{-1}\sum_{q=1}^{k}(Kn_1+1)^q\Lambda^{-qn_1} \leq \sum_{q=1}^{k}\varepsilon^q \leq \frac{\varepsilon}{1-\varepsilon} .
\end{aligned}$$

□

The following corollary is used in Corollary 7.9 and in Lemma 7.7.

Corollary 5.3. *There exists $C_2 > 0$ such that for any ε, δ, and n_1 as in Lemma* 5.2*,*

$$\frac{\# L_n^\delta(W)}{\# \mathcal{G}_n^\delta(W)} \geq \frac{1-3\varepsilon}{1-\varepsilon} \quad \forall W \in \widehat{\mathcal{W}}^s \;\; \forall n \geq C_2 n_1 \frac{|\log(|W|/\delta)|}{|\log \varepsilon|} .$$

Proof. The proof is essentially the same as that for Lemma 5.2, except that for curves shorter than length $\delta/3$ one must wait $n \sim |\log(|W|/\delta)|$ for at least one component of $\mathcal{G}_n^\delta(W)$ to belong to $L_n^\delta(W)$.

More precisely, fix $\varepsilon > 0$ and the corresponding δ and n_1 from Lemma 5.2. Let $W \in \widehat{\mathcal{W}}^s$ with $|W| < \delta/3$ and take $n > n_1$. Decomposing $\mathcal{G}_n^\delta(W)$ as in Lemma 5.2, we estimate the second term of (5.5) as before.

[23]We only consider what happens at the beginning of a block of length n_1. It does not affect our argument if W_i belongs to a long piece at an intermediate time, since we only consider the cardinality of short pieces that can be created in each block of length n_1 according to our choice of δ.

For the first term of (5.5), $\#\mathcal{I}_n^\delta(W)/\#\mathcal{G}_n^\delta(W)$, for δ sufficiently small, notice that since the flow is continuous, either $\#\mathcal{G}_\ell^\delta(W) \le K\ell + 1$ by (5.1) or at least one element of $\mathcal{G}_\ell^\delta(W)$ has length at least $\delta/3$. Let n_2 denote the first iterate ℓ at which $\mathcal{G}_\ell^\delta(W)$ contains at least one element of length more than $\delta/3$. By the complexity estimate (5.1) and the fact that $|T^{-n_2}W| \ge C_1\Lambda^{n_2}|W|$ by (3.1), there exists $\bar{C}_2 > 0$, independent of $W \in \widehat{\mathcal{W}}^s$, such that $n_2 \le \bar{C}_2|\log(|W|/\delta)|$.

Now for $n \ge n_2$, and some $W' \in \mathcal{G}_{n_2}^\delta(W)$,

$$\#\mathcal{I}_n^\delta(W) \le (Kn_2+1)\#\mathcal{I}_{n-n_2}^\delta(W') \le (Kn_2+1)(Kn_1+1)^{\lfloor (n-n_2)/n_1 \rfloor},$$

while

$$\#\mathcal{G}_n^\delta(W) \ge C_1\Lambda^{n-n_2}/3.$$

Putting these together, we have,

$$\frac{\#\mathcal{I}_n^\delta(W)}{\#\mathcal{G}_n^\delta(W)} \le \frac{(Kn_2+1)(Kn_1+1)^{\lfloor n/n_1 \rfloor}}{C_1\Lambda^n/3}\Lambda^{n_2} \le \varepsilon^{\lfloor n/n_1 \rfloor}(Kn_2+1)\Lambda^{n_2}.$$

Since $n_2 \le \bar{C}_2|\log(|W|/\delta)|$, we may make this expression $< \varepsilon$ by choosing n so large that $n/n_1 \ge C_2\frac{\log(|W|/\delta)}{\log\varepsilon}$ for some $C_2 > 0$. For such n, the estimate (5.5) is bounded by $\varepsilon + \frac{\varepsilon}{1-\varepsilon} \le \frac{2\varepsilon}{1-\varepsilon}$, which completes the proof of the corollary. □

Choose $\varepsilon = 1/4$ and let $\delta_1 \le \delta_0$ and n_1 be the corresponding δ and n_1 from Lemma 5.2. With this choice, we have

$$\text{(5.6)} \quad \#L_n^{\delta_1}(W) \ge \tfrac{2}{3}\#\mathcal{G}_n^{\delta_1}(W) \qquad \text{for all } W \in \widehat{\mathcal{W}}^s \text{ with } |W| \ge \delta_1/3 \text{ and } n \ge n_1.$$

Notice that for $W \in \mathcal{W}^s$, each element $V \in \mathcal{G}_n^{\delta_1}(W)$ is contained in one element of $\mathcal{M}_0^n$ and its image $T^nV \subset W$ is contained in one element of $\mathcal{M}_{-n}^0$. Indeed, there is a one-to-one correspondence between elements of $\mathcal{M}_0^n$ and elements of $\mathcal{M}_{-n}^0$.

The boundary of the partition formed by $\mathcal{M}_{-n}^0$ is comprised of unstable curves belonging to $\mathcal{S}_{-n} = \bigcup_{j=0}^n T^j(\mathcal{S}_0)$. Let $L_u(\mathcal{M}_{-n}^0)$ denote the elements of $\mathcal{M}_{-n}^0$ whose unstable diameter[24] is at least $\delta_1/3$. Similarly, let $L_s(\mathcal{M}_0^n)$ denote the elements of $\mathcal{M}_0^n$ whose stable diameter is at least $\delta_1/3$.

The following lemma will be used to get both lower and upper bounds on the spectral radius via Proposition 5.5.

Lemma 5.4. *Let δ_1 and n_1 be associated with $\varepsilon = 1/4$ by Lemma 5.2. There exist $C_{n_1} > 0$ and $n_2 \ge n_1$ such that for all $n \ge n_2$,*

$$\#L_u(\mathcal{M}_{-n}^0) \ge C_{n_1}\delta_1\#\mathcal{M}_{-n}^0 \quad \text{and} \quad \#L_s(\mathcal{M}_0^n) \ge C_{n_1}\delta_1\#\mathcal{M}_0^n.$$

Proof. We prove the lower bound for $L_u(\mathcal{M}_{-n}^0)$. The lower bound for $L_s(\mathcal{M}_0^n)$ then follows by time reversal.

Let $I_u(\mathcal{M}_{-n}^0)$ denote the elements of $\mathcal{M}_{-n}^0$ whose unstable diameter is less than $\delta_1/3$. Clearly, $I_u(\mathcal{M}_{-n}^0) \cup L_u(\mathcal{M}_{-n}^0) = \mathcal{M}_{-n}^0$. Similarly, let $I_u(T^j\mathcal{S}_0)$ denote the set of unstable curves in $T^j(S_0)$ whose length is less than $\delta_1/3$.

We first prove the following claim: $\#I_u(\mathcal{M}_{-n}^0) \le 2\sum_{j=1}^n \#I_u(T^j\mathcal{S}_0) + K_2n$. Recall that the boundaries of elements of $\mathcal{M}_{-n}^0$ are comprised of elements of $\mathcal{S}_{-n} = \bigcup_{i=0}^n T^i\mathcal{S}_0$, which are unstable curves for $i \ge 1$. We use the following property established in Lemma 3.1: If a smooth unstable curve $U_i \subset T^i\mathcal{S}_0$ intersects a smooth curve $U_j \subset T^j\mathcal{S}_0$ for $i < j$, then U_j must terminate on U_i. Thus if $A \in I_u(\mathcal{M}_{-n}^0)$,

[24]Recall from Section 3 that the unstable diameter of a set is the length of the longest unstable curve contained in that set.

then either the boundary of A contains a short curve in $T^j(\mathcal{S}_0)$ for some $1 \le j \le n$, or ∂A contains an intersection point of two curves in $T^j(\mathcal{S}_0)$ for some $1 \le j \le n$ (see Figure 3). But such intersections of curves within $T^j(\mathcal{S}_0)$ are images of intersections of curves within $T(\mathcal{S}_0)$, and the cardinality of cells created by such intersections is bounded by some uniform constant $K_2 > 0$ depending only on $T(\mathcal{S}_0)$. Then, since each short curve in $T^j(\mathcal{S}_0)$ belongs to the boundary of at most two $A \in I_u(\mathcal{M}^0_{-n})$, the claim follows.

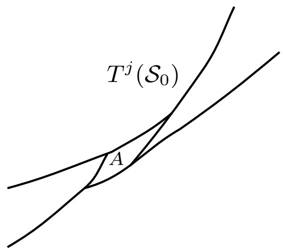


FIGURE 3. A short cell $A \in I_u(\mathcal{M}^0_{-n})$ created by long elements of $T^j(\mathcal{S}_0)$.

Next, subdivide $\mathcal{S}_0$ into ℓ_0 horizontal segments U_i such that TU_i is an unstable curve of length between $\delta_1/3$ and δ_1 for each i. Analogous to stable curves, let $\mathcal{G}^{\delta_1}_j(U)$ denote the decomposition of the union of unstable curves comprising T^jU at length scale δ_1. Then for $j \ge n_1$ using the time reversal of (5.6), we have

$$\#I_u(T^j\mathcal{S}_0) = \sum_{i=1}^{\ell_0} \#I_u(\mathcal{G}^{\delta_1}_{j-1}(TU_i)) \le \tfrac{1}{2}\sum_{i=1}^{\ell_0} \#L_u(\mathcal{G}^{\delta_1}_{j-1}(TU_i))\,. \tag{5.7}$$

Using the claim and (5.7) we split the sum over j into 2 parts,

$$\#I_u(\mathcal{M}^0_{-n}) \le K_2 n + 2\sum_{j=1}^{n_1-1} \#I_u(T^j\mathcal{S}_0) + \sum_{j=n_1}^{n}\sum_{i=1}^{\ell_0} \#L_u(\mathcal{G}^{\delta_1}_{j-1}(TU_i))\,. \tag{5.8}$$

The cardinality of the sum over the first n_1 terms is bounded by a fixed constant depending on n_1, but not on n; let us call it $\bar{C}_{n_1}$. We want to relate the sum over the terms for $j \ge n_1$ to $L_u(\mathcal{M}^0_{-n})$. To this end, we follow the proof of Lemma 5.2 and split $n-j$ into blocks of length n_1.

For each $n_1 \le j \le n - n_1$, write $n - j = kn_1 + \ell$ for some $k \ge 1$. If $V \in L_u(\mathcal{G}^{\delta_1}_{j-1}(TU_i))$, then $|T^{n-j}V| \ge C_1\Lambda^{n-j}\delta_1/3$, while $T^{n-j}V$ can be cut into at most $(Kn_1+1)^k$ pieces. Since we have chosen $\varepsilon = 1/4$ in the application of Lemma 5.2, by choice of n_1,

$$\#L_u(\mathcal{G}^{\delta_1}_{n-1}(TU_i)) \ge 4^k \#L_u(\mathcal{G}^{\delta_1}_{j-1}(TU_i)) \text{ for each } n_1 \le j \le n-n_1 \text{ and } k = \left\lfloor \frac{(n-j)}{n_1} \right\rfloor.$$

For $n - n_1 < j \le n$, we perform the same estimate, but relating j with $j + n_1$,

$$\#L_u(\mathcal{G}^{\delta_1}_{j+n_1-1}(TU_i)) \ge 4\#L_u(\mathcal{G}^{\delta_1}_{j-1}(TU_i)) \text{ for each } n - n_1 + 1 \le j \le n.$$

Gathering these estimates together and using (5.8), we obtain,

(5.9)

$$
\begin{aligned}
&\#I_u(\mathcal{M}^0_{-n}) \\
&\qquad \le K_2 n + \bar{C}_{n_1} + \sum_{j=n_1}^{n-n_1} 4^{-\lfloor (n-j)/n_1 \rfloor} \#L_u(T^n \mathcal{S}_0) + \sum_{j=n-n_1+1}^{n} \tfrac{1}{4} \#L_u(T^{j+n_1}\mathcal{S}_0) \\
&\qquad \le 2K_2 n + \bar{C}_{n_1} + C\delta_1^{-1} n_1 \#L_u(\mathcal{M}^0_{-n}) + \sum_{j=n-n_1+1}^{n} C\delta_1^{-1} \#L_u(\mathcal{M}^0_{-j-n_1}),
\end{aligned}
$$

where the second inequality uses $\#L_u(T^\ell \mathcal{S}_0) \le C\delta_1^{-1} L_u(\mathcal{M}^0_{-\ell}) + K_2$ for $\ell \ge n$, which stems from the same noncrossing property used earlier: a curve in $T^\ell(\mathcal{S}_0)$ must terminate on a curve in $T^i(\mathcal{S}_0)$ if the two intersect for $i < \ell$.

To estimate the final sum in (5.9), note that if $A \in L_u(\mathcal{M}^0_{-n-1})$, then $A \subseteq A' \in L_u(\mathcal{M}^0_{-n})$. Moreover, there exists a constant $B > 0$, independent of n, such that each $A' \in L_u(\mathcal{M}^0_{-n})$ can contain at most B elements of $L_u(\mathcal{M}^0_{-n-1})$. (Indeed by Lemma 3.3, B is at most $|\mathring{\mathcal{P}}|$, and depends only on $\mathcal{S}_1$.) Inductively then,

$$
\sum_{j=1}^{n_1} \#L_u(\mathcal{M}^0_{-n-j}) \le \sum_{j=1}^{n_1} B^j \#L_u(\mathcal{M}^0_{-n}) \le C B^{n_1} \#L_u(\mathcal{M}^0_{-n}).
$$

Putting this estimate together with (5.9) yields,

$$
\#I_u(\mathcal{M}^0_{-n}) \le \#L_u(\mathcal{M}^0_{-n}) C\delta_1^{-1}(n_1 + B^{n_1}) + C_{n_1} + 2K_2 n.
$$

Using $\#\mathcal{M}^0_{-n} = \#L_u(\mathcal{M}^0_{-n}) + \#I_u(\mathcal{M}^0_{-n})$, this implies,

$$
\#L_u(\mathcal{M}^0_{-n}) \ge \frac{\#\mathcal{M}^0_{-n} - C_{n_1} - 2K_2 n}{1 + C\delta_1^{-1}(n_1 + B^{n_1})}.
$$

Since $\#\mathcal{M}^0_{-n}$ increases at an exponential rate and n_1 is fixed, there exists $n_2 \in \mathbb{N}$ such that $\#\mathcal{M}^0_{-n} - \bar{C}_{n_1} - 2K_2 n \ge \frac{1}{2}\#\mathcal{M}^0_{-n}$ for $n \ge n_2$. Thus there exists $C_{n_1} > 0$ such that for $n \ge n_2$, $\#L_u(\mathcal{M}^0_{-n}) \ge C_{n_1}\delta_1 \#\mathcal{M}^0_{-n}$, as required. □

5.3. **Exact exponential growth of $\#\mathcal{M}_0^n$—Cantor rectangles.** It follows from submultiplicativity of $\#\mathcal{M}_0^n$ that $e^{nh_*} \le \#\mathcal{M}_0^n$ for all n. In this subsection, we shall prove a supermultiplicativity statement (Lemma 5.6) from which we deduce the upper bound for $\#\mathcal{M}_0^n$ in Proposition 4.6 giving the upper bound in Proposition 4.7, and ultimately the upper bound on the spectral radius of $\mathcal{L}$ on $\mathcal{B}$.

The following key estimate is a lower bound on the rate of growth of stable curves having a certain length. The proof will crucially use the fact that the SRB measure is mixing in order to bootstrap from Lemma 5.4.

Proposition 5.5. *Let δ_1 be the value of δ from Lemma 5.2 associated with $\varepsilon = 1/4$ (see (5.6)). There exists $c_0 > 0$ such that for all $W \in \widehat{\mathcal{W}}^s$ with $|W| \ge \delta_1/3$ and $n \ge 1$, we have $\#\mathcal{G}_n(W) \ge c_0 \#\mathcal{M}_0^n$. The constant c_0 depends on δ_1.*

This will be used for the lower bound in Section 6.3. It also has the following important consequence.

Lemma 5.6 (Supermultiplicativity)**.** *There exists $c_1 > 0$ such that $\forall n, j \in \mathbb{N}$, with $j \le n$, we have*

$$
\#\mathcal{M}_0^n \ge c_1 \#\mathcal{M}_0^{n-j} \#\mathcal{M}_0^j.
$$

We next introduce Cantor rectangles. Let $W^s(x)$ and $W^u(x)$ denote the maximal smooth components of the local stable and unstable manifolds of $x \in M$.

Definition 5.7 ((Locally maximal) Cantor rectangles)**.** A solid rectangle D in M is a closed region whose boundary comprises precisely four nontrivial curves: two stable manifolds and two unstable manifolds. Given a solid rectangle D, the *locally maximal Cantor rectangle* R in D is formed by taking the union of all points in D whose local stable and unstable manifolds completely cross D. Locally maximal Cantor rectangles have a natural product structure: for any $x, y \in R$, $W^s(x) \cap W^u(y) \in R$, where $W^{s/u}(x)$ is the local stable/unstable manifold containing x. It is proved in [CM, Section 7.11] that such rectangles are closed and as such contain their outer boundaries, which coincide with the boundary of D. We shall refer to this pair of stable and unstable manifolds as the stable and unstable boundaries of R. In this case, we denote D by $D(R)$ to emphasize that it is the smallest solid rectangle containing R. We shall sometimes drop the words "locally maximal" referring simply to Cantor rectangles R.

Definition 5.8 (Properly crossing a (locally maximal) Cantor rectangle)**.** For a (locally maximal) Cantor rectangle R such that

$$\inf_{x \in R} \frac{m_{W^u}(W^u(x) \cap R)}{m_{W^u}(W^u(x) \cap D(R))} \geq 0.9 \,, \tag{5.10}$$

we[25] say a stable curve $W \in \widehat{\mathcal{W}}^s$ *properly crosses* R if

a) W crosses both unstable sides of R;

b) for every $x \in R$, the intersection $W \cap W^s(x) \cap D(R) = \emptyset$, i.e., W does not cross any stable manifolds in R;

c) for all $x \in R$, the point $W \cap W^u(x)$ divides the curve $W^u(x) \cap D(R)$ in a ratio between 0.1 and 0.9, i.e., W does not come too close to either unstable boundary of R.

Remark 5.9. The (unstable analogue of) condition b) is not needed in its full strength, even in the proof of [CM, Lemma 7.90]. What is used there is that the fake unstable is trapped between two real unstable that it does not cross. Since the real unstable intersect and fully cross the target rectangle, this forces the fake unstable to do so as well. For us, we reverse time and consider stable manifolds. For real stable manifolds, condition b) is not needed at all: If a real stable fully crosses the initial rectangle, then, when it intersects the target rectangle under iteration by T^{-n}, it must intersect a real stable manifold, and it must fully cross. (Otherwise, the preimage of a singularity would lie on a real stable manifold in the interior of the target rectangle. But this cannot be since real stable manifolds are never cut going forward and so do not intersect the preimages of singularity curves except at their end points.) When discussing proper crossing for real stable manifolds, we will drop condition b) and allow $W \in \mathcal{W}^s$ to be one of the stable manifolds defining R.

Proof of Proposition 5.5. Using [CM, Lemma 7.87], we may cover M by Cantor rectangles $R_1, \dots, R_k$ satisfying (5.10) whose stable and unstable boundaries have length at most $\frac{1}{10}\delta_1$, with the property that any stable curve of length at least $\delta_1/3$

[25] This is a version of Definition 7.85 of [CM] formulated with stable (instead of unstable) curves crossing R. We have also dropped any mention of homogeneous components, which are used in the construction in [CM].

properly crosses at least one of them. The cardinality k is fixed, depending only on δ_1.

Recall that $L_u(\mathcal{M}_{-n}^0)$ denotes the elements of $\mathcal{M}_{-n}^0$ whose unstable diameter is longer than $\delta_1/3$. We claim that for all $n \in \mathbb{N}$, at least one R_i is fully crossed in the unstable direction by at least $\frac{1}{k}\#L_u(\mathcal{M}_{-n}^0)$ elements of $\mathcal{M}_{-n}^0$. Notice that if $A \in \mathcal{M}_{-n}^0$, then ∂A is comprised of unstable curves belonging to $\bigcup_{i=1}^n T^i\mathcal{S}_0$, and possibly $\mathcal{S}_0$. By definition of unstable manifolds, $T^i\mathcal{S}_0$ cannot intersect the unstable boundaries of the R_i; thus if $A \cap R_i \neq \emptyset$, then either ∂A terminates inside R_i or A fully crosses R_i. Thus elements of $L_u(\mathcal{M}_{-n}^0)$ fully cross at least one R_i and so at least one R_i must be fully crossed by $1/k$ of them, proving the claim.

For each $n \in \mathbb{N}$, denote by i_n the index of a rectangle R_{i_n} which is fully crossed by at least $\frac{1}{k}\#L_u(\mathcal{M}_{-n}^0)$ elements of $\mathcal{M}_{-n}^0$. The main idea at this point will be to force every stable curve to properly cross R_{i_n} in a bounded number of iterates and so to intersect all elements of $\mathcal{M}_{-n}^0$ that fully cross R_{i_n}.

To this end, fix $\delta_* \in (0, \delta_1/10)$ and for $i = 1, \ldots, k$, choose a "high density" subset $R_i^* \subset R_i$ satisfying the following conditions: R_i^* has nonzero Lebesgue measure, and for any unstable manifold W^u such that $W^u \cap R_i^* \neq \emptyset$ and $|W^u| < \delta_*$, we have $\frac{m_{W^u}(W^u \cap R_i^*)}{|W^u|} \geq 0.9$. (Such a δ_* and R_i^* exist due to the fact that m_{W^u}-almost every $y \in R_i$ is a Lebesgue density point of the set $W^u(y) \cap R_i$ and the unstable foliation is absolutely continuous with respect to μ_{SRB} or, equivalently, Lebesgue.)

Due to the mixing property of μ_{SRB} and the finiteness of the number of rectangles R_i, there exist $\varepsilon > 0$ and $n_3 \in \mathbb{N}$ such that for all $1 \leq i, j \leq k$ and all $n \geq n_3$, $\mu_{\mathrm{SRB}}(R_i^* \cap T^{-n}R_j) \geq \varepsilon$. If necessary, we increase n_3 so that the unstable diameter of the set $T^{-n}R_i$ is less than δ_* for each i, and $n \geq n_3$.

Now let $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$ be arbitrary. Let R_j be a Cantor rectangle that is properly crossed by W. Let $n \in \mathbb{N}$ and let i_n be as above. By mixing, $\mu_{\mathrm{SRB}}(R_{i_n}^* \cap T^{-n_3}R_j) \geq \varepsilon$. By [CM, Lemma 7.90], there is a component of $T^{-n_3}W$ that fully crosses $R_{i_n}^*$ in the stable direction. Call this component $V \in \mathcal{G}_{n_3}^{\delta_1}(W)$. By choice of R_{i_n}, this implies that $\#\mathcal{G}_n(V) \geq \frac{1}{k}\#L_u(\mathcal{M}_{-n}^0)$, and thus

$$\#\mathcal{G}_{n+n_3}(W) \geq \tfrac{1}{k}\#L_u(\mathcal{M}_{-n}^0) \implies \#\mathcal{G}_n(W) \geq \tfrac{C'}{k}\#L_u(\mathcal{M}_{-n}^0)\,,$$

where C' is a constant depending only on n_3 since at each refinement of $\mathcal{M}_{-j}^0$ to $\mathcal{M}_{-j-1}^0$, the cardinality of the partition increases by a factor which is at most $|\mathring{\mathcal{P}}|$, as noted in the proof of Lemma 5.4. The final estimate needed is $\#L_u(\mathcal{M}_{-n}^0) \geq C_{n_1}\delta_1\#\mathcal{M}_{-n}^0$ for $n \geq n_2$ from Lemma 5.4. Thus the proposition holds for $n \geq \max\{n_2, n_3\}$. It extends to all $n \in \mathbb{N}$ since $\#\mathcal{M}_0^n \leq (\#\mathcal{M}_0^1)^n$ and there are only finitely many values of n to correct for. $\square$

Proof of Lemma 5.6. Recall the singularity sets defined for $n, k \in \mathbb{N}$ by $\mathcal{S}_n = \bigcup_{i=0}^n T^{-i}\mathcal{S}_0$ and $\mathcal{S}_{-k} = \bigcup_{i=0}^k T^i\mathcal{S}_0$. Due to the relation, $T^{-k}(\mathcal{S}_{-k} \cup \mathcal{S}_n) = \mathcal{S}_k \cup T^{-k}\mathcal{S}_n = \mathcal{S}_{n+k}$, we have a one-to-one correspondence between elements of $\mathcal{M}_{-k}^n$ and $\mathcal{M}_0^{n+k}$.

Now fix $n, j \in \mathbb{N}$ with $j < n$. Using the above relation, we have

$$\#\mathcal{M}_0^n = \#\mathcal{M}_{-j}^{n-j} = \#\big(\mathcal{M}_0^{n-j} \vee \mathcal{M}_{-j}^0\big)\,.$$

In order to prove the lemma, it suffices to show that a positive fraction (independent of n and j) of elements of $\mathcal{M}_0^{n-j}$ intersect a positive fraction of elements of

$\mathcal{M}^0_{-j}$. Note that $\partial\mathcal{M}_0^{n-j}$ is comprised of stable curves, while $\partial\mathcal{M}^0_{-j}$ is comprised of unstable curves.

Recall that $L_u(\mathcal{M}^0_{-j})$ denotes the elements of $\mathcal{M}^0_{-j}$ whose unstable diameter is longer than $\delta_1/3$. Similarly, $L_s(\mathcal{M}_0^{n-j})$ denotes those elements of $\mathcal{M}_0^{n-j}$ whose stable diameter is longer than $\delta_1/3$. By Lemma 5.4,

$$\#L_s(\mathcal{M}_0^{n-j}) \geq C_{n_1}\delta_1\#\mathcal{M}_0^{n-j} \quad \text{for } n-j \geq n_2 .$$

Let $A \in L_s(\mathcal{M}_0^{n-j})$ and let $V \in \widehat{\mathcal{W}}^s$ be a stable curve in A with length at least $\delta_1/3$. By Proposition 5.5, $\#\mathcal{G}_j(V) \geq c_0\#\mathcal{M}_0^j$. Each component of $\mathcal{G}_j(V)$ corresponds to one component of $V \setminus \mathcal{S}_{-j}$ (up to subdivision of long pieces in $\mathcal{G}_j(V)$). Thus V intersects at least $c_0\#\mathcal{M}_0^j = c_0\#\mathcal{M}^0_{-j}$ elements of $\mathcal{M}^0_{-j}$. Since this holds for all $A \in L_s(\mathcal{M}_0^{n-j})$, we have

$$\#\mathcal{M}_0^n = \#\big(\mathcal{M}_0^{n-j} \vee \mathcal{M}^0_{-j}\big) \geq \#L_s(\mathcal{M}_0^{n-j}) \cdot c_0\#\mathcal{M}_0^j \geq C_{n_1}\delta_1 c_0\#\mathcal{M}_0^{n-j}\#\mathcal{M}_0^j ,$$

proving the lemma with $c_1 = c_0 C_{n_1}\delta_1$ when $n-j \geq n_2$. For $n-j \leq n_2$, since $\#\mathcal{M}_0^{n-j} \leq (\#\mathcal{M}_0^1)^{n-j}$, we obtain the lemma by decreasing c_1 since there are only finitely many values to correct for. $\square$

Proof of Proposition 4.6. Define $\psi(n) = \#\mathcal{M}_0^n e^{-nh_*}$, and note that $\psi(n) \geq 1$ for all n. From Lemma 5.6 it follows that

$$\psi(n) \geq c_1\psi(j)\psi(n-j) \quad \text{for all } n \in \mathbb{N} \text{ and } 0 \leq j \leq n. \tag{5.11}$$

Suppose there exists $n_1 \in \mathbb{N}$ such that $\psi(n_1) \geq 2/c_1$. Then using (5.11), we have

$$\psi(2n_1) \geq c_1\psi(n_1)\psi(n_1) \geq \frac{4}{c_1} .$$

Iterating this bound, we have inductively for any $k \geq 1$,

$$\psi(2kn_1) \geq c_1\psi(2n_1)\psi(2(k-1)n_1) \geq c_1\frac{4}{c_1}\frac{4^{k-1}}{c_1} = \frac{4^k}{c_1} .$$

This implies that $\lim_{k\to\infty}\frac{1}{2kn_1}\log\psi(2kn_1) \geq \frac{\log 4}{2n_1}$, which contradicts the definition of $\psi(n)$ (since $\lim_{n\to\infty}\frac{1}{n}\log\psi(n) = 0$). We conclude that $\psi(n) \leq 2/c_1$ for all $n \geq 1$. $\square$

Our final result of this section demonstrates the uniform exponential rate of growth enjoyed by all stable curves of length at least $\delta_1/3$.

Corollary 5.10. *For all stable curves $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$ and all $n \geq n_1$, we have*

$$\frac{2\delta_1 c_0}{9}e^{nh_*} \leq |T^{-n}W| \leq \frac{4}{c_1}e^{nh_*} .$$

Proof. For $W \in \widehat{\mathcal{W}}^s$ with $|W| \leq \delta_1/3$, Lemma 5.1b) with $\bar\gamma = 0$ together with Propositions 4.6 and 5.5 yield,

$$c_0e^{nh_*} \leq c_0\#\mathcal{M}_0^n \leq \#\mathcal{G}_n(W) \leq 2\delta_0^{-1}\#\mathcal{M}_0^n \leq \tfrac{4}{c_1\delta_0}e^{nh_*} .$$

The upper bound of the corollary is completed by noting that

$$|T^{-n}W| = \sum_{W_i\in\mathcal{G}_n(W)}|W_i| \leq \delta_0\#\mathcal{G}_n(W) .$$

The lower bound follows using (5.6) since $\#\mathcal{G}_n^{\delta_1}(W) \geq \#\mathcal{G}_n(W)$,

$$|T^{-n}W| = \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |W_i| \geq \frac{\delta_1}{3} \# L_n^{\delta_1}(W) \geq \tfrac{2\delta_1}{9} \# \mathcal{G}_n^{\delta_1}(W) \geq \tfrac{2\delta_1 c_0}{9} e^{nh_*} . \tag{5.12}$$

□

6. Proof of the "Lasota–Yorke" Proposition 4.7—spectral radius

6.1. **Weak norm and strong stable norm estimates.** We start with the weak norm estimate (4.9). Let $f \in C^1(M)$, $W \in \mathcal{W}^s$, and $\psi \in C^\alpha(W)$ be such that $|\psi|_{C^\alpha(W)} \leq 1$. For $n \geq 0$ we use the definition of the weak norm on each $W_i \in \mathcal{G}_n(W)$ to estimate

(6.1)
$$\int_W \mathcal{L}^n f \, \psi \, dm_W = \sum_{W_i \in \mathcal{G}_n(W)} \int_{W_i} f \, \psi \circ T^n \, dm_W \leq \sum_{W_i \in \mathcal{G}_n(W)} |f|_w |\psi \circ T^n|_{C^\alpha(W_i)} .$$

Clearly, $\sup |\psi \circ T^n|_{W_i} \leq \sup_W |\psi|$. For $x, y \in W_i$, we have,

$$\frac{|\psi(T^n x) - \psi(T^n y)|}{d_W(T^n x, T^n y)^\alpha} \cdot \frac{d_W(T^n x, T^n y)^\alpha}{d_W(x,y)^\alpha} \leq C |\psi|_{C^\alpha(W)} |J^s T^n|^\alpha_{C^0(W_i)} \tag{6.2}$$
$$\leq C \Lambda^{-\alpha n} |\psi|_{C^\alpha(W)} ,$$

so that $H^\alpha_{W_i}(\psi \circ T^n) \leq C \Lambda^{-\alpha n} H^\alpha_W(\psi)$ and thus $|\psi \circ T^n|_{C^\alpha(W_i)} \leq C |\psi|_{C^\alpha(W)}$. Using this estimate and Lemma 5.1b) with $\bar\gamma = 0$ in equation (6.1), we obtain

$$\int_W \mathcal{L}^n f \, \psi \, dm_W \ \leq \sum_{W_i \in \mathcal{G}_n(W)} C |f|_w \leq C \delta_0^{-1} |f|_w (\# \mathcal{M}_0^n) .$$

Taking the supremum over $W \in \mathcal{W}^s$ and $\psi \in C^\alpha(W)$ with $|\psi|_{C^\alpha(W)} \leq 1$ yields (4.9), using the upper bound on $\#\mathcal{M}_0^n$ in Proposition 4.6.

We now prove the strong stable norm estimate (4.10). Recall that our choice of m in (5.4) implies $2^{s_0 \gamma}(Km+1)^{1/m} < e^{h_*}$, where K is from (5.1). Define

$$D_n = D_n(m, \gamma) := 2^{2\gamma+1} \delta_0^{-1} \sum_{j=1}^n 2^{j s_0 \gamma} (Km+1)^{j/m} \# \mathcal{M}_0^{n-j} . \tag{6.3}$$

We claim that it follows from Proposition 4.6 that

$$D_n \leq C e^{nh_*} . \tag{6.4}$$

Indeed, by choice of γ and m, setting $\varepsilon_1 := h_* - \log(2^{s_0\gamma}(Km+1)^{1/m}) > 0$, we have

$$D_n = 2^{2\gamma+1} \delta_0^{-1} \sum_{j=1}^n 2^{j s_0 \gamma} (Km+1)^{j/m} \# \mathcal{M}_0^{n-j} \leq 2^{2\gamma+1} \delta_0^{-1} \sum_{j=1}^n e^{(h_* - \varepsilon_1) j} \frac{2}{c_1} e^{(n-j)h_*}$$
$$\leq 2^{2\gamma+1} \delta_0^{-1} \frac{2}{c_1} e^{nh_*} \sum_{j=1}^n e^{-\varepsilon_1 j} .$$

To prove the strong stable bound, let $W \in \mathcal{W}^s$ and $\psi \in C^\beta(W)$ with $|\psi|_{C^\beta(W)} \leq |\log |W||^\gamma$. Using equation (6.1), and applying the strong stable norm on each

$W_i \in \mathcal{G}_n(W)$, we write

$$\int_W \mathcal{L}^n f \, \psi \, dm_W = \sum_i \int_{W_i} f \, \psi \circ T^n \, dm_W \leq \sum_i \|f\|_s |\log |W_i||^{-\gamma} |\psi \circ T^n|_{C^\beta(W_i)} \, .$$

From the estimate analogous to (6.2), we have $|\psi \circ T^n|_{C^\beta(W_i)} \leq C|\psi|_{C^\beta(W)} \leq C|\log |W||^\gamma$. (Note that the contraction coming from the negative power of Λ in (6.2) cannot be exploited; see footnote 20 and the comments after Remark 4.8.)

Thus,

$$\int_W \mathcal{L}^n f \, \psi \, dm_W \leq C\|f\|_s \sum_{W_i \in \mathcal{G}_n(W)} \left(\frac{\log |W|}{\log |W_i|} \right)^\gamma \leq C\|f\|_s D_n \, ,$$

where we have used Lemma 5.1b) with $\bar{\gamma} = \gamma$.

Taking the supremum over W and ψ and recalling (6.4) proves (4.10), since we have shown that $\|\mathcal{L}^n f\|_s \leq C D_n \|f\|_s$.

6.2. **Unstable norm estimate.** Fix $\varepsilon \leq \varepsilon_0$ and consider two curves $W^1, W^2 \in \mathcal{W}^s$ with $d_{\mathcal{W}^s}(W^1, W^2) \leq \varepsilon$. For $n \geq 1$, we describe how to partition $T^{-n}W^\ell$ into "matched" pieces U_j^ℓ and "unmatched" pieces V_i^ℓ, $\ell = 1, 2$.

Let ω be a connected component of $W^1 \setminus \mathcal{S}_{-n}$. To each point $x \in T^{-n}\omega$, we associate a vertical line segment γ_x of length at most $C\Lambda^{-n}\varepsilon$ such that its image $T^n\gamma_x$, if not cut by a singularity, will have length $C\varepsilon$. By [CM, §4.4], all the tangent vectors to $T^i\gamma_x$ lie in the unstable cone $C^u(T^i x)$ for each $i \geq 1$ so that they remain uniformly transverse to the stable cone and enjoy the minimum expansion given by Λ.

Doing this for each connected component of $W^1 \setminus \mathcal{S}_{-n}$, we subdivide $W^1 \setminus \mathcal{S}_{-n}$ into a countable collection of subintervals of points for which $T^n\gamma_x$ intersects $W^2 \setminus \mathcal{S}_{-n}$ and subintervals for which this is not the case. This in turn induces a corresponding partition on $W^2 \setminus \mathcal{S}_{-n}$.

We denote by V_i^ℓ the pieces in $T^{-n}W^\ell$ which are not matched up by this process and note that the images $T^n V_i^\ell$ occur either at the endpoints of W^ℓ or because the vertical segment γ_x has been cut by a singularity. In both cases, the length of the curves $T^n V_i^\ell$ can be at most $C\varepsilon$ due to the uniform transversality of $\mathcal{S}_{-n}$ with the stable cone and of $C^s(x)$ with $C^u(x)$.

In the remaining pieces the foliation $\{T^n\gamma_x\}_{x \in T^{-n}W^1}$ provides a one-to-one correspondence between points in W^1 and W^2. We further subdivide these pieces in such a way that the lengths of their images under T^{-i} are less than δ_0 for each $0 \leq i \leq n$ and the pieces are pairwise matched by the foliation $\{\gamma_x\}$. We call these matched pieces U_j^ℓ. Since the stable cone is bounded away from the vertical direction, we can adjust the elements of $\mathcal{G}_n(W^\ell)$ created by artificial subdivisions due to length so that $U_j^\ell \subset W_i^\ell$ and $V_k^\ell \subset W_{i'}^\ell$ for some $W_i^\ell, W_{i'}^\ell \in \mathcal{G}_n(W^\ell)$ for all $j, k \geq 1$ and $\ell = 1, 2$, without changing the cardinality of the bound on $\mathcal{G}_n(W^\ell)$. There is at most one U_j^ℓ and two V_j^ℓ per $W_i^\ell \in \mathcal{G}_n(W^\ell)$.

In this way we write $W^\ell = (\bigcup_j T^n U_j^\ell) \cup (\bigcup_i T^n V_i^\ell)$. Note that the images $T^n V_i^\ell$ of the unmatched pieces must be short while the images of the matched pieces U_j^ℓ may be long or short.

We have arranged a pairing of the pieces $U_j^\ell = G_{U_j^\ell}(I_j)$, $\ell = 1, 2$, with the property:

$$\text{If } U_j^1 = \{(r, \varphi_{U_j^1}(r)) : r \in I_j\}, \text{ then } U_j^2 = \{(r, \varphi_{U_j^2}(r)) : r \in I_j\}, \tag{6.5}$$

so that the point $x = (r, \varphi_{U_j^1}(r))$ is associated with the point $\bar{x} = (r, \varphi_{U_j^2}(r))$ by the vertical segment $\gamma_x \subset \{(r,s)\}_{s \in [-\pi/2, \pi/2]}$ for each $r \in I_j$.

Given ψ_ℓ on W^ℓ with $|\psi_\ell|_{C^\alpha(W^\ell)} \le 1$ and $d(\psi_1, \psi_2) \le \varepsilon$, we must estimate

(6.6)
$$\left| \int_{W^1} \mathcal{L}^n f\, \psi_1\, dm_W - \int_{W^2} \mathcal{L}^n f\, \psi_2\, dm_W \right| \le \sum_{\ell, i} \left| \int_{V_i^\ell} f\, \psi_\ell \circ T^n\, dm_W \right| + \sum_j \left| \int_{U_j^1} f\, \psi_1 \circ T^n\, dm_W - \int_{U_j^2} f\, \psi_2 \circ T^n\, dm_W \right| .$$

We first estimate the differences of matched pieces U_j^ℓ. The function $\phi_j = \psi_1 \circ T^n \circ G_{U_j^1} \circ G_{U_j^2}^{-1}$ is well-defined on U_j^2, and we can estimate,

(6.7)
$$\left| \int_{U_j^1} f\, \psi_1 \circ T^n - \int_{U_j^2} f\, \psi_2 \circ T^n \right| \le \left| \int_{U_j^1} f\, \psi_1 \circ T^n - \int_{U_j^2} f\, \phi_j \right| + \left| \int_{U_j^2} f (\phi_j - \psi_2 \circ T^n) \right| .$$

We bound the first term in equation (6.7) using the strong unstable norm. As before, (6.2) implies $|\psi_1 \circ T^n|_{C^\alpha(U_j^1)} \le C |\psi_1|_{C^\alpha(W^1)} \le C$. We have $|G_{U_j^1} \circ G_{U_j^2}^{-1}|_{C^1} \le C_g$ for some $C_g > 0$ due to the fact that each curve U_j^ℓ has uniformly bounded curvature and slopes bounded away from infinity. Thus

$$|\phi_j|_{C^\alpha(U_j^2)} \le C C_g |\psi_1|_{C^\alpha(W^1)} . \tag{6.8}$$

Moreover, $d(\psi_1 \circ T^n, \phi_j) = \left| \psi_1 \circ T^n \circ G_{U_j^1} - \phi_j \circ G_{U_j^2} \right|_{C^0(I_j)} = 0$ by the definition of ϕ_j.

To complete the bound on the first term of (6.7), we need the following estimate from [DZ1, Lemma 4.2]: There exists $C > 0$, independent of W^1 and W^2, such that

$$d_{\mathcal{W}^s}(U_j^1, U_j^2) \le C \Lambda^{-n} n \varepsilon =: \varepsilon_1 \qquad \forall j . \tag{6.9}$$

In view of (6.8), we renormalize the test functions by CC_g. Then we apply the definition of the strong unstable norm with ε_1 in place of ε. Thus,

$$\sum_j \left| \int_{U_j^1} f\, \psi_1 \circ T^n - \int_{U_j^2} f\, \phi_j \right| \le (CC_g) C \delta_0^{-1} |\log \varepsilon_1|^{-\varsigma} \|f\|_u (\#\mathcal{M}_0^n) , \tag{6.10}$$

where we used Lemma 5.1b) with $\bar{\gamma} = 0$ since there is at most one matched piece U_j^1 corresponding to each component $W_i^1 \in \mathcal{G}_n(W^1)$ of $T^{-n} W^1$.

It remains to estimate the second term in (6.7) using the strong stable norm

$$\left| \int_{U_j^2} f (\phi_j - \psi_2 \circ T^n) \right| \le \|f\|_s |\log |U_j^2||^{-\gamma} \, |\phi_j - \psi_2 \circ T^n|_{C^\beta(U_j^2)} . \tag{6.11}$$

In order to estimate the C^β-norm of the function in (6.11), we use that $|G_{U_j^2}|_{C^1} \le C_g$ and $|G^{-1}_{U_j^2}|_{C^1} \le C_g$ to write

$$|\phi_j - \psi_2 \circ T^n|_{C^\beta(U_j^2)} \;\le\; C_g|\psi_1 \circ T^n \circ G_{U_j^1} - \psi_2 \circ T^n \circ G_{U_j^2}|_{C^\beta(I_j)}\,. \tag{6.12}$$

The difference can now be bounded by the following estimate from [DZ1, Lemma 4.4]:

$$|\psi_1 \circ T^n \circ G_{U_j^1} - \psi_2 \circ T^n \circ G_{U_j^2}|_{C^\beta(I_j)} \le C\varepsilon^{\alpha-\beta}\,. \tag{6.13}$$

Indeed, using (6.13) together with (6.12) yields by (6.11)

$$\begin{aligned}&\sum_j \Big| \int_{U_j^2} f(\phi_j - \psi_2 \circ T^n)\, dm_W \Big| \\ &\qquad \le C\|f\|_s \sum_j |\log|U_j^2||^{-\gamma}\, \varepsilon^{\alpha-\beta} \le C|\log\delta_0|^{-\gamma}\|f\|_s\varepsilon^{\alpha-\beta}2\delta_0^{-1}(\#\mathcal{M}_0^n)\,,\end{aligned} \tag{6.14}$$

where used (as in (6.10)) Lemma 5.1b) with $\bar\gamma = 0$ since there is at most one matched piece U_j^2 corresponding to each component $W_i^2 \in \mathcal{G}_n(W^2)$ of $T^{-n}W^2$. Since $\delta_0 < 1$ is fixed, this completes the estimate on the second term of matched pieces in (6.7).

We next estimate over the unmatched pieces V_i^ℓ in (6.6), using the strong stable norm. Note that by (6.2), $|\psi_\ell \circ T^n|_{C^\beta(V_i^\ell)} \le C|\psi_\ell|_{C^\alpha(W^\ell)} \le C$. The relevant sum for unmatched pieces in $\mathcal{G}_n(W^1)$ is

$$\sum_i \int_{V_i^1} f\psi_1 \circ T^n\, dm_{V_i^1} \tag{6.15}$$

with a similar sum for unmatched pieces in $\mathcal{G}_n(W^2)$.

We say an unmatched curve V_i^1 is created at time j, $1 \le j \le n$, if j is the first time that $T^{n-j}V_i^1$ is not part of a matched element of $\mathcal{G}_j(W^1)$. Indeed, there may be several curves V_i^1 (in principle exponentially many in $n-j$) such that $T^{n-j}V_i^1$ belongs to the same unmatched element of $\mathcal{G}_j(W^1)$. Define

$$\begin{aligned}A_{j,k} = \{i : V_i^1 &\text{ is created at time } j \\ &\text{and } T^{n-j}V_i^1 \text{ belongs to the unmatched curve } W_k^1 \subset T^{-j}W^1\}\,.\end{aligned}$$

Due to the uniform hyperbolicity of T, and, again, uniform transversality of $\mathcal{S}_{-n}$ with the stable cone and of $C^s(x)$ with $C^u(x)$, we have $|W_k^1| \le C\Lambda^{-j}\varepsilon$.

Let δ_1 be the value of $\delta \le \delta_0$ from Lemma 5.2 associated with $\varepsilon = 1/4$ (recall (5.6)). For a certain time, the iterate $T^{-q}W_k^1$ remains shorter than length δ_1. In this case, by Lemma 5.1a) for $\bar\gamma = 0$, its complexity grows subexponentially,

$$\#\mathcal{G}_q(W_k^1) \le (Km+1)^{q/m}\,. \tag{6.16}$$

We would like to establish the maximal value of q as a function of j.

More precisely, we want to find $q(j)$ so that any $q \le q(j)$ satisfies the conditions:

(a) $T^{-q}W_k^1$ remains shorter than length δ_1;

(b) $\dfrac{|\log|T^{-q}W_k^1||^{-\gamma}}{|\log\varepsilon|^{-\varsigma}} \le 1$.

For (a), we use (5.3) together with the fact that $|W_k^1| \le C\Lambda^{-j}\varepsilon$ to estimate

$$|T^{-q}W_k^1| \le \delta_1 \;\Longleftarrow\; C''|W_k^1|^{2^{-s_0 q}} \le \delta_1 \;\Longleftarrow\; C''\Lambda^{-j2^{-s_0 q}}\varepsilon^{2^{-s_0 q}} \le \delta_1\,.$$

Omitting the $\varepsilon^{2^{-s_0 q}}$ factor and solving the last inequality for q yields,

$$q \le \frac{\log j}{s_0 \log 2} + C_2\,, \text{ where } C_2 = \frac{\log(\frac{\log \Lambda}{|\log(\delta_1/C'')|})}{s_0 \log 2}\,. \tag{6.17}$$

For (b), we again use (5.3) to bound $|T^{-q}W_k^1| \le C''(\Lambda^{-j}\varepsilon)^{2^{-s_0 q}}$, so that

$$\frac{|\log(\Lambda^{-j}\varepsilon)^{2^{-s_0 q}}|^{-\gamma}}{|\log \varepsilon|^{-\varsigma}} \le 1 \implies 2^{\gamma s_0 q}|\log \varepsilon|^{\varsigma} \le (|\log \varepsilon| + j \log \Lambda)^{\gamma} \tag{6.18}$$

implies (b). In turn, (6.18) is implied by

$$q \le \frac{(\gamma - \varsigma)\log j}{\gamma s_0 \log 2}\,. \tag{6.19}$$

Since the bound in (6.19) is smaller than that in (6.17) for j larger than some fixed constant depending only on δ_1, s_0, and C'', we will use (6.19) to define $q(j)$.

Now we return to the estimate in (6.15). Grouping the unmatched pieces V_i^1 by their creation times j, we estimate, [26]

$$\begin{aligned}
&\sum_i \int_{V_i^1} f\,\psi_1 \circ T^n\, dm_{V_i^1} \\
&\quad = \sum_{j=1}^n \sum_{i \in A_{j,k}} \int_{T^{n-j}V_i^1} (\mathcal{L}^{n-j} f)\, \psi \circ T^j = \sum_{j=1}^n \sum_k \int_{W_k^1} (\mathcal{L}^{n-j} f)\, \psi \circ T^j \\
&\quad \le \sum_{j=1}^n \sum_k \sum_{V_\ell \in \mathcal{G}_{q(j)}(W_k^1)} \int_{V_\ell} (\mathcal{L}^{n-j-q(j)} f)\, \psi \circ T^{j+q(j)} \\
&\quad \le \sum_{j=1}^n \sum_k \sum_{V_\ell \in \mathcal{G}_{q(j)}(W_k^1)} \|\mathcal{L}^{n-j-q(j)} f\|_s C |\log|V_\ell||^{-\gamma} \\
&\quad \le C\|f\|_s \sum_{j=1}^n \#\mathcal{M}_0^j \#\mathcal{M}_0^{n-j-q(j)} (Km+1)^{q(j)/m} |\log(\Lambda^{-j}\varepsilon)|^{2^{-s_0 q(j)}})^{-\gamma}\,,
\end{aligned}$$

where we have used (6.16) to bound $\#\mathcal{G}_{q(j)}(W_k^1)$, the cardinality $\#\mathcal{M}_0^j$ to bound the cardinality of the possible pieces $W_k^1 \subset T^{-j}W^1$, the estimate $\|\mathcal{L}^{n-j-q(j)} f\|_s \le C\#\mathcal{M}_0^{n-j-q(j)}\|f\|_s$, and, again $|T^{-q}W_k^1| \le C''(\Lambda^{-j}\varepsilon)^{2^{-s_0 q}}$. We also have, by the supermultiplicativity Lemma 5.6,

$$\#\mathcal{M}_0^j \#\mathcal{M}_0^{n-j-q(j)} \le C e^{-q(j)h_*} \#\mathcal{M}_0^n\,.$$

Thus using (b) in the definition of $q(j)$ (or, more precisely, (6.18)), we estimate

$$\sum_i \int_{V_i^1} f\psi_1 \circ T^n\, dm_{V_i^1} \le C\|f\|_s |\log \varepsilon|^{-\varsigma} \#\mathcal{M}_0^n \sum_{j=1}^n (Km+1)^{q(j)/m} e^{-q(j)h_*}\,. \tag{6.20}$$

[26] When we sum the integrals in the first line over the different $T^{n-j}V_i^1$, we find the integral over W_k^1 since the union of those pieces is precisely W_k^1.

For the final sum over j, we let $\varepsilon_2 = \frac{1}{m}\log(Km+1)$ and use (6.19),

$$\sum_{j=1}^{n}(Km+1)^{q(j)/m}e^{-q(j)h_*} = \sum_{j=1}^{n} e^{-q(j)(h_*-\varepsilon_2)} \le \sum_{j=1}^{n} e^{-(h_*-\varepsilon_2)\frac{(\gamma-\varsigma)\log j}{\gamma s_0 \log 2}}$$
$$= \sum_{j=1}^{n} j^{-(h_*-\varepsilon_2)\frac{\gamma-\varsigma}{\gamma s_0 \log 2}} .$$

Then by (6.20), since the exponent of j in the above sum is strictly negative by choice of m (see (5.4)), there exist $C<\infty$ and $\varpi \in [0,1)$ such that the contribution to $\|\mathcal{L}^n f\|_u$ of the unmatched pieces is bounded by

$$\sum_{\ell,i}\left|\int_{V_i^\ell} f\,\psi_\ell \circ T^n\, dm_W\right| \le C|\log\varepsilon|^{-\varsigma} n^{\varpi}\#\mathcal{M}_0^n \|f\|_s . \tag{6.21}$$

Now we use (6.21) together with (6.10) and (6.14) to estimate (6.6)

$$\left|\int_{W^1}\mathcal{L}^n f\,\psi_1\, dm_W - \int_{W^2}\mathcal{L}^n f\,\psi_2\, dm_W\right|$$
$$\le C\delta_0^{-1}\|f\|_u|\log\varepsilon_1|^{-\varsigma}\#\mathcal{M}_0^n + C\delta_0^{-1}(n^{\varpi}\|f\|_s|\log\varepsilon|^{-\varsigma} + \|f\|_s\varepsilon^{\alpha-\beta})\#\mathcal{M}_0^n .$$

Dividing through by $|\log\varepsilon|^{-\varsigma}$ and taking the appropriate suprema, we complete the proof of (4.11), recalling Proposition 4.6.

Finally, we study the consequences of the additional assumption $h_* > s_0\log 2$ on the estimate over unmatched pieces. In this case, again recalling (5.4) and following, we may choose $\varsigma > 0$ small enough such that

$$\varepsilon_1 := h_* - \frac{1}{m}\log(Km+1) - \frac{\gamma}{\gamma-\varsigma}s_0\log 2 > 0 .$$

Then

$$\sum_{j=1}^{n} j^{-(h_*-\varepsilon_2)\frac{\gamma-\varsigma}{\gamma s_0\log 2}} = \sum_{j=1}^{n} j^{-1-\varepsilon_1\frac{\gamma-\varsigma}{\gamma s_0\log 2}} < \infty .$$

Thus, by (6.20), the contribution to $\|\mathcal{L}^n f\|_u$ of the unmatched pieces is bounded by

$$\sum_{\ell,i}\left|\int_{V_i^\ell} f\,\psi_\ell \circ T^n\, dm_W\right| \le C|\log\varepsilon|^{-\varsigma}\#\mathcal{M}_0^n\|f\|_s \tag{6.22}$$

if $h_* > s_0\log 2$. So we find (4.12) for $h_* > s_0\log 2$ by replacing (6.21) with (6.22).

6.3. **Upper and lower bounds on the spectral radius.** We now deduce the bounds of Theorem 4.10 from the inequalities of Proposition 4.7 and the rate of growth of stable curves proved in Proposition 5.5.

Proof of Theorem 4.10. The upper bounds (4.13) and (4.15) are immediate consequences of Proposition 4.7. To prove the lower bound on $|\mathcal{L}^n 1|_w$, recall the choice of $\delta_1 = \delta > 0$ from Lemma 5.2 for $\varepsilon = 1/4$, giving (5.6). Let $W \in \mathcal{W}^s$ with $|W| \ge \delta_1/3$ and set the test function $\psi \equiv 1$. For $n \ge n_1$,

$$\int_W \mathcal{L}^n 1\, dm_W = \sum_{W_i\in\mathcal{G}_n^{\delta_1}(W)}\int_{W_i} 1\, dm_{W_i} = \sum_{W_i\in\mathcal{G}_n^{\delta_1}(W)}|W_i| \ge \frac{2\delta_1}{9}c_0 e^{nh_*} , \tag{6.23}$$

by (5.12). Thus,

$$\|\mathcal{L}^n 1\|_s \geq |\mathcal{L}^n 1|_w \geq \frac{2\delta_1}{9} c_0 e^{nh_*} . \tag{6.24}$$

Letting n tend to infinity, one obtains $\lim_{n\to\infty} \|\mathcal{L}^n\|_{\mathcal{B}}^{1/n} \geq e^{h_*}$. □

6.4. **Compact embedding.** The following compact embedding property is crucial to exploit Proposition 4.7 in order to construct μ_* in Section 7.1.

Proposition 6.1 (Compact embedding)**.** *The embedding of the unit ball of $\mathcal{B}$ in $\mathcal{B}_w$ is compact.*

Proof. Consider the set $\widehat{\mathcal{W}}^s$ of (not necessarily homogeneous) cone-stable curves with uniformly bounded curvature and the distance $d_{\mathcal{W}^s}(\cdot,\cdot)$ between them defined in Section 4.1. According to (4.3), each of these curves can be viewed as graphs of C^2 functions of the position coordinate r with uniformly bounded second derivative, $W = \{G_W(r)\}_{r\in I_w} = \{(r, \varphi_W(r))\}_{r\in I_W}$. Thus they are compact in the C^1 distance $d_{\mathcal{W}^s}$. Given $\varepsilon > 0$, we may choose finitely many $V_i \in \widehat{\mathcal{W}}^s$, $i = 1, \dots, N_\varepsilon$, such that the balls of radius $\varepsilon/2$ in the $d_{\mathcal{W}^s}$ metric centered at the curves $\{V_i\}_{i=1}^{N_\varepsilon}$ form a covering of $\widehat{\mathcal{W}}^s$.

Since $\mathcal{W}^s \subset \widehat{\mathcal{W}}^s$, we proceed as follows. In each ball $B_{\varepsilon/2}(V_i)$ centered at V_i in the space of C^1 graphs, if $B_{\varepsilon/2}(V_i) \cap \mathcal{W}^s \neq \emptyset$, then we choose one representative $W_i \in B_{\varepsilon/2}(V_i) \cap \mathcal{W}^s$. Otherwise, we discard $B_{\varepsilon/2}(V_i)$. The balls of radius ε in the $d_{\mathcal{W}^s}$ metric centered at the curves $\{W_i\}_{i=1}^{N_\varepsilon}$ constructed in this way form a covering of $\mathcal{W}^s$. (There may be fewer than N_ε such curves due to some balls having been discarded, but we will continue to use the symbol N_ε in any case.)

We now argue one component of the phase space, $M_\ell = \partial B_\ell \times [-\pi/2, \pi/2]$, at a time. Define $\mathbb{S}^1_\ell$ to be the circle of length $|\partial B_\ell|$ and let C_g be the graph constant from (6.8). Since the ball of radius C_g in the $C^\alpha(\mathbb{S}^1_\ell)$ norm is compactly embedded in $C^\beta(\mathbb{S}^1_\ell)$, we may choose finitely many functions $\overline{\psi}_j \in C^\alpha(\mathbb{S}^1_\ell)$ such that the balls of radius ε in the $C^\beta(\mathbb{S}^1_\ell)$ metric centered at the functions $\{\overline{\psi}_j\}_{j=1}^{L_\varepsilon}$ form a covering of the ball of radius C_g in $C^\alpha(\mathbb{S}^1_\ell)$.

Now let $W = G_W(I_W) \in \mathcal{W}^s$, and $\psi \in C^\alpha(W)$ with $|\psi|_{C^\alpha(W)} \leq 1$. Viewing I_W as a subset of $\mathbb{S}^1_\ell$, we define the push down of ψ to I_W by $\overline{\psi} = \psi \circ G_W$. We extend $\overline{\psi}$ to $\mathbb{S}^1_\ell$ by linearly interpolating between its two endpoint values on the complement of I_W in $\mathbb{S}^1_\ell$. Since I_W is much shorter than $\mathbb{S}^1_\ell$, this can be accomplished while maintaining $|\overline{\psi}|_{C^\alpha(\mathbb{S}^1_\ell)} \leq C_g$.

Choose $W_i = G_{W_i}(I_{W_i})$ such that $d_{\mathcal{W}^s}(W, W_i) < \varepsilon$ and $\overline{\psi}_j$ such that $|\overline{\psi} - \overline{\psi}_j|_{C^\beta(\mathbb{S}^1_\ell)} < \varepsilon$. Define $\psi_j = \overline{\psi}_j \circ G_{W_i}^{-1}$ and $\widetilde{\psi}_j = \overline{\psi}_j \circ G_W^{-1}$ to be the lifts of $\overline{\psi}_j$ to W_i and W, respectively. Note that $|\psi_j|_{C^\beta(W_i)} \leq C_g$, $|\widetilde{\psi}_j|_{C^\beta(W)} \leq C_g$, while

$$d(\psi_j, \widetilde{\psi}_j) = |\psi_j \circ G_{W_i} - \widetilde{\psi}_j \circ G_W|_{C^0(I_{W_i}\cap I_W)} = 0 \quad \text{and} \quad |\psi - \widetilde{\psi}_j|_{C^\beta(W)} \leq C_g \varepsilon .$$

Thus,

$$\begin{aligned}
&\left| \int_W f\psi \, dm_W - \int_{W_i} f\psi_j \, dm_{W_i} \right| \\
&\qquad \leq \left| \int_W f(\psi - \widetilde{\psi}_j) \, dm_W \right| + \left| \int_W f\widetilde{\psi}_j \, dm_W - \int_{W_i} f\psi_j \, dm_{W_i} \right| \\
&\qquad \leq \|f\|_s |\log|W||^{-\gamma} |\psi - \widetilde{\psi}_j|_{C^\beta(W)} + |\log \varepsilon|^{-\varsigma} \|f\|_u C_g \leq 2C_g \|f\|_{\mathcal{B}} |\log \varepsilon|^{-\varsigma} .
\end{aligned}$$

We have proved that for each $\varepsilon > 0$, there exist finitely many bounded linear functionals $\ell_{i,j}(\cdot) = \int_{W_i} \cdot\, \psi_j \, dm_{W_i}$, such that for all $f \in \mathcal{B}$,

$$|f|_w \le \max_{i \le N_\varepsilon, j \le L_\varepsilon} \ell_{i,j}(f) + 2C_g \|f\|_{\mathcal{B}} |\log \varepsilon|^{-\varsigma},$$

which implies the relative compactness of $\mathcal{B}$ in $\mathcal{B}_w$. $\square$

7. The measure μ_*

In this section, we assume throughout that $h_* > s_0 \log 2$ (with $s_0 < 1$ defined by (1.4)).

7.1. Construction of the measure μ_*—measure of singular sets (Theorem 2.6).

In this section, we construct a T-invariant probability measure μ_* on M by combining in (7.1) a maximal eigenvector of $\mathcal{L}$ on $\mathcal{B}$ and a maximal eigenvector of its dual obtained in Proposition 7.1. In addition, the information on these left and right eigenvectors will give Lemma 7.3 and Corollary 7.4, which immediately imply Theorem 2.6.

We first show that such maximal eigenvectors exist and are in fact nonnegative Radon measures (i.e., elements of the dual of $C^0(M)$).

Proposition 7.1. *If $h_* > s_0 \log 2$, then there exist $\nu \in \mathcal{B}_w$ and $\tilde\nu \in \mathcal{B}_w^*$ such that $\mathcal{L}\nu = e^{h_*}\nu$ and $\mathcal{L}^*\tilde\nu = e^{h_*}\tilde\nu$. In addition*[27] *$\nu$ and $\tilde\nu$ take nonnegative values on nonnegative C^1 functions on M and are thus nonnegative Radon measures. Finally, $\tilde\nu(\nu) \neq 0$ and $\|\nu\|_u \le \bar C$.*

Remark 7.2. The norm of the space $\mathcal{B}$ depends on the parameter γ and is used in the proof of the proposition. However, this proof provides ν and $\tilde\nu$ which do not depend on γ (as soon as $2^{s_0\gamma} < e^{h_*}$), and do not depend on the parameters β and ς of $\mathcal{B}$.

It is easy to see that $|f\varphi|_w \le |\varphi|_{C^1}|f|_w$ (use $|\varphi\psi|_{C^\alpha(W)} \le |\varphi|_{C^1}|\psi|_{C^\alpha(W)}$). Clearly, if $f \in C^1$ and $\varphi \in C^1$, then $f\varphi \in C^1$. Therefore, if $h_* > s_0 \log 2$, a bounded linear map μ_* from $C^1(M)$ to $\mathbb{C}$ can be defined by taking ν and $\tilde\nu$ from Proposition 7.1 and setting

$$\mu_*(\varphi) = \frac{\tilde\nu(\nu\varphi)}{\tilde\nu(\nu)}. \tag{7.1}$$

This map is nonnegative for all nonnegative φ and thus defines a nonnegative measure $\mu_* \in (C^0)^*$ with $\mu_*(1) = 1$. Clearly, μ_* is a T invariant probability measure since for every $\varphi \in C^1$ we have

$$\tilde\nu(\nu\varphi) = e^{-h_*}\tilde\nu(\varphi\mathcal{L}(\nu)) = e^{-h_*}\tilde\nu(\mathcal{L}(\nu(\varphi\circ T))) = \tilde\nu(\nu(\varphi\circ T)) = \tilde\nu(\nu)\mu_*(\varphi\circ T).$$

Proof of Proposition 7.1. Let 1 denote the constant function[28] equal to one on M. We will take this as a seed in our construction of a maximal eigenvector. From (4.14) in Theorem 4.10 we see that $\|\mathcal{L}^n 1\|_{\mathcal{B}} \ge \|\mathcal{L}^n 1\|_s \ge |\mathcal{L}^n 1|_w \ge C\#\mathcal{M}_0^n \ge Ce^{nh_*}$. Now, consider

$$\nu_n = \frac{1}{n}\sum_{k=0}^{n-1} e^{-kh_*}\mathcal{L}^k 1 \in \mathcal{B}, \quad n \ge 1. \tag{7.2}$$

[27] Recall Proposition 4.2 and Remark 4.3.

[28] We could replace the seed function 1 by any C^1 positive function f on M.

By construction the ν_n are nonnegative, and thus Radon measures. By our assumption on h_* and (4.15) in Theorem 4.10 they satisfy $\|\nu_n\|_{\mathcal{B}} \le \bar{C}$, so using the relative compactness of $\mathcal{B}$ in $\mathcal{B}_w$ (Proposition 6.1), we extract a subsequence (n_j) such that $\lim_j \nu_{n_j} = \nu$ is a nonnegative measure, and the convergence is in $\mathcal{B}_w$. (Changing the value of γ does not affect ν since $\mathcal{B}_w$ does not depend on γ.) Since $\mathcal{L}$ is continuous on $\mathcal{B}_w$, we may write,

$$\begin{aligned}\mathcal{L}\nu &= \lim_{j\to\infty} \frac{1}{n_j}\sum_{k=0}^{n_j-1} e^{-kh_*}\mathcal{L}^{k+1}1 \\ &= \lim_{j\to\infty}\left(\frac{e^{h_*}}{n_j}\sum_{k=0}^{n_j-1} e^{-kh_*}\mathcal{L}^k 1 - \frac{1}{n_j}e^{h_*}1 + \frac{1}{n_j}e^{-(n_j-1)h_*}\mathcal{L}^{n_j}1\right) = e^{h_*}\nu\,,\end{aligned}$$

where we used that the second and third terms go to 0 (in the $\mathcal{B}$-norm). We thus obtain a nonnegative measure $\nu \in \mathcal{B}_w$ such that $\mathcal{L}\nu = e^{h_*}\nu$.

Although ν is not a priori an element of $\mathcal{B}$, it does inherit bounds on the unstable norm from the sequence ν_n. The convergence of (ν_{n_j}) to ν in $\mathcal{B}_w$ implies that

$$\lim_{j\to\infty}\sup_{W\in\mathcal{W}^s}\sup_{\substack{\psi\in C^\alpha(W)\\ |\psi|_{C^\alpha(W)}\le 1}}\left(\int_W \nu\,\psi\,dm_W - \int_W \nu_{n_j}\,\psi\,dm_W\right) = 0\,. \tag{7.3}$$

Since $\|\nu_{n_j}\|_u \le \bar{C}$, it follows that $\|\nu\|_u \le \bar{C}$, as claimed.

Next, recalling the bound $|\int f\,d\mu_{\mathrm{SRB}}| \le \hat{C}|f|_w$ from Proposition 4.2, setting $d\mu_{\mathrm{SRB}} \in (\mathcal{B}_w)^*$ to be the functional defined on $C^1(M) \subset \mathcal{B}_w$ by $d\mu_{\mathrm{SRB}}(f) = \int f\,d\mu_{\mathrm{SRB}}$ and extended by density, we define[29]

$$\tilde{\nu}_n = \frac{1}{n}\sum_{k=0}^{n-1} e^{-kh_*}(\mathcal{L}^*)^k(d\mu_{\mathrm{SRB}})\,. \tag{7.4}$$

Then, we have $|\tilde{\nu}_n(f)| \le C|f|_w$ for all n and all $f \in \mathcal{B}_w$. So $\tilde{\nu}_n$ is bounded in $(\mathcal{B}_w)^* \subset \mathcal{B}^*$. By compactness of this embedding (Proposition 6.1), we can find a subsequence $\tilde{\nu}_{\tilde{n}_j}$ converging to $\tilde{\nu} \in \mathcal{B}^*$. By the argument above, we have $\mathcal{L}^*\tilde{\nu} = e^{h_*}\tilde{\nu}$. The nonnegativity claim on $\tilde{\nu}$ follows by construction.[30]

We next check that $\tilde{\nu}$, which in principle lies in the dual of $\mathcal{B}$, is in fact an element of $(\mathcal{B}_w)^*$. For this, it suffices to find $\tilde{C} < \infty$ so that for any $f \in \mathcal{B}$ we have

$$\tilde{\nu}(f) \le \tilde{C}|f|_w\,. \tag{7.5}$$

Now, for $f \in \mathcal{B}$ and any $n \ge 1$, we have

$$|\tilde{\nu}(f)| \le |(\tilde{\nu} - \tilde{\nu}_n)(f)| + |\tilde{\nu}_n(f)| \le |(\tilde{\nu} - \tilde{\nu}_n)(f)| + |f|_w\,.$$

Since $\tilde{\nu}_n \to \tilde{\nu}$ in $\mathcal{B}^*$, we conclude $|\tilde{\nu}(f)| \le |f|_w$ for all $f \in \mathcal{B}$. Since $\mathcal{B}$ is dense in $\mathcal{B}_w$, by [RS, Thm I.7] $\tilde{\nu}$ extends uniquely to a bounded linear functional on $\mathcal{B}_w$, satisfying (7.5). It only remains to see that $\tilde{\nu}(\nu) > 0$.

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[29] We could again replace the seed μ_{SRB} by $f\mu_{\mathrm{SRB}}$ for any C^1 positive function f on M.

[30] To check γ-independence of $\tilde{\nu}$, note that if $\tilde{\gamma} > \gamma$, then, since the dual norms satisfy $\|\tilde{\nu}_{\tilde{n}_j} - \tilde{\nu}\|_{*,\tilde{\gamma}} \le \|\tilde{\nu}_{\tilde{n}_j} - \tilde{\nu}\|_{*,\gamma}$, the subsequence converges to $\tilde{\nu}$ in the $\|\cdot\|_{*,\tilde{\gamma}}$-norm as well. If $\tilde{\gamma} < \gamma$, then a further subsequence of $\tilde{n}_j$ must converge to some $\tilde{\nu}_{\tilde{\gamma}}$ in the $\|\cdot\|_{*,\tilde{\gamma}}$ norm. The domination then implies $\tilde{\nu} = \tilde{\nu}_{\tilde{\gamma}}$.

Let (n_j) (resp., $(\tilde n_j)$) denote the subsequence such that $\nu = \lim_j \nu_{n_j}$ (resp., $\tilde\nu = \lim_j \tilde\nu_{\tilde n_j}$). Since $\tilde\nu$ is continuous on $\mathcal{B}_w$, we have on the one hand

$$(7.6)\quad \tilde\nu(\nu) = \lim_{j\to\infty} \tilde\nu(\nu_{n_j}) = \lim_j \frac{1}{n_j}\sum_{k=0}^{n_j-1} e^{-kh_*}\tilde\nu(\mathcal{L}^k 1) = \lim_j \frac{1}{n_j}\sum_{k=0}^{n_j-1}\tilde\nu(1) = \tilde\nu(1)\,,$$

where we have used that $\tilde\nu$ is an eigenvector for $\mathcal{L}^*$. On the other hand,

$$(7.7)\ \tilde\nu(1) = \lim_{j\to\infty}\frac{1}{\tilde n_j}\sum_{k=0}^{\tilde n_j-1} e^{-kh_*}(\mathcal{L}^*)^k d\mu_{\mathrm{SRB}}(1) = \lim_j \frac{1}{\tilde n_j}\sum_{k=0}^{\tilde n_j-1} e^{-kh_*}\int \mathcal{L}^k 1\, d\mu_{\mathrm{SRB}}\,.$$

Next, we disintegrate μ_{SRB} as in the proof of Lemma 4.4 into conditional measures $\mu_{\mathrm{SRB}}^{W_\xi}$ on maximal homogeneous stable manifolds $W_\xi \in \mathcal{W}^s_{\mathbb{H}}$ and a factor measure $d\hat\mu_{\mathrm{SRB}}(\xi)$ on the index set Ξ of stable manifolds. Recall that $\mu_{\mathrm{SRB}}^{W_\xi} = |W_\xi|^{-1}\rho_\xi dm_W$, where ρ_ξ is uniformly log-Hölder continuous so that

$$(7.8)\qquad 0 < c_\rho \le \inf_{\xi\in\Xi}\inf_{W_\xi}\rho_\xi \le \sup_{\xi\in\Xi}|\rho_\xi|_{C^\alpha(W_\xi)} \le C_\rho < \infty\,.$$

Let Ξ^{δ_1} denote those $\xi\in\Xi$ such that $|W_\xi| \ge \delta_1/3$ and note that $\hat\mu_{\mathrm{SRB}}(\Xi^{\delta_1}) > 0$. Then, disintegrating as usual, we get by (6.23) for $k \ge n_1$,

$$\begin{aligned}\int \mathcal{L}^k 1\, d\mu_{\mathrm{SRB}} &= \int_\Xi\int_{W_\xi}\mathcal{L}^k 1\,\rho_\xi|W_\xi|^{-1}\,dm_{W_\xi}d\hat\mu_{\mathrm{SRB}}(\xi)\\ &\ge \int_{\Xi^{\delta_1}}\int_{W_\xi}\mathcal{L}^k 1\, dm_{W_\xi}c_\rho 3\delta_1^{-1}d\hat\mu_{\mathrm{SRB}}(\xi) \ge c_\rho\frac{2c_0}{3}e^{kh_*}\hat\mu_{\mathrm{SRB}}(\Xi^{\delta_1})\,.\end{aligned}$$

Combining this with (7.6) and (7.7) yields $\tilde\nu(\nu) = \tilde\nu(1) \ge \frac{2c_\rho c_0}{3}\hat\mu_{\mathrm{SRB}}(\Xi^{\delta_1}) > 0$, as required. □

We next study the measure of neighbourhoods of singularity sets and stable manifolds, in order to establish (2.2) in Theorem 2.6.

Lemma 7.3. *For any $\gamma > 0$ such that $2^{s_0\gamma} < e^{h_*}$ and any $k\in\mathbb{Z}$, there exists $C_k > 0$ such that*

$$\mu_*(\mathcal{N}_\varepsilon(\mathcal{S}_k)) \le C_k|\log\varepsilon|^{-\gamma}\qquad \forall\varepsilon > 0\,.$$

In particular, for any $p > 1/\gamma$ (one can choose $p < 1$ if $\gamma > 1$), $\eta > 0$, and $k\in\mathbb{Z}$, for μ_*-almost every $x\in M$, there exists $C > 0$ such that

$$(7.9)\qquad d(T^n x, \mathcal{S}_k) \ge Ce^{-\eta n^p}\qquad \forall n \ge 0\,.$$

Proof. First, for each $k \ge 0$, we claim that there exists $C_k > 0$ such that for all $\varepsilon > 0$,

$$(7.10)\qquad |\nu(\mathcal{N}_\varepsilon(\mathcal{S}_{-k}))| \le C|1_{k,\varepsilon}\nu|_w \le C_k|\log\varepsilon|^{-\gamma}\,,$$

where $1_{k,\varepsilon}$ is the indicator function of the set $\mathcal{N}_\varepsilon(\mathcal{S}_{-k})$. To prove the first inequality in (7.10), first note that since $\mathcal{S}_{-k}$ comprises finitely many smooth curves, uniformly transverse to the stable cone, this also holds for the boundary curves of the set $\mathcal{N}_\varepsilon(\mathcal{S}_{-k})$. By [DZ3, Lemma 5.3], we have $1_{k,\varepsilon}f\in\mathcal{B}$ for $f\in\mathcal{B}$; similarly (and by a simpler approximation) if $f\in\mathcal{B}_w$, then $1_{k,\varepsilon}f\in\mathcal{B}_w$. So the first inequality in (7.10) follows from Lemma 4.4.

We next prove the second inequality in (7.10). Let $W\in\mathcal{W}^s$ and $\psi\in C^\alpha(W)$ with $|\psi|_{C^\alpha(W)}\le 1$. Due to the uniform transversality of the curves in $\mathcal{S}_{-k}$ with the stable cone, the intersection $W\cap\mathcal{N}_\varepsilon(\mathcal{S}_{-k})$ can be expressed as a finite union with cardinality bounded by a constant A_k (depending only on $\mathcal{S}_{-k}$) of stable manifolds $W_i \in \mathcal{W}^s$, of lengths at most $C\varepsilon$. Therefore, for any $f \in C^1$,

$$(7.11) \qquad \int_{W_\xi} f\, 1_{k,\varepsilon}\, \psi \, dm_W = \sum_i \int_{W_i} f \, \psi \, dm_{W_i} \le \sum_i |f|_w |\psi|_{C^\alpha(W_i)} \le C A_k |f|_w \, .$$

It follows that $|1_{k,\varepsilon} f|_w \le A_k |f|_w$ for all $f \in \mathcal{B}_w$. Similarly, we have $|1_{k,\varepsilon} f|_w \le A_k \|f\|_s |\log \varepsilon|^{-\gamma}$ for all $f \in \mathcal{B}$. Now recalling ν_n from (7.2), we estimate,

$$|1_{k,\varepsilon}\nu|_w \le |1_{k,\varepsilon}(\nu - \nu_n)|_w + |1_{k,\varepsilon}\nu_n|_w \le A_k|\nu - \nu_n|_w + C'_k |\log \varepsilon|^{-\gamma} \|\nu_n\|_{\mathcal{B}} \, .$$

Since $\|\nu_n\|_{\mathcal{B}} \le \bar{C}$ for all $n \ge 1$, we take the limit as $n \to \infty$ to conclude that $|1_{k,\varepsilon}\nu|_w \le C_k |\log \varepsilon|^{-\gamma}$, concluding the proof of (7.10).

Next, applying (7.5), we have

$$\mu_*(\mathcal{N}_\varepsilon(\mathcal{S}_{-k})) = \tilde{\nu}(1_{k,\varepsilon}\nu) \le \tilde{C} |1_{k,\varepsilon}\nu|_w \le \tilde{C} C_k |\log \varepsilon|^{-\gamma} \qquad \forall k \ge 0 \, .$$

To obtain the analogous bound for $\mathcal{N}_\varepsilon(\mathcal{S}_k)$, for $k > 0$, we use the invariance of μ_*. It follows from the time reversal of (5.2) that $T(\mathcal{N}_\varepsilon(\mathcal{S}_1)) \subset \mathcal{N}_{C\varepsilon^{1/2}}(\mathcal{S}_{-1})$. Thus,

$$\mu_*(\mathcal{N}_\varepsilon(\mathcal{S}_1)) \le \mu_*(\mathcal{N}_{C\varepsilon^{1/2}}(\mathcal{S}_{-1})) \le C_1 |\log(C\varepsilon^{1/2})|^{-\gamma} \le C'_1 |\log \varepsilon|^{-\gamma}.$$

The estimate for $\mathcal{N}_\varepsilon(\mathcal{S}_k)$, for $k \ge 2$, follows similarly since $T^k \mathcal{S}_k = \mathcal{S}_{-k}$.

Finally, fix $\eta > 0$, $k \in \mathbb{Z}$ and $p > 1/\gamma$. Since

$$(7.12) \qquad \sum_{n \ge 0} \mu_*(\mathcal{N}_{e^{-\eta n^p}}(\mathcal{S}_k)) \le \tilde{C} C_k \eta^{-\gamma} \sum_{n \ge 1} n^{-p\gamma} < \infty,$$

by the Borel–Cantelli lemma, μ_*-almost every $x \in M$ visits $\mathcal{N}_{e^{-\eta n^p}}(\mathcal{S}_k)$ only finitely many times, and the last statement of the lemma follows. □

Lemma 7.3 will imply the following.

Corollary 7.4. a) *For any $\gamma > 0$ so that $2^{s_0\gamma} < e^{h_*}$ and any C^1 curve S uniformly transverse to the stable cone, there exists $C > 0$ such that $\nu(\mathcal{N}_\varepsilon(S)) \le C|\log \varepsilon|^{-\gamma}$ and $\mu_*(\mathcal{N}_\varepsilon(S)) \le C|\log \varepsilon|^{-\gamma}$ for all $\varepsilon > 0$.*

b) *The measures ν and μ_* have no atoms, and $\mu_*(W) = 0$ for all $W \in \mathcal{W}^s$ and $W \in \mathcal{W}^u$.*

c) *$\int |\log d(x, \mathcal{S}_{\pm 1})| \, d\mu_* < \infty$.*

d) *μ_*-almost every point in M has a stable and unstable manifold of positive length.*

Proof. a) This follows immediately from the bounds in the proof of Lemma 7.3 since the only property required of $\mathcal{S}_{-k}$ is that it comprises finitely many smooth curves uniformly transverse to the stable cone.

b) That ν and μ_* have no atoms follows from part a). If $\mu_*(W) = a > 0$, then by invariance, $\mu_*(T^n W) = a$ for all $n > 0$. Since μ_* is a probability measure and T^n is continuous on stable manifolds, $\bigcup_{n \ge 0} T^n W$ must be the union of only finitely many smooth curves. Since $|T^n W| \to 0$ there is a subsequence (n_j) such that $\cap_{j \ge 0} T^{n_j} W = \{x\}$. Thus $\mu_*(\{x\}) = a$, which is impossible. A similar argument applies to $W \in \mathcal{W}^u$, using the fact that T^{-n} is continuous on such manifolds.

c) Choose $\gamma > 1$ and $p > 1/(\gamma - 1)$. Then by Lemma 7.3,

$$\int |\log d(x, \mathcal{S}_1)|\, d\mu_* = \sum_{n\geq 0} \int_{\mathcal{N}_{e^{-n^p}}(\mathcal{S}_1)\setminus \mathcal{N}_{e^{-(n+1)^p}}(\mathcal{S}_1)} |\log d(x, \mathcal{S}_1)|\, d\mu_*$$
$$\leq \sum_{n\geq 0} (n+1)^p \mu_*(\mathcal{N}_{e^{-n^p}}(\mathcal{S}_1)) \leq 1 + \sum_{n\geq 1} C_1 n^{p(1-\gamma)}(1+1/n)^p < \infty.$$

A similar estimate holds for $\int |\log d(x, \mathcal{S}_{-1})|\, d\mu_*$.

d) The existence of stable and unstable manifolds for μ_*-almost every x follows from the Borel–Cantelli estimate (7.12) by a standard argument if we choose $\gamma > 1$, $p = 1$, and $e^\eta < \Lambda$ (see, for example, [CM, Sect. 4.12]). □

Lemma 7.3 and Corollary 7.4 prove all the items of Theorem 2.6.

7.2. **ν-almost everywhere positive length of unstable manifolds.** We establish almost everywhere positive length of unstable manifolds in the sense of the measure ν (the maximal eigenvector of $\mathcal{L}$). The proof of this fact, as well as some arguments in subsequent sections, will require viewing elements of $\mathcal{B}_w$ as *leafwise distributions*; see Definition 7.5 below. Indeed, to prove Lemma 7.6, we make in Lemma 7.7 an explicit connection[31] between the element $\nu \in \mathcal{B}_w$ viewed as a measure on M, and the family of leafwise measures defined on the set of stable manifolds $\mathcal{W}^s$.

While ν is not an invariant measure, the almost everywhere existence of positive length unstable manifolds on *every* stable manifold $W \in \mathcal{W}^s$ follows from the regularity inherited from the strong stable norm. This property may have some independent interest as it has not been proved in previous uses of this type of norm [DZ1, DZ3], and it will be important for proving the absolute continuity of the unstable foliation for μ_* (Corollary 7.9), which relies on the analogous property for the measure ν (Proposition 7.8). Lemmas 7.6 and 7.7 will also be useful to obtain that μ_* has full support (Proposition 7.11).

Definition 7.5 (Leafwise distributions and leafwise measures)**.** For $f \in C^1(M)$ and $W \in \mathcal{W}^s$, the map defined on $C^\alpha(W)$ by

$$\psi \mapsto \int_W f\, \psi\, dm_W$$

can be viewed as a distribution of order α on W. Since we have the bound $|\int_W f\,\psi\, dm_W| \leq |f|_w |\psi|_{C^\alpha(W)}$, the map sending $f \in C^1$ to this distribution of order α on W can be extended to $f \in \mathcal{B}_w$. We denote this extension by $\int_W \psi\, f$ or $\int_W f\,\psi\, dm_W$, and we call the corresponding family of distributions (indexed by W) the leafwise distribution $(f, W)_{W\in\mathcal{W}^s}$ associated with $f \in \mathcal{B}_w$. Note that if $f \in \mathcal{B}_w$ is such that $\int_W \psi\, f \geq 0$ for all $\psi \geq 0$, then using again [Sch, §I.4], the leafwise distribution on W extends to a bounded linear functional on $C^0(W)$, i.e., it is a Radon measure. If this holds for all $W \in \mathcal{W}^s$, the leafwise distribution is called a leafwise measure.

Lemma 7.6 (Almost everywhere positive length of unstable manifolds for ν)**.** *For ν-almost every $x \in M$ the stable and unstable manifolds have positive length. Moreover, viewing ν as a leafwise measure, for every $W \in \mathcal{W}^s$, ν-almost every $x \in W$ has an unstable manifold of positive length.*

[31] This connection is used in Section 7.3.

Recall the disintegration of $\mu_{\rm SRB}$ into conditional measures $\mu_{\rm SRB}^{W_\xi}$ on maximal homogeneous stable manifolds $W_\xi \in \mathcal{W}^s_{\mathbb{H}}$ and a factor measure $d\hat\mu_{\rm SRB}(\xi)$ on the index set Ξ of homogeneous stable manifolds, with $d\mu_{\rm SRB}^{W_\xi} = |W_\xi|^{-1}\rho_\xi dm_W$, where ρ_ξ is uniformly log-Hölder continuous as in (7.8).

Lemma 7.7. *Let ν^{W_ξ} and $\hat\nu$ denote the conditional measures and factor measure obtained by disintegrating ν on the set of homogeneous stable manifolds $W_\xi \in \mathcal{W}^s_{\mathbb{H}}$, $\xi\in\Xi$. Then for any $\psi \in C^\alpha(M)$,*

$$\int_{W_\xi} \psi\, d\nu^{W_\xi} = \frac{\int_{W_\xi}\psi\,\rho_\xi\,\nu}{\int_{W_\xi}\rho_\xi\,\nu} \quad \forall \xi\in\Xi \quad \text{and} \quad d\hat\nu(\xi) = |W_\xi|^{-1}\Big(\int_{W_\xi}\rho_\xi\,\nu\Big)\, d\hat\mu_{\rm SRB}(\xi)\,.$$

Moreover, viewed as a leafwise measure, $\nu(W)>0$ for all $W\in\mathcal{W}^s$.

Proof. First, we we establish the following claim: For $W \in \mathcal{W}^s$, we let $n_2 \le \bar C_2|\log(|W|/\delta_1)|$ be the constant from the proof of Corollary 5.3. (This is the first time ℓ such that $\mathcal{G}_\ell(W)$ has at least one element of length at least $\delta_1/3$.) Then there exists $\bar C>0$ such that for all $W\in\mathcal{W}^s$,

$$\int_W \nu \ge \bar C |W|^{h_*\bar C_2}\,. \tag{7.13}$$

Indeed, recalling (7.2) and using (6.23), we have for $\bar C = \frac{2c_0}{9}\delta_1^{1-h_*\bar C_2}$,

$$\begin{aligned}
\int_W \nu &= \lim_{n_j}\frac{1}{n_j}\sum_{k=0}^{n_j-1} e^{-kh_*}\int_W \mathcal{L}^k 1 dm_W \\
&\ge \lim_{n_j}\frac{1}{n_j}\sum_{k=n_2}^{n_j-1} e^{-kh_*}\sum_{W_i\in\mathcal{G}_{n_2}(W)}\int_{W_i}\mathcal{L}^{k-n_2}1dm_{W_i} \\
&\ge \lim_{n_j}\frac{1}{n_j}\sum_{k=n_2}^{n_j-1} e^{-kh_*}\tfrac{2\delta_1}{9}c_0e^{h_*(k-n_2)} \ge \tfrac{2\delta_1}{9}c_0e^{-h_*n_2} \ge \bar C|W|^{h_*\bar C_2}\,.
\end{aligned}$$

This proves the last statement of the lemma.

Next, for any $f\in C^1(M)$, according to our convention, we view f as an element of $\mathcal{B}_w$ by considering it as a measure integrated against $\mu_{\rm SRB}$. Now suppose $(\nu_n)_{n\in\mathbb{N}}$ is the sequence of functions from (7.2) such that $|\nu_n-\nu|_w\to 0$. For any $\psi\in C^\alpha(M)$, we have

$$\begin{aligned}
\nu_n(\psi) &= \int_M \nu_n\,\psi\, d\mu_{\rm SRB} = \int_\Xi\int_{W_\xi}\nu_n\,\psi\,\rho_\xi\, dm_{W_\xi}|W_\xi|^{-1}d\hat\mu_{\rm SRB}(\xi) \\
&= \int_\Xi \frac{\int_{W_\xi}\nu_n\,\psi\,\rho_\xi\,dm_{W_\xi}}{\int_{W_\xi}\nu_n\,\rho_\xi\,dm_{W_\xi}}\, d(\hat\mu_{\rm SRB})_n(\xi)\,,
\end{aligned} \tag{7.14}$$

where $d(\hat\mu_{\rm SRB})_n(\xi) = |W_\xi|^{-1}\int_{W_\xi}\nu_n\,\rho_\xi\,dm_{W_\xi}\,d\hat\mu_{\rm SRB}(\xi)$. By definition of convergence in $\mathcal{B}_w$ (see for example (7.3)) since $\psi,\rho_\xi\in C^\alpha(W_\xi)$, the ratio of integrals converges (uniformly in ξ) to $\int_{W_\xi}\psi\,\rho_\xi\,\nu/\int_{W_\xi}\rho_\xi\,\nu$, and the factor measure converges to $|W_\xi|^{-1}\int_{W_\xi}\rho_\xi\,d\nu\,d\hat\mu_{\rm SRB}(\xi)$. Note that since ρ_ξ is uniformly log-Hölder, and due to (7.13), we have $\int_{W_\xi}\nu\,\rho_\xi\,dm_{W_\xi}>0$ with lower bound depending only on the length of W_ξ.

Finally, by Proposition 4.2 and Lemma 4.4, we have $\nu_n(\psi)$ converging to $\nu(\psi)$. Disintegrating ν according to the statement of the lemma yields the claimed identifications. $\square$

Proof of Lemma 7.6. The statement about stable manifolds of positive length follows from the characterization of $\hat\nu$ in Lemma 7.7, since the set of points with stable manifolds of zero length has zero $\hat\mu_{\mathrm{SRB}}$-measure [CM].

We fix $W \in \mathcal{W}^s$ and prove the statement about ν as a leafwise measure. This will imply the statement regarding unstable manifolds for the measure ν by Lemma 7.7.

Fix $\varepsilon > 0$ and $\hat\Lambda \in (\Lambda, 1)$, and define $O = \bigcup_{n\ge 1} O_n$, where

$$O_n = \{x \in W : n = \min j \text{ such that } d_u(T^{-j}x, \mathcal{S}_1) < \varepsilon C_e \hat\Lambda^{-j}\},$$

and d_u denotes distance restricted to the unstable cone. By [CM, Lemma 4.67], any $x \in W \setminus O$ has unstable manifold of length at least 2ε. We proceed to estimate $\nu(O) = \sum_{n\ge1} \nu(O_n)$, where equality holds since the O_n are disjoint. In addition, since O_n is a finite union of open subcurves of W, we have

$$\int_W 1_{O_n}\,\nu = \lim_{j\to\infty} \int_W 1_{O_n}\,\nu_{\ell_j} = \lim_{j\to\infty} \ell_j^{-1} \sum_{k=0}^{\ell_j-1} e^{-kh_*} \int_W 1_{O_n}\,\mathcal{L}^k 1\, dm_W\,. \tag{7.15}$$

We estimate two cases.

Case I ($k < n$). Write $\int_{W\cap O_n} \mathcal{L}^k 1\, dm_W = \sum_{W_i \in \mathcal{G}_k(W)} \int_{W_i \cap T^{-k}O_n} 1\, dm_{W_i}$.

If $x \in T^{-k}O_n$, then $y = T^{-n+k}x$ satisfies $d_u(y, \mathcal{S}_1) < \varepsilon C_e \hat\Lambda^{-n}$ and thus we have $d_u(Ty, \mathcal{S}_{-1}) \le C\varepsilon^{1/2}\hat\Lambda^{-n/2}$. Due to the uniform transversality of stable and unstable cones, as well as the fact that elements of $\mathcal{S}_{-1}$ are uniformly transverse to the stable cone, we have $d_s(Ty, \mathcal{S}_{-1}) \le C\varepsilon^{1/2}\hat\Lambda^{-n/2}$ as well, with possibly a larger constant C.

Let $r^s_{-j}(x)$ denote the distance from $T^{-j}x$ to the nearest endpoint of $W^s(T^{-j}x)$, where $W^s(T^{-j}x)$ is the maximal local stable manifold containing $T^{-j}x$. From the above analysis, we see that $W_i \cap T^{-k}O_n \subseteq \{x \in W_i : r^s_{-n+k+1}(x) \le C\varepsilon^{1/2}\hat\Lambda^{-n/2}\}$. The time reversal of the growth lemma [CM, Thm 5.52] gives $m_{W_i}(r^s_{-n+k+1}(x) \le C\varepsilon^{1/2}\hat\Lambda^{-n/2}) \le C'\varepsilon^{1/2}\hat\Lambda^{-n/2}$ for a constant C' that is uniform in n and k. Thus, using Proposition 4.6, we find

$$\int_{W\cap O_n} \mathcal{L}^k 1\, dm_W \le \#\mathcal{G}_k(W) C'\varepsilon^{1/2}\hat\Lambda^{-n/2} \le Ce^{kh_*}\varepsilon^{1/2}\hat\Lambda^{-n/2}\,.$$

Case II ($k \ge n$). Using the same observation as in Case I, if $x \in T^{-n+1}O_n$, then x satisfies $d_s(x, \mathcal{S}_{-1}) \le C\varepsilon^{1/2}\hat\Lambda^{-n/2}$. We change variables to estimate the integral precisely at time $-n+1$, again using Proposition 4.6,

$$\begin{aligned}
\int_{W\cap O_n} \mathcal{L}^k 1\, dm_W &= \sum_{W_i\in\mathcal{G}_{n-1}(W)} \int_{W_i\cap T^{-n+1}O_n} \mathcal{L}^{k-n+1} 1\, dm_{W_i} \\
&\le \sum_{W_i\in\mathcal{G}_{n-1}(W)} |\log|W_i \cap T^{-n+1}O_n||^{-\gamma} \|\mathcal{L}^{k-n+1}1\|_s \\
&\le \sum_{W_i\in\mathcal{G}_{n-1}(W)} |\log(C\varepsilon^{1/2}\hat\Lambda^{-n/2})|^{-\gamma} Ce^{(k-n+1)h_*} \le |\log(C\varepsilon^{1/2}\hat\Lambda^{-n/2})|^{-\gamma} Ce^{kh_*}\,.
\end{aligned}$$

Using the estimates of Cases I and II in (7.15) and using the weaker bound, we see that,

$$\int_W 1_{O_n}\,\nu_{\ell_j} \le C|\log(C\varepsilon^{1/2}\hat\Lambda^{-n/2})|^{-\gamma}\,.$$

Summing over n, we have, $\int_W 1_O\,\nu_{\ell_j} \le C'|\log\varepsilon|^{1-\gamma}$, uniformly in j. Since ν_{ℓ_j} converges to ν in the weak norm, this bound carries over to ν. Since $\gamma > 1$ and $\varepsilon > 0$ was arbitrary, this implies $\nu(O) = 0$, completing the proof of the lemma. $\square$

7.3. **Absolute continuity of μ_*—full support.** *In this subsection, we assume throughout that $\gamma > 1$ (this is possible since we assumed $h_* > s_0 \log 2$ to construct μ_*).*

Our proof of the Bernoulli property relies on showing first that μ_* is K-mixing (Proposition 7.16). As a first step, we will prove that μ_* is ergodic (see the Hopf-type Lemma 7.15). This will require establishing absolute continuity of the unstable foliation for μ_* (Corollary 7.9), which will be deduced from the following absolute continuity result for ν.

Proposition 7.8. *Let R be a Cantor rectangle. Fix $W^0 \in \mathcal{W}^s(R)$ and for $W \in \mathcal{W}^s(R)$, let Θ_W denote the holonomy map from $W^0 \cap R$ to $W \cap R$ along unstable manifolds in $\mathcal{W}^u(R)$. Then Θ_W is absolutely continuous with respect to the leafwise measure ν.*

Proof. Since by Lemma 7.6 unstable manifolds comprise a set of full ν-measure, it suffices to fix a set $E \subset W^0 \cap R$ with ν-measure zero, and prove that the ν-measure of $\Theta_W(E) \subset W$ is also zero.

Since ν is a regular measure on W^0, for $\varepsilon > 0$, there exists an open set $O_\varepsilon \subset W^0$, $O_\varepsilon \supset E$, such that $\nu(O_\varepsilon) \le \varepsilon$. Indeed, since W^0 is compact, we may choose O_ε to be a finite union of intervals. Let ψ_ε be a smooth function which is 1 on O_ε and 0 outside of an ε-neighbourhood of O_ε. We may choose ψ_ε so that $|\psi_\varepsilon|_{C^1(W^0)} \le 2\varepsilon^{-1}$.

Using (6.2), we choose $n = n(\varepsilon)$ such that $|\psi_\varepsilon \circ T^n|_{C^1(T^{-n}W^0)} \le 1$. Note this implies in particular that $\Lambda^{-n} \le \varepsilon$. Following the procedure described at the beginning of Section 6.2, we subdivide $T^{-n}W^0$ and $T^{-n}W$ into matched pieces U_j^0, U_j and unmatched pieces V_i^0, V_i. With this construction, none of the unmatched pieces $T^nV_i^0$ intersect an unstable manifold in $\mathcal{W}^u(R)$ since unstable manifolds are not cut under T^{-n}.

Indeed, on matched pieces, we may choose a foliation $\Gamma_j = \{\gamma_x\}_{x\in U_j^0}$ such that:

i) $T^n\Gamma_j$ contains all unstable manifolds in $\mathcal{W}^u(R)$ that intersect $T^nU_j^0$;

ii) between unstable manifolds in $\Gamma_j \cap T^{-n}(\mathcal{W}^u(R))$, we interpolate via unstable curves;

iii) the resulting holonomy Θ_j from $T^nU_j^0$ to T^nU_j has uniformly bounded Jacobian[32] with respect to arc-length, with bound depending on the unstable diameter of $D(R)$, by [BDL, Lemmas 6.6, 6.8];

iv) pushing forward Γ_j to $T^n\Gamma_j$ in $D(R)$, we interpolate in the gaps using unstable curves; call $\overline{\Gamma}$ the resulting foliation of $D(R)$;

v) the associated holonomy map $\overline{\Theta}_W$ extends Θ_W and has uniformly bounded Jacobian, again by [BDL, Lemmas 6.6 and 6.8].

[32]Indeed, [BDL] shows the Jacobian is Hölder continuous, but we shall not need this here.

Using the map $\overline{\Theta}_W$, we define $\widetilde{\psi}_\varepsilon = \psi_\varepsilon \circ \overline{\Theta}_W^{-1}$, and note that $|\widetilde{\psi}_\varepsilon|_{C^1(W)} \le C|\psi_\varepsilon|_{C^1(W^0)}$, where we write $C^1(W)$ for the set of Lipschitz functions on W, i.e., C^α with $\alpha = 1$.

Next, we modify ψ_ε and $\widetilde{\psi}_\varepsilon$ as follows: We set them equal to 0 on the images of unmatched pieces, $T^n V_i^0$ and $T^n V_i$, respectively. Since these curves do not intersect unstable manifolds in $\mathcal{W}^u(R)$, we still have $\psi_\varepsilon = 1$ on E and $\widetilde{\psi}_\varepsilon = 1$ on $\Theta_W(E)$. Moreover, the set of points on which $\psi_\varepsilon > 0$ (resp., $\widetilde{\psi}_\varepsilon > 0$) is a finite union of open intervals that cover E (resp., $\Theta_W(E)$).

Following Section 6.2, we estimate

(7.16)
$$\begin{aligned}\int_{W^0} \psi_\varepsilon \, \nu - \int_W \widetilde{\psi}_\varepsilon \, \nu &= e^{-nh_*} \left(\int_{W^0} \psi_\varepsilon \, \mathcal{L}^n \nu - \int_W \widetilde{\psi}_\varepsilon \, \mathcal{L}^n \nu \right) \\ &= e^{-nh_*} \sum_j \int_{U_j^0} \psi_\varepsilon \circ T^n \, \nu - \int_{U_j} \phi_j \, \nu + \int_{U_j} (\phi_j - \widetilde{\psi}_\varepsilon \circ T^n) \, \nu \, ,\end{aligned}$$

where $\phi_j = \psi_\varepsilon \circ T^n \circ G_{U_j^0} \circ G_{U_j}^{-1}$, and $G_{U_j^0}$ and G_{U_j} represent the functions defining U_j^0 and U_j, respectively, defined as in (6.5). Next, since $d(\psi_\varepsilon \circ T^n, \phi_j) = 0$ by construction, and using (6.9) and the assumption that $\Lambda^{-n} \le \varepsilon$, we have by (6.10),

$$e^{-nh_*} \left| \sum_j \int_{U_j^0} \psi_\varepsilon \circ T^n \, \nu - \int_{U_j} \phi_j \, \nu \right| \le C |\log \varepsilon|^{-\varsigma} \|\nu\|_u \, . \tag{7.17}$$

It remains to estimate the last term in (7.16). This we do using the weak norm,

$$\int_{U_j} (\phi_j - \widetilde{\psi}_\varepsilon \circ T^n) \, \nu \le |\phi_j - \widetilde{\psi}_\varepsilon \circ T^n|_{C^\alpha(U_j)} \, |\nu|_w \, . \tag{7.18}$$

By (6.12), we have

$$|\phi_j - \widetilde{\psi}_\varepsilon \circ T^n|_{C^\alpha(U_j)} \le C|\psi_\varepsilon \circ T^n \circ G_{U_j^0} - \widetilde{\psi}_\varepsilon \circ T^n \circ G_{U_j}|_{C^\alpha(I_j)} \, ,$$

where I_j is the common r-interval on which $G_{U_j^0}$ an G_{U_j} are defined.

Fix $r \in I_j$, and let $x = G_{U_j^0}(r) \in U_j$ and $\bar{x} = G_{U_j}(r)$. Since U_j^0 and U_j are matched, there exists $y \in U_j^0$ and an unstable curve $\gamma_y \in \Gamma_j$ such that $\gamma_y \cap U_j = \bar{x}$. By definition of $\widetilde{\psi}_\varepsilon$, we have $\widetilde{\psi}_\varepsilon \circ T^n(\bar{x}) = \psi_\varepsilon \circ T^n(y)$. Thus,

$$\begin{aligned}&|\psi_\varepsilon \circ T^n \circ G_{U_j^0}(r) - \widetilde{\psi}_\varepsilon \circ T^n \circ G_{U_j}(r)| \\ &\qquad \le |\psi_\varepsilon \circ T^n(x) - \psi_\varepsilon \circ T^n(y)| + |\psi_\varepsilon \circ T^n(y) - \widetilde{\psi}_\varepsilon \circ T^n(\bar{x})| \\ &\qquad \le |\psi_\varepsilon \circ T^n|_{C^1(U_j^0)} d(x,y) \le C\Lambda^{-n} \le C\varepsilon \, ,\end{aligned}$$

where we have used the fact that $d(x,y) \le C\Lambda^{-n}$ due to the uniform transversality of stable and unstable curves.

Now given $r, s \in I_j$, we have on the one hand,

$$|\psi_\varepsilon \circ T^n \circ G_{U_j^0}(r) - \widetilde{\psi}_\varepsilon \circ T^n \circ G_{U_j}(r) - \psi_\varepsilon \circ T^n \circ G_{U_j^0}(s) + \widetilde{\psi}_\varepsilon \circ T^n \circ G_{U_j}(s)| \le 2C\varepsilon \, ,$$

while on the other hand,

$$\begin{aligned}&|\psi_\varepsilon \circ T^n \circ G_{U_j^0}(r) - \widetilde{\psi}_\varepsilon \circ T^n \circ G_{U_j}(r) - \psi_\varepsilon \circ T^n \circ G_{U_j^0}(s) + \widetilde{\psi}_\varepsilon \circ T^n \circ G_{U_j}(s)| \\ &\qquad \le (|\psi_\varepsilon \circ T^n|_{C^1(U_j^0)} + |\widetilde{\psi}_\varepsilon \circ T^n|_{C^1(U_j)}) C|r-s| \, ,\end{aligned}$$

where we have used the fact that $G_{U_j^0}^{-1}$ and $G_{U_j}^{-1}$ have bounded derivatives since the stable cone is bounded away from the vertical.

The difference is bounded by the minimum of these two expressions. This is greatest when the two are equal, i.e., when $|r-s| = C\varepsilon$. Thus

$$H^\alpha(\psi_\varepsilon \circ T^n \circ G_{U_j^0} - \widetilde{\psi}_\varepsilon \circ T^n \circ G_{U_j}) \le C\varepsilon^{1-\alpha},$$

and so $|\phi_j - \widetilde{\psi}_\varepsilon \circ T^n|_{C^\alpha(U_j)} \le C\varepsilon^{1-\alpha}$. Putting this estimate together with (7.17) and (7.18) in (7.16), we conclude,

$$\left|\int_{W^0} \psi_\varepsilon\, \nu - \int_W \widetilde{\psi}_\varepsilon\, \nu\right| \le C|\log\varepsilon|^{-\varsigma}\|\nu\|_u + C\varepsilon^{1-\alpha}|\nu|_w\,. \tag{7.19}$$

Now since $\int_{W^0} \psi_\varepsilon\, \nu \le 2\varepsilon$, we have

$$\int_W \widetilde{\psi}_\varepsilon\, \nu \le C'|\log\varepsilon|^{-\varsigma}\,, \tag{7.20}$$

where C' depends on ν. Since $\widetilde{\psi}_\varepsilon = 1$ on $\Theta_W(E)$ and $\widetilde{\psi}_\varepsilon > 0$ on an open set containing $\Theta_W(E)$ for every $\varepsilon > 0$, we have $\nu(\Theta_W(E)) = 0$, as required. □

We next state our main absolute continuity result.

Corollary 7.9 (Absolute continuity of μ_* with respect to unstable foliations)**.** *Let R be a Cantor rectangle with $\mu_*(R) > 0$. Fix $W^0 \in \mathcal{W}^s(R)$ and for $W \in \mathcal{W}^s(R)$, let Θ_W denote the holonomy map from $W^0 \cap R$ to $W \cap R$ along unstable manifolds in $\mathcal{W}^u(R)$. Then Θ_W is absolutely continuous with respect to the measure μ_*.*

To deduce the corollary from Proposition 7.8, we shall introduce a set M^{reg} of regular points and a countable cover of this set by Cantor rectangles. The set M^{reg} is defined by

$$M^{reg} = \{x \in M : d(x, \partial W^s(x)) > 0\,,\ \ d(x, \partial W^u(x)) > 0\}\,.$$

At each $x \in M^{reg}$, by [CM, Prop 7.81], we construct a (closed) locally maximal[33] Cantor rectangle R_x, containing x, which is the direct product of local stable and unstable manifolds (recall Section 5.3). By trimming the sides, we may arrange it so that $\frac{1}{2}\mathrm{diam}^s(R_x) \le \mathrm{diam}^u(R_x) \le 2\mathrm{diam}^s(R_x)$.

Lemma 7.10 (Countable cover of M^{reg} by Cantor rectangles)**.** *There exists a countable set $\{x_j\}_{j\in\mathbb{N}} \subset M^{reg}$, such that $\bigcup_j R_{x_j} = M^{reg}$ and each $R_j := R_{x_j}$ satisfies* (5.10)*.*

Proof. Let $n_\delta \in \mathbb{N}$ be such that $1/n_\delta \le \delta_0$. As already mentioned, in the proof of Proposition 5.5, for each $n \ge n_\delta$, by [CM, Lemma 7.87], there exists a finite number of R_x such that any stable manifold of length at least $1/n$ properly crosses one of the R_x (see Section 5.3 for the definition of proper crossing, recalling that each R_x must satisfy (5.10)). This fact follows from the compactness of the set of stable curves in the Hausdorff metric. Call this finite set of rectangles $\{R_{n,i}\}_{i\in\tilde{I}_n}$.

Fix $y \in M^{reg}$ and define $\epsilon = \min\{d(y, \partial W^s(y)), d(y, \partial W^u(y))\} > 0$. Choose $n \ge n_\delta$ such that $2/n < \epsilon$. By construction, there exists $i \in \tilde{I}_n$ such that $W^s(y)$ properly crosses $R_{n,i}$. Now $\mathrm{diam}^s(R_{n,i}) \le 1/n$, which implies $\mathrm{diam}^u(R_{n.i}) \le 2/n < \epsilon$. Thus $W^u(y)$ crosses $R_{n,i}$ as well. By maximality, $y \in R_{n,i}$. □

[33] Recall that, as in Section 5.3, by locally maximal we mean that $y \in R_x$ if and only if $y \in D(R_x)$ and y has stable and unstable manifolds that completely cross $D(R_x)$.

Let $\{R_{n,i} : n \geq n_\delta, i \in \tilde{I}_n\}$ be the Cantor rectangles constructed in the proof of Lemma 7.10. Since $\mu_*(M^{reg}) = 1$, by discarding any $R_{n,i}$ of zero measure, we obtain a countable collection of Cantor rectangles

$$\{R_j\}_{j\in\mathbb{N}} := \{R_{n,i} : n \geq n_\delta, i \in I_n\} \tag{7.21}$$

such that $\mu_*(R_j) > 0$ for all j and $\mu_*(\bigcup_j R_j) = 1$. In the rest of the paper we shall work with this countable collection of rectangles.

Given a Cantor rectangle R, define $\mathcal{W}^s(R)$ to be the set of stable manifolds that completely cross $D(R)$, and similarly for $\mathcal{W}^u(R)$.

Proof of Corollary 7.9. In order to prove absolute continuity of the unstable foliation with respect to μ_*, we will show that the conditional measures μ_*^W of μ_* are equivalent to ν on μ_*-almost every $W \in \mathcal{W}^s(R)$.

Fix a Cantor rectangle R satisfying (5.10) with $\mu_*(R) > 0$, and W^0 as in the statement of the corollary. Let $E \subset W^0 \cap R$ satisfy $\nu(E) = 0$, for the leafwise measure ν.

For any $W \in \mathcal{W}^s(R)$, we have the holonomy map $\Theta_W : W^0 \cap R \to W \cap R$ as in the proof of Proposition 7.8. For $\varepsilon > 0$, we approximate E, choose n, and construct a foliation $\overline{\Gamma}$ of the solid rectangle $D(R)$ as before. Define ψ_ε and use the foliation $\overline{\Gamma}$ to define $\widetilde{\psi}_\varepsilon$ on $D(R)$. We have $\widetilde{\psi}_\varepsilon = 1$ on $\bar{E} = \bigcup_{x\in E} \bar{\gamma}_x$, where $\bar{\gamma}_x$ is the element of $\overline{\Gamma}$ containing x. We extend $\widetilde{\psi}_\varepsilon$ to M by setting it equal to 0 on $M \setminus D(R)$.

It follows from the proof of Proposition 7.8, in particular (7.20), that $\widetilde{\psi}_\varepsilon \nu \in \mathcal{B}_w$, and $|\widetilde{\psi}_\varepsilon \nu|_w \leq C' |\log \varepsilon|^{-\varsigma}$. Now,

$$\begin{aligned}
\mu_*(\widetilde{\psi}_\varepsilon) = \tilde{\nu}(\widetilde{\psi}_\varepsilon \nu) &= \lim_{j\to\infty} \frac{1}{n_j} \sum_{k=0}^{n_j-1} e^{-kh_*} (\mathcal{L}^*)^k d\mu_{\mathrm{SRB}}(\widetilde{\psi}_\varepsilon \nu) \\
&= \lim_{j\to\infty} \frac{1}{n_j} \sum_{k=0}^{n_j-1} e^{-kh_*} \mu_{\mathrm{SRB}}(\mathcal{L}^k(\widetilde{\psi}_\varepsilon \nu)) .
\end{aligned} \tag{7.22}$$

For each k, using the disintegration of μ_{SRB} as in the proof of Lemma 7.7 with the same notation as there, we estimate,

$$\begin{aligned}
\mu_{\mathrm{SRB}}(\mathcal{L}^k(\widetilde{\psi}_\varepsilon \nu)) &= \int_\Xi \int_{W_\xi} \mathcal{L}^k(\widetilde{\psi}_\varepsilon \nu)\, \rho_\xi \, dm_{W_\xi} \, |W_\xi|^{-1} \, d\hat{\mu}_{\mathrm{SRB}}(\xi) \\
&\leq C \int_\Xi |\mathcal{L}^k(\widetilde{\psi}_\varepsilon \nu)|_w \, |W_\xi|^{-1} \, d\hat{\mu}_{\mathrm{SRB}}(\xi) \\
&\leq C e^{kh_*} |\widetilde{\psi}_\varepsilon \nu|_w \leq C e^{kh_*} |\log \varepsilon|^{-\varsigma} ,
\end{aligned}$$

where we have used (4.9) in the last line. Thus $\mu_*(\widetilde{\psi}_\varepsilon) \leq C |\log \varepsilon|^{-\varsigma}$, for each $\varepsilon > 0$, so that $\mu_*(\bar{E}) = 0$.

Disintegrating μ_* into conditional measures $\mu_*^{W_\xi}$ on $W_\xi \in \mathcal{W}^s$ and a factor measure $d\hat{\mu}_*(\xi)$ on the index set Ξ_R of stable manifolds in $\mathcal{W}^s(R)$, it follows that $\mu_*^{W_\xi}(\bar{E}) = 0$ for $\hat{\mu}_*$-almost every $\xi \in \Xi_R$. Since E was arbitrary, the conditional measures of μ_* on $\mathcal{W}^s(R)$ are absolutely continuous with respect to the leafwise measure ν.

To show that in fact μ_*^W is equivalent to ν, suppose now that $E \subset W^0$ has $\nu(E) > 0$. For any $\varepsilon > 0$ such that $C' |\log \varepsilon|^{-\varsigma} < \nu(E)/2$, where C' is from (7.20), choose $\psi_\varepsilon \in C^1(W^0)$ such that $\nu(|\psi_\varepsilon - 1_E|) < \varepsilon$, where 1_E is the indicator function of the set E. As above, we extend ψ_ε to a function $\widetilde{\psi}_\varepsilon$ on $D(R)$ via the foliation $\overline{\Gamma}$, and then to M by setting $\widetilde{\psi}_\varepsilon = 0$ on $M \setminus D(R)$.

We have $\widetilde{\psi}_\varepsilon \nu \in \mathcal{B}_w$ and by (7.19)

$$\nu(\widetilde{\psi}_\varepsilon 1_W) \geq \nu(\psi_\varepsilon 1_{W^0}) - C'|\log \varepsilon|^{-\varsigma} \qquad \text{for all } W \in \mathcal{W}^s(R)\,. \tag{7.23}$$

Now following (7.22) and disintegrating μ_{SRB} as usual, we obtain,

$$\begin{aligned} \mu_*(\widetilde{\psi}_\varepsilon) &= \lim_n \frac{1}{n} \sum_{k=0}^{n-1} e^{-kh_*} \int_\Xi \int_{W_\xi} \mathcal{L}^k(\widetilde{\psi}_\varepsilon \nu)\, \rho_\xi \, dm_{W_\xi} \, d\hat{\mu}_{\mathrm{SRB}}(\xi) \\ &= \lim_n \frac{1}{n} \sum_{k=0}^{n-1} e^{-kh_*} \int_\Xi \left(\sum_{W_{\xi,i} \in \mathcal{G}_k(W_\xi)} \int_{W_{\xi,i}} \widetilde{\psi}_\varepsilon \, \rho_\xi \circ T^k \, \nu \right) d\hat{\mu}_{\mathrm{SRB}}(\xi)\,. \end{aligned} \tag{7.24}$$

To estimate this last expression, we estimate the cardinality of the curves $W_{\xi,i}$ which properly cross the rectangle R.

By Corollary 5.3 and the choice of δ_1 in (5.6), there exists k_0, depending only on the minimum length of $W \in \mathcal{W}^s(R)$, such that $\# L_k^{\delta_1}(W_\xi) \geq \frac{1}{3} \# \mathcal{G}_k(W_\xi)$ for all $k \geq k_0$.

By choice of our covering $\{R_i\}$ from (7.21), all $W_{\xi,j} \in L_k^{\delta_1}(W_\xi)$ properly cross one of finitely many R_i. By the topological mixing property of T, there exists n_0, depending only on the length scale δ_1, such that some smooth component of $T^{-n_0} W_{\xi,j}$ properly crosses R. Thus, letting $\mathcal{C}_k(W_\xi)$ denote those $W_{\xi,i} \in \mathcal{G}_k(W_\xi)$ which properly cross R, we have

$$\# \mathcal{C}_k(W_\xi) \geq \# L_{k-n_0}^{\delta_1}(W_\xi) \geq \tfrac{1}{3} \# \mathcal{G}_{k-n_0}(W_\xi) \geq \tfrac{1}{3} c e^{(k-n_0)h_*}$$

for all $k \geq k_0 + n_0$, where $c > 0$ depends on c_0 from Proposition 5.5 as well as the minimum length of $W \in \mathcal{W}^s(R)$.

Using this lower bound on the cardinality together with (7.23) yields,

$$\mu_*(\widetilde{\psi}_\varepsilon) \geq \tfrac{1}{3} c e^{-n_0 h_*} \big(\nu(\psi_\varepsilon) - C'|\log \varepsilon|^{-\varsigma} \big) \geq C'' \big(\nu(E) - |\log \varepsilon|^{-\varsigma} \big)\,.$$

Taking $\varepsilon \to 0$, we have $\mu_*(\bar{E}) \geq C'' \nu(E)$, and so $\mu_*^W(\bar{E}) > 0$ for almost every $W \in \mathcal{W}^s(R)$. $\square$

A consequence of the proof of Corollary 7.9 is the positivity of μ_* on open sets.

Proposition 7.11 (Full support)**.** *We have $\mu_*(O) > 0$ for any open set O.*

Proof. Suppose R is a Cantor rectangle with index set of stable leaves Ξ_R. We call $I \subset \Xi_R$ an interval if $a, b \in I$ implies that $c \in I$ for all $c \in \Xi_R$ such that W_c lies between W_a and W_b.[34] It follows from the proof of Corollary 7.9 that for any interval $I \subset \Xi_R$ such that $\hat{\mu}_{\mathrm{SRB}}(I) > 0$, then $\mu_*(\bigcup_{\xi \in I} W_\xi) > 0$. Indeed, by Lemma 7.7, $\hat{\nu}$ is equivalent to $\hat{\mu}_{\mathrm{SRB}}$ (since $\nu(W) > 0$ for all $W \in \mathcal{W}^s$, when ν is viewed as a leafwise measure), so that $\hat{\mu}_{\mathrm{SRB}}(I) > 0$ implies $\hat{\nu}(I) > 0$. Then by Lemma 7.6 there exists a Cantor rectangle R' with $D(R') \subset D(R)$ and $\Xi_{R'} \subset I$ such that $\nu(R') > 0$. Then we simply apply (7.24) and the argument following it with $\widetilde{\psi}_\varepsilon$ replaced by the characteristic function of $\bigcup_{\xi \in \Xi_{R'}} W_\xi$.

[34]Notice that if $I \subset \Xi_j$ is an interval such that $\hat{\mu}_{\mathrm{SRB}}(I) > 0$, then $\bigcup_{\xi \in I} W_\xi \cap R_j$ is a Cantor rectangle which contains a subset satisfying the high density condition (5.10), so we can talk about proper crossings.

Then if O is an open set in M, it contains a Cantor rectangle R such that $D(R) \subset O$ and $\mu_{\mathrm{SRB}}(R) > 0$. It follows that $\hat{\mu}_{\mathrm{SRB}}(\Xi_R) > 0$, and so $\mu_*(\bigcup_{\xi \in \Xi_R} W_\xi) > 0$. □

7.4. **Bounds on dynamical Bowen balls—comparing μ_* and μ_{SRB}.** In this section we show upper and lower bounds on the μ_*-measure of dynamical Bowen balls, from which we establish a necessary condition for μ_* and μ_{SRB} to coincide. (The lower bound will use results from Section 7.3.)

For $\epsilon > 0$ and $x \in M$, we denote by $B_n(x, \epsilon)$ the dynamical (Bowen) ball at x of length $n \geq 1$ for T^{-1}, i.e.,

$$B_n(x, \epsilon) = \{y \in M \mid d(T^{-j}(y), T^{-j}(x)) \leq \epsilon \ \forall \, 0 \leq j \leq n\}.$$

For $\eta, \delta > 0$ and $p \in (1/\gamma, 1]$, let $M^{reg}(\eta, p, \delta)$ denote those $x \in M^{reg}$ such that $d(T^{-n}x, \mathcal{S}_{-1}) \geq \delta e^{-\eta n^p}$. It follows from Lemma 7.3 that $\mu_*(\bigcup_{\delta > 0} M^{reg}(\eta, p, \delta)) = 1$.

Proposition 7.12 (Topological entropy and measure of dynamical balls)**.** *Assume that $h_* > s_0 \log 2$. There exists $A < \infty$ such that for all $\epsilon > 0$ sufficiently small, $x \in M$, and $n \geq 1$, the measure μ_* constructed in* (7.1) *satisfies*

$$\mu_*(B_n(x, \epsilon)) \leq \mu_*(\overline{B_n(x, \epsilon)}) \leq A e^{-n h_*} . \tag{7.25}$$

Moreover, for all $\eta, \delta > 0$ and $p \in (1/\gamma, 1]$, for each $x \in M^{reg}(\eta, p, \delta)$, and all $\varepsilon > 0$ sufficiently small, there exists $C(x, \epsilon, \eta, p, \delta) > 0$ such that for all $n \geq 1$,

$$C(x, \epsilon, \eta, p, \delta) \, e^{-n h_* - \eta h_* \bar{C}_2 n^p} \leq \mu_*(B_n(x, \epsilon)), \tag{7.26}$$

where $\bar{C}_2 > 0$ is the constant from the proof of Corollary 5.3.

Proof. Assume $\gamma > 1$. Fix $\epsilon > 0$ such that $\epsilon \leq \min\{\delta_0, \varepsilon_0\}$, where ε_0 is from the proof of Lemma 3.4. For $x \in M$ and $n \geq 0$, define $1^B_{n,\epsilon}$ to be the indicator function of the dynamical ball $B_n(x, \epsilon)$.

Since ν is attained as the (averaged) limit of $\mathcal{L}^n 1$ in the weak norm and since we have $\int_W (\mathcal{L}^n 1) \, \psi dm_W \geq 0$ whenever $\psi \geq 0$, it follows that, viewing ν as a leafwise distribution,

$$\int_W \psi \, \nu \geq 0 \quad \text{ for all } \psi \geq 0. \tag{7.27}$$

Then the inequality $|\int_W \psi \, \nu| \leq \int_W |\psi| \, \nu$ implies that the supremum in the weak norm can be obtained by restricting to $\psi \geq 0$.

Let $W \in \mathcal{W}^s$ be a curve intersecting $B_n(x, \epsilon)$, and let $\psi \in C^\alpha(W)$ satisfy $\psi \geq 0$ and $|\psi|_{C^\alpha(W)} \leq 1$. Then, since $\mathcal{L}\nu = e^{h_*}\nu$, we have

$$\int_W \psi \, 1^B_{n,\epsilon} \, \nu = \int_W \psi \, 1^B_{n,\epsilon} \, e^{-n h_*} \mathcal{L}^n \nu = e^{-n h_*} \sum_{W_i \in \mathcal{G}_n(W)} \int_{W_i} \psi \circ T^n \, 1^B_{n,\epsilon} \circ T^n \, \nu \, . \tag{7.28}$$

We claim that $1^B_{n,\varepsilon} \nu \in \mathcal{B}_w$ (and indeed in $\mathcal{B}$). To see this, note that

$$1^B_{n,\varepsilon} = \prod_{j=0}^{n} 1_{\mathcal{N}_\varepsilon(T^{-j}x)} \circ T^{-j} = \prod_{j=0}^{n} \mathcal{L}^j_{\mathrm{SRB}}(1_{\mathcal{N}_\varepsilon(T^{-j}x)}) \, ,$$

where, as in Section 1.3, $\mathcal{L}_{\mathrm{SRB}}$ denotes the transfer operator with respect to μ_{SRB}. Since $\mathcal{L}_{\mathrm{SRB}}$ preserves $\mathcal{B}$ and $\mathcal{B}_w$ ([DZ3, Lemma 3.6]), it suffices to show that $1_{\mathcal{N}_\varepsilon(T^{-j}x)}$ satisfies the assumptions of [DZ3, Lemma 5.3]. This follows from the fact that $\partial\mathcal{N}_\varepsilon(T^{-j}x)$ comprises a single circular arc, possibly together with a segment of $\mathcal{S}_0$, which satisfies the weak transversality condition of that lemma with $t_0 = 1/2$. Then applying [DZ3, Lemma 5.3] successively for each j yields the claim.

In the proof of Lemma 3.4, it was shown that if x, y lie in different elements of $\mathcal{M}_0^n$, then $d_n(x,y) \geq \varepsilon_0$, where $d_n(\cdot,\cdot)$ is the dynamical distance defined in (2.1). Since $B_n(x,\epsilon)$ is defined with respect to T^{-1}, we will use the time reversal counterpart of this property. Thus since $\epsilon < \varepsilon_0$, we conclude that $B_n(x,\epsilon)$ is contained in a single component of $\mathcal{M}_{-n}^0$, i.e., $B_n(x,\epsilon)\cap\mathcal{S}_{-n} = \emptyset$, so that T^{-n} is a diffeomorphism of $B_n(x,\epsilon)$ onto its image. Note that $1_{n,\epsilon}^B \circ T^n = 1_{T^{-n}(B_n(x,\epsilon))}$ and that $T^{-n}(B_n(x,\epsilon))$ is contained in a single component of $\mathcal{M}_0^n$, denoted $A_{n,\epsilon}$.

It follows that for each $W_i \in \mathcal{G}_n(W)$ we have $W_i \cap A_{n,\epsilon} = W_i$. By (7.27), we have

$$\int_{W_i} (\psi\circ T^n)\, 1_{T^{-n}(B_n(x,\epsilon))}\, \nu \leq \int_{W_i} \psi\circ T^n\, \nu\,.$$

Moreover, there can be at most two $W_i \in \mathcal{G}_n(W)$ having nonempty intersection with $T^{-n}(B_n(x,\epsilon))$. This follows from the facts that $\epsilon \leq \delta_0$, and that, in the absence of any cuts due to singularities, the only subdivisions occur when a curve has grown to length longer than δ_0 and is subdivided into two curves of length at least $\delta_0/2$.

Using these facts together with (6.2), we sum over $W_i' \in \mathcal{G}_n(W)$ such that $W_i' \cap T^{-n}(B_n(x,\epsilon)) \neq 0$, to obtain

$$\int_W \psi\, 1_{n,\epsilon}^B\, \nu \leq e^{-nh_*}\sum_i \int_{W_i'} \psi\circ T^n\,\nu \leq 2Ce^{-nh_*}|\nu|_w\,.$$

This implies that $|1_{n,\varepsilon}^B \nu|_w \leq 2Ce^{-nh_*}|\nu|_w$. Applying (7.5), implies (7.25).

Next we prove (7.26). Fix $\eta, \delta > 0$ with $e^\eta < \Lambda$ and $p \in (1/\gamma, 1]$, and let $x \in M^{reg}(\eta,p,\delta)$. By [CM, Lemma 4.67] the length of the local stable manifold containing x is at least δC_1, where C_1 is from (3.1). So by [CM, Lemma 7.87], there exists a Cantor rectangle R_x containing x such that $\mu_{\mathrm{SRB}}(R_x) > 0$ and whose diameter depends only on the length scale δC_1. By the proof of Proposition 7.11, we also have $\mu_*(R_x) > 0$. In particular, $\hat\mu_*(\Xi_{R_x}) = c_x > 0$, where Ξ_{R_x} is the index set of stable manifolds comprising R_x. Let $\delta' > 0$ denote the minimum length of $W_\xi \cap D(R_x)$ for $\xi \in \Xi_{R_x}$, where $D(R_x)$ is the smallest solid rectangle containing R_x, as in Definition 5.7.

Choose $\epsilon > 0$ such that $\epsilon \leq \min\{\delta_0, \varepsilon_0, \delta', \delta\}$. As above, we note that $B_n(x,\epsilon)$ is contained in a single component of $\mathcal{M}_{-n}^0$, and thus $T^{-n}(B_n(x,\epsilon))$ is contained in a single component of $\mathcal{M}_0^n$. Moreover, T^{-n} is smooth on $W^u(x)\cap D(R_x)$. Now suppose $y \in W^u(x)\cap R_x$. Then since $x \in M^{reg}(\eta,p,\delta)$,

$$d(T^{-n}y, \mathcal{S}_{-1}) \geq d(T^{-n}x,\mathcal{S}_{-1}) - d(T^{-n}y, T^{-n}x) \geq \delta e^{-\eta n^p} - C_1\Lambda^{-n} \geq \tfrac{\delta}{2}e^{-\eta n^p}$$

for n sufficiently large. It follows that for each $\xi \in \Xi_{R_x}$, there exists $W_{\xi,i} \in \mathcal{G}_n(W_\xi)$ such that $W_{\xi,i}' = W_{\xi,i}\cap T^{-n}(B_n(x,\epsilon))$ is a single curve and $|W_{\xi,i}'| \geq \min\{\frac{\delta}{2}e^{-\eta n^p}, \epsilon\} \geq \frac{\epsilon}{2}e^{-\eta n^p}$. Thus recalling (7.13) and following (7.28) with $\psi \equiv 1$,

$$\int_{W_\xi} 1_{n,\epsilon}^B\,\nu \geq e^{-nh_*}\int_{W_{\xi,i}'} \nu \geq \bar C e^{-nh_*}|W_{\xi,i}'|^{h_*\bar C_2} \geq C' e^{-nh_* - \eta h_* \bar C_2 n^p}\,,$$

where C' depends on ϵ.

Finally, using the fact from the proof of Corollary 7.9 that μ_*^W is equivalent to ν on μ_*-a.e. $W \in \mathcal{W}^s$, we estimate,

$$\mu_*(B_n(x,\epsilon)) \geq \mu_*(B_n(x,\epsilon) \cap D(R_x)) = \int_{\Xi_{R_x}} \mu_*^{W_\xi}(B_n(x,\epsilon))\, d\hat{\mu}_*(\xi)$$
$$\geq C \int_{\Xi_{R_x}} \nu(B_n(x,\epsilon) \cap W_\xi)\, d\hat{\mu}_*(\xi) \geq C'' e^{-nh_* - \eta h_* \bar{C}_2 n^p} \hat{\mu}_*(\Xi_{R_x})\,.$$

□

Periodic points whose orbit do not have grazing collisions belong to M^{reg}. We call them *regular.*

Proposition 7.13 (μ_* and μ_{SRB})**.** *Assume $h_* > s_0 \log 2$. If there exists a regular periodic point x of period p such that $\lambda_x = \frac{1}{p}\log|\det(DT^{-p}|_{E^s}(x))| \neq h_*$, then $\mu_* \neq \mu_{\text{SRB}}$.*

Although h_* may not be known a priori, using Proposition 7.13 it suffices to find two regular periodic points x, y such that $\lambda_x \neq \lambda_y$, to conclude that $\mu_* \neq \mu_{\text{SRB}}$. (All known examples of dispersing billiard tables satisfy this condition.)

Proposition 7.13 relies on the following lemma.

Lemma 7.14. *Let $x \in M^{reg}$ be a regular periodic point. There exists $A > 0$ such that for all $\epsilon > 0$ sufficiently small, there exists $C(x,\epsilon) > 0$ such that for all $n \geq 1$,*

$$C(x,\epsilon) e^{-n\lambda_x} \leq \mu_{\text{SRB}}(B_n(x,\epsilon)) \leq A e^{-n\lambda_x}\,.$$

Proof. Let x be a regular periodic point for T of period p. For ϵ sufficiently small, $T^{-i}(\mathcal{N}_\epsilon(x))$ belongs to a single homogeneity strip for $i = 0, 1, \ldots, p$. Thus if $y \in B_n(x,\epsilon) \cap W^s(x)$, then the stable Jacobians $J^sT^n(x)$ and $J^sT^n(y)$ satisfy the bounded distortion estimate, $|\log \frac{J^sT^n(x)}{J^sT^n(y)}| \leq C_d d(x,y)^{1/3}$, for a uniform $C_d > 0$ [CM, Lemma 5.27]. It follows that the conditional measure on $W^s(x)$ satisfies

$$C_x^{-1} \epsilon e^{-n\lambda_x} \leq \mu_{\text{SRB}}^{W^s(x)}(B_n(x,\epsilon)) \leq C_x \epsilon e^{-n\lambda x} \tag{7.29}$$

for some $C_x \geq 1$, depending on the homogeneity strips in which the orbit of x lies.

Next, using again [CM, Prop 7.81], we can find a Cantor rectangle $R_x \subset \mathcal{N}_\epsilon(x)$ with diameter at most $\varepsilon/(2C_1)$ and $\mu_{\text{SRB}}(R_x) \geq C\mu_{\text{SRB}}(\mathcal{N}_\epsilon(x))/(2C_1)^2$, for a constant $C > 0$ depending on the distortion of the measure. Note that $W^u(x) \cap D(R_x)$ is never cut by $\mathcal{S}_{-n}$ and lies in $B_n(x,\epsilon)$ by (3.1). Thus each $W \in \mathcal{W}^s(R_x)$ has a component in $B_n(x,\varepsilon)$ and this component has length satisfying the same bounds as (7.29). Integrating over $B_n(x,\epsilon)$ as in the proof of Proposition 7.12 proves the lemma. An inspection of the proof shows that the constant in the upper bound can be chosen independently of x when ϵ is sufficiently small, while the constant in the lower bound cannot. □

Proof of Proposition 7.13. If x is a regular periodic point, then the upper and lower bounds on $\mu_*(B_n(x,\epsilon))$ from Proposition 7.12 hold with[35] $\eta = 0$ for ϵ sufficiently small. If $\lambda_x \neq h_*$, these do not match the exponential rate in the bounds on $\mu_{\text{SRB}}(B_n(x,\epsilon))$ from Lemma 7.14. Thus for n sufficiently large, $\mu_*(B_n(x,\epsilon)) \neq \mu_{\text{SRB}}(B_n(x,\epsilon))$. □

[35] Here, it is convenient to have the role of η explicit in (7.26).

7.5. **K-mixing and maximal entropy of μ_*—Bowen–Pesin–Pitskel Theorem 2.5.** In this section we use the absolute continuity results from Section 7.3 to establish K-mixing of μ_*. We also show that μ_* has maximal entropy, exploiting the upper bound from Section 7.4. Finally, we show that h_* coincides with the Bowen–Pesin–Pitskel entropy.

Lemma 7.15 (Single ergodic component)**.** *If R is a Cantor rectangle with $\mu_*(R) > 0$, then the set of stable manifolds $\mathcal{W}^s(R)$ belongs to a single ergodic component of μ_*.*

Proof. We follow the well-known Hopf strategy outlined in [CM, Section 6.4] of smooth ergodic theory to show that μ_*-almost every stable and unstable manifold has a full measure set of points belonging to a single ergodic component: Given a continuous function φ on M, let $\overline{\varphi}_+$, $\overline{\varphi}_-$ denote the forward and backward ergodic averages of φ, respectively. Let $M_\varphi = \{x \in M^{reg} : \overline{\varphi}_+(x) = \overline{\varphi}_-(x)\}$. When the two functions agree, denote their common value by $\overline{\varphi}$.

Now fix a Cantor rectangle R with $\mu_*(R) > 0$. By Corollary 7.4, if $\gamma > 1$, then $\mu_*(M^{reg}) = 1$. So, by the Birkhoff ergodic theorem, $\mu_*(M_\varphi) = 1$. Thus for μ_* almost every $W \in \mathcal{W}^s(R)$, the conditional measure μ_*^W satisfies $\mu_*^W(M_\varphi) = 1$. Due to the fact that forward ergodic averages are the same for any two points in W, it follows that $\overline{\varphi}$ is constant on $W \cap M_\varphi$. The analogous fact holds for unstable manifolds in $\mathcal{W}^u(R)$.

Let

$$G_\varphi = \{x \in M_\varphi : \overline{\varphi} \text{ is constant on a full measure subset of } W^u(x) \text{ and } W^s(x)\}.$$

Clearly, $\mu_*(G_\varphi) = 1$, so the same facts apply to G_φ as M_φ.

Let $W^0, W \in \mathcal{W}^s(R)$ be stable manifolds with $\mu_*^{W0}(G_\varphi) = \mu_*^W(G_\varphi) = 1$. Let Θ_W denote the holonomy map from $W^0 \cap R$ to $W \cap R$. By absolute continuity, Corollary 7.9, $\mu_*^W(\Theta_W(W^0 \cap G_\varphi)) > 0$. Thus $\overline{\varphi}$ is constant for almost every point in $\Theta_W(W^0 \cap G_\varphi)$. Let y be one such point and let $x = \Theta_W^{-1}(y)$. Then since $x \in W^u(y) \cap G_\varphi$,

$$\overline{\varphi}(x) = \overline{\varphi}_-(x) = \overline{\varphi}_-(y) = \overline{\varphi}(y),$$

so that the values of $\overline{\varphi}$ on a positive measure set of points in W^0 and W agree. Since $\overline{\varphi}$ is constant on G_φ, the values of $\overline{\varphi}$ on a full measure set of points in W and W^0 are equal. Since this applies to any W with $\mu_*^W(G_\varphi) = 1$, we conclude that $\overline{\varphi}$ is constant almost everywhere on the set $\bigcup_{W \in \mathcal{W}^s(R)} W$. Finally, since φ was an arbitrary continuous function, the set $\mathcal{W}^s(R)$ belongs (mod 0) to a single ergodic component of μ_*. □

We are now ready to prove the K-mixing property of μ_*.

Proposition 7.16. *(T, μ_*) is K-mixing.*

Proof. We begin by showing that (T^n, μ_*) is ergodic for all $n \geq 1$. Recall the countable set of (locally maximal) Cantor rectangles $\{R_i\}_{i\in\mathbb{N}}$ with $\mu_*(R_i) > 0$, such that $\bigcup_i R_i = M^{reg}$ from (7.21).

We fix n and let R_1 and R_2 be two such Cantor rectangles. By Lemma 7.15, $\mathcal{W}^s(R_i)$ belongs (mod 0) to a single ergodic component of μ_*. Since T is topologically mixing, and using [CM, Lemma 7.90], there exists $n_0 > 0$ such that for any $k \geq n_0$, a smooth component of $T^{-k}(D(R_1))$ properly crosses $D(R_2)$. Let us call D_k the part of this smooth component lying in $D(R_2)$.

Since the set of stable manifolds is invariant under T^{-k}, by the maximality of the set $\mathcal{W}^s(R_2)$, we have that $T^{-k}(\mathcal{W}^s(R_1)) \cap D_k \supseteq \mathcal{W}^s(R_2) \cap D_k$. And since this set of stable manifolds in R_1 has positive measure with respect to $\hat{\mu}_*$, it follows that $\mu_*(T^{-k}(\mathcal{W}^s(R_1)) \cap \mathcal{W}^s(R_2)) > 0$. Thus R_1 and R_2 belong to the same ergodic component of T. Indeed, since we may choose $k = jn$ for some $j \in \mathbb{N}$, R_1 and R_2 belong to the same ergodic component of T^n. Since this is true for each pair of Cantor rectangles R_i, R_j in our countable collection, and $\mu_*(\bigcup_i R_i) = 1$, we conclude that T^n is ergodic.

We shall use the Pinsker partition

$$\pi(T) = \bigvee \{\xi : \xi \text{ finite partition of } M, h_{\mu_*}(T, \xi) = 0\}\,.$$

Since T is an automorphism, the sigma-algebra generated by $\pi(T)$ is T-invariant.

Given two measurable partitions ξ_1 and ξ_2, the meet of the two partitions $\xi_1 \wedge \xi_2$ is defined as the finest measurable partition with the property that $\xi_1 \wedge \xi_2 \le \xi_j$ for $j = 1, 2$. All definitions of measurable partitions and inequalities between them are taken to be mod 0, with respect to the measure μ_*. It is a standard fact in ergodic theory (see, e.g., [RoS]) that if ξ is a partition of M such that (i) $T\xi \ge \xi$ and (ii) $\bigvee_{n=0}^\infty T^n \xi = \epsilon$, where ϵ is the partition of M into points, then $\bigwedge_{n=0}^\infty T^{-n}\xi \ge \pi(T)$ (mod 0).

Define ξ^s to be the partition of M into maximal local stable manifolds. If $x \in M$ has no stable manifold or x is an endpoint of a stable manifold, then define $\xi^s(x) = \{x\}$. Similarly, define ξ^u to be the partition of M into maximal local unstable manifolds. Note that ξ^s is a measurable partition of M since it is generated by the countable family of finite partitions given by the elements of $\mathcal{M}_0^n$ and their closures. Similarly, $\mathcal{M}_{-n}^0$ provides a countable generator for ξ^u.

It is a consequence of the uniform hyperbolicity of T that ξ^s satisfies (i) and (ii) above. Also, ξ^u satisfies these conditions with respect to T^{-1}, i.e., $T^{-1}\xi^u \ge \xi^u$ and $\bigvee_{n=0}^\infty T^{-n}\xi^u = \epsilon$. Thus $\bigwedge_{n=0}^\infty T^n \xi^u \ge \pi(T)$.

Define $\eta_\infty = \bigwedge_{n=0}^\infty (T^n\xi^u \wedge T^{-n}\xi^s)$, and notice that $\eta_\infty \ge \pi(T)$ by the above. Then since $\xi^s \wedge \xi^u \ge \eta_\infty$, we have $\xi^s \wedge \xi^u \ge \pi(T)$ as well.

We will show that each Cantor rectangle in our countable family belongs to one element of $\xi^s \wedge \xi^u$ (mod 0). This will follow from the product structure of each R_i coupled with the absolute continuity of the holonomy map given by Corollary 7.9.

For brevity, let us fix i and set $R = R_i$. We index the curves $W^s_\zeta \in \mathcal{W}^s(R)$ by $\zeta \in Z$. Define $\mu_R = \frac{\mu_*|_R}{\mu_*(R)}$. We disintegrate the measure μ_R into a family of conditional probability measures $\mu_R^{W^s}$, $W^s \in \mathcal{W}^s(R)$, and a factor measure $\hat{\mu}_R$ on the set Z. Then

$$\mu_R(A) = \int_{\zeta \in Z} \mu_R^{W^s_\zeta}(A)\, d\hat{\mu}_R(\zeta) \quad \text{for all measurable sets } A\,.$$

The set R belongs to a single element of $\xi^s \wedge \xi^u$ if a full measure set of points can be connected by elements of ξ^s and ξ^u even after the removal of a set of μ_*-measure 0. Let $N \subset M$ be such that $\mu_*(N) = 0$. By the above disintegration, it follows that for $\hat{\mu}_R$-almost every $\zeta \in Z$, we have $\mu_R^{W^s_\zeta}(N) = 0$.

Let W^s_1 and W^s_2 be two elements of $\mathcal{W}^s(R)$ such that $\mu_R^{W^s_j}(N) = 0$ for $j = 1, 2$. For all $x \in W^s_1 \cap R$, $\xi^u(x)$ intersects W^s_2, and vice versa. Let Θ denote the holonomy map from W^s_1 to W^s_2. Then by Corollary 7.9, we have $\mu_R^{W^s_2}(\Theta(W^s_1 \cap N)) = 0$ and

$\mu_R^{W_1^s}(\Theta^{-1}(W_2^s \cap N)) = 0$. Thus the set $\Theta(W_1^s \setminus N)$ has full measure in W_2^s and vice versa. It folllows that W_1^s and W_2^s belong to one element of $\xi^s \wedge \xi^u$. This proves that R belongs to a single element of $\xi^s \wedge \xi^u$ (mod 0).

Since $\xi^s \wedge \xi^u \geq \pi(T)$, we have shown that each R_i belongs to a single element of $\pi(T)$, mod 0. Since $\mu_*(R_i) > 0$ and $\mu_*(\cup_i R_i) = 1$, the ergodicity of T and the invariance of $\pi(T)$ imply that $\pi(T)$ contains finitely many elements, all having the same measure, whose union has full measure. The action of T is simply a permutation of these elements. Since (T^n, μ_*) is ergodic for all n, it follows that $\pi(T)$ is trivial. Thus (T, μ_*) is K-mixing. $\square$

Now that we know that μ_* is ergodic, the upper bound in Proposition 7.12 will easily[36] imply that $h_{\mu_*}(T) = h_*$.

Corollary 7.17 (Maximum entropy). *For μ_* defined as in* (7.1), *we have $h_{\mu_*}(T) = h_*$.*

Proof. Since $\int |\log d(x, \mathcal{S}_{\pm 1})| \, d\mu_* < \infty$ by Theorem 2.6, and μ_* is ergodic, we may apply [DWY, Prop 3.1][37] to T^{-1}, which states that for μ_*-almost every $x \in M$,

$$\lim_{\epsilon \to 0} \liminf_{n \to \infty} -\tfrac{1}{n} \log \mu_*(B_n(x, \epsilon)) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} -\tfrac{1}{n} \log \mu_*(B_n(x, \epsilon)) = h_{\mu_*}(T^{-1}) \, .$$

Using (7.25) and (7.26) with $p < 1$, it follows that $\lim_{n\to\infty} -\frac{1}{n} \log \mu_*(B_n(x, \epsilon)) = h_*$, for any $\varepsilon > 0$ sufficiently small. Thus $h_{\mu_*}(T) = h_{\mu_*}(T^{-1}) = h_*$. $\square$

Corollary 7.17 next allows us to prove Theorem 2.5 about the Bowen–Pesin–Pitskel entropy.

Proof of Theorem 2.5. To show $h_* \leq h_{\mathrm{top}}(T|_{M'})$, we first use Corollary 7.17 and the fact that $\mu_*(M') = 1$ (since $\mu_*(\mathcal{S}_n) = 0$ for every n by Theorem 2.6) to see that

$$h_* = h_{\mu_*}(T) = \sup_{\mu : \mu(M')=1} h_\mu(T) \, .$$

Then we apply the bound [Pes, (A.2.1)] or [PP, Thm 1] (by Remarks I and II there, T need not be continuous on M) to get

$$\sup_{\mu : \mu(M')=1} h_\mu(T) \leq h_{\mathrm{top}}(T|_{M'}) \, .$$

To show $h_{\mathrm{top}}(T|_{M'}) \leq h_*$, we use that [Pes, (11.12)] implies[38] $h_{\mathrm{top}}(T|_{M'}) \leq Ch_{M'}(T)$, where $Ch_{M'}(T)$ denotes the capacity topological entropy of the (invariant) set M'. Now, for any $\delta > 0$, the elements of $\mathring{\mathcal{P}}_{-k}^k = \mathcal{M}_{-k-1}^{k+1}$ form an open cover of M' of diameter $< \delta$, if k is large enough (see the proof of Lemma 3.4). By adding finitely many open sets, we obtain an open cover $\mathcal{U}_\delta$ of M of diameter $< \delta$. Next [Pes, (11.13)] gives that

$$Ch_{M'}(T) = \lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \log \Lambda(M', \mathcal{U}_\delta, n) \, ,$$

[36] It is not much harder to deduce this fact in the absence of ergodicity, using only (7.26) with Theorem 2.3.

[37] This is a slight generalization of the Brin-Katok local theorem [BK], using [M, Lemma 2]. Continuity of the map is not used in the proof of the theorem, and so it applies to our setting.

[38] Just like in [PP, I and II], it is essential that M is compact, but the fact that T is not continuous on M is irrelevant. Note also that [Pes, (A.3'), p. 66] should be corrected, replacing "any $\varepsilon > \epsilon > 0$" by "any $\varepsilon > 1/m > 0$".

where $\Lambda(M',\mathcal{U}_\delta,n)$ is the smallest cardinality of a cover of M' by elements of $\bigvee_{j=0}^n T^{-j}\mathcal{U}_\delta$. Since for any $n \geq 1$, the sets of $\bigvee_{j=0}^n T^{-j}\mathring{\mathcal{P}}_{-k}^k$ form a cover of M', the second equality of Lemma 3.3 (i.e., $\lim_n \frac{1}{n}\log \#\mathring{\mathcal{P}}_{-k}^{k+n} = h_*$) implies that $Ch_{M'}(T) \leq h_*$. □

7.6. **Bernoulli property of μ_*.** In this section, we prove that μ_* is Bernoulli by bootstrapping from K-mixing. The key ingredients of the proof, in addition to K-mixing, are Cantor rectangles with a product structure of stable and unstable manifolds, the absolute continuity of the unstable foliation with respect to μ_*, and the bounds (2.2) on the neighbourhoods of the singularity sets. First, we recall some definitions, following Chernov–Haskell [ChH] and the notion of very weak Bernoulli partitions introduced by Ornstein [O].

Let (X,μ_X) and (Y,μ_Y) be two nonatomic Lebesgue probability spaces. A *joining* λ of the two spaces is a measure on $X\times Y$ whose marginals on X and Y are μ_X and μ_Y, respectively. Given finite partitions[39] $\alpha = \{A_1,\dots,A_k\}$ of X and $\beta = \{B_1,\dots,B_k\}$ of Y, let $\alpha(x)$ denote the element of α containing $x\in X$ (and similarly for β). Moreover, if $x\in A_j$ and $y \in B_j$ for the same value of j (which depends on the order in which the elements are listed), then we will write $\alpha(x)=\beta(y)$.

The distance $\bar d$ defined below considers two partitions to be close if there is a joining λ such that most of the measure lies on the set of points (x,y) with $\alpha(x)=\beta(y)$: given two finite sequences of partitions $\{\alpha_i\}_{i=1}^n$ of X and $\{\beta_i\}_{i=1}^n$ of Y, define

$$\bar d(\{\alpha_i\},\{\beta_i\}) = \inf_\lambda \int_{X\times Y} h(x,y)\,d\lambda\,,$$

where λ is a joining of X and Y and h is defined by

$$h(x,y) = \frac{1}{n}\#\{i\in[1,\dots,n] : \alpha_i(x)\neq\beta_i(y)\}\,.$$

We will adopt the following notation: If $E\subset X$, then $\alpha|E$ denotes the partition α conditioned on E, i.e., the partition of E given by elements of the form $A\cap E$ for $A\in\alpha$. Similarly, $\mu_X(\,\cdot\,|E)$ is the measure μ_X conditioned on E. If a property holds for all atoms of α except for a collection whose union has measure less than ε, then we say the property holds for ε-almost every atom of α.

If $f : X\to X$ is an invertible, measure preserving transformation of (X,μ_X), and α is a finite partition of X, then α is said to be *very weak Bernoullian* (vwB) if for all $\varepsilon>0$, there exists $N>0$ such that for every $n>0$ and N_0,N_1 with $N<N_0<N_1$, and for ε-almost every atom A of $\bigvee_{N_0}^{N_1} f^i\alpha$, we have

$$\bar d(\{f^{-i}\alpha\}_{i=1}^n, \{f^{-i}\alpha|A\}_{i=1}^n) < \varepsilon\,. \tag{7.30}$$

The following theorem from [OW] provides the essential connection between the Bernoulli property and vwB partitions. (See also Theorems 4.1 and 4.2 in [ChH].)

Theorem 7.18. *If a partition α of X is vwB, then $(X,\bigvee_{n=-\infty}^\infty f^{-n}\alpha,\mu_X,f)$ is a Bernoulli shift. Moreover, if $\bigvee_{n=-\infty}^\infty f^{-n}\alpha$ generates the whole σ-algebra of X, then (X,μ_X,f) is a Bernoulli shift.*

We are ready to state and prove the main result of this section.

[39] As we shall not need the norms of $\mathcal{B}$ and $\mathcal{B}_w$ in this section, we are free to use the letters α and β to denote partitions instead of real parameters.

Proposition 7.19. *The measure μ_* is Bernoulli.*

Proof. First notice that since f is measure preserving in (7.30), then to prove that a partition α is vwB, it suffices to show that for every $\varepsilon > 0$, there exist integers m and $N > 0$ such that for every n, N_0, N_1 with $N < N_0 < N_1$, and for ε-almost every atom A of $\bigvee_{N_0-m}^{N_1-m} f^i\alpha$,

$$\bar{d}(\{f^{-i}\alpha\}_{i=1+m}^{n+m}, \{f^{-i}\alpha|A\}_{i=1+m}^{n+m}) < \varepsilon\,. \tag{7.31}$$

To prove Proposition 7.19, we will follow the arguments in Sections 5 and 6 of [ChH], only indicating where modifications should be made.

First, we remark that [ChH] decomposes the measure μ_{SRB} into conditional measures on unstable manifolds and a factor measure on the set of unstable leaves. Due to Corollary 7.9, we prefer to decompose μ_* into conditional measures on stable manifolds and the factor measure $\hat{\mu}_*$. For this reason, we exchange the roles of stable and unstable manifolds throughout the proofs of [ChH].

To this end, we take $f = T^{-1}$ in the set-up presented above, and $X = M$. Moreover, we set $\alpha = \mathcal{M}_{-1}^1$, since this (mod 0) partition generates the full σ-algebra on M. We will follow the proof of [ChH] to show that α is vwB, and so by Theorem 7.18, μ_* will be Bernoulli with respect to T^{-1}, and therefore with respect to T. The proof in [ChH] proceeds in two steps.

Step 1 (Construction of δ-regular coverings). Given $\delta > 0$, the idea is to cover M, up to a set of μ_*-measure at most δ, by Cantor rectangles of stable and unstable manifolds such that μ_* restricted to each rectangle is arbitrarily close to a product measure. This is very similar to our covering $\{R_i\}_{i\in\mathbb{N}}$ from (7.21); however, some adjustments must be made in order to guarantee uniform properties for the Jacobian of the relevant holonomy map.

On a Cantor rectangle R with $\mu_*(R) > 0$, we can define a product measure as follows.[40] Fix a point $z \in R$, and consider R as the product of $R \cap W^s(z)$ with $R \cap W^u(z)$, where $W^{s/u}(z)$ are the local stable and unstable manifolds of z, respectively. As usual, we disintegrate μ_* on R into conditional measures $\mu_{*,R}^W$, on $W \cap R$, where $W \in \mathcal{W}^s(R)$, and a factor measure $\hat{\mu}_*$ on the index set Ξ_R of the curves $\mathcal{W}^s(R)$.

Define $\mu_{*,R}^p = \mu_{*,R}^{W^s(z)} \times \hat{\mu}_*$ and note that we can view $\hat{\mu}_*$ as inducing a measure on $W^u(z)$. Corollary 7.9 implies that $\mu_{*,R}^p$ is absolutely continuous with respect to μ_*. The following definition is taken from [ChH] (as mentioned above, a δ-regular covering of M is a collection of rectangles which covers M *up to a set of measure* δ).

Definition 7.20. For $\delta > 0$, a δ-regular covering of M is a finite collection of disjoint Cantor rectangles $\mathcal{R}$ for which,[41]

a) $\mu_*(\bigcup_{R\in\mathcal{R}} R) \geq 1 - \delta$.

b) Every $R \in \mathcal{R}$ satisfies $\left|\frac{\mu_{*,R}^p(R)}{\mu_*(R)} - 1\right| < \delta$. Moreover, there exists $G \subset R$, with $\mu_*(G) > (1-\delta)\mu_*(R)$, such that $\left|\frac{d\mu_{*,R}^p}{d\mu_*}(x) - 1\right| < \delta$ for all $x \in G$.

[40] We follow the definition in [ChH, Section 5.1], exchanging the roles of stable and unstable manifolds.

[41] The corresponding definition in [ChH] has a third condition, but this is trivially satisfied in our setting since our stable and unstable manifolds are one-dimensional and have uniformly bounded curvature.

By [ChH, Lemma 5.1], such coverings exist for any $\delta > 0$. The proof essentially uses the covering from (7.21), and then subdivides the rectangles into smaller ones on which the Jacobian of the holonomy between stable manifolds is nearly 1, in order to satisfy item b) above. This argument relies on Lusin's theorem and goes through in our setting with no changes. Indeed, the proof in our case is simpler since the angles between stable and unstable subspaces are uniformly bounded away from zero, and the hyperbolicity constants in (3.1) are uniform for all $x \in M$.

Step 2 (Proof that $\alpha = \mathcal{M}_{-1}^1$ is vwB). Indeed, [ChH] prove that any α with piecewise smooth boundary is vwB, but due to Theorem 7.18, it suffices to prove it for a single partition which generates the σ-algebra on M. Moreover, using $\alpha = \mathcal{M}_{-1}^1$ allows us to apply the bounds (2.2) directly since $\partial\alpha = \mathcal{S}_1 \cup \mathcal{S}_{-1}$.

Fix $\varepsilon > 0$, and define

$$\delta = e^{-(\varepsilon/C')^{2/(1-\gamma)}} ,$$

where $C' > 0$ is the constant from (7.33).

Let $\mathcal{R} = \{R_1, R_2, \ldots, R_k\}$ be a δ-regular cover of M such that the diameters of the R_i are less than δ. Define the partition $\pi = \{R_0, R_1, \ldots, R_k\}$, where $R_0 = M \setminus \bigcup_{i=1}^k R_i$. For each $i \ge 1$, let $G_i \subset R_i$ denote the set identified in Definition 7.20b).

Since T^{-1} is K-mixing, there exists an even integer $N = 2m$, such that for any integers N_0, N_1 such that $N < N_0 < N_1$, δ-almost every atom A of $\bigvee_{N_0-m}^{N_1-m} T^{-i}\alpha$ satisfies,

$$\left| \frac{\mu_*(R|A)}{\mu_*(R)} - 1 \right| < \delta \qquad \text{for all } R \in \pi\,. \tag{7.32}$$

Now let n, N_0, N_1 be given as above, and define $\omega = \bigvee_{N_0-m}^{N_1-m} T^{-i}\alpha$. [ChH] proceeds to show that $c\varepsilon$-almost every atom of ω satisfies (7.31) with ε replaced by $c\varepsilon$ for some uniform constant $c > 0$. The first set of estimates in the proof is to bound the measure of bad sets which must be thrown out, and to show that these add up to at most $c\varepsilon$.

The first set is $\hat{F}_1$, which is the union of all atoms in ω, which do not satisfy (7.32). By choice of N, we have $\mu_*(\hat{F}_1) < \delta$.

The second set is $\hat{F}_2$. Let $F_2 = \bigcup_{i=1}^k R_i \setminus G_i$, and define $\hat{F}_2$ to be the union of all atoms $A \in \omega$, for which either $\mu_*(F_2|A) > \delta^{1/2}$, or

$$\sum_{i=1}^k \frac{\mu_{*,R_i}^p(A \cap F_2)}{\mu_*(A)} > \delta^{1/2}\,.$$

It follows as in [ChH, p. 38], with no changes, that $\mu_*(\hat{F}_2) < c\delta^{1/2}$ for some $c > 0$ independent of δ and k.

Define F_3 to be the set of all points $x \in M \setminus R_0$ such that $W^s(x)$ intersects the boundary of the element $\omega(x)$ before it fully crosses the rectangle $\pi(x)$. Thus if $x \in F_3$, there exists a subcurve of $W^s(x)$ connecting x to the boundary of $(T^{-i}\alpha)(x)$ for some $i \in [N_0 - m, N_1 - m]$. Then since $\pi(x)$ has diameter less than δ, $T^i(x)$ lies within a distance $C_1\Lambda^{-i}\delta$ of the boundary of α, where C_1 is from (3.1). Using (2.2), the total measure of such points must add up to at most

$$\sum_{i=N_0-m}^{N_1-m} \frac{C}{|\log(C_1\Lambda^{-i}\delta)|^\gamma} \le C'|\log\delta|^{1-\gamma} \tag{7.33}$$

for some $C' > 0$. Letting $\hat{F}_3$ denote the union of atoms $A \in \omega$ such that $\mu_*(F_3|A) > |\log\delta|^{\frac{1-\gamma}{2}}$, it follows that $\mu_*(\hat{F}_3) \le C'|\log\delta|^{\frac{1-\gamma}{2}}$. This is at most ε by choice of δ.

Define F_4 (following [ChH, Section 6.1], and not [ChH, Section 6.2]) to be the set of all $x \in M \setminus R_0$ for which there exists $y \in W^u(x) \cap \pi(x)$ such that $h(x,y) > 0$. This implies that $W^u(x)$ intersects the boundary of the element $(T^i\alpha)(x)$ for some $i \in [1+m, n+m]$, remembering (7.31), and the definition of h. Using again the uniform hyperbolicity (3.1), this implies that $T^{-i}(x)$ lies in a $C_1\Lambda^{-i}\delta$-neighbourhood of the boundary of α. Thus the same estimate as in (7.33) implies $\mu_*(F_4) \le C'|\log\delta|^{1-\gamma}$. Finally, letting $\hat{F}_4$ denote the union of all atoms $A \in \omega$ such that $\mu_*(F_4|A) > |\log\delta|^{\frac{1-\gamma}{2}}$, it follows as before that $\mu_*(\hat{F}_4) \le C'|\log\delta|^{\frac{1-\gamma}{2}}$.

Finally, the bad set to be avoided in the construction of the joining λ is $R_0 \cup (\bigcup_{i=1}^4 \hat{F}_i)$. Its measure is less than $c\varepsilon$ by choice of δ. From this point, once the measure of the bad set is controlled, the rest of the proof in Section 6.2 of [ChH] can be repeated verbatim. This proves that (7.31) holds for $c\varepsilon$-almost every atom of ω, and thus that α is vwB.

□

7.7. **Uniqueness of the measure of maximal entropy.** This subsection is devoted to the following proposition.

Proposition 7.21. *The measure μ_* is the unique measure of maximal entropy.*

The proof of uniqueness relies on exploiting the fact that while the lower bound on Bowen balls (or elements of $\mathcal{M}_{-n}^0$) cannot be improved for μ_*-almost every x, yet if one fixes n, most elements of $\mathcal{M}_{-n}^0$ should either have unstable diameter of a fixed length, or have previously been contained in an element of $\mathcal{M}_{-j}^0$ with this property for some $j < n$. Such elements collectively satisfy stronger lower bounds on their measure. Since we have established good control of the elements of $\mathcal{M}_{-n}^0$ and $\mathcal{M}_0^n$ in the fragmentation lemmas of Section 5, we will work with these partitions instead of Bowen balls.

Recalling (5.1), choose m_1 such that $(Km_1+1)^{1/m_1} < e^{h_*/4}$. Now choose $\delta_2 > 0$ sufficiently small that for all $n, k \in \mathbb{N}$, if $A \in \mathcal{M}_{-n}^k$ is such that

$$\max\{\mathrm{diam}^u(A), \mathrm{diam}^s(A)\} \le \delta_2\,,$$

then $A \setminus \mathcal{S}_{\pm m_1}$ consists of no more than $Km_1 + 1$ connected components.

For $n \ge 1$, define

$$B_{-2n}^0 = \{A \in \mathcal{M}_{-2n}^0 : \forall\, j, 0 \le j \le n/2, \\ T^{-j}A \subset E \in \mathcal{M}_{-n+j}^0 \text{ such that } \mathrm{diam}^u(E) < \delta_2\}\,,$$

with the analogous definition for $B_0^{2n} \subset \mathcal{M}_0^{2n}$ replacing unstable diameter by stable diameter. Next, set $B_{2n} = \{A \in \mathcal{M}_{-2n}^0 :$ either $A \in B_{-2n}^0$ or $T^{-2n}A \in B_0^{2n}\}$. Define $G_{2n} = \mathcal{M}_{-2n}^0 \setminus B_{2n}$.

Our first lemma shows that the set B_{2n} is small relative to $\#\mathcal{M}_{-2n}^0$ for large n. Let $n_1 > 2m_1$ be chosen so that for all $A \in \mathcal{M}_{-n}^0$, $\mathrm{diam}^s(A) \le C\Lambda^{-n} \le \delta_2$ for all $n \ge n_1$.

Lemma 7.22. *There exists $C > 0$ such that for all $n \ge n_1$,*

$$\#B_{2n} \le Ce^{3nh_*/2}(Km_1+1)^{\frac{n}{m_1}+1} \le Ce^{7nh_*/4}\,.$$

Proof. Fix $n \geq n_1$ and suppose $A \in B^0_{-2n} \subset \mathcal{M}^0_{-2n}$. For $0 \leq j \leq \lfloor n/2 \rfloor$, define $A_j \in \mathcal{M}^0_{-\lceil 3n/2 \rceil - j}$ to be the element containing $T^{-(\lfloor n/2 \rfloor - j)}A$ (note that $T^{-k}A \in \mathcal{M}^k_{-2n+k}$ for each $k \leq 2n$).

By definition of B^0_{-2n} and choice of n_1, we have $\max\{\mathrm{diam}^u(A_j), \mathrm{diam}^s(A_j)\} \leq \delta_2$. Thus the number of connected components of $\mathcal{M}^{m_1}_{-\lceil 3n/2 \rceil}$ in A_0 is at most $Km_1 + 1$. Thus the number of connected components of $T^{m_1}A_0$ (one of which is A_{m_1}) is at most $Km_1 + 1$. Since the stable and unstable diameters of A_{m_1} are again both shorter than δ_2 (since $A \in B^0_{-2n}$) and $n > 2m_1$, we may apply this estimate inductively. Thus writing $\lfloor n/2 \rfloor = \ell m_1 + i$ for some $i < m_1$, we have that $\#\{A' \in B^0_{-2n} : T^{-\lfloor n/2 \rfloor} A' \subset A_0\} \leq (Km_1 + 1)^{\ell+1}$. Summing over all possible $A_0 \in \mathcal{M}^0_{-\lceil 3n/2 \rceil}$ yields by Proposition 4.6 and choice of m_1,

$$\#B^0_{-2n} \leq \#\mathcal{M}^0_{-\lceil 3n/2 \rceil}(Km_1 + 1)^{n/m_1 + 1} \leq Ce^{7nh_*/4} .$$

A similar estimate holds for $\#B^{2n}_0$. Given the one-to-one correspondence between elements of $\mathcal{M}^0_{-2n}$ and $\mathcal{M}^{2n}_0$, it follows that $\#B_{2n} \leq 2\#B^0_{-2n}$, proving the required estimate. □

Next, the following lemma establishes the importance of long pieces in providing good lower bounds on the measure of partition elements.

Lemma 7.23. *There exists $C_{\delta_2} > 0$, such that for all $j \geq 1$ and all $A \in \mathcal{M}^0_{-j}$ such that* $\mathrm{diam}^u(A) \geq \delta_2$ *and* $\mathrm{diam}^s(T^{-j}A) \geq \delta_2$, *we have*[42]

$$\mu_*(A) \geq C_{\delta_2} e^{-jh_*} .$$

Proof. As in the proof of Proposition 5.5, by [Ch1, Lemma 7.87], we may choose finitely many (maximal) Cantor rectangles, $R_1, R_2, \ldots, R_k$, with $\mu_*(R_i) > 0$, and having the property that every unstable curve of length at least δ_2 properly crosses at least one of them in the unstable direction, and every stable curve of length at least δ_2 properly crosses at least one of them in the stable direction. Let $\mathcal{R}_{\delta_2} = \{R_1, \ldots, R_k\}$.

Now let $j \in \mathbb{N}$, and $A \in \mathcal{M}^0_{-j}$ with $\mathrm{diam}^u(A) \geq \delta_2$ and $\mathrm{diam}^s(T^{-j}A) \geq \delta_2$. Notice that $T^{-j}A \in \mathcal{M}^j_0$. By construction, A properly crosses one rectangle $R_i \in \mathcal{R}_\delta$, and $T^{-j}A$ properly crosses another rectangle $R_{i'} \in \mathcal{R}_\delta$. Let Ξ_i denote the index set for the family of stable manifolds comprising R_i. For $\xi \in \Xi_i$, let $W_{\xi,A} = W_\xi \cap A$. Since $T^{-j}A$ properly crosses $R_{i'}$ in the stable direction and T^{-j} is smooth on A, it follows that $T^{-j}(W_{\xi,A})$ is a single curve that contains a stable manifold in the family comprising $R_{i'}$.

Let ℓ_{δ_2} denote the length of the shortest stable manifold in the finite set of rectangles comprising $\mathcal{R}_{\delta_2}$. Then using (7.28) and (7.13), we have for all $\xi \in \Xi_i$,

$$\int_{W_{\xi,A}} \nu = e^{-jh_*} \int_{T^{-j}(W_{\xi,A})} \nu \geq e^{-jh_*} \bar{C} \ell_{\delta_2}^{h_* \bar{C}_2} ,$$

where $\bar{C}, \bar{C}_2 > 0$ are independent of δ and j.

Lastly, denoting by $D(R_i)$ the smallest solid rectangle containing R_i (as in Definition 5.7) and using the fact from the proof of Corollary 7.9 that μ^W_* is equivalent

[42] It also follows from the proof of Proposition 7.12 that the upper bound $\mu_*(A) \leq Ce^{-jh_*}$ holds for all $A \in \mathcal{M}^0_{-j}$ for some constant $C > 0$ independent of j and δ_2, but we shall not need this here.

to ν on μ_*-a.e. $W \in \mathcal{W}^s$, we estimate,

$$\mu_*(A) \geq \mu_*(A \cap D(R_i)) \geq \int_{\Xi_i} \mu_*^{W_\xi}(A)\, d\hat{\mu}_*(\xi)$$
$$\geq C \int_{\Xi_i} \nu(A \cap W_\xi)\, d\hat{\mu}_*(\xi) \geq C'_{\delta_2} e^{-jh_*} \hat{\mu}_*(\Xi_i)\,,$$

which proves the lemma since the family $\mathcal{R}_{\delta_2}$ is finite. □

We may finally prove Proposition 7.21.

Proof. This follows from the previous two lemmas, adapting Bowen's proof of uniqueness of equilibrium states (see the use of [KH, Lemma 20.3.4] in [KH, Thm 20.3.7], as observed in the proof of [GL, Thm 6.4], noting that there is no need to check that boundaries have zero measure).

Since μ_* is ergodic, it suffices by a standard argument (see, e.g., the beginning of the proof of [KH, Thm 20.1.3]) to check that if μ is a T-invariant probability measure so that there exists a Borel set $F \subset M$ with $T^{-1}(F) = F$ and $\mu_*(F) = 0$ but $\mu(F) = 1$ (that is, μ is singular with respect to μ_*), then $h_\mu(T) < h_{\mu_*}(T)$.

Observe first that the billiard map T (as well as its inverse T^{-1}) is expansive, that is, there exists $\varepsilon_0 > 0$ so that if $d(T^j(x), T^j(y)) < \varepsilon_0$ for some $x, y \in M$ and all $j \in \mathbb{Z}$, then $x = y$. (Indeed, if $x \neq y$, then there is $n \geq 1$ and an element of either $\mathcal{S}_n$ or $\mathcal{S}_{-n}$ that separates them. So x and y get mapped to different sides of a singularity line and by (3.3) are separated by a minimum distance ε_0, depending on the table.)

For each $n \in \mathbb{N}$, we consider the partition $\mathcal{Q}_n$ of maximal connected components of M on which T^{-n} is continuous. By Lemmas 3.2 and 3.3, $\mathcal{Q}_n$ is $\mathcal{M}^0_{-n}$ plus isolated points whose cardinality grows at most linearly with n. Thus $G_n \subset \mathcal{Q}_n$ for each n. Define $\tilde{B}_n = \mathcal{Q}_n \setminus G_n$. The set $\tilde{B}_n$ contains B_n plus isolated points, and so its cardinality is bounded by the expression in Lemma 7.22, by possibly adjusting the constant C.

By the uniform hyperbolicity of T, the diameters of the elements of $T^{-\lfloor n/2 \rfloor}(\mathcal{Q}_n)$ tend to zero as $n \to \infty$. This implies the following fact.

Sublemma 7.24. *For each $n \geq n_1$ there exists a finite union $\mathcal{C}_n$ of elements of $\mathcal{Q}_n$ so that*

$$\lim_{n\to\infty} (\mu + \mu_*)((T^{-\lfloor n/2 \rfloor}\mathcal{C}_n) \triangle F) = 0\,.$$

Proof. See [Bo3, Lemma 2]: Let $\bar{\mu} = \mu + \mu_*$ and $\tilde{\mathcal{Q}}_n = T^{-\lfloor n/2 \rfloor}(\mathcal{Q}_n)$. For $\delta > 0$ pick compact sets $K_1 \subset F$ and $K_2 \subset M \setminus F$ so that $\max\{\bar{\mu}(F \setminus K_1), \bar{\mu}((M \setminus F) \setminus K_2)\} < \delta$. We have $\eta = \eta_\delta := d(K_1, K_2) > 0$. If $\mathrm{diam}(\tilde{Q}) < \eta/2$, then either $\tilde{Q} \cap K_1 = \emptyset$ or $\tilde{Q} \cap K_2 = \emptyset$. Let $n = n_\delta$ be so that the diameter of $\tilde{\mathcal{Q}}_n$ is $< \eta_\delta/2$. Set $\tilde{\mathcal{C}}_n = \bigcup\{\tilde{Q} \in \tilde{\mathcal{Q}}_n : Q \cap K_1 \neq \emptyset\}$. Then $K_1 \subset \tilde{\mathcal{C}}_n$ and $\tilde{\mathcal{C}}_n \cap K_2 = \emptyset$. Hence, $\bar{\mu}(\tilde{\mathcal{C}}_n \triangle F) \leq \delta + \bar{\mu}(\tilde{\mathcal{C}}_n \triangle K_1) \leq \delta + \bar{\mu}(M \setminus (K_1 \cup K_2)) \leq 3\delta$. Defining $\mathcal{C}_n = T^{\lfloor n/2 \rfloor}\tilde{\mathcal{C}}_n$ completes the proof. □

We remark that, since $T^{-1}(F) = F$, it follows that

$$(\mu + \mu_*)(\mathcal{C}_n \triangle F) = (\mu + \mu_*)((T^{\lfloor n/2 \rfloor}\mathcal{C}_n) \triangle F)$$

also tends to zero as $n \to \infty$.

Since $\mathcal{Q}_{2n}$ is generating for T^{2n}, we have

$$h_\mu(T^{2n}) = h_\mu(T^{2n}, \mathcal{Q}_{2n}) \le H_\mu(\mathcal{Q}_{2n}) = -\sum_{Q\in\mathcal{Q}_{2n}} \mu(Q)\log\mu(Q)\,.$$

By the proof of Sublemma 7.24, for each n, there exists a compact set $K_1(n)$ that defines $\tilde{\mathcal{C}}_n = T^{-\lfloor n/2\rfloor}\mathcal{C}_n$, and satisfying $K_1(n) \nearrow F$ as $n\to\infty$. Next, we group elements $Q\in\mathcal{Q}_{2n}$ according to whether $T^{-n}Q\subset\tilde{\mathcal{C}}_n$ or $T^{-n}Q\cap\tilde{\mathcal{C}}_n=\emptyset$. Note that if Q is not an isolated point, and if $T^{-n}Q\cap\tilde{\mathcal{C}}_n\neq\emptyset$, then $T^{-n}Q\in\mathcal{M}_{-n}^n$ is contained in an element of $\mathcal{M}_{-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor}$ that intersects $K_1(n)$. Thus $Q\subset T^n\tilde{\mathcal{C}}_n = T^{\lceil n/2\rceil}\mathcal{C}_n$. Therefore,

$$\begin{aligned}
2nh_\mu(T) = h_\mu(T^{2n}) &\le -\sum_{Q\in\mathcal{Q}_{2n}}\mu(Q)\log\mu(Q)\\
&\le -\sum_{Q\subset T^n\tilde{\mathcal{C}}_n}\mu(Q)\log\mu(Q) - \sum_{Q\in\mathcal{Q}_{2n}\setminus(T^n\tilde{\mathcal{C}}_n)}\mu(Q)\log\mu(Q)\\
&\le \frac{2}{e} + \mu(T^n\tilde{\mathcal{C}}_n)\log\#(\mathcal{Q}_{2n}\cap T^n\tilde{\mathcal{C}}_n) + \mu(M\setminus(T^n\tilde{\mathcal{C}}_n))\log\#(\mathcal{Q}_{2n}\setminus(T^n\tilde{\mathcal{C}}_n))\,,
\end{aligned}$$

where we used in the last line that the convexity of $x\log x$ implies that, for all $p_j>0$ with $\sum_{j=1}^N p_j\le 1$, we have (see, e.g., [KH, (20.3.5)])

$$-\sum_{j=1}^N p_j\log p_j \le \frac{1}{e} + (\log N)\sum_{j=1}^N p_j\,.$$

Then, since $-h_{\mu_*}(T) = \big(\mu(T^n\tilde{\mathcal{C}}_n)+\mu(M\setminus(T^n\tilde{\mathcal{C}}_n))\big)\log e^{-h_*}$ for $n\ge n_1$, we write

(7.34)
$$\begin{aligned}
&2n(h_\mu(T)-h_{\mu_*}(T)) - \frac{2}{e}\\
&\le \mu(T^n\tilde{\mathcal{C}}_n)\log\sum_{Q\in\mathcal{Q}_{2n}:Q\subset T^n\tilde{\mathcal{C}}_n} e^{-2nh_*} + \mu(M\setminus(T^n\tilde{\mathcal{C}}_n))\log\sum_{Q\in\mathcal{Q}_{2n}\setminus(T^n\tilde{\mathcal{C}}_n)} e^{-2nh_*}\\
&\le \mu(\mathcal{C}_n)\log\left(\sum_{Q\in G_{2n}:Q\subset T^n\tilde{\mathcal{C}}_n} e^{-2nh_*} + \sum_{Q\in\tilde{B}_{2n}:Q\subset T^n\tilde{\mathcal{C}}_n} e^{-2nh_*}\right)\\
&\qquad + \mu(M\setminus\mathcal{C}_n)\log\left(\sum_{Q\in G_{2n}\setminus(T^n\tilde{\mathcal{C}}_n)} e^{-2nh_*} + \sum_{Q\in\tilde{B}_{2n}\setminus(T^n\tilde{\mathcal{C}}_n)} e^{-2nh_*}\right)\,,
\end{aligned}$$

where we have used the invariance of μ in the last inequality. By Lemma 7.22, both sums over elements in $\tilde{B}_{2n}$ are bounded by $Ce^{-nh_*/4}$. It remains to estimate the sum over elements of G_{2n}.

First we provide the following characterization of elements of G_{2n}. Let $Q\in G_{2n}\subset\mathcal{M}_{-2n}^0$. Since $Q\notin B_{-2n}^0$, there exists $0\le j\le\lfloor n/2\rfloor$ such that $T^{-j}Q\subset E_j\in\mathcal{M}_{-2n+j}^0$ and $\mathrm{diam}^u(E_j)\ge\delta_2$. We claim that there exists $k\le\lfloor n/2\rfloor$ and $\bar{E}\in\mathcal{M}_{-2n+j+k}^0$ such that $E_j\subset\bar{E}$ and $\mathrm{diam}^s(T^{-2n+j+k}\bar{E})\ge\delta_2$.

The claim follows from the fact that $T^{-2n}Q\notin B_0^{2n}$. Thus there exists $k\le\lfloor n/2\rfloor$ such that $T^{-2n+k}Q\subset\tilde{E}_k\in\mathcal{M}_0^{2n-k}$ with $\mathrm{diam}^s(\tilde{E}_k)\ge\delta_2$. But notice that

$T^{-2n+j+k}E_j \in \mathcal{M}_{-k}^{2n-j-k}$ contains $T^{-2n+k}Q$. Thus letting $\tilde{E}$ denote the unique element of $\mathcal{M}_0^{2n-j-k}$ containing both $T^{-2n+j+k}E_j$ and $\tilde{E}_k$, we define $\bar{E} = T^{2n-j-k}\tilde{E} \in \mathcal{M}^0_{-2n+j+k}$, and $\bar{E}$ has the required property since $T^{-2n+j+k}\bar{E} \supset \tilde{E}_k$.

By construction, $\bar{E}$ satisfies the assumptions of Lemma 7.23 since $\bar{E} \in \mathcal{M}^0_{-2n+j+k}$ with $\operatorname{diam}^u(\bar{E}) \geq \delta_2$, and $\operatorname{diam}^s(T^{-2n+j+k}\bar{E}) \geq \delta_2$. Thus,

$$\mu_*(\bar{E}) \geq C_{\delta_2} e^{(-2n+j+k)h_*} . \tag{7.35}$$

We call $(\bar{E}, j, k)$ an *admissible triple* for $Q \in G_{2n}$ if $0 \leq j, k \leq \lfloor n/2 \rfloor$ and $\bar{E} \in \mathcal{M}^0_{-2n+j+k}$, with $T^{-j}Q \subset \bar{E}$ and $\min\{\operatorname{diam}^u(\bar{E}), \operatorname{diam}^s(T^{-2n+j+k}\bar{E})\} \geq \delta_2$. Obviously, there may be many admissible triples associated to a given $Q \in G_{2n}$; however, we define the unique *maximal triple* for Q by taking first the maximum j, and then the maximum k over all admissible triples for Q.

Let $\mathcal{E}_{2n}$ be the set of maximal triples obtained in this way from elements of G_{2n}. For $(\bar{E}, j, k) \in \mathcal{E}_{2n}$, let $\mathcal{A}_M(\bar{E}, j, k)$ denote the set of $Q \in G_{2n}$ for which $(\bar{E}, j, k)$ is the maximal triple. The importance of the set $\mathcal{E}_{2n}$ lies in the following property.

Sublemma 7.25. *Suppose that $(\bar{E}_1, j_1, k_1)$, $(\bar{E}_2, j_2, k_2) \in \mathcal{E}_{2n}$ with $j_2 \geq j_1$ and $(\bar{E}_1, j_1, k_1) \neq (\bar{E}_2, j_2, k_2)$. Then $T^{-(j_2-j_1)}\bar{E}_1 \cap \bar{E}_2 = \emptyset$.*

Proof. Suppose, to the contrary, that there exist $(\bar{E}_1, j_1, k_1)$, $(\bar{E}_2, j_2, k_2) \in \mathcal{E}_{2n}$ with $j_2 \geq j_1$ and $T^{-(j_2-j_1)}\bar{E}_1 \cap \bar{E}_2 \neq \emptyset$. Note that $T^{-(j_2-j_1)}\bar{E}_1 \in \mathcal{M}^{j_2-j_1}_{-2n+j_2+k_1}$ while $\bar{E}_2 \in \mathcal{M}^0_{-2n+j_2+k_2}$.

Thus if $k_1 \leq k_2$, then $T^{-(j_2-j_1)}\bar{E}_1 \subset \bar{E}_2$, and so $(\bar{E}_1, j_1, k_1)$ is not a maximal triple for all $Q \in \mathcal{A}_M(\bar{E}_1, j_1, k_1)$, a contradiction.

If, on the other hand, $k_1 > k_2$, then both $T^{-(j_2-j_1)}\bar{E}_1$ and $\bar{E}_2$ are contained in a larger element $\bar{E}' \in \mathcal{M}^0_{-2n+j_2+k_1}$. Since $\bar{E}' \supset \bar{E}_2$, we have $\operatorname{diam}^u(\bar{E}') \geq \delta_2$, and since $T^{-2n+j_2+k_1}\bar{E}' \supset T^{-2n+j_1+k_1}\bar{E}_1$, we have $\operatorname{diam}^s(T^{-2n+j_2+k_1}\bar{E}') \geq \delta_2$. Thus neither $(\bar{E}_1, j_1, k_1)$ nor $(\bar{E}_2, j_2, k_2)$ is a maximal triple, also a contradiction. □

Note that by definition, if $Q \in T^n\tilde{\mathcal{C}}_n \cap \mathcal{A}_M(\bar{E}, j, k)$, then $T^{-n+j}\bar{E} \in \mathcal{M}^{n-j}_{-n+k}$ contains $T^{-n}Q$. Also, since $j, k \leq \lfloor n/2 \rfloor$, $T^{-n+j}\bar{E}$ is contained in the same element of $\mathcal{M}^{\lfloor n/2 \rfloor}_{-\lfloor n/2 \rfloor}$ that contains $T^{-n}Q$ and intersects $K_1(n)$. Thus $T^{-n+j}\bar{E} \subset \tilde{\mathcal{C}}_n$ whenever $T^n\tilde{\mathcal{C}}_n \cap \mathcal{A}_M(\bar{E}, j, k) \neq \emptyset$. This also implies that $\mathcal{A}_M(\bar{E}, j, k) \subset T^n\tilde{\mathcal{C}}_n$ whenever $T^n\tilde{\mathcal{C}}_n \cap \mathcal{A}_M(\bar{E}, j, k) \neq \emptyset$.

Next, for a fixed $(\bar{E}, j, k) \in \mathcal{E}_{2n}$, by submultiplicativity, since $\bar{E} \in \mathcal{M}^0_{-2n+j+k}$ and $G_{2n} \subset \mathcal{M}^0_{-2n}$, we have $\#\mathcal{A}_M(\bar{E}, j, k) \leq \#\mathcal{M}_0^{j+k}$. Now using Proposition 4.6 and (7.35), we estimate

$$\begin{aligned}
\sum_{Q \in G_{2n} : Q \subset T^n\tilde{\mathcal{C}}_n} e^{-2nh_*} &\leq \sum_{(\bar{E},j,k) \in \mathcal{E}_{2n} : \bar{E} \subset T^{n-j}\tilde{\mathcal{C}}_n} \; \sum_{Q \in \mathcal{A}_M(\bar{E},j,k)} e^{-2nh_*} \\
&\leq \sum_{(\bar{E},j,k) \in \mathcal{E}_{2n} : \bar{E} \subset T^{n-j}\tilde{\mathcal{C}}_n} C e^{(-2n+j+k)h_*} \leq \sum_{(\bar{E},j,k) \in \mathcal{E}_{2n} : \bar{E} \subset T^{n-j}\tilde{\mathcal{C}}_n} C' \mu_*(\bar{E}) \\
&\leq \sum_{(\bar{E},j,k) \in \mathcal{E}_{2n} : \bar{E} \subset T^{n-j}\tilde{\mathcal{C}}_n} C' \mu_*(T^{-n+j}\bar{E}) \leq C' \mu_*(\tilde{\mathcal{C}}_n) = C' \mu_*(\mathcal{C}_n) ,
\end{aligned}$$

where the constant C' depends on δ_2, but not on n. We have also used that $T^{-n+j_1}\bar{E}_1 \cap T^{-n+j_2}\bar{E}_2 = \emptyset$ for all distinct triples $(\bar{E}_1, j_1, k_1), (\bar{E}_2, j_2, k_2) \in \mathcal{E}_{2n}$, by Sublemma 7.25, in order to sum over the elements of $\mathcal{E}_{2n}$. A similar bound holds for the sum over $Q \in G_{2n} \setminus (T^n\tilde{\mathcal{C}}_n)$ since $T^{-n+j}\bar{E} \subset M \setminus \tilde{\mathcal{C}}_n$ whenever $T^n\tilde{\mathcal{C}}_n \cap \mathcal{A}(\bar{E}, j, k) = \emptyset$. Putting these bounds together allows us to complete our estimate of (7.34),

$$2n(h_\mu(T) - h_{\mu_*}(T)) - \frac{2}{e} \le \mu(\mathcal{C}_n) \log \left(C'\mu_*(\mathcal{C}_n) + Ce^{-nh_*/4} \right) \\ + \mu(M \setminus \mathcal{C}_n) \log \left(C'\mu_*(M \setminus \mathcal{C}_n) + Ce^{-nh_*/4} \right) .$$

Since $\mu(\mathcal{C}_n)$ tends to 1 as $n \to \infty$ while $\mu_*(\mathcal{C}_n)$ tends to 0 as $n \to \infty$ the limit of the right-hand side tends to $-\infty$. This yields a contradiction unless $h_\mu(T) < h_{\mu_*}(T)$. □

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