



Local Solution to the Multi-layer KPZ Equation

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Abstract

In this article we prove local well-posedness of the system of equations $\partial_t h_i = \sum_{j=1}^i \partial_x^2 h_j + (\partial_x h_i)^2 + \xi$ on the circle where $1 \leq i \leq N$ and ξ is a space-time white noise. We attempt to generalize the renormalization procedure which gives the Hopf-Cole solution for the single layer equation and our h_1 (solution to the first layer) coincides with this solution. However, we observe that cancellation of logarithmic divergences that occurs at the first layer does not hold at higher layers and develop explicit combinatorial formulae for them.

Keywords Renormalization · Regularity structures · Stochastic partial differential equations

1 Introduction

The aim of this paper is to introduce a system of equations called the multi-layer KPZ equation, formally given by

$$\partial_t h_i = \sum_{j=1}^i \partial_x^2 h_j + (\partial_x h_i)^2 + \xi \quad i \in \{1, 2, \dots, N\}, \quad (1.1)$$

where ξ is a space-time white noise, starting from initial condition $(h_1^{(0)}, \dots, h_N^{(0)})$. We study the local well-posedness and renormalization of this system on the circle. This system is interesting for several reasons.

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First, there has been increasing interest recently in the study of coupled KPZ systems. In [4], the local solution to the following coupled system is studied

$$\partial_t h^\alpha = \frac{1}{2} \partial_x^2 h^\alpha + \frac{1}{2} \sum_{\beta, \gamma=1}^n \Gamma_{\beta\gamma}^\alpha \partial_x h^\beta \partial_x h^\gamma + \sum_{\beta=1}^n \sigma_\beta^\alpha \xi^\beta \quad \alpha \in \{1, 2, \dots, N\}. \quad (1.2)$$

Here $\{\xi^\beta\}_{\beta=1}^n$ are space-time white noises. This is an $\mathbb{R}^n$ vector valued KPZ system driven by n independent space-time white noises, with constant coefficients $\Gamma_{\beta\gamma}^\alpha$ and σ_β^α . When $n = 1$, this then becomes the KPZ equation. In [7], Hairer studied the equation describing the random motion of a string in an n dimensional manifold driven by m independent space-time white noises, written as

$$\partial_t h^\alpha = \partial_x^2 h^\alpha + \sum_{\beta, \gamma=1}^N \Gamma_{\beta\gamma}^\alpha(h) \partial_x h^\beta \partial_x h^\gamma + \sum_{i=1}^m \sigma_i^\alpha(h) \xi_i \quad \alpha \in \{1, 2, \dots, N\}, \quad (1.3)$$

where $\Gamma_{\beta\gamma}^\alpha(h)$ are the Christoffel symbols for the Levi-Civita connection, and $\{\sigma_i(h)\}$ is a collection of vector fields on the manifold. In these works, the coupling between the equations in the system arises from the nonlinearities. In the present article, we investigate a new situation, that is, one has a *matrix* in front of the diffusive term. In particular, the matrix is the lower triangle matrix with all the lower triangular entries being 1. Moreover, the whole system is driven by only *one single* white noise ξ . The system is thus such that the first m equations evolve independently of the m' th equation for all $m' > m$. As a matter of fact, one could also think of the i th equation as a usual KPZ equation driven by ξ together with “noises” $\partial_x^2 h_1, \dots, \partial_x^2 h_{i-1}$.

Another motivation of studying the present problem, especially the particular choice of the lower triangular diffusive matrix, is the following. In the paper [10] the authors introduced a system of (continuum) multi-layer partition functions Z_N for $N \in \mathbb{N}$ in order to better understand the integrability structure behind the discrete or semi-discrete polymer models, and their connection with quantum Toda lattice and geometric RSK correspondence:

$$Z_N(t, x, y) = p(t, x, y)^N \left(1 + \sum_{k=1}^{\infty} \int R_k^n((t_1, x_1), \dots, (t_k, x_k)) \xi^{\otimes k}(dt_1, \dots, dx_k) \right)$$

where $p(t, x, y)$ denotes the heat kernel (we will write $p(t, x) := p(t, 0, x)$ below), and R_k^n is the k -point correlation for a collection of n non-intersecting Brownian bridges starting at x at time 0 and ending at y at time t , and the integrals are stochastic iterated integrals against the white noise ξ over $(x_1, \dots, x_k) \in \mathbb{R}^k$ and $0 < t_1 < \dots < t_k < t$. It was shown in [10] that this series representation for each Z_n is L^2 convergent. In particular, when $N = 1$, R_k is the transition density of the Brownian motion passing through the points $(t_1, x_1), \dots, (t_k, x_k)$, and the above series becomes the chaos series solution to the stochastic heat equation $Z = \frac{1}{2} \partial_y^2 Z + Z \xi$ with Dirac initial data. The processes Z_N are related with a system of stochastic PDEs in the following way. Define $u = u_1 = Z_1$ and for $n \geq 2$ define $u_n = Z_n / Z_{n-1}$. Then pretending that ξ is smooth, [10] derived via a Karlin-McGregor formula and Darboux transformation a coupled system of stochastic (nonlinear) heat equations for u_n :¹

$$\partial_t u_n = \frac{1}{2} \partial_x^2 u_n + \left(\xi + \partial_x^2 \log(Z_{n-1} / p^{n-1}) \right) u_n \quad (1.4)$$

¹ In [10] the processes depend on two spatial parameters x and y , but here we set x to be zero and write x for y for notational simplicity.

with initial conditions $u_n(0, x) = \delta(x)$. Then in [10, Proof of Prop. 3.3 and Prop. 3.7], the authors introduced $h_n = \log u_n$ which formally satisfy

$$\partial_t h_n = \frac{1}{2} \partial_x^2 h_n + \frac{1}{2} (\partial_x h_n)^2 + \xi + \partial_x^2 \log(Z_{n-1}/p^{n-1}).$$

A direct computation shows that

$$\partial_x^2 \log(1/p(t, x)^n) = -n \partial_x^2 \log p(t, x) = -n \cdot \frac{(\partial_x^2 p)p - (\partial_x p)^2}{p^2} = n/t$$

So we arrive at the equation $\partial_t h_n = \frac{1}{2} \partial_x^2 h_n + \frac{1}{2} (\partial_x h_n)^2 + \xi + \sum_{j=1}^{n-1} \partial_x^2 h_j + \frac{n-1}{t}$. Of course, the system (1.1) we study in this article is slightly different from the setting in [10]; for instance we are on a finite interval with periodic boundary condition instead of the full line, and we will not consider Dirac initial data, and we also drop the term $\frac{n-1}{t}$ (which arises from the Dirac initial condition).

Before we formulate our main result we fix an even, smooth, compactly supported function $\varrho : \mathbb{R}^2 \rightarrow \mathbb{R}$, integrating to one and we set for $\varepsilon \in (0, 1]$,

$$\varrho_\varepsilon(t, x) = \varepsilon^{-3} \varrho(\varepsilon^{-2}t, \varepsilon^{-1}x), \quad \text{and} \quad \xi_\varepsilon = \xi * \varrho_\varepsilon. \quad (1.5)$$

Let S^1 be the unit circle and $C^\beta(S^1)$ be the space of β -Hölder continuous functions on the circle S^1 . We also denote by $C^\eta((0, T] \times S^1)$ the space of η -Hölder continuous functions on space-time, defined by a parabolic scaling as in [6]. Our main result is the following.

Theorem 1.1 *Fix $N \in \mathbb{N}$ and $(h_1^{(0)}, \dots, h_N^{(0)}) \in (C^\beta(S^1))^{\otimes N}$ with $\beta \in (0, 1)$. For each $i \in \{1, 2, \dots, N\}$, there exists a sequence of constants $(C_{\varepsilon,i} : 1 \leq i \leq N)$ tending to infinity as ε tends to zero, such that the solutions $(h_{\varepsilon,1}, \dots, h_{\varepsilon,N})$ to the sequence of systems of N equations*

$$\partial_t h_{\varepsilon,i} = \sum_{j=1}^i \partial_x^2 h_{\varepsilon,j} + (\partial_x h_{\varepsilon,i})^2 + \xi_\varepsilon - C_{\varepsilon,i} \quad (1.6)$$

starting from initial condition $(h_1^{(0)}, \dots, h_N^{(0)})$ converge in probability to N random fields $(h_1, \dots, h_N)$ in $C^\eta((0, T] \times S^1)^{\otimes N}$ for any $\eta \in (0, \beta \wedge \frac{1}{2})$. Here, each of the constants $(C_{\varepsilon,i} : 1 \leq i \leq N)$ is given by a sum of three terms whose precise expressions are described by Proposition 4.4 and they are such that

$$\begin{aligned} C_{\varepsilon,1} &= \varepsilon^{-1} C_1 + c_1 \\ C_{\varepsilon,i} &= \varepsilon^{-1} C_i + \tilde{C}_i \log \varepsilon + c_i \quad (2 \leq i \leq N) \end{aligned}$$

for finite constants $C_i, c_i, \tilde{C}_j$ ($1 \leq i \leq N, 2 \leq j \leq N$). Moreover, if C_1 is chosen to suitably depend on ϱ (defined in (1.5)) then the limit $(h_1, \dots, h_N)$ does not depend on the choice of the mollifier ϱ and h_1 agrees with the Hole-Copf solution of the KPZ equation driven by space-time white noise.

Recall that for the standard KPZ equation [5], and the manifold-valued KPZ type equation (1.3) in the geometric setting, and the vector valued KPZ system (1.2) under the so called “trilinear” condition therein, the logarithmic renormalization constants cancel out. In our case, we expect that $\tilde{C}_j$ in general do not vanish for $j \geq 2$, but we have not found a general argument to prove this (however, see Remark 5.2 for the first several layers.) For the standard KPZ equation, the cancellation of logarithmic renormalization is “expected” because the Itô term appearing from the Itô calculus for the Cole-Hopf transform behaves as $1/\varepsilon$, and

the linear stochastic heat equation is defined in Itô sense. For the multi-layer KPZ however, there is no Cole-Hopf transform that linearizes the system, and the corresponding nonlinear multi-layer stochastic heat equation can not be simply defined in the Itô sense. Requiring non-vanishing logarithmic renormalizations seems not surprising in our case.

We think that it may be of interest to study a general type of KPZ system

$$\partial_t h^\alpha = \frac{1}{2} \sum_{\beta=1}^{\alpha} v_{\beta}^{\alpha} \partial_x^2 h^{\beta} + \frac{1}{2} \sum_{\beta, \gamma=1}^n \Gamma_{\beta\gamma}^{\alpha} \partial_x h^{\beta} \partial_x h^{\gamma} + \sum_{\beta=1}^n \sigma_{\beta}^{\alpha} \xi^{\beta} \quad \alpha \in \{1, 2, \dots, N\}, \quad (1.7)$$

where $v_{\alpha}^{\alpha} > 0$ and $v_{\beta}^{\alpha} = 0$ when $\beta > \alpha$. In particular, finding the condition for the coefficients under which the logarithmic renormalizations cancel would be very interesting.

2 Notation and Formulation

Notation We will denote by $\binom{j}{i}$ the binomial coefficients, with convention $\binom{j}{i} = 0$ when $i, j \geq 0$ and $j < i$. We sometimes also need the case that i or j is -1 : we have $\binom{n}{-1} = 0$ for $n \geq 0$ and $\binom{-1}{-1} = \binom{0}{0} = 1$. Also, for $N \in \mathbb{Z}_+$, we write $[N] \stackrel{\text{def}}{=} \{1, 2, \dots, N\}$. Moreover we introduce the distance $\|(t, x)\|_{\mathbb{S}} = \sqrt{|t|} + |x|$ on $\mathbb{R}^2$.

We start by rewriting (1.6) in a mild formulation. Let

$$G(t, x) \stackrel{\text{def}}{=} \sum_{k \in 2\pi\mathbb{Z}} \frac{1}{\sqrt{4\pi t}} \exp \left[-\frac{(x-k)^2}{4t} \right]$$

for $t > 0$ and $G(t, x) = 0$ for $t \leq 0$ be the heat kernel on S^1 . For each $i \in \mathbb{N}$, we recursively define kernels G_i by setting $G_0 \stackrel{\text{def}}{=} G$ and for $i > 0$ setting $G_i \stackrel{\text{def}}{=} G * \partial_x^2 G_{i-1}$, so that

$$G_i = G * (\partial_x^2 G)^{*i}$$

where $*$ stands for convolution over space and time variables, and for any function H over space-time H^{*i} stands for the convolution of i copies of H . We also define for $j \geq 1$,

$$\overline{G}_j \stackrel{\text{def}}{=} G * \partial_x^2 G * (\delta + \partial_x^2 G)^{*(j-1)} = \sum_{i=1}^j \binom{j-1}{i-1} G_i \quad (2.1)$$

where δ is the Dirac distribution, and for any function F it is understood that $F^{*0} = \delta$. The last equality of (2.1) follows by a binomial identity. Moreover we define $\overline{G}_0 \stackrel{\text{def}}{=} G$. For $\varepsilon \in (0, 1]$ and ϱ_{ε} as in (1.5), we introduce the shorthands

$$G_{i,\varepsilon} = G_i * \varrho_{\varepsilon}, \quad \overline{G}_{i,\varepsilon} = \overline{G}_i * \varrho_{\varepsilon}.$$

Furthermore, if h is a function *only* defined in space, then with a slight abuse of notation we also write

$$(\overline{G}_j * h)(t, x) = \int \overline{G}_j(t, x - y) h(y) dy. \quad (2.2)$$

We then have the following lemma, where the term unrenormalized refers to Eq. (1.6) with all the constants $C_{\varepsilon,i}$ set to zero.

Lemma 2.1 For any $N \in \mathbb{N}$ the mild formulation of the (unrenormalized) system (1.6) is given by

$$h_{\varepsilon,j} = \sum_{i=1}^j \overline{G}_{j-i} * \left[(\partial_x h_{\varepsilon,i})^2 + \xi_\varepsilon + h_i^{(0)} \right] \quad \text{for } 1 \leq j \leq N. \quad (2.3)$$

Proof We proceed by induction in j . The case $j = 1$ is immediate since $\overline{G}_0 = G$. Suppose that the Eq. (2.3) holds for all $j \leq k$. Then by the induction assumption we have

$$\begin{aligned} h_{\varepsilon,k+1} &= G * \left[(\partial_x h_{\varepsilon,k+1})^2 + \xi_\varepsilon + h_{k+1}^{(0)} \right] + G * \sum_{j=1}^k \partial_x^2 h_{\varepsilon,j} \\ &= G * \left[(\partial_x h_{\varepsilon,k+1})^2 + \xi_\varepsilon + h_{k+1}^{(0)} \right] + G * \sum_{j=1}^k \sum_{i=1}^j \partial_x^2 \overline{G}_{j-i} * \left[(\partial_x h_{\varepsilon,i})^2 + \xi_\varepsilon + h_i^{(0)} \right] \\ &= G * \left[(\partial_x h_{\varepsilon,k+1})^2 + \xi_\varepsilon + h_{k+1}^{(0)} \right] + \sum_{i=1}^k \left(\sum_{j=i}^k G * \partial_x^2 \overline{G}_{j-i} \right) * \left[(\partial_x h_{\varepsilon,i})^2 + \xi_\varepsilon + h_i^{(0)} \right]. \end{aligned}$$

Note that for $1 \leq i \leq k$, one has by $\overline{G}_0 = G$ and (2.1)

$$\begin{aligned} \sum_{j=i}^k G * \partial_x^2 \overline{G}_{j-i} &= G * \partial_x^2 G + \sum_{j=i+1}^k G * (\partial_x^2 G)^{*2} * (\delta + \partial_x^2 G)^{*(j-i-1)} \\ &= G * \partial_x^2 G + G * \partial_x^2 G * \left((\delta + \partial_x^2 G)^{*(k-i)} - \delta \right) \\ &= \overline{G}_{k+1-i} \end{aligned}$$

where we used the following identity

$$(H - \delta) * \sum_{j=0}^{m-1} H^{*j} = H^{*m} - \delta$$

with $H = \delta + \partial_x^2 G$ and $m = k - i$. Therefore the statement holds for $k + 1$. $\square$

We fix for the remainder of the entire paper the number of layers $N \in \mathbb{N}$ in the statement of Theorem 1.1.

3 Applying the Theory of Regularity Structures

In this section we describe how we recast our problem in the language of the theory of regularity structures [6]. We will invoke the machinery of [1, 3] to define a regularity structure and renormalization group rich enough to solve (2.3) and show convergence of appropriately renormalized models.

3.1 Construction of the Multi-layer KPZ Regularity Structure

We fix a parabolic space-time scaling $\mathfrak{s}$ on $\mathbb{R}^2$ by setting $\mathfrak{s} = (2, 1)$.

We introduce two finite, abstract, sets $\mathfrak{L}_+ \stackrel{\text{def}}{=} \{t_{(j,i)} : 1 \leq i \leq j \leq N\}$ and $\mathfrak{L}_- = \{\mathfrak{l}\}$ which, respectively, index the regularizing kernels and driving noises $\mathfrak{l}$ appearing in our problem.

Here, $\mathfrak{l}$ represents an occurrence of the driving noise ξ and $\mathfrak{t}_{(j,i)}$ represents an occurrence of (a truncation of) the kernel $\overline{G}_{j-i}$. Clearly there is some redundancy in our labeling of kernels in the sense that if $j - i = j' - i' = k \in \mathbb{N}$ then $\mathfrak{t}_{(j,i)}$ and $\mathfrak{t}_{(j',i')}$ both represent an occurrence of the same kernel $\overline{G}_k$. However, for fixed k , the kernel $\overline{G}_k$ can appear in many different equations. Keeping track of these separate occurrences is convenient when defining our regularity structure when it comes to describing the set of allowable trees in terms of a rule, thus we write $\mathfrak{t}_{(j,i)}$ to represent the specific occurrence of $\overline{G}_{j-i}$ in the equation for *the j th layer equation*.

Next, we must specify a homogeneity assignment $|\cdot|_{\mathfrak{s}} : \mathfrak{L}_- \sqcup \mathfrak{L}_+ \rightarrow \mathbb{R}$. To that end we set $|\mathfrak{l}|_{\mathfrak{s}} \stackrel{\text{def}}{=} -\frac{3}{2} - \kappa$ and $|\mathfrak{t}|_{\mathfrak{s}} \stackrel{\text{def}}{=} 2$ for any $\mathfrak{t} \in \mathfrak{L}_+$. In the sequel we will use the abbreviation $\mathfrak{L} = \mathfrak{L}_- \sqcup \mathfrak{L}_+$. Here $\kappa > 0$ is a sufficiently small parameter in the construction of our regularity structure which is fixed throughout the paper (see for instance the proof of Proposition 3.2 for a necessary bound on κ).

3.1.1 The Multi-layer KPZ Rule and Regularity Structure

Invoking the machinery of [1] The relevant reference for this section is [1, Section 5]. We define, as in [1, Eq (5.5)], $\mathfrak{E} \stackrel{\text{def}}{=} \mathfrak{L} \times \mathbb{N}^2$ to be the set of edge types and $\mathcal{N}$ to be the set node types, namely the set of all *multi-sets*²

We adopt a shorthand for $\mathfrak{E}$, writing $\mathcal{I}_{(j,i)}$ to represent $(\mathfrak{t}_{(j,i)}, 0) \in \mathfrak{E}$, $\mathcal{I}'_{(j,i)}$ to represent $(\mathfrak{t}_{(j,i)}, (0, 1)) \in \mathfrak{E}$, and Ξ to represent $(\mathfrak{l}, 0)$.

We now define a rule $R : \mathfrak{L} \rightarrow \mathcal{P}\mathcal{N} \setminus \{\emptyset\}$ as in [1, Definition 5.7], where $\mathcal{P}\mathcal{N}$ denotes the power set of $\mathcal{N}$. For any $\mathfrak{l} \in \mathfrak{L}_-$ we set $R[\mathfrak{l}] \stackrel{\text{def}}{=} \{\emptyset\}$. For $\mathfrak{t}_{(j,i)} \in \mathfrak{L}_+$ we set

$$R[\mathfrak{t}_{(j,i)}] \stackrel{\text{def}}{=} \{(\mathcal{I}'_{(i,k)}, \mathcal{I}'_{(i,k')}) : 1 \leq k, k' \leq i\} \sqcup \{(\mathcal{I}'_{(i,k)}) : 1 \leq k \leq i\} \sqcup \{(\Xi), ()\}$$

This rule is clearly normal and if we define $\text{reg} : \mathfrak{L} \rightarrow \mathbb{R}$ via $\text{reg}(\mathfrak{l}) \stackrel{\text{def}}{=} -3/2 - 2\kappa$ for $\mathfrak{l} \in \mathfrak{L}_-$ and $\text{reg}(\mathfrak{t}) \stackrel{\text{def}}{=} -1/2 - 3\kappa$ for $\mathfrak{t} \in \mathfrak{L}_+$ then we see that R is subcritical in the sense of [1, Definition 5.13].

Finally, let $\bar{R}$ be the completion of R as constructed in [1, Proposition 5.20], and $\mathcal{T} = (\mathcal{T}, G)$ be the associated *reduced* regularity structure truncated above homogeneity $\gamma \stackrel{\text{def}}{=} \frac{3}{2} + 3\kappa$.

The trees of our regularity structure For those readers not familiar with [1] we now try to give some more intuition for what our regularity structure looks like. Recall that a key ingredient of a regularity structure is a graded vector space and in most applications to SPDEs (including this paper) this graded vector space is a free vector space generated by a family of rooted decorated trees. We describe how to think of these trees in a certain formal symbolic notation where edges are seen as unary operators.

The nodes of these trees correspond to symbols of the form Ξ representing an occurrence of the driving noise or to $\mathbf{X}^k$ for some $k \in \mathbb{N}^d$ which represents a polynomial. Then, $\mathbf{X}^2 \mathcal{I}'_{(3,2)}[\Xi] \mathcal{I}'_{(3,2)}[\Xi]$ is an example of a tree with three nodes (corresponding to $\mathbf{X}^2$, the root, and two nodes of the form Ξ) and two edges (the two instances of $\mathcal{I}'_{(3,2)}$). Our “rule” described above just enforces that if there are $\mathcal{I}$ or $\mathcal{I}'$ operators inside some of the $\mathcal{I}'_{(j,i)}[\cdot]$ or $\mathcal{I}_{(j,i)}[\cdot]$ then they must be of the form $\mathcal{I}'_{(i,k)}$.

² A multi-set is like a set where one allows for multiplicity. Using the notation of [1], one sets $\mathcal{N} \stackrel{\text{def}}{=} \sqcup_{n \geq 0} [\mathfrak{E}]^n$ where $[\mathfrak{E}]^n$ consists of the equivalence classes of elements of $\mathfrak{E}^n$ where one identifies tuples in $\mathfrak{E}^n$ which are related by a permutation.

We will often just write expressions like $\mathbb{V}[(j_k, i_k)_{k=1}^6]$ to save space. When we write $(j_k, i_k)_{k=1}^l$ it is understood we are referring to a tuple of pairs $(j_1, i_1), \dots, (j_l, i_l)$ with $1 \leq i_k \leq j_k \leq N$ for each $1 \leq k \leq l$.

As a final example, for $\mathbb{V}[(j_k, i_k)_{k=1}^3]$ the constraint on indices is $j_1 = j_2 = i_3$.

3.3 Models for the Multi-layer KPZ Equation

We fix kernels $K_0, \dots, K_{N-1}$ mapping $\mathbb{R}^2$ into $\mathbb{R}$ such that each K_i (a) is compactly supported in the ball of radius one around the origin, (b) is symmetric in space, (c) coincides with $\overline{G}_i$ on the ball of radius $1/2$, and (d) for every polynomial of $\mathfrak{s}$ -degree 2 one has $\int_{\mathbb{R}^2} K_i(z) Q(z) dz = 0$.

By [6, Lemma 5.5] and Proposition A.1 the K_i 's can be additionally chosen such that one has the identity $\overline{G}_i = K_i + R_i$, for a smooth remainder R_i .

The assignment of kernels to elements of $\mathfrak{L}_+$ is then given by $t_{(j,i)} \in \mathfrak{L}_+$, $K_{t_{(j,i)}} \stackrel{\text{def}}{=} K_{j-i}$.

We write $\mathcal{M}_\infty(\mathcal{T})$ and $\mathcal{M}_0(\mathcal{T})$ for the space of smooth admissible models on $\mathcal{T}$ and its closure, respectively. Here, the closure is taken with respect to the family of seminorms defined in [6, Section 2].

Definition 3.1 Given any stationary $\mathcal{C}(\mathbb{R}^2)$ -valued random variable ζ we denote by Z_{BPHZ}^ζ the $\mathcal{M}_\infty(\mathcal{T})$ -valued random variable called the BPHZ lift of ζ , see [1, Theorem 6.17] for the definition of the BPHZ lift.

The following proposition produces the random admissible model that will allow us to describe a solution to the multi-layer KPZ equation.

Proposition 3.2 *Let ξ denote space-time white noise. The BPHZ lift admits an extension to ξ in the following sense – there exists an $\mathcal{M}_0(\mathcal{T})$ -valued random variable Z_{BPHZ}^ξ such that for any smooth compactly supported function $\varrho : \mathbb{R}^2 \rightarrow \mathbb{R}$ with $\int_{\mathbb{R}^2} \varrho(z) dz = 1$ one has that the random models $Z_{\text{BPHZ}}^{\xi * \varrho_\varepsilon}$ converges in probability to Z_{BPHZ}^ξ on $\mathcal{M}_0(\mathcal{T})$ as $\varepsilon \downarrow 0$ where ϱ_ε is defined as in (1.5).*

Proof This proposition follows from [3, Theorem 2.15]. In addition to subcriticality [3, Theorem 2.15] enforces certain assumptions on the homogeneities of $\tau \in \mathcal{T}$ with $\tau \neq \Xi$. In our setting all one must check is that for every such τ one has $|\tau|_\mathfrak{s} > -|\mathfrak{s}|/2$. The second is that any such τ must have $|\tau|_\mathfrak{s} + |\Xi|_\mathfrak{s} + |\mathfrak{s}| > 0$. Both of these conditions guarantee that none of the random fields in our local expansions will have diverging variances.

One can easily check that the worst such trees are those of the form $\mathcal{I}'_{(j,i)}[\Xi]^2$ which have homogeneity $-1 - 2\kappa$. We see that $\kappa \in (0, 1/6)$ suffices to guarantee the two conditions mentioned above. $\square$

Remark 3.3 Note that the solution we refer to in Theorem 1.1 will be a finite shift of the solution obtained from the BPHZ model.

3.4 Abstract Fixed Point Problem

In view of Lemma 2.1 the fixed point problem associated to (1.6) can be written as

$$\vec{H} = \mathcal{P}((\vec{\mathcal{D}}H)^2 + \vec{\Xi}) + Ph^{(0)}. \quad (3.2)$$

The above fixed point problem should be seen as a vector fixed point problem where the components of vector are elements of an appropriate $\mathcal{D}^{\gamma,\eta}$ -space, see [6, Section 6] for a definition of these spaces. It can be compared to [9, Eq (2.12)]. We now make it more precise by specifying the notation used above.

$\mathcal{P}$ is an N by N matrix of operators on a given $\mathcal{D}^{\gamma,\eta}$ space. Writing the matrix components $\mathcal{P} = (\mathcal{P}_{j,i})_{i,j=1}^N$ one has $\mathcal{P}_{j,i} = 0$ for $j < i$ and for $j \geq i$ we set $\mathcal{P}_{j,i}$ to be an abstract integration operator corresponding to $\bar{G}_{j-i}$. In particular, we use the decomposition (K_{j-i}, R_{j-i}) to specify the local rough part and non-local smooth part of the operator - see [6, Section 5]. Then $\vec{H}$, $(\mathcal{D}\vec{H})^2$ and $P\vec{h}_0$ are N component vectors of elements of $\mathcal{D}^{\gamma,\eta}$. Writing $\vec{H} = (H_i)_{i=1}^N$, $H_i \in \mathcal{D}^{\gamma,\eta}$ is the modelled distribution describing the i th layer of the equation. We then set

$$(\vec{H})^2 \stackrel{\text{def}}{=} \left((\mathcal{D}H_i)^2 \right)_{i=1}^N$$

Finally, one has $P\vec{h}^{(0)} \stackrel{\text{def}}{=} (Ph_i^{(0)})_{i=1}^N$ where $h_i^{(0)}$ is the initial data for the i th layer as referenced in Theorem 1.1 and $Ph_i^{(0)}$ is the result of applying the heat kernel G to this initial data and lifting it to $\mathcal{D}^{\gamma,\eta}$ as in [6, Lemma 7.5].

We have the following lemma.

Lemma 3.4 Choose $\gamma > 3/2 + \kappa$ and $\eta > 0$ such that each $h_{i,0} \in \mathcal{C}^\eta$. Then, for every admissible model, there exists $T > 0$ such that the fixed point problem (3.2) has a unique solution in $\vec{H} \in \mathcal{D}^{\gamma,\eta}([0, T] \times S^1)$.

Proof We can proceed by induction since $\mathcal{P}_{j,i} = 0$ for $j < i$. The fixed point problem associated to H_1 (the first coordinate of $\vec{H}$) coincides with the fixed point problem of the KPZ equation, see also [9, Section 2] and it follows for instance from [6, Theorem 7.8] that, for every admissible model, there exists $T_1 > 0$ such that the above fixed point problem for H_1 has a unique solution in $\mathcal{D}^{\gamma,\eta}([0, T_1] \times S^1)$. With the fixed point problems for $H_1, \dots, H_i$ solved, we can then again invoke [6, Theorem 7.8] to show that there is $T_{i+1} > 0$ such that the fixed point problem for H_{i+1} has a unique solution in $\mathcal{D}^{\gamma,\eta}([0, T_{i+1}] \times S^1)$; this is because the operator $\mathcal{P}_{j,i}$ does enjoy the desired Schauder estimate by Proposition A.1. This finishes the argument with $T = \min_i T_i$. $\square$

In order to later derive renormalised equations it will be useful to know what the solution promised in Lemma 3.4 looks like.

Lemma 3.5 Let $\gamma \in (3/2 + \kappa, 3/2 + 2\kappa)$, $\eta > 0$, and $\vec{H} = (H_l)_{l=1}^N \in \mathcal{D}^{\gamma,\eta}([0, T] \times S^1)$ be the solution to the abstract fixed point problem (3.2) promised by Lemma 3.4. Then $\vec{H}$ is of the following form: for $1 \leq l \leq N$ one has

$$\begin{aligned} H_l(z) &\stackrel{\text{def}}{=} H_{l,I}(z)I + H_{l,X_l}(z)X_1 + \sum_{\substack{(j_1, i_1) \\ j_1=l}} \mathfrak{I}[(j_1, i_1)] \\ &+ \sum_{\substack{((j_k, i_k))_{k=1}^3 \\ j_3=l}} \mathfrak{Y}[(j_k, i_k)_{k=1}^3] + 2 \sum_{\substack{((j_k, i_k))_{k=1}^5 \\ j_5=l}} \mathfrak{Y}[(j_k, i_k)_{k=1}^5] \\ &+ 2H_{l,X_l}(z) \sum_{\substack{((j_k, i_k))_{k=1}^2 \\ j_2=l}} \mathfrak{Z}[(j_k, i_k)_{k=1}^2]. \end{aligned} \quad (3.3)$$

Proof Our choice of γ constraints the monomials that are kept in any $\mathcal{D}^{\gamma, \eta}$ expansion. The fact that the solution $\tilde{H}$ must be of this form can be verified by observing that it reproduces itself upon being inserted on the right hand side of (3.2). $\square$

4 Renormalization

If $Z^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon)$ is the canonical lift of ξ_ε into a smooth model then the BPHZ lift $\hat{Z}^\varepsilon \stackrel{\text{def}}{=} (\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$ of ξ_ε can be described as a shift of Z^ε by an element of the renormalization group $\mathfrak{R}$, a group of linear operators from $\mathcal{T}$ to itself that was characterized in [1].

In terms of the algebraic theory of renormalization of regularity structures the situation here is essentially⁴ the same as those found in previous treatments of the KPZ equation [8, 9]. A full specification of the BPHZ model is unnecessary for our purposes, we will only need to understand its action on certain elements of the regularity structure present so that we can derive the renormalized equation.

4.1 Definition of Renormalization Constants

We first specify the renormalization constants of the BPHZ model. We set, for collections of indices compatible with the corresponding symbol,

$$\begin{aligned} \ell_{\varepsilon, \mathbb{V}}[(j_k, i_k)_{k=1}^2] &\stackrel{\text{def}}{=} \mathbf{E} \left[\int_{(\mathbb{R}^2)^2} dz_1 dz_2 \prod_{p=1}^2 K'_{j_p - i_p}(-z_p) \xi_\varepsilon(z_p) \right] \\ \ell_{\varepsilon, \mathbb{V}}[(j_k, i_k)_{k=1}^6] &\stackrel{\text{def}}{=} \mathbf{E} \left[\int_{(\mathbb{R}^2)^2} dz_5 dz_6 K'_{j_5 - i_5}(-z_5) K'_{j_6 - i_6}(-z_6) \right. \\ &\quad \times \left(\int_{(\mathbb{R}^2)^2} dz_1 dz_2 \prod_{p=1}^2 K'_{j_p - i_p}(z_5 - z_p) \xi_\varepsilon(z_p) \right) \\ &\quad \times \left. \left(\int_{(\mathbb{R}^2)^2} dz_3 dz_4 \prod_{p=3}^4 K'_{j_p - i_p}(z_6 - z_p) \xi_\varepsilon(z_p) \right) \right] \\ \ell_{\varepsilon, \mathbb{V}}[(j_k, i_k)_{k=1}^6] &\stackrel{\text{def}}{=} \mathbf{E} \left[\int_{(\mathbb{R}^2)^2} dz_5 dz_6 K'_{j_5 - i_5}(-z_5) K'_{j_6 - i_6}(-z_6) \xi_\varepsilon(z_6) \right. \\ &\quad \times \int_{(\mathbb{R}^2)^2} dz_3 dz_4 K'_{j_3 - i_3}(z_5 - z_3) K'_{j_4 - i_4}(z_5 - z_4) \xi_\varepsilon(z_4) \\ &\quad \times \left. \left(\int_{(\mathbb{R}^2)^2} dz_1 dz_2 \prod_{p=1}^2 K'_{j_p - i_p}(z_3 - z_p) \xi_\varepsilon(z_p) \right) \right] \quad (4.1) \end{aligned}$$

Here, for $j \geq 0$, $K'_j(t, x) \stackrel{\text{def}}{=} \partial_x K_j(t, x)$ where K_j is as in Sect. 3.3.

We also define $\bar{\ell}_{\varepsilon, \mathbb{V}}[(j_k, i_k)_{k=1}^2]$, $\bar{\ell}_{\varepsilon, \mathbb{V}}[(j_k, i_k)_{k=1}^6]$, and $\bar{\ell}_{\varepsilon, \mathbb{V}}[(j_k, i_k)_{k=1}^6]$ to be defined analogously as to the constants in (4.1) except every instance of K'_j , $j \geq 0$ should be replaced by $\bar{G}'_j$.

⁴ The only difference being the the (j, i) decorations.

We again adopt the convention that $\ell_{\varepsilon, \mathbb{V}}, \ell_{\varepsilon, \mathbb{V}\mathbb{V}}, \ell_{\varepsilon, \mathbb{V}\mathbb{V}\mathbb{V}}, \bar{\ell}_{\varepsilon, \mathbb{V}}, \bar{\ell}_{\varepsilon, \mathbb{V}\mathbb{V}}, \bar{\ell}_{\varepsilon, \mathbb{V}\mathbb{V}\mathbb{V}}$ vanish if given a set of indices that do not satisfy the constraint coming from the rule R .

The constants of (4.1) are those used to define the BPHZ model but this choice is not completely canonical since what we call the BPHZ model depends on the choice of kernel truncation K_j made in Sect. 3.3.

Using instead the renormalization constants defined in terms of $\bar{G}_j$ in our renormalised equation will yield the solution referred to in Theorem 1.1. However, the following lemma tells us that the discrepancy between the two sets of constants amounts to a finite shift.

Lemma 4.1 *One has the convergence*

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \bar{\ell}_{\varepsilon, \mathbb{V}}[(j_k, i_k)_{k=1}^2] - \ell_{\varepsilon, \mathbb{V}}[(j_k, i_k)_{k=1}^2] &= \bar{c}_{\mathbb{V}}[(j_k, i_k)_{k=1}^2] \\ \lim_{\varepsilon \downarrow 0} \bar{\ell}_{\varepsilon, \mathbb{V}\mathbb{V}}[(j_k, i_k)_{k=1}^6] - \ell_{\varepsilon, \mathbb{V}\mathbb{V}}[(j_k, i_k)_{k=1}^6] &= \bar{c}_{\mathbb{V}\mathbb{V}}[(j_k, i_k)_{k=1}^6] \\ \lim_{\varepsilon \downarrow 0} \bar{\ell}_{\varepsilon, \mathbb{V}\mathbb{V}\mathbb{V}}[(j_k, i_k)_{k=1}^6] - \ell_{\varepsilon, \mathbb{V}\mathbb{V}\mathbb{V}}[(j_k, i_k)_{k=1}^6] &= \bar{c}_{\mathbb{V}\mathbb{V}\mathbb{V}}[(j_k, i_k)_{k=1}^6], \end{aligned}$$

where the three constants on the right hand sides above are all finite and we restrict the indices $((j_k, i_k)_{k=1}^2)$ or $((j_k, i_k)_{k=1}^6)$ are chosen to be compatible with the corresponding symbol.

Proof For the discrepancy between the constants $\bar{\ell}_{\varepsilon, \mathbb{V}}[(j_k, i_k)_{k=1}^2]$ and $\ell_{\varepsilon, \mathbb{V}}[(j_k, i_k)_{k=1}^2]$ one can show that by scale invariance of $\bar{G}_m$ under parabolic scaling one has

$$\bar{\ell}_{\varepsilon, \mathbb{V}}[(j_k, i_k)_{k=1}^2] = \frac{1}{\varepsilon} \bar{\ell}_{1, \mathbb{V}}[(j_k, i_k)_{k=1}^2].$$

Then using that the kernels $K_{j_1-i_1}$ and $K_{j_2-i_2}$ are just truncations of $\bar{G}_{j_1-i_1}, \bar{G}_{j_2-i_1}$ one can show

$$\ell_{\varepsilon, \mathbb{V}}[(j_k, i_k)_{k=1}^2] = \frac{1}{\varepsilon} \bar{\ell}_{1, \mathbb{V}}[(j_k, i_k)_{k=1}^2] + c$$

for some finite constant c .

One needs more detailed analysis for the discrepancy for the symbols $\mathbb{V}\mathbb{V}$ and $\mathbb{V}\mathbb{V}\mathbb{V}$ since these in general produce logarithmic divergences. However, very similar analysis was done in the context of a single layer KPZ equation in [8, Section 6.3]. There one can show that appropriate rescalings of the kernels $K_0 * \varrho$ and $\partial_x K_0 * \varrho$ always converge to the corresponding untruncated kernels $G_0 * \varrho$ and $\partial_x G_0 * \varrho$ in semi-normed spaces of kernels that give good control over convolution on large and small scales—see [8, Lemma 6.8]—and this can be used to control the error in renormalization constants introduced by choosing the truncation K_0 of G_0 in the integrals defining $\ell_{\varepsilon, \mathbb{V}\mathbb{V}}[(0, 0), \dots, (0, 0)]$ and $\ell_{\varepsilon, \mathbb{V}\mathbb{V}\mathbb{V}}[(0, 0), \dots, (0, 0)]$ —see [8, Lemma 6.11].

However, it is straightforward to check that one has convergence, under the same rescaling, of the kernels $K_j * \varrho$ and $\partial_x K_j * \varrho$ to $\bar{G}_j * \varrho$ and $\partial_x \bar{G}_j * \varrho$ and this combined with the same arguments as before gives the desired result. $\square$

4.2 Renormalization of Symbols and Renormalised Equation

In the following lemma the canonical lift of a smooth noise ξ_ε refers to the canonical model constructed in [6, Section 8.2].

Lemma 4.2 Let $Z^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon)$ be the canonical lift of ξ_ε into a smooth model and $\hat{Z}^\varepsilon \stackrel{\text{def}}{=} (\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$ be the BPHZ lift of ξ_ε .

Then one has

$$\begin{aligned}\hat{\Pi}_z^\varepsilon \mathbf{I} &= \Pi_z^\varepsilon \mathbf{I} \\ \hat{\Pi}_z^\varepsilon \mathfrak{I}[(j_1, i_1)] &= \Pi_z^\varepsilon \mathfrak{I}[(j_1, i_1)] \\ \hat{\Pi}_z^\varepsilon \mathfrak{V}[(j_k, i_k)_{k=1}^2] &= \Pi_z^\varepsilon \mathfrak{V}[(j_k, i_k)_{k=1}^2] - \ell_{\varepsilon, \mathfrak{V}}[(j_k, i_k)_{k=1}^2] \\ \hat{\Pi}_z^\varepsilon \mathfrak{V}\mathfrak{V}[(j_k, i_k)_{k=1}^6] &= \Pi_z^\varepsilon \mathfrak{V}\mathfrak{V}[(j_k, i_k)_{k=1}^6] - \ell_{\varepsilon, \mathfrak{V}\mathfrak{V}}[(j_k, i_k)_{k=1}^6] \\ \hat{\Pi}_z^\varepsilon \mathfrak{V}\mathfrak{V}\mathfrak{V}[(j_k, i_k)_{k=1}^6] &= \Pi_z^\varepsilon \mathfrak{V}\mathfrak{V}\mathfrak{V}[(j_k, i_k)_{k=1}^6] - \ell_{\varepsilon, \mathfrak{V}\mathfrak{V}\mathfrak{V}}[(j_k, i_k)_{k=1}^6],\end{aligned}$$

where all the indices above are chosen to satisfy the constraint of the corresponding symbol.

Proof Starting with the generic formulation of BPHZ renormalization as a starting point, the key observations needed to justify the simple formulas above are that

- Additional subtractions for symbols of the type $\mathfrak{V}\mathfrak{V}$ or $\mathfrak{V}\mathfrak{V}\mathfrak{V}$ due to the occurrence of $\mathfrak{V}$ vanish since our truncated kernel K integrates to 0.
- We are in the Gaussian case, and so symbols with an odd number of noises do not produce any renormalization.⁵
- Symbols of the type $\mathfrak{V}\mathfrak{V}\mathfrak{V}$ produce vanishing renormalization constants, this is by parity under the reflection $(t, x) \mapsto (t, -x)$ (since we imposed that our mollifier is even in space ξ_ε is invariant in law under this reflection).

□

The next lemma presents the renormalized equation obtained by solving the equation driven by $\hat{Z}^\varepsilon$.

Lemma 4.3 Let $\hat{Z}^\varepsilon$ be the BPHZ lift of ξ_ε . Let $\gamma > 3/2 + \kappa$, $\eta > 0$, and $\bar{H}_\varepsilon$ be the solution to the abstract fixed point problem (3.2) in the $\mathcal{D}^{\gamma, \eta}([0, T] \times S^1)$ space over $\hat{Z}^\varepsilon$. Let $\bar{u}_\varepsilon = (u_{\varepsilon, l})_{l=1}^N$ be the reconstruction of $\bar{H}_\varepsilon$ given by $\hat{\mathcal{R}}^\varepsilon \bar{H}$, where $\hat{\mathcal{R}}^\varepsilon$ denotes the reconstruction operator associated to $\hat{Z}^\varepsilon$ which is applied componentwise to $\bar{H}$.

Then $\bar{u}_\varepsilon$ solves, on $[0, T] \times S^1$, the initial value problem

$$\partial_t u_{\varepsilon, l} = \partial_x^2 u_{\varepsilon, l} + (\partial_x u_{\varepsilon, l})^2 + \xi_\varepsilon - \sum_{q=1}^3 \tilde{C}_{\varepsilon, l}^{(q)} + \sum_{j=1}^{l-1} \partial_x^2 u_{\varepsilon, j} \quad (4.2)$$

starting from initial condition $(h_1^{(0)}, \dots, h_N^{(0)})$, where

$$\begin{aligned}\tilde{C}_{\varepsilon, l}^{(1)} &\stackrel{\text{def}}{=} \sum_{\substack{(j_k, i_k)_{k=1}^2 \\ j_1=j_2=l}} \ell_{\varepsilon, \mathfrak{V}}[(j_k, i_k)_{k=1}^2] \\ \tilde{C}_{\varepsilon, l}^{(2)} &\stackrel{\text{def}}{=} \sum_{\substack{(j_k, i_k)_{k=1}^6 \\ j_5=j_6=l}} \ell_{\varepsilon, \mathfrak{V}\mathfrak{V}}[(j_k, i_k)_{k=1}^6] \\ \tilde{C}_{\varepsilon, l}^{(3)} &\stackrel{\text{def}}{=} 4 \sum_{\substack{(j_k, i_k)_{k=1}^6 \\ j_5=j_6=l}} \ell_{\varepsilon, \mathfrak{V}\mathfrak{V}\mathfrak{V}}[(j_k, i_k)_{k=1}^6].\end{aligned} \quad (4.3)$$

⁵ In contrast to the situation in [9].

Moreover, the $(u_{1,\varepsilon}, \dots, u_{N,\varepsilon})$ converge in probability to N random fields $(u_1, \dots, u_N) \in \mathcal{C}^\eta([0, T], S^1)^{\otimes N}$ for any $\eta \in (0, \beta \wedge \frac{1}{2})$.

Proof Applying $\hat{\mathcal{R}}^\varepsilon$ to both sides of (3.2) gives

$$u_{\varepsilon,j} = \sum_{i=1}^j \bar{G}_{j-i} * \left[\hat{\mathcal{R}}^\varepsilon[(\mathcal{D}H_i)^2] + \xi_\varepsilon + h_i^{(0)} \right] \quad \text{for } 1 \leq j \leq N. \quad (4.4)$$

Now, using the identity $(\hat{\mathcal{R}}^\varepsilon F)(z) = (\hat{\Pi}_z F(z))(z)$, see for instance [6, Remark 3.15], one can verify, using Lemma 4.2,

$$(\hat{\mathcal{R}}^\varepsilon(\mathcal{D}H_i)^2)(z) = (\partial_x u_{\varepsilon,i})^2(z) - \sum_{q=1}^3 \tilde{C}_{\varepsilon,i}^{(q)}.$$

Then after inserting the above into (4.4) one can rewrite the resulting system in the form (4.2) (just as (2.3) was a rewriting itself of the non-renormalized equation).

The statement of convergence as $\varepsilon \downarrow 0$ follows from the convergence of the models $\hat{Z}^\varepsilon$, the continuity of the abstract solution map for (3.2), and the continuity of reconstruction. $\square$

The above proposition gives a notion of local solution to the multi-layer KPZ equation but it is not completely canonical since it depends on a truncation of the heat kernel and it will not give the Hopf-Cole solution for the first layer so it differs from the solution described in our main theorem but only by a finite shift.

Proof Theorem 1.1 For $i \in [N]$, $q = 1, 2, 3$, we choose $C_{\varepsilon,i}^{(q)}$ to be defined as $\tilde{C}_{\varepsilon,i}^{(q)}$ but with $\ell_\bullet[\bullet]$ replaced by $\bar{\ell}_\bullet[\bullet]$. Convergence of the $\vec{h}_\varepsilon$ then follows from that of the $\vec{u}_\varepsilon$ of (4.2) since one has

$$h_{\varepsilon,i}(t, x) - u_{\varepsilon,i}(t, x) = t \sum_{q=1}^3 (\tilde{C}_{\varepsilon,i}^{(q)} - C_{\varepsilon,i}^{(q)})$$

and the right hand side above converges to a finite limit by Lemma 4.1.

Identification of h_1 with the corresponding Hopf-Cole solution can be performed just as in the proof of [8, Proposition 7.1] in the simpler setting where one just takes $F(u) = u^2$ there.

The solution h constructed there is obtained by taking renormalized solutions where $\sum_{q=1}^3 \tilde{C}_{\varepsilon,1}^{(q)}$ is replaced by $C_{\varepsilon,1}^{(1)} + \tilde{C}_{\varepsilon,i}^{(2)} + \tilde{C}_{\varepsilon,i}^{(3)}$, then the constant c referenced in [8, Proposition 7.1] corresponds precisely to $(\tilde{C}_{\varepsilon,i}^{(2)} + \tilde{C}_{\varepsilon,i}^{(3)}) - (C_{\varepsilon,i}^{(2)} + C_{\varepsilon,i}^{(3)})$. $\square$

We now proceed to obtain the promised combinatorial formula for the renormalization constants appearing in Theorem 1.1.

Proposition 4.4 $C_{\varepsilon,i}^{(1)}$ is given by (4.6) below, $C_{\varepsilon,i}^{(2)}$ is given by (5.5) where $\mathcal{G}_{m_3,m_5,k_1,k_2}^{(\varepsilon)}$ therein is given by (5.12)–(5.16), and $C_{\varepsilon,i}^{(3)}$ is given by (5.7) where $\tilde{\mathcal{G}}_{m_5;k_1,k_2;m_6}^{(\varepsilon)}$ therein is given by (5.18).

Note that although formulae for $C_{\varepsilon,i}^{(2)}$ and $C_{\varepsilon,i}^{(3)}$ are very sophisticated, they are all written in terms of integrals of Hermite polynomials and heat kernels (with many combinatoric coefficients).

We consider the constant $C_{\varepsilon,i}^{(1)}$ in this section, and will study $C_{\varepsilon,i}^{(2)}$ and $C_{\varepsilon,i}^{(3)}$ in the next section.

Consider $C_{\varepsilon,i}^{(1)}$. Using the definition (2.1) one has

$$\tilde{\ell}_{\varepsilon,\mathbb{V}}[(i,j), (i,k)] = \sum_{m_1=0}^{i-j} \sum_{m_2=0}^{i-k} \binom{i-j-1}{m_1-1} \binom{i-k-1}{m_2-1} G'_{m_1,\varepsilon} * \tilde{G}'_{m_2,\varepsilon}(0)$$

where tilde denotes the reflection, i.e. $\tilde{F}(z) := F(-z)$. In view of (4.3) we need to sum over j and k . To this end we perform a re-summing, using

$$\sum_{j=1}^i \sum_{m=0}^{i-j} \binom{i-j-1}{m-1} H_m = \sum_{m=0}^{i-1} \sum_{j=1}^{i-m} \binom{i-j-1}{m-1} H_m = \sum_{m=0}^{i-1} \binom{i-1}{m} H_m$$

where H_m is a generic quantity which depends on m but not j . We get

$$C_{\varepsilon,i}^{(1)} = \sum_{m_1,m_2=0}^{i-1} \binom{i-1}{m_1} \binom{i-1}{m_2} G'_{m_1,\varepsilon} * \tilde{G}'_{m_2,\varepsilon}(0) \quad (4.5)$$

Note that here we have used our convention for binomial constants with entries allowed to be -1 .

Lemma 4.5 Fix $i \in \{1, 2, \dots, N\}$. Define $\varrho_\varepsilon^{(2)} \stackrel{\text{def}}{=} \varrho_\varepsilon * \varrho_\varepsilon$. For each $k \in \{0, 1, \dots, i-1\}$, with constant

$$C(\varepsilon, k) = \frac{1}{k!} (t^k G^{(2k)}) * \varrho_\varepsilon^{(2)}(0) \simeq 1/\varepsilon$$

one has

$$C_{\varepsilon,i}^{(1)} = \sum_{k=0}^{i-1} (-2)^k T(i-1, k) C(\varepsilon, k), \quad (4.6)$$

where for each i and each k ,

$$T(i, k) = \sum_{m_2=0}^i \sum_{m_1=k}^i (-2)^{-(m_1+m_2)} \binom{i}{m_1} \binom{i}{m_2} \binom{m_1-k+m_2}{m_2}. \quad (4.7)$$

Proof Let tilde denote the reflection, i.e. $\tilde{F}(z) := F(-z)$. Recall (4.5). We claim that

$$\begin{aligned} & G'_{m_1,\varepsilon} * \tilde{G}'_{m_2,\varepsilon}(0) \\ &= - \sum_{k=0}^{m_1} (-2)^{-(m_1+m_2-k+1)} \binom{m_1+m_2-k}{m_2} (\tilde{G}_k * \varrho_\varepsilon^{(2)})(0) \\ &\quad - \sum_{k=0}^{m_2} (-2)^{-(m_1+m_2-k+1)} \binom{m_1+m_2-k}{m_1} (G_k * \varrho_\varepsilon^{(2)})(0), \end{aligned} \quad (4.8)$$

where $\varrho_\varepsilon^{(2)} = \varrho_\varepsilon * \varrho_\varepsilon$. Indeed, let $m_1, m_2 \geq 0$, we define

$$D_{m_1,m_2}^{(\varepsilon)} = G'_{m_1,\varepsilon} * \tilde{G}'_{m_2,\varepsilon}, \quad (4.9)$$

and by (B.2) we see that $D_{m_1,m_2}^{(\varepsilon)} = D_{m_1,m_2} * \varrho_\varepsilon^{(2)}$. Using (B.4) we conclude the proof of (4.8).

We define

$$C(\varepsilon, k) \stackrel{\text{def}}{=} (G_k * \varrho_\varepsilon^{(2)})(0) = (\widetilde{G}_k * \varrho_\varepsilon^{(2)})(0) \quad (4.10)$$

where the last equality is a consequence of the fact that ϱ is an even function. Using (4.5) together with (4.8) gives (4.6). This is done by performing a resummation: we fix the k in (4.8), and sum m_1 from k to $i-1$ (or sum m_2 from k to $i-1$, which yields the same result by (4.10) and invariance of swapping m_1 and m_2 - thus we get a factor 2 which cancels a factor $-(-2)^{-1}$).

Moreover, integrating for each of the convolutions that follow first over space, making use of the semi-group property, and then integrating over time we see that

$$G^{*n} = \frac{t^{n-1}}{(n-1)!} G. \quad (4.11)$$

Hence, we can then write

$$C(\varepsilon, k) = (G_k * \varrho_\varepsilon^{(2)})(0) = \frac{1}{k!} \left(t^k G^{(2k)} \right) * \varrho_\varepsilon^{(2)}(0).$$

It now follows immediately that $C(\varepsilon, k) \sim \varepsilon^{-1}$. $\square$

Remark 4.6 One could also derive an alternative formula for $C(\varepsilon, k)$. Let

$$H_n(x) = (-1)^n e^{x^2/2} \partial_x^n e^{-x^2/2} =: (-1)^n e^{x^2/2} \partial_x^n \tilde{H}(x) \quad (4.12)$$

be the n th Hermite polynomial where $\tilde{H}(x) = e^{-x^2/2}$. It then follows that

$$\partial_x^n G(x, t) = (-1)^n e^{x^2/4t} \partial_x^n \tilde{H}(x/\sqrt{2t}) (-1)^n G(x, t) = \frac{1}{(2t)^{n/2}} (-1)^n H_n(x/\sqrt{2t}) G(x, t). \quad (4.13)$$

Hence, we can write $C(\varepsilon, k) = \frac{t^k}{k!} \frac{1}{(2t)^k} F_k G * \varrho_\varepsilon^{(2)}(0) = \frac{1}{2^k k!} (F_k G * \varrho_\varepsilon^{(2)})(0)$ where $F_k(x, t) = H_{2k}(x/\sqrt{2t})$.

5 Logarithmic Renormalization

We derive formulae for the other two constants in Proposition 4.4 and thus complete the proof of this Proposition.

5.1 Combinatoric Coefficients, and Doing Contractions

Now consider the n th equation. Using (2.1) for $\tilde{G}$, the first tree in the description of h_n , which we denote by $h_n^{(0)}$, is equal to

$$\begin{aligned} \sum_{i=1}^n \tilde{G}_{n-i} \Xi &= \sum_{i=1}^n \sum_{\ell=0}^{n-i} \binom{n-i-1}{\ell-1} G_\ell \Xi = \sum_{\ell=0}^{n-1} \sum_{i=1}^{n-\ell} \binom{n-i-1}{\ell-1} \mathfrak{I}[\ell] \\ &= \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} \mathfrak{I}[\ell] \end{aligned} \quad (5.1)$$

Note that $\sum_{\ell=0}^{n-1}$ can be rewritten as $\sum_{\ell \geq 0}$ since for ℓ out of that range the binomial coefficients are understood (with usual convention) as 0. The next tree in the description of h_n , denoted by $h_n^{(1)}$, is equal to

$$\begin{aligned}
\sum_{i=1}^n \bar{G}_{n-i} (\partial_x h_i^{(0)})^2 &= \sum_{i=1}^n \sum_{\ell=0}^{n-i} \binom{n-i-1}{\ell-1} G_\ell \sum_{m_1, m_2=0}^{i-1} \binom{i-1}{m_1} \binom{i-1}{m_2} \mathbb{Y}[m_1, m_2] \\
&= \sum_{\ell, m_1, m_2=0}^{n-1} \left\{ \sum_{i=m_1 \vee m_2 + 1}^{n-\ell} \binom{n-i-1}{\ell-1} \binom{i-1}{m_1} \binom{i-1}{m_2} \right\} \mathbb{Y}[m_1, m_2, \ell]
\end{aligned} \quad (5.2)$$

where we have performed resummation,⁶ and as above $\sum_{i=m_1 \vee m_2 + 1}^{n-\ell}$ could be replaced by $\sum_{i \geq 0}$ since for i out of that range the binomial coefficients are understood as 0.

Turning to the next tree in the description of h_n , by (5.1) and (5.2)

$$\begin{aligned}
h_n^{(2)} &= 2 \sum_{i=1}^n \bar{G}_{n-i} (\partial_x h_i^{(0)} \partial_x h_i^{(1)}) \\
&= 2 \sum_{i=1}^n \sum_{\ell=0}^{n-i} \binom{n-i-1}{\ell-1} G_\ell \sum_{m_1, m_2, m_3, m_4=0}^{i-1} \sum_{j=m_1 \vee m_2 + 1}^{i-m_3} \\
&\quad \times \binom{j-1}{m_1} \binom{j-1}{m_2} \binom{i-1}{m_4} \binom{i-j-1}{m_3-1} \mathbb{Y}[m_1, m_2, m_3, m_4] \\
&= 2 \sum_{\ell, m_1, m_2, m_3, m_4=0}^{n-1} c_{m_1, m_2, m_3, m_4, \ell}^{\mathbb{Y}} \mathbb{Y}[m_1, m_2, m_3, m_4, \ell]
\end{aligned}$$

where we have performed resummation and

$$c_{m_1, m_2, m_3, m_4, \ell}^{\mathbb{Y}} \stackrel{\text{def}}{=} \sum_{i=\max(m_1, m_2, m_3, m_4)+1}^{n-\ell} \sum_{j \geq 0} \binom{n-i-1}{\ell-1} \binom{j-1}{m_1} \binom{j-1}{m_2} \binom{i-1}{m_4} \binom{i-j-1}{m_3-1}$$

So in the expansion of the right hand of the n th layer equation, we have the following term

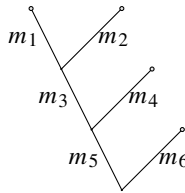
$$4 \sum_{m_1, \dots, m_6=0}^{n-1} c_{m_1, m_2, m_3, m_4, m_5, m_6}^{\mathbb{Y}} \mathbb{Y}[m_1, m_2, m_3, m_4, m_5, m_6] \quad (5.3)$$

where

$$c_{m_1, m_2, m_3, m_4, m_5, m_6}^{\mathbb{Y}} \stackrel{\text{def}}{=} \sum_{i \in I} \sum_{j \geq 0} \binom{n-i-1}{m_5-1} \binom{j-1}{m_1} \binom{j-1}{m_2} \binom{i-1}{m_4} \binom{i-j-1}{m_3-1} \binom{n-1}{m_6}$$

⁶ The coefficient in the parenthesis $\{\dots\}$ automatically restricts the admissible labels for trees $\mathbb{Y}[m_1, m_2, \ell]$. For example, $n = 2$ (2nd layer), if $\ell = 1$ then m_1, m_2 must both be 0, otherwise this coefficient is zero.

where $I \stackrel{\text{def}}{=} [\max(m_1, m_2, m_3, m_4) + 1, n - m_5] \cap \mathbb{Z}$. In particular when $n = 1$, $c_{0,0,0,0,0,0} = 1$ which is consistent with the expansion for single-layer KPZ equation.



Now we do contractions for the noises. There are two ways of contraction (see the above figure): the first way is to contract the pair (m_1, m_6) and the pair (m_2, m_4) ; the second way is to contract the pair (m_1, m_4) and the pair (m_2, m_6) .

Write

$$G_k^{(\varepsilon)} \stackrel{\text{def}}{=} G_k * \varrho_\varepsilon^{(2)}.$$

For the first contraction one has

$$D_{m_1, m_6}^{(\varepsilon)} = - \sum_{k=0}^{m_6} (-2)^{-(m_1+m_6-k+1)} \binom{m_1+m_6-k}{m_1} G_k^{(\varepsilon)} + \dots$$

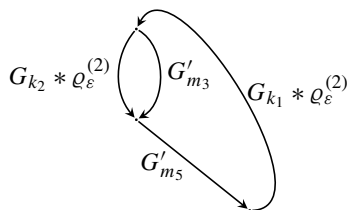
$$D_{m_2, m_4}^{(\varepsilon)} = - \sum_{k=0}^{m_4} (-2)^{-(m_2+m_4-k+1)} \binom{m_2+m_4-k}{m_2} G_k^{(\varepsilon)} + \dots$$

For the second contraction one has

$$D_{m_1, m_4}^{(\varepsilon)} = - \sum_{k=0}^{m_4} (-2)^{-(m_1+m_4-k+1)} \binom{m_1+m_4-k}{m_1} G_k^{(\varepsilon)} + \dots$$

$$D_{m_2, m_6}^{(\varepsilon)} = - \sum_{k=0}^{m_6} (-2)^{-(m_2+m_6-k+1)} \binom{m_2+m_6-k}{m_2} G_k^{(\varepsilon)} + \dots$$

Here “...” stand for terms with $\widetilde{G}_k$; they are omitted, because they only contribute a finite constant to the contracted graphs. Indeed, with a reflected G_k , one has a contracted graph which contains a directed loop, for instance



In this situation since $\varrho_\varepsilon^{(2)}$ is supported in a region of size of order ε , and the heat kernel vanishes at negative time variables, the integral corresponding to the graph is finite as $\varepsilon \rightarrow 0$.

Putting together these two ways of contractions and (5.3), one has

$$\begin{aligned}
 & 4 \sum_{m_1, \dots, m_6=0}^{n-1} \sum_{i \in I, j \geq 0} \binom{n-i-1}{m_5-1} \binom{j-1}{m_1} \binom{j-1}{m_2} \binom{i-1}{m_4} \binom{i-j-1}{m_3-1} \binom{n-1}{m_6} \\
 & \times \sum_{k_1=0}^{m_6} \sum_{k_2=0}^{m_4} (-2)^{-(m_1+m_2+m_4+m_6-k_1-k_2+2)} \left[\binom{m_1+m_6-k_1}{m_1} \binom{m_2+m_4-k_2}{m_2} \right. \\
 & \left. + \binom{m_1+m_4-k_2}{m_1} \binom{m_2+m_6-k_1}{m_2} \right] \times \mathcal{G}_{m_3, m_5, k_1, k_2}^{(\varepsilon)}
 \end{aligned}$$

where

$$\mathcal{G}_{m_3, m_5, k_1, k_2}^{(\varepsilon)} \stackrel{\text{def}}{=} \begin{array}{c} \begin{array}{c} \text{Diagram showing a loop structure with vertices labeled } G_{k_2}^{(\varepsilon)}, G'_{m_3}, G_{k_1}^{(\varepsilon)}, \text{ and } G'_{m_5}. \end{array} \end{array} \quad (5.4)$$

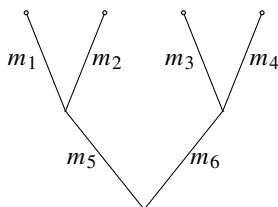
A resummation yields

$$\begin{aligned}
 & 4 \sum_{m_3, m_5, k_1, k_2=0}^{n-1} \left\{ \sum_{m_1, m_2=0}^{n-1} \sum_{m_4=k_2}^{n-1} \sum_{m_6=k_1}^{n-1} (-2)^{-(m_1+m_2+m_4+m_6-k_1-k_2+2)} \right. \\
 & \times \sum_{i \in I, j \geq 0} \binom{n-i-1}{m_5-1} \binom{j-1}{m_1} \binom{j-1}{m_2} \binom{i-1}{m_4} \binom{i-j-1}{m_3-1} \binom{n-1}{m_6} \\
 & \times \left[\binom{m_1+m_6-k_1}{m_1} \binom{m_2+m_4-k_2}{m_2} + \binom{m_1+m_4-k_2}{m_1} \binom{m_2+m_6-k_1}{m_2} \right] \left. \right\} \\
 & \times \mathcal{G}_{m_3, m_5, k_1, k_2}^{(\varepsilon)} \quad (5.5)
 \end{aligned}$$

Now we consider the trees $\mathbb{V}^{\circ\circ}[m_1, \dots, m_6]$. By (5.2), in the expansion of the right hand of the n th layer equation, we have the following term

$$\begin{aligned}
 & \sum_{m_1, \dots, m_6=0}^{n-1} \sum_{i=m_1 \vee m_2+1}^{n-m_5} \sum_{j=m_3 \vee m_4+1}^{n-m_6} \binom{n-i-1}{m_5-1} \binom{i-1}{m_1} \binom{i-1}{m_2} \\
 & \binom{n-j-1}{m_6-1} \binom{j-1}{m_3} \binom{j-1}{m_4} \mathbb{V}^{\circ\circ}[m_1, \dots, m_6]
 \end{aligned}$$

where i, j can equivalently sum over $i, j \geq 0$.



There are two ways of contraction (see the above figure for indices): the first is (m_1, m_4) and (m_2, m_3) ; the other way is (m_1, m_3) and (m_2, m_4) . After contraction

$$\begin{aligned} & \sum_{m_1, \dots, m_6=0}^{n-1} \sum_{i, j \geq 0} \binom{n-i-1}{m_5-1} \binom{i-1}{m_1} \binom{i-1}{m_2} \binom{n-j-1}{m_6-1} \binom{j-1}{m_3} \binom{j-1}{m_4} \\ & \times \left\{ \sum_{k_1=0}^{m_1} \sum_{k_2=0}^{m_2} (-2)^{-(m_1+m_2+m_3+m_4-k_1-k_2+2)} \left[\binom{m_1+m_4-k_1}{m_4} \binom{m_2+m_3-k_2}{m_3} \right. \right. \\ & + \left. \binom{m_1+m_3-k_1}{m_3} \binom{m_2+m_4-k_2}{m_4} \right] \times \bar{\mathcal{G}}_{m_5; k_1, k_2; m_6}^{(\varepsilon)} \\ & + \sum_{k_1=0}^{m_4} \sum_{k_2=0}^{m_3} (-2)^{-(m_1+m_2+m_3+m_4-k_1-k_2+2)} \left[\binom{m_1+m_4-k_1}{m_1} \binom{m_2+m_3-k_2}{m_2} \right. \\ & + \left. \left. \binom{m_1+m_3-k_2}{m_1} \binom{m_2+m_4-k_1}{m_2} \right] \times \bar{\mathcal{G}}_{m_5; k_1, k_2; m_6}^{(\varepsilon)} \right\} \end{aligned}$$

where we have omitted the terms leading to graphs with “directed loops” which only contribute finite constants, as explained above, and

$$\bar{\mathcal{G}}_{m_5; k_1, k_2; m_6}^{(\varepsilon)} \stackrel{\text{def}}{=} \begin{array}{c} \text{Graph with vertices } m_5, m_6 \text{ and } k_1, k_2. \\ \text{Edges: } m_5 \xrightarrow{G'_{m_5}} k_1, m_5 \xrightarrow{G'_{m_5}} k_2, m_6 \xrightarrow{G'_{m_6}} k_1, m_6 \xrightarrow{G'_{m_6}} k_2. \\ \text{Internal edges: } k_1 \xrightarrow{G_{k_1}^{(\varepsilon)}} k_2, k_2 \xrightarrow{G_{k_2}^{(\varepsilon)}} k_1. \end{array} \quad (5.6)$$

Now note that in the parenthesis $\{\dots\}$ the first sum is equal to the second sum, if we switch $m_1 \leftrightarrow m_3, m_2 \leftrightarrow m_4, m_5 \leftrightarrow m_6, i \leftrightarrow j, k_1 \leftrightarrow k_2$. We only consider the second sum (namely the one with $\bar{\mathcal{G}}_{m_5; k_1, k_2; m_6}^{(\varepsilon)}$), which will cause a factor 2. (Of course for $k_1 \neq k_2$, swapping k_1 and k_2 results in the same graph $\bar{\mathcal{G}}_{m_5; k_1, k_2; m_6}^{(\varepsilon)}$, but we do not combine these identical graphs for now.)

Combining the two sums in the parenthesis $\{\dots\}$ and re-summing, one has

$$\begin{aligned} & 2 \sum_{m_5, m_6, k_1, k_2=0}^{n-1} \left\{ \sum_{m_1, m_2=0}^{n-1} \sum_{m_3=k_2}^{n-1} \sum_{m_4=k_1}^{n-1} (-2)^{-(m_1+m_2+m_3+m_4-k_1-k_2+2)} \right. \\ & \sum_{i, j \geq 0} \binom{n-i-1}{m_5-1} \binom{i-1}{m_1} \binom{i-1}{m_2} \binom{n-j-1}{m_6-1} \binom{j-1}{m_3} \binom{j-1}{m_4} \\ & \left. \left[\binom{m_1+m_4-k_1}{m_1} \binom{m_2+m_3-k_2}{m_2} + \binom{m_1+m_3-k_2}{m_1} \binom{m_2+m_4-k_1}{m_2} \right] \right\} \\ & \times \bar{\mathcal{G}}_{m_5; k_1, k_2; m_6}^{(\varepsilon)} \end{aligned} \quad (5.7)$$

5.2 Computation of Contracted Graphs

Recall that the n th Hermite polynomial is $H_n(x) = (-1)^n e^{\frac{x^2}{2}} \partial_x^n e^{-\frac{x^2}{2}}$. Let $\tilde{H}(x) = e^{-\frac{x^2}{2}}$. Note that $\tilde{H}(x/\sqrt{2t}) = e^{-\frac{x^2}{4t}}$ and $\partial_x^n \tilde{H}(x/\sqrt{2t}) = \frac{1}{(2t)^{n/2}} H^{(n)}(x/\sqrt{2t})$.

We have the following formula for derivatives of heat kernel

$$G^{(n)}(t, x) \stackrel{\text{def}}{=} \partial_x^n G(t, x) = \frac{(-1)^n}{(2t)^{n/2}} H_n(x/\sqrt{2t}) G(t, x) \quad (5.8)$$

Another general formula:

$$G_n(t, x) = G \star G'' \star \dots \star G'' = \partial_x^{2n} G \star \dots \star G = \frac{t^n}{n!} G^{(2n)} \quad (5.9)$$

We compute $\mathcal{G}_{m_3, m_5, k_1, k_2}^{(\varepsilon)}$ as defined in (5.4).

Case 1 We first consider the case $m_3 - k_2 \in \{-1, 0\}$, which means

$$|2m_3 + 1 - 2k_2| = 1.$$

The two “parallel” edges G'_{m_3} and $G_{k_2}^{(\varepsilon)}$ in (5.4) can be dealt with as follows: by (5.9)

$$G'_{m_3} G_{k_2} = \frac{t^{m_3}}{m_3!} G^{(2m_3+1)} \frac{t^{k_2}}{k_2!} G^{(2k_2)} = \frac{1}{2} \frac{t^{m_3+k_2}}{m_3! k_2!} \partial_x \left((G^{(2m_3+1) \wedge (2k_2)})^2 \right)$$

where the last step used the assumption $|2m_3 + 1 - 2k_2| = 1$. Replacing $G'_{m_3} \cdot G_{k_2}^{(\varepsilon)}$ in $\mathcal{G}_{m_3, m_5, k_1, k_2}^{(\varepsilon)}$ by

$$\frac{1}{2} \frac{t^{m_3+k_2}}{m_3! k_2!} \partial_x \left((G^{(2m_3+1) \wedge (2k_2)})^2 * \varrho_\varepsilon^{(2)} \right)$$

only causes a finite difference and does not change the logarithmically divergent part of $\mathcal{G}_{m_3, m_5, k_1, k_2}^{(\varepsilon)}$.

With an integration by parts, we shift the x -derivative in the above expression to $G_{k_1}^{(\varepsilon)}$ in (5.4), which produces a negative sign. The pair of edges connected to the bottom vertex in (5.4) is then D_{m_5, k_1} . We have

$$\begin{aligned} D_{m_5, k_1} &= - \sum_{\ell=0}^{k_1} (-2)^{-(m_5+k_1-\ell+1)} \binom{m_5+k_1-\ell}{m_5} G_\ell + \dots \\ &= - \sum_{\ell=0}^{k_1} (-2)^{-(m_5+k_1-\ell+1)} \binom{m_5+k_1-\ell}{m_5} \frac{t^\ell}{\ell!} G^{(2\ell)} + \dots \end{aligned}$$

where the terms “...” are omitted because they will only contribute finite constants as explained above. So

$$\begin{aligned} \mathcal{G}_{m_3, m_5, k_1, k_2}^{(\varepsilon)} &= \int_{|t| \geq \varepsilon^2} \frac{1}{2} \sum_{\ell=0}^{k_1} (-2)^{-(m_5+k_1-\ell+1)} \binom{m_5+k_1-\ell}{m_5} \\ &\quad \times \frac{t^{\ell+k_2+m_3}}{\ell!k_2!m_3!} G^{(2\ell)} \cdot (G^{(2m_3+1) \wedge (2k_2)})^2 dt dx + O(1) \end{aligned}$$

where functions in the integrand depend on (t, x) , and we have replaced mollifiers by a small scale cutoff in the integral which only causes an $O(1)$ constant. Using (5.8), we have

$$\begin{aligned} &t^{\ell+k_2+m_3} G^{(2\ell)} \cdot (G^{(2m_3+1) \wedge (2k_2)})^2 \\ &= \frac{t^{\ell+k_2+m_3}}{(2t)^\ell (2t)^{(2m_3+1) \wedge (2k_2)}} H_{2\ell}(x/\sqrt{2t}) H_{(2m_3+1) \wedge (2k_2)}(x/\sqrt{2t})^2 G(t, x)^3. \end{aligned}$$

Note that the entire power of t does not depend on whether $m_3 - k_2 = -1$ or 0 , i.e., the powers of t cancel out.

Case 2 Let's turn to the case of $m_3 - k_2 \geq 1$. We first write the two “parallel” edges in (5.4) as

$$G'_{m_3} G_{k_2} = \frac{t^{m_3+k_2}}{m_3!k_2!} G^{(2m_3+1)} G^{(2k_2)}$$

Now the idea is the following: if we integrate by parts w.r.t. the vertex v (see (5.10)), a derivative will be shifted from $G^{(2m_3+1)}$ to either G'_{m_5} or $G^{(2k_2)}$. The good situations are that either we have G''_{m_5} (one of these two derivatives can be shifted to G_{k_1}) so that we can apply the identity for $D_{\cdot}$ to convolve the bottom vertex (as first term on RHS of (5.10)), or the two parallel edges represent derivatives of G of orders that only differ by one (so that it can be written as $\frac{1}{2} \partial_x ((\dots)^2)$ as in Case 1). To be more precise, starting from (5.4), and ignoring for the moment all powers of t , using (5.9) the above leads to the analysis of

$$(5.10)$$

Here and below we omit the mollifiers to make the graphic notation simpler. If $m_3 - k_2 \in \{-1, 0\}$ we stop at this point because the second term above can be studied as in Case 1, otherwise we apply another integration by parts to the second term above and we repeat this process until we can make use of the techniques used in the case $m_3 - k_2 \in \{-1, 0\}$. In this way we see that the graph becomes

$$\begin{aligned}
& G^{(2k_2)} \begin{array}{c} \text{Diagram 1: Loop with } G^{(2m_3)} \text{ and } G_{k_1} \text{ edges, } G''_{m_5} \text{ bottom edge} \end{array} - G^{(2k_2+1)} \begin{array}{c} \text{Diagram 2: Loop with } G^{(2m_3-1)} \text{ and } G_{k_1} \text{ edges, } G''_{m_5} \text{ bottom edge} \end{array} + G^{(2k_2+2)} \begin{array}{c} \text{Diagram 3: Loop with } G^{(2m_3-1)} \text{ and } G_{k_1} \text{ edges, } G'_{m_5} \text{ bottom edge} \end{array} \\
& = \dots = \sum_{\ell=1}^{m_3-k_2} (-1)^{\ell-1} G^{(2k_2+\ell-1)} \begin{array}{c} \text{Diagram 4: Loop with } G^{(2m_3+1-\ell)} \text{ and } G_{k_1} \text{ edges, } G''_{m_5} \text{ bottom edge} \end{array} + (-1)^{m_3-k_2} G^{(m_3+k_2)} \begin{array}{c} \text{Diagram 5: Loop with } G^{(m_3+k_2+1)} \text{ and } G_{k_1} \text{ edges, } G'_{m_5} \text{ bottom edge} \end{array} \\
& := A + B.
\end{aligned} \tag{5.11}$$

Now for the term A we convolve the origin

$$D_{m_5, k_1} = - \sum_{s=0}^{k_1} (-2)^{-(m_5+k_1-s+1)} \binom{m_5+k_1-s}{m_5} \frac{t^s}{s!} G^{(2s)} + \dots$$

where “...” are again terms that will only contribute finite constants. For each fixed s we need to integrate

$$\frac{t^{s+m_3+k_2}}{s!m_3!k_2!} G^{(2s)} G^{(2m_3+1-\ell)} G^{(2k_2+\ell-1)}$$

Using (5.8) we get

$$\frac{1}{s!m_3!k_2!} \frac{1}{2^{s+m_3+k_2}} \int_{|t| \geq \varepsilon^2} H_{2s}(x/\sqrt{2t}) H_{2m_3+1-\ell}(x/\sqrt{2t}) H_{2k_2+\ell-1}(x/\sqrt{2t}) G(t, x)^3 dt dx$$

Regarding B , note that $G^{(m_3+k_2+1)} G^{(m_3+k_2)} = \frac{1}{2} \partial_x ((G^{(m_3+k_2)})^2)$. Shifting this ∂_x to G_{k_1} and integrating out the bottom vertex, we can proceed as before.

Case 3. We can apply the same strategy for the case $m_3 - k_2 \leq -2$. Integrate by parts until one can write the product of the parallel edges as $1/2$ times the spacial derivative of a square.

Below we summarize the formulas we found.

1. If $m_3 - k_2 \in \{-1, 0\}$. Then the tall tree equals

$$\begin{aligned}
& \frac{1}{2} \sum_{s=0}^{k_1} (-2)^{-(m_5+k_1-s+1)} \binom{m_5+k_1-s}{m_5} \frac{1}{s!k_2!m_3!2^{s+(2m_3+1) \wedge (2k_2)}} \\
& \times \int H_{2s}(x/\sqrt{2t}) H_{(2m_3+1) \wedge (2k_2)}(x/\sqrt{2t})^2 G(x, t)^3.
\end{aligned} \tag{5.12}$$

2. If $m_3 - k_2 \geq 1$. In this case the tall tree can be written as $A + B$ with

$$A = \sum_{\ell=1}^{m_3-k_2} (-1)^{\ell+1} \sum_{s=0}^{k_1} (-2)^{-(m_5+k_1-s+1)} \binom{m_5+k_1-s}{m_5} \frac{1}{s!m_3!k_2!2^{s+m_3+k_2}} \\ \times \int H_{2s}(x/\sqrt{2t}) H_{(2m_3+1-\ell)}(x/\sqrt{2t}) H_{(2k_2+\ell-1)}(x/\sqrt{2t}) G(x, t)^3 \quad (5.13)$$

and

$$B = (-1)^{m_3-k_2} \frac{1}{2} \sum_{s=0}^{k_1} (-2)^{-(m_5+k_1-s+1)} \binom{m_5+k_1-s}{m_5} \frac{1}{s!m_3!k_2!2^{s+m_3+k_2}} \\ \times \int H_{2s}(x/\sqrt{2t}) H_{m_3+k_2}(x/\sqrt{2t})^2 G(x, t)^3. \quad (5.14)$$

3. If $m_3 - k_2 \leq -2$ we should get that the tall tree is again of the form $A + B$ with

$$A = \sum_{\ell=1}^{k_2-m_3-1} (-1)^{\ell+1} \sum_{s=0}^{k_1} (-2)^{-(m_5+k_1-s+1)} \binom{m_5+k_1-s}{m_5} \frac{1}{s!m_3!k_2!2^{s+m_3+k_2}} \\ \times \int H_{2s}(x/\sqrt{2t}) H_{(2m_3+\ell)}(x/\sqrt{2t}) H_{(2k_2-\ell)}(x/\sqrt{2t}) G(x, t)^3 \quad (5.15)$$

and

$$B = (-1)^{k_2-m_3-1} \frac{1}{2} \sum_{s=0}^{k_1} (-2)^{-(m_5+k_1-s+1)} \binom{m_5+k_1-s}{m_5} \frac{1}{s!m_3!k_2!2^{s+m_3+k_2}} \\ \times \int H_{2s}(x/\sqrt{2t}) H_{m_3+k_2}(x/\sqrt{2t})^2 G(x, t)^3. \quad (5.16)$$

Let's describe a general formula for the logarithmic constant from the trees $\mathbb{V}^{\circ\circ}$. Recall $\bar{\mathcal{G}}_{m_5; k_1, k_2; m_6}^{(\varepsilon)}$ defined in (5.6).

The two edges pointing to the bottom vertex carry the kernels G'_{m_5} and G'_{m_6} . This is precisely our kernel D_{m_5, m_6} . One has

$$D_{m_5, m_6} = - \sum_{s=0}^{m_5} (-2)^{-(m_5+m_6-s+1)} \binom{m_5+m_6-s}{m_6} G_s + \dots, \quad (5.17)$$

where $\dots$ is the part that only contribute finite constants. Fix s . Then, we need to evaluate $G_s G_{k_1} G_{k_2}$. Recall that $G_k(x, t) = \frac{t^k}{k!} G^{(2k)}(x, t) = \frac{t^k}{k!(2t)^k} H_{2k}(x/\sqrt{2t}) G(x, t)$. Thus, we get

$$\bar{\mathcal{G}}_{m_5; k_1, k_2; m_6}^{(\varepsilon)} = - \sum_{s=0}^{m_5} (-2)^{-(m_6+m_5-s+1)} \binom{m_6+m_5-s}{m_6} \frac{1}{k_1!k_2!s!2^{k_1+k_2+s}} \\ \times \int_{|t| \geq \varepsilon^2} H_{2k_1}(x/\sqrt{2t}) H_{2k_2}(x/\sqrt{2t}) H_{2s}(x/\sqrt{2t}) G(x, t)^3 dt dx. \quad (5.18)$$

It is possible to explicitly carry out the integrals of the type

$$\int \left(\prod_{j=1}^3 H_{n_j}(x/\sqrt{2t}) \right) G(t, x)^3 dx dt$$

Indeed, one has the following lemma. However, having the values of these integrals does not seem to lead us to more instructive formulae. On the other hand, with the following lemma we may compute the logarithmic constants on a computer, see Remark 5.2.

Lemma 5.1 *Define, for $n_1, n_2, n_3 \in \mathbb{Z}_{\geq 0}$*

$$A_{n_1, n_2, n_3} \stackrel{\text{def}}{=} \int dx \left(\prod_{j=1}^3 H_{n_j}(x/\sqrt{2t}) \right) G(t, x)^3$$

Then $A_{n_1, n_2, n_3} = 0$ if $n_1 + n_2 + n_3$ is odd and otherwise one has

$$\begin{aligned} A_{n_1, n_2, n_3} &= \frac{2^{-(n_1+n_2+n_3)/2}}{3^{1/2} 4\pi t} \sum_{\substack{r_1, r_2, r_3 \\ r_1+r_2+r_3 \leq (n_1+n_2+n_3)/2}} \left(\prod_{j=1}^3 (-1)^{r_j} 3^{r_j-n_j/2} \frac{n_j!}{r_j!(n_j-2r_j)!} \right) \\ &\quad \times \frac{\left(\sum_{j=1}^3 n_j - 2r_j \right)!}{\left(\sum_{j=1}^3 n_j/2 - r_j \right)!} \end{aligned}$$

Proof This is a straightforward computation using a change of variable $\sqrt{\frac{3}{2t}}x \mapsto y$ and an expansion for a triple product of scaled Hermite polynomials [2]. $\square$

Remark 5.2 With Mathematica we found that the logarithmic renormalization constants do sum to a non-zero constant for the second, third and fourth layer. In fact, with help of Mathematica we have $C_{\varepsilon,1}^{(2)} = -\frac{1}{2} \frac{1}{4\sqrt{3\pi}} \log \varepsilon$ and $C_{\varepsilon,1}^{(3)} = \frac{1}{2} \frac{1}{4\sqrt{3\pi}} \log \varepsilon$ (consistent with [9, Section 3.2]), which cancel out; we have

$$\begin{aligned} C_{\varepsilon,2}^{(2)} &= -\frac{85}{288} \frac{1}{4\sqrt{3\pi}} \log \varepsilon & C_{\varepsilon,2}^{(3)} &= \frac{47}{144} \frac{1}{4\sqrt{3\pi}} \log \varepsilon \\ C_{\varepsilon,3}^{(2)} &= -\frac{995}{6912} \frac{1}{4\sqrt{3\pi}} \log \varepsilon & C_{\varepsilon,3}^{(3)} &= \frac{445}{3456} \frac{1}{4\sqrt{3\pi}} \log \varepsilon \\ C_{\varepsilon,4}^{(2)} &= -\frac{5129851}{53747712} \frac{1}{4\sqrt{3\pi}} \log \varepsilon & C_{\varepsilon,4}^{(3)} &= \frac{1018585}{13436928} \frac{1}{4\sqrt{3\pi}} \log \varepsilon \end{aligned}$$

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Appendix A: A Short Analysis of G_i

Let $\zeta \in \mathbb{R}$. We say that a kernel G defined on $\mathbb{R}^d \setminus \{0\}$ or on a subset thereof is of order ζ if for all multiindices k there exists a constant $C > 0$ such that

$$\sup_{\|x\|_S \leq 1} |D^k G(x)| \leq C \|x\|_S^{\zeta - |k|_S}. \quad (\text{A.1})$$

Proposition A.1 *The kernel G_i is of order -1 for every $i \geq 0$.*

Proof The result is well known for the usual heat kernel, see for instance [6, Lemma 7.4]. If $i = 1$, then note that after a partial integration we have the identity

$$G_1 = \partial_x G * \partial_x G, \quad (\text{A.2})$$

and both kernels on the right hand side are of order -2 . The claim therefore follows from [6, Lemma 10.14]. Note that in that result the kernels are supposed to be compactly supported. However, since the heat kernel decays superexponentially fast at infinity it is possible to adapt this result to the present setting. For $i > 1$ we can write $G_i = G_{i-1} * \partial_x^2 G$ so that we can conclude as in the case $i = 1$. $\square$

Appendix B: Useful Identity

The identity

$$2G' * \tilde{G}' = G + \tilde{G} \quad (\text{B.1})$$

was used in [8, Sect. 6] where the tilde is the reflection, i.e. $\tilde{F}(z) := F(-z)$, and the derivative above is with respect to the spatial variable. Iteratively applying this identity we can get useful identities for convolutions of (derivatives of) G_m . For instance we have $4G'_0 * \tilde{G}'_1 = 2\tilde{G}_1 - G_0 - \tilde{G}_0$, since

$$4G' * \tilde{G}' * \tilde{G}'' = 2(G + \tilde{G}) * \tilde{G}'' = 2\tilde{G} * \tilde{G}'' - 2G' * \tilde{G}' = 2\tilde{G} * \tilde{G}'' - G - \tilde{G}$$

where we used integration by parts to shift a derivative $2G * \tilde{G}'' = -2G' * \tilde{G}'$ in the second step. When the indices are large this calculation can get more involved, for instance, we have

$4G'_1 * \tilde{G}'_1 = -G_1 - \tilde{G}_1 + G_0 + \tilde{G}_0$, because

$$\begin{aligned} 4G'' * G' * \tilde{G}' * \tilde{G}'' &= 2G'' * (G + \tilde{G}) * \tilde{G}'' = 2G'' * G * \tilde{G}'' + 2G'' * \tilde{G} * \tilde{G}'' \\ &= -2G'' * G' * \tilde{G}' - 2G' * \tilde{G}' * \tilde{G}'' = -G'' * (G + \tilde{G}) - (G + \tilde{G}) * \tilde{G}'' \\ &= -G * G'' - \tilde{G} * \tilde{G}'' - G'' * \tilde{G} - G * \tilde{G}'' = -G * G'' - \tilde{G} * \tilde{G}'' + 2G' * \tilde{G}' \\ &= -G * G'' - \tilde{G} * \tilde{G}'' + G + \tilde{G}. \end{aligned}$$

In order to obtain a general set of identities we define kernels $D_{i,j}$, for $(i, j) \in \Lambda \stackrel{\text{def}}{=} \{0, 1, 2, \dots\}^2$ via

$$D_{i,j} \stackrel{\text{def}}{=} G'_i * \tilde{G}'_j. \quad (\text{B.2})$$

For $i, j > 0$, one has the recursion relation

$$D_{i,j} = -\frac{1}{2} \left(D_{i-1,j} + D_{i,j-1} \right).$$

Indeed, making use of (B.1), we can write

$$\begin{aligned} D_{i,j} &= (\partial_x^2 G)^{*i} * G' * \tilde{G}' * (\partial_x^2 \tilde{G})^{*j} \\ &= \frac{1}{2} \left((\partial_x^2 G)^{*i} * G * (\partial_x^2 \tilde{G})^{*j} + (\partial_x^2 G)^{*i} * \tilde{G} * (\partial_x^2 \tilde{G})^{*j} \right) \\ &= -\frac{1}{2} \left((\partial_x^2 G)^{*i} * G * \partial_x^2 \tilde{G} * (\partial_x^2 \tilde{G})^{*(j-1)} + (\partial_x^2 G)^{*i} * \partial_x^2 G * \tilde{G} * (\partial_x^2 \tilde{G})^{*j} \right) \end{aligned} \quad (\text{B.3})$$

and by shifting a derivative the above recursion relation follows.

When $i = 0, j > 0$ one has the recursion

$$D_{0,j} = -\frac{1}{2}D_{0,j-1} + \frac{1}{2}G_j.$$

Since $\widetilde{D_{i,j}} = D_{j,i}$ we get, when $j = 0$ and $i > 0$

$$D_{i,0} = -\frac{1}{2}D_{i-1,0} + \frac{1}{2}\widetilde{G_i}.$$

Finally, $D_{0,0} = \frac{1}{2}[G_0 + \widetilde{G_0}]$.

We now use these recursions to find formula for $D_{i,j}$. A lattice path is a sequence of nearest neighbor edges (steps) of Λ which satisfy the natural adjacency relation.

Let $B \stackrel{\text{def}}{=} \{(x, y) \in \Lambda, i = 0 \text{ or } j = 0\}$ be the boundary of the discrete first quadrant. We denote by $W(i, j)$ the set of all lattice paths γ which

- Start at (i, j)
- Only move down or to the left
- Terminate at a site of B (note, paths are allowed to travel along B for some time, but they always end at a site of B , they can't go negative.)

For a lattice path γ we denote by $l(\gamma)$ the number of steps in γ . One then has the following formula

$$D_{i,j} = \sum_{\gamma \in W(i,j)} (-2)^{-l(\gamma)} F(\gamma_{\text{end}})$$

where γ_{end} is the final site visited by γ and F is a map from the sites of B to kernels given as follows: $F(0, 0) = \frac{1}{2}(G + \widetilde{G})$ and

$$F(0, j) = \frac{1}{2}G_j \quad \text{and} \quad F(i, 0) = \frac{1}{2}\widetilde{G_i}.$$

For fixed $(i, j) \in \Lambda$ let $B(i, j)$ be the set of sites of B one can reach via walks in $W(i, j)$. We then get

$$D_{i,j} = \sum_{(x,y) \in B(i,j)} (-2)^{-(i+j-x-y)} \binom{i+j-x-y}{i-x} F(x, y),$$

where $\binom{i+j-x-y}{i-x}$ counts the number of paths from (i, j) to (x, y) . Equivalently,

$$D_{i,j} = - \sum_{k=0}^i (-2)^{-(i+j-k+1)} \binom{i+j-k}{j} \widetilde{G_k} - \sum_{k=0}^j (-2)^{-(i+j-k+1)} \binom{i+j-k}{i} G_k. \quad (\text{B.4})$$

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