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NONCOMPACT COMPLETE RIEMANNIAN MANIFOLDS WITH SINGULAR CONTINUOUS SPECTRUM EMBEDDED INTO THE ESSENTIAL SPECTRUM OF THE LAPLACIAN, I. THE HYPERBOLIC CASE

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Abstract. We construct Riemannian manifolds with singular continuous spectrum embedded in the absolutely continuous spectrum of the Laplacian. Our manifolds are asymptotically hyperbolic with sharp curvature bounds.

1. Introduction and main results

Let (M_n, g), $n \geq 2$, be an n-dimensional connected noncompact complete Riemannian manifold. The Laplace-Beltrami operator $\Delta := \Delta_g$ on $M := (M_n, g)$, is essentially self-adjoint on $C_0^\infty(M_n)$. We also denote by Δ its unique self-adjoint extension to $L^2(M_n, dv_g)$.

We refer the readers to [9] for a review of results on the spectral theory of Laplacians on noncompact manifolds. Most of the past work has been focused on proofs of the purity of absolutely continuous spectrum, guaranteed by the asymptotic curvature conditions, going back to [6, 26]. Several extensions of purity results have also appeared recently [10, 11, 24, ?Liu1]. Lately, some attention has turned to the opposite phenomenon. Kumura [20] constructed manifolds with an eigenvalue embedded in the spectrum of the Laplacian. In [13] we constructed manifolds with arbitrary finite or countable subset of the essential spectrum embedded as eigenvalues. This brings a natural question whether singular continuous spectrum can also be embedded in the essential (absolutely continuous) spectrum of the Laplace-Beltrami operator. The goal of this paper is to construct such manifolds. We prove the following.

Theorem 1.1. *For $K_0 < 0$, there exist smooth simply connected n-dimensional Riemannian manifolds such that*

(1) $\sigma_{\text{ess}}(-\Delta) = \sigma_{\text{ac}}(-\Delta) = \left[\frac{|K_0|}{4}(n-1)^2, \infty\right)$,
(2) $\sigma_{\text{sc}}(-\Delta) \neq \emptyset$.[1]

Despite a significant interest in the Schrödinger operators community in the last 30 years and various ubiquitous results (initiated by [32]) singular continuous spectrum remains rather mysterious in the spectral theory and has been virtually unseen

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[1] It is then automatically embedded in $\left[\frac{|K_0|}{4}(n-1)^2, \infty\right)$.

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and unstudied in spectral geometry. In particular, to the best of our knowledge, there have been no previous constructions of Riemannian manifolds with embedded singular continuous spectrum of the Laplacian. The only appearance of the singular continuous spectrum in the context of Laplace-Beltrami operators we are aware of is [33] where Simon proved topological genericity of manifolds with purely singular continuous spectrum in a class of metrics on the 2D infinite cylinder (so not simply connected, and without an explicit construction).

Singular continuous spectral measures are supported on zero measure sets, yet give zero weight to every point, making them particularly difficult to control explicitly. Quite often singular continuous spectrum is proved by ruling out existence of absolutely continuous and point components (or turning the reasoning above on its head as in [32]). Clearly, this is not going to work for singular continuous spectrum *embedded* into an absolutely continuous one, making corresponding questions especially hard.

In this paper we study the asymptotically hyperbolic case: Riemannian manifolds with the radial curvature $K_{\rm rad}(r)$ (sectional curvature with one fixed direction ∇r) approaching $K_0 < 0$ as $r \to \infty$. If $K_{\rm rad}(r) = K_0 < 0$ is constant, it is well known that

$$\sigma(-\Delta) = \sigma_{\rm ac}(-\Delta_g) = \sigma_{\rm ess}(-\Delta_g) = \left[\frac{|K_0|}{4}(n-1)^2, \infty\right)$$

and the singular spectrum (the union of point and singular continuous spectra) is empty. The essential spectrum is preserved under decaying perturbations and it is natural to expect that no embedded singular spectrum will persist when $K_{\rm rad}(r)$ approaches K_0 sufficiently fast, but point and singular continuous spectra can be embedded into the essential (absolutely continuous) spectrum for slower rates of decay of $|K_{\rm rad}(r) - K_0|$. Note that compact perturbations of constant curvature can only lead to eigenvalues below the essential spectrum, so embedding questions are naturally tied to the rate of decay. Sharp decay thresholds have been established for existence of metrics with an embedded eigenvalue [20] (see also [?Liu1] for a simple proof of sharpness) and with an embedded arbitrary countable (in particular, dense) set [13] (also for the flat, i.e., $K_0 = 0$, case). Here we prove a correspondingly more precise version of Theorem 1.1.

Theorem 1.2. *Suppose $K_0 < 0$. Let $h(r) > 0$ be any function on $(0,\infty)$ with $\lim_{r\to\infty} h(r) = \infty$. Then there exist smooth simply connected Riemannian manifolds (M_n, g) such that*

(1) $|K_{\rm rad}(r) - K_0| = O\left(\frac{h(r)}{1+r}\right)$,
(2) $\sigma_{\rm ess}(-\Delta) = \sigma_{\rm ac}(-\Delta) = \left[\frac{|K_0|}{4}(n-1)^2, \infty\right)$,
(3) $\sigma_{\rm sc}(-\Delta) \neq \emptyset$.

Remark 1.3. Modifying our construction, the spectral measure of the Laplacian can have both pure point and singular continuous components on $\left(\frac{|K_0|}{4}(n-1)^2, \infty\right)$.

We expect that our result is sharp, that is, like in the 1D case discussed below, $\frac{O(1)}{1+r}$ provides a threshold for existence of asymptotically hyperbolic metrics with embedded singular continuous spectrum: for manifolds with $|K_{\rm rad}(r) - K_0| < C(1+r)^{-1}$, the essential spectrum should be purely absolutely continuous. So far it has been established under somewhat more restrictive conditions. Kumura

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[21] proved absolute continuity of the Laplacian by the limiting absorption principle (originally from Agmon's theory [1]) under the condition $|K_{\mathrm{rad}}(r) - K_0| = \frac{O(1)}{r^{1+\delta}}, \delta > 0$ and assuming convexity of the Hessian of r. Donnelly used the exhaustion function to investigate the spectral structure of the Laplacian, which can also show the absence of singular continuous spectrum for some manifolds [7, 8].

There is a remarkable similarity between results on curvature thresholds for embedded eigenvalues for the noncompact manifolds in arbitrary dimension and for 1D Schrödinger operators with decaying potentials. This leads to a natural conjecture that the curvature threshold for existence of metrics with embedded singular continuous spectrum is also going to be the same as in the 1D Schrödinger case, where this was a known difficult problem, popularized by B. Simon in the 1990s and included in his list of 15 Schrödinger operator problems for the twenty-first century [34]. Unlike for the manifolds, for Schrödinger operators, existence of *some* potentials with prescribed spectral behavior is guaranteed by the inverse spectral theory [22, 25], so the issue is potentials with certain decay. Existence of L^2 potentials with embedded singular continuous spectrum was proved by Denisov [5] and followed from Killip-Simon's criterion [14] but in an implicit way. Potentials with power decaying solutions on a set of expected Hausdorff dimension [3, 30] were constructed by Remling [18, 31], but this was insufficient to infer existence of an embedded singular continuous component. Decaying potentials with purely singular continuous spectrum were constructed in [17, 28]. An explicit construction of potential that has singular continuous spectrum *embedded* into absolutely continuous (and a sharp result in terms of decay) was given by Kiselev [16], therefore solving Simon's problem. He proved that if the potential $V(x) = \frac{O(1)}{1+x}$, then the singular continuous spectrum of $-D^2 + V$ is empty, but given any positive function $h(x)$ tending to infinity as x grows, there exist potentials $V(x)$ such that $|V(x)| \leq \frac{h(x)}{1+x}$ and the operator $-D^2 + V$ has a nonempty singular continuous spectrum on $(0, \infty)$ [16]. By the Weyl theorem and classical results in [2, 4, 29], both the essential spectrum and absolutely continuous spectrum of $-D^2 + V$ constructed by Kiselev are $(0, \infty)$.

In this paper, we use Kiselev's potentials to construct our manifolds. The Riemannian manifolds (M, g) we construct are rotationally symmetric, and we effectively reduce the problem to a 1D Schrödinger operator, with the main work needed to guarantee the existence of smooth metrics leading to a 1D potential with desired properties. It turns out this is possible to do in the asymptotically hyperbolic case, using Kiselev's construction almost as a black box. The asymptotically flat case (i.e., $K_0 = 0$) however turns out to be more difficult, with the corresponding problem unsolvable without further assumptions on the potential, thus requiring us to significantly modify Kiselev's construction to guarantee the additional desired structure of the potential. This will be done in [12].

To construct a rotationally symmetric manifold, we fix some $O \in M_n$ as the origin. Using the radial coordinates (from O) we construct a Riemannian manifold with the structure of the form

$$(M_n, g) = \left(\mathbb{R}^n, dr^2 + f_1^2(r) g_{S^{n-1}(1)}\right),$$

where $g_{S^{n-1}(1)}$ is the standard Riemannian metric on the unit sphere, and we need to construct f_1 so that the Laplacian has the desired properties. To determine the

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spectral representation of the Laplacian on a rotationally symmetric manifold, one can use separation of variables .

Let $Y_{i,j}(\theta)$, $\theta \in S^{n-1}$, $i \geq 0$, and $j = 1, 2, \cdots, q_i$, be the spherical harmonics. They form a complete orthonormal basis for $L^2(S^{n-1})$ [36]. Each $Y_{i,j}(\theta)$ belongs to a q_i-dimensional eigenspace of the spherical Laplacian with corresponding eigenvalue λ_i. One may expand $\phi(r,\theta) \in L^2(M_n, g)$ as

$$\phi(r,\theta) = \sum_{i=0}^{\infty}\sum_{j=1}^{q_i} \phi_{i,j}(r) Y_{i,j}(\theta).$$

A computation gives

$$-\Delta\phi = \sum_{i=0}^{\infty}\sum_{j=1}^{q_i} (-\Delta_i)\phi_{i,j}(r) Y_{i,j}(\theta),$$

where $-\Delta_i$ is defined on $L^2(\mathbb{R}^+, f_1^{n-1}dr)$, by

$$-\Delta_i v = -\left(\frac{\partial^2}{\partial r^2} + (n-1)\frac{f_1'(r)}{f_1(r)}\frac{\partial}{\partial r}\right)v + \frac{\lambda_i}{f_1^2}v. \tag{1}$$

Notice that $v(r)$ is a function on M only depending on the radius r. Thus Δ is decomposed into a direct sum of 1D operators Δ_i with multiplicity q_i.

We now renormalize the measure to Lebesgue. Let $U(v) = f_1^{\frac{n-1}{2}} v$ and

$$L_i = U(-\Delta_i)U^{-1}.$$

U is clearly unitary, making $-\Delta_i$ on $L^2((0,\infty), f_1^{n-1}dr)$ unitarily equivalent to operator L_i on $L^2((0,\infty), dr)$. Straightforward calculations give

$$L_i u = -D^2 u + V_i u, \tag{2}$$

where

$$D^2 u = u'', V_i = \frac{(n-1)(n-3)}{4}\left(\frac{f_1'}{f_1}\right)^2 + \frac{n-1}{2}\frac{f_1''}{f_1} + \frac{\lambda_i}{f_1^2}. \tag{3}$$

The formulas (1) and (3) are quite standard. We refer readers to [9] and the references therein for details.

The proof now almost reduces to showing the existence of a singular continuous component for some L_i, which is a 1D problem. However, in order to make the manifold smooth in the neighborhood of O, $f_1^{(2k)}$ must vanish at 0, $k \in \mathbb{Z}$, and one must have $f_1'(0) \neq 0$, where $f_1^{(m)}$ is the mth derivative of f_1 and $f_1^{(0)} = f_1$. This makes $\frac{f_1'}{f_1}(r)$ and $V(r)$ singular at the point $r = 0$, so we need to deal with 1D Schrödinger operator (1) or (2) with singularities at both 0 and ∞.

It is well known that we have

$$K_{\text{rad}}(r) = -\frac{f_1''(r)}{f_1(r)}. \tag{4}$$

Our goal therefore is to construct $f_1(r)$ such that the 1D Schrödinger operator given by (2) has nonempty singular continuous spectrum, and the radial curvature (4) eventually satisfies

$$|K_{\text{rad}}(r) - K_0| \leq \frac{h(r)}{1+r}. \tag{5}$$

Here is the sketch of our construction.

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In the neighborhood of O (i.e., $r = 0$), we use the Euclidean metric. Then the Schrödinger operator (2) is a limit point at the left singular point $r = 0$. For the Euclidean space, the spectral analysis can proceed by the generalized eigen-expansion, which is well known for the Hankel transformation (Bessel-type functions). Our first step is to obtain similar results by the generalized eigen-expansion for $-D+V$ where V is generated by the Euclidean metrics only for small values of r.

For large r (neighborhood of $r = \infty$), we will adapt Kiselev's construction [16], which originally was done for a Schrödinger operator without a singular point at $r = 0$. There are two difficulties here. First, we need to construct f_1 such that the 1D potential given by (3) is what one gets from the Kiselev's construction and the radial curvature given by (4) satisfies (5). It is this step that becomes impossible in the asymptotically flat case without further requirements on the 1D potential. Second, f_1 constructed here for large r should "match" f_1 in the neighborhood of $r = 0$ so that we can use the generalized eigen-expansion to complete the spectral analysis.

This paper is the second in a series started with [13], where we study curvature thresholds for and construct Riemannian manifolds with exotic spectral behavior of the Laplace-Beltrami operator. An important setup for relating rotationally symmetric metrics to 1D potentials while dealing with the singularities was presented in [13], but this only takes us so far because not every potential "at infinity" leads to a metric with the desired properties, and further delicate arguments/constructions are usually needed to make the proof work. Embedded singular continuous spectrum is one of the most delicate spectral phenomena, and the achievement of this paper is both in developing the setup for proving it for Laplace-Beltrami operators, and in making the corresponding construction work for the asymptotically hyperbolic case. The technical difficulties of the latter are well illustrated by the fact that Kiselev's potentials [16] do not in general lead to the desired construction for the asymptotically flat case.

The rest of the paper is organized as follows: In §2, we set up the spectral analysis of Bessel-type potentials. In §3, we give all the remaining technical preparations. In §4, we complete the proof of Theorem 1.2.

2. Spectral analysis of Bessel-type potentials

As mentioned in the introduction, for $r < 1$, we define the metric to be Euclidean, that is, $f_1 = r$. Thus, for $r < 1$, the potential V_i given by (3) is

$$V_i(r) = \left(\frac{(n-1)(n-3)}{4} + \lambda_i \right) \frac{1}{r^2}. \tag{6}$$

By the fact that $\lambda_i \to \infty$, we can choose some i so that

$$\frac{(n-1)(n-3)}{4} + \lambda_i \geq 1. \tag{7}$$

In the following, we fix such λ_i and let

$$\nu^2 = \frac{(n-1)(n-3)}{4} + \lambda_i + \frac{1}{4} \tag{8}$$

so that $\nu > 1$. Now we only consider the operator L_i on $L^2(\mathbb{R}^+, dr)$. We omit the dependence on i for simplicity.

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Thus we have

$$Lu = -D^2u + Vu \tag{9}$$

and by (6)

$$V(r) = \frac{\nu^2 - \frac{1}{4}}{r^2} \text{ for } r < 1. \tag{10}$$

Assume $V \in C^\infty[0,\infty)$ and there is some constant $a \geq 0$ such that

$$V - a \in L^2[1,\infty). \tag{11}$$

Since $-D^2 + V$ is unitarily equivalent to a component of $-\Delta$ on a noncompact manifold, it is nonnegative and 0 is not an eigenvalue.

Assumption (11) will be easily satisfied by our construction. Actually, we will prove $|V(r) - \frac{|K_0|}{4}(n-1)^2| \leq \frac{h(r)}{1+r}$, therefore $a = \frac{|K_0|}{4}(n-1)^2$ works.

In this section, we will set up a generalized eigenfunction expansion for Schrödinger operator (9). L given by (9) is a Bessel differential operator for $r < 1$. V has two singular points: $r = 0$ and $r = \infty$. Since $\nu > 1$ by [27, Theorems X.10], L is in the limit point case at 0, and since $V - a \in L^2[1,\infty)$, by [27, Theorems X.28] L is in the limit point case at ∞. So by Weyl's criterion, L is essentially self-adjoint on $C_0^\infty(0,\infty)$.

Let us consider the eigen-equation

$$Lu = zu \tag{12}$$

with $z \in \mathbb{C}$ and $z \neq 0$. Let $u(r) = \sqrt{r}y(r)$. (12) becomes

$$y''(r) + \frac{y'(r)}{r} + (z - \frac{\nu^2}{r^2})y(r) = 0. \tag{13}$$

Let $\sqrt{z}r = x$. (13) becomes

$$y''(x) + \frac{y'(x)}{x} + (1 - \frac{\nu^2}{x^2})y(x) = 0. \tag{14}$$

(14) is a standard Bessel equation and it has a solution $y(x) = J_\nu(x)$ (see, e.g., Chapter 17 in [36]), where

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(\nu+n+1)} \left(\frac{x}{2}\right)^{2n+\nu}.$$

Thus the Bessel differential equation (12) then has a solution

$$u(r) = \sqrt{r}J_\nu(\sqrt{z}r)$$

for $r < 1$. It is easy to see that $u \in L^2((0,1))$ and since L is in the limit point case at 0, it is unique up to a normalization constant.

Now we extend the solution u to $r \geq 1$ with u still solving (12). For convenience, denote

$$\tilde{J}_\nu(r,z) := u(r). \tag{15}$$

We emphasize that

$$\tilde{J}_\nu(r,z) = \sqrt{r}J_\nu(\sqrt{z}r) \tag{16}$$

for $r < 1$. Thus $\tilde{J}_\nu(z,r)$ is the unique eigen-solution of (12), such that $\tilde{J}_\nu(z,r) \in L^2((0,1))$. Notice that $\tilde{J}_\nu(z,r)$ may not be in $L^2([1,\infty))$.

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Our main result in this section is the following.

Theorem 2.1. *Suppose V satisfies* (10) *and* (11). *Assume $-D^2+V$ is a non-negative operator and 0 is not an eigenvalue. Then there exists a monotone measurable function $\rho(\lambda)$ on $\mathbb{R}^+$ of locally bounded variation on $(0,\infty)$ such that the following statements hold:*

I. *For any $f \in L^2(\mathbb{R}^+, dr)$ there exists a unique $\hat{f} \in L^2(\mathbb{R}^+, d\rho)$ such that*

$$\hat{f}(\lambda) = \int_{\mathbb{R}^+} f(r)\tilde{J}_\nu(r,\lambda)dr.$$

Conversely, for any $g \in L^2(\mathbb{R}^+, d\rho)$, there exists a unique $f \in L^2(\mathbb{R}^+, dr)$ such that $g = \hat{f}$.

II. *For any $f_1, f_2 \in L^2(\mathbb{R}^+, dr)$, we have*

$$\int_{\mathbb{R}^+} f_1 f_2 dr = \int_{\mathbb{R}^+} \hat{f}_1 \hat{f}_2 d\rho.$$

III. *For any $f \in L^2(\mathbb{R}^+, dr)$, let $g = \hat{f}$. Then we have*

$$f(r) = \int_{\mathbb{R}^+} g(\lambda)\tilde{J}_\nu(r,\lambda)d\rho(\lambda).$$

IV. *Define the unitary operator U from $L^2(\mathbb{R}^+, dr)$ to $L^2(\mathbb{R}^+, d\rho)$ by*

$$Uf = \hat{f}$$

which is called the generalized Fourier transform. Then we have $\hat{L} = ULU^{-1}$ is the multiplication operator on $L^2(\mathbb{R}^+, d\rho)$, that is,

$$\mathfrak{D}(\hat{L}) = \{g : g(\lambda), \lambda g(\lambda) \in L^2(\mathbb{R}^+, d\rho)\}$$

and

$$(\hat{L}g)(\lambda) = \lambda g(\lambda)$$

for $g \in \mathfrak{D}(\hat{L})$.

The proof is based on the Titchmarsh expansion techniques in [35]. While they are rather standard, full details are needed to prove Theorem 2.1 in its full strength, so we list them here. We go over the classical Weyl theory first. Suppose differential operator $T = -D^2 + q$ on $L^2(\mathbb{R}^+)$ is in the limit point case on both sides 0 and ∞. Thus T is essentially self-adjoint. We assume $z \in \mathbb{C}^+$. Let $\theta(x,z)$ and $\phi(x,z)$ be the solutions of

$$\begin{cases} -y'' + qy = zy, \\ y(1) = 1, \\ y'(1) = 0, \end{cases} \quad \text{and} \quad \begin{cases} -y'' + qy = zy, \\ y(1) = 0, \\ y'(1) = 1, \end{cases} \tag{17}$$

respectively. Since both 0 and ∞ are limit points, for $\Im z > 0$, there exist unique $M_-(z)$ and $M_+(z)$ so that

$$\psi_1(x,z) = \theta(x,z) + M_-(z)\phi(x,z) \in L^2(0,1] \tag{18}$$

and

$$\psi_2(x,z) = \theta(x,z) + M_+(z)\phi(x,z) \in L^2[1,\infty). \tag{19}$$

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By the Weyl theory [35, Formula 2.18.3], we have

$$\int_0^1 |\varphi(t,z)+M_-(z)\psi(t,z)|^2dt=\frac{\Im M_-(z)}{\Im z} \tag{20}$$

and

$$\int_1^\infty |\varphi(t,z)+M_+(z)\psi(t,z)|^2dt=-\frac{\Im M_+(z)}{\Im z}. \tag{21}$$

Let

$$M_{11}(z) = -\frac{1}{M_-(z)-M_+(z)}, \tag{22}$$

$$M_{12}(z) = M_{21}(z)=-\frac{M_-(z)}{M_-(z)-M_+(z)}, \tag{23}$$

$$M_{22}(z) = -\frac{M_-(z)M_+(z)}{M_-(z)-M_+(z)}. \tag{24}$$

All of M_{jk}, $j,k=1,2$ are Herglotz functions from $\mathbb{C}^+$ to $\mathbb{C}^+$.

Thus we can define monotone functions ρ_{jk}, $j,k=1,2$, (with locally bounded variation on $(-\infty,\infty)$; see p. 58 in [35]) such that

$$\frac{1}{\pi}M_{jk}(z)=\int_{\mathbb{R}}\frac{d\rho_{jk}(x)}{x-z}$$

for $\Im z>0$. Each $\rho_{jk}(x)$ is unique up to a constant. Let $z=x+iy$ with $y>0$. Then (formula 3.5.3 on p. 58 of [35])

$$\rho_{jk}(u_2)-\rho_{jk}(u_1)=\lim_{y\to 0}\frac{1}{\pi}\int_{u_1}^{u_2}\Im(M_{jk}(x+iy))dx. \tag{25}$$

Denote by ρ the matrix with coefficients ρ_{jk}, $j,k=1,2$ and let

$$L^2_\rho=\left\{g=\left(\begin{array}{c} g_1\\ g_2\end{array}\right): ||g||_\rho^2=\int_{\mathbb{R}}\sum_{j,k=1}^2 g_jg_kd\rho_{jk}(\lambda)<\infty\right\}.$$

The inner product on L^2_ρ is given by

$$(g,h)_\rho=\int_{\mathbb{R}}\sum_{j,k=1}^2 g_jh_kd\rho_{jk}(\lambda).$$

Theorem 2.2 ([35, formulas 3.1.8-3.1.11 in Chapter 3])**.** *The following statements hold:*

I. *For any $f\in L^2(\mathbb{R}^+,dx)$ there exists a unique $g=\binom{g_1}{g_2}\in L^2_\rho$ such that*

$$g_1(\lambda)=\int_{\mathbb{R}^+}f(x)\theta(x,\lambda)dx \tag{26}$$

and

$$g_2(\lambda)=\int_{\mathbb{R}^+}f(x)\phi(x,\lambda)dx. \tag{27}$$

Denote $g=\hat{f}$ for simplicity. Conversely, for any $g=\binom{g_1}{g_2}\in L^2_\rho$, there exists a unique $f\in L^2(\mathbb{R}^+,dx)$ such that $g=\hat{f}$.

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II. *For any* $f_1, f_2 \in L^2(\mathbb{R}^+, dx)$*, we have*

$$\int_{\mathbb{R}^+} f_1 f_2 dx = (\hat{f}_1, \hat{f}_2)_\rho.$$

III. *For any* $f \in L^2(\mathbb{R}^+, dx)$*, let* $g = \left(\begin{smallmatrix} g_1 \\ g_2 \end{smallmatrix}\right) = \hat{f}$*. Then we have*

$$\begin{aligned} f(x) \quad = \quad & \int_{\mathbb{R}} \theta(x,\lambda) g_1(\lambda) d\rho_{11}(\lambda) + \theta(x,\lambda) g_2(\lambda) d\rho_{12}(\lambda) \\ & + \int_{\mathbb{R}} \phi(x,\lambda) g_1(\lambda) d\rho_{21}(\lambda) + \phi(x,\lambda) g_2(\lambda) d\rho_{22}(\lambda). \end{aligned} \tag{28}$$

IV. *Define the unitary operator* U *from* $L^2(\mathbb{R}^+, dx)$ *to* L^2_ρ *by*

$$Uf = \hat{f}$$

which is called the generalized Fourier transform. Then we have $\hat{L} = ULU^{-1}$ *is the multiplication operator on* L^2_ρ*, that is,*

$$\mathfrak{D}(\hat{L}) = \{g : g(\lambda), \lambda g(\lambda) \in L^2_\rho\}$$

and

$$(\hat{L}g)(\lambda) = \lambda g(\lambda)$$

for $g \in \mathfrak{D}(\hat{L})$*.*

Theorem 2.3 ([35, formula 3.1.12 in Chapter 3])**.** *Suppose* $\lim_{\Im z \to 0} \Im M_-(z) = 0$*. Then*

$$d\rho_{12}(\lambda) = M_-(\lambda) d\rho_{11}(\lambda), d\rho_{22}(\lambda) = M_-^2(\lambda) d\rho_{11}(\lambda). \tag{29}$$

Moreover, the following statements hold:

I. *For any* $f \in L^2(\mathbb{R}^+, dr)$ *there exists a unique* $\hat{f} \in L^2(\mathbb{R}^+, d\rho_{11})$ *such that*

$$\hat{f}(\lambda) = \int_{\mathbb{R}^+} f(r) \psi_1(r,\lambda) dr.$$

Conversely, for any $g \in L^2(\mathbb{R}, d\rho_{11})$*, there exists a unique* $f \in L^2(\mathbb{R}^+, dr)$ *such that* $g = \hat{f}$*.*

II. *For any* $f_1, f_2 \in L^2(\mathbb{R}^+, dr)$*, we have*

$$\int_{\mathbb{R}^+} f_1 f_2 dr = \int_{\mathbb{R}} \hat{f}_1 \hat{f}_2 d\rho_{11}.$$

III. *For any* $f \in L^2(\mathbb{R}^+, dr)$*, let* $g = \hat{f}$*. Then we have*

$$f(r) = \int_{\mathbb{R}} g(\lambda) \psi_1(\lambda, r) d\rho_{11}(\lambda).$$

IV. *Define the unitary operator* U *from* $L^2(\mathbb{R}^+, dr)$ *to* $L^2(\mathbb{R}, d\rho_{11})$ *by*

$$Uf = \hat{f}$$

which is called the generalized Fourier transformation. Then we have $\hat{L} = ULU^{-1}$ *is the multiplication operator on* $L^2(\mathbb{R}^+, d\rho)$*, that is,*

$$\mathfrak{D}(\hat{L}) = \{g : g(\lambda), \lambda g(\lambda) \in L^2(\mathbb{R}, d\rho_{11})\}$$

and

$$(\hat{L}g)(\lambda) = \lambda g(\lambda)$$

for $g \in \mathfrak{D}(\hat{L})$*.*

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We remark that $\psi_1(r,\lambda)$ is given by (18) and ρ_{11} is given by (25).

Proof of Theorem 2.1. We will use Theorems 2.2 and 2.3 to prove Theorem 2.1. Applying Theorem 2.3 to operator (12), we obtain M_{jk} and ρ_{jk}. By the assumption that $-D^2+V$ is nonnegative and 0 is not an eigenvalue, $d\rho_{jk}(\lambda)$ is supported on $(0,\infty)$, for $j,k=1,2$.

Recall that $\tilde{J}_\nu(z,r)$, $z\in\mathbb{C}$, is the unique solution of (12) in $L^2(0,1]$. Thus one has for $r>0$,

$$\psi_1(r,z)=\theta(r,z)+M_-(z)\phi(r,z)=C\tilde{J}_\nu(r,z). \tag{30}$$

Let $r=1$ in (30); using the boundary condition of θ,ϕ at $r=1$ and $\tilde{J}_\nu(r,z)=\sqrt{r}J_\nu(\sqrt{z}r)$, one has

$$C=\frac{1}{J_\nu(\sqrt{z})}$$

and

$$M_-(z)=\frac{\tilde{J}_\nu'(1,z)}{\tilde{J}_\nu(1,z)}=\frac{1}{2}+\frac{\sqrt{z}}{J_\nu(\sqrt{z})}J_\nu'(\sqrt{z}) \tag{31}$$

for $\Im z>0$. It implies

$$\psi_1(r,z)=\frac{\tilde{J}_\nu(r,z)}{J_\nu(\sqrt{z})}. \tag{32}$$

Thus $M_-(z)$ can be extended to $\mathbb{R}$ except for the zeros of $J_\nu(\sqrt{z})$, that is,

$$M_-(z)=\frac{1}{2}+\frac{\sqrt{z}}{J_\nu(\sqrt{z})}J_\nu'(\sqrt{z}) \tag{33}$$

for $\Im z\geq 0$. Moreover,

$$\Im M_-(\lambda)=0 \tag{34}$$

for $\lambda\geq 0$ and $J_\nu(\sqrt{\lambda})\neq 0$. (34) is true because $J_\nu(x)\in\mathbb{R}$ for $x\in\mathbb{R}$.

Let

$$d\rho(\lambda)=\frac{1}{J_\nu^2(\sqrt{\lambda})}d\rho_{11}(\lambda). \tag{35}$$

By (32) and Theorem 2.3, we obtain Theorem 2.1 except for the local boundedness of variation of ρ on $(0,\infty)$. To prove the latter, fix $[a,b]\subset(0,\infty)$. For any given $\lambda_0\in[a,b]$, ρ is of bounded variation in a neighborhood of λ_0 if $J_\nu(\sqrt{\lambda_0})\neq 0$. Suppose $J_\nu(\sqrt{\lambda_0})=0$. It is easy to see that $\tilde{J}_\nu'(1,\lambda_0)\neq 0$ so that $|\tilde{J}_\nu'(1,\lambda)|>\delta>0$ for $\lambda\in(\lambda_0-\epsilon,\lambda_0+\epsilon)$ with some $\epsilon,\delta>0$.

By (29) and (31), one has

$$d\rho_{22}=\frac{(\tilde{J}_\nu'(1,\lambda))^2}{J_\nu^2(\sqrt{\lambda})}d\rho_{11}.$$

Thus

$$d\rho=\frac{1}{(\tilde{J}_\nu'(1,\lambda))^2}d\rho_{22}.$$

By the fact that ρ_{22} is of bounded variation on $[a.b]$ and $|\tilde{J}_\nu'(1,\lambda)|>\delta>0$ for $\lambda\in(\lambda_0-\epsilon,\lambda_0+\epsilon)$, we have that ρ is of bounded variation on $\lambda\in(\lambda_0-\epsilon,\lambda_0+\epsilon)$. Since there are finitely many zeros of $J_\nu(\sqrt{\lambda})$ in $[a,b]$, this completes the proof. $\square$

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Lemma 2.4. *Under the condition of Theorem* 2.1, *suppose* $\lambda_0 > 0$ *is an eigenvalue. Then* $\rho(\lambda_0) = \|\tilde{J}_\nu(\cdot, \lambda_0)\|^{-2}_{L^2(\mathbb{R}^+)}$.

Proof. Suppose λ_0 is an eigenvalue. Then $f = \tilde{J}_\nu(r, \lambda_0)$ is the corresponding eigenfunction. The Fourier transform $\hat{f}$ of f is well defined, and

$$\hat{f}(\lambda) = \int_{\mathbb{R}^+} f(r)\tilde{J}_\nu(r, \lambda)dr = \int_{\mathbb{R}^+} \tilde{J}_\nu(r, \lambda_0)\tilde{J}_\nu(r, \lambda)dr.$$

It leads to

$$\hat{f}(\lambda_0) = \int_{\mathbb{R}^+} \tilde{J}^2_\nu(r, \lambda_0)dr. \tag{36}$$

By the fact that λ_0 is an eigenvalue and Theorem 2.1, one has

$$\hat{f} = C\chi_{\{\lambda_0\}}(\lambda),$$

where C is a constant and $\chi_{\{\lambda_0\}}(\lambda)$ is the characteristic function of λ_0. It implies

$$||\hat{f}||^2_\rho = |\hat{f}(\lambda_0)|^2\rho(\lambda_0), \tag{37}$$

where $||\hat{f}||^2_\rho = (\int_{\mathbb{R}^+} |\hat{f}(\lambda)|^2 d\rho)^{1/2}$. By II of Theorem 2.1, we have

$$||\hat{f}||^2_\rho = \int_{\mathbb{R}^+} f(r)^2 dr = \int_{\mathbb{R}^+} \tilde{J}^2_\nu(r, \lambda_0)dr. \tag{38}$$

Now the lemma follows from (36), (37), and (38). □

3. Preparations

In this section, we will use Kiselev's construction [16] to prove the following.

Theorem 3.1. *Fix any* $\tau \geq 0$, $0 < \delta < 1/2$, $\nu > 1$, *and positive function* $h(r)$ *such that* $\lim_{r\to\infty} h(r) = \infty$. *Suppose* $\tilde{V}(r) \in C^\infty(0, b)$ *with* $b \geq 10$ *and for* $r \leq 1$, $\tilde{V}(r) = \frac{\nu^2 - \frac{1}{4}}{r^2}$. *Then there exists a potential* $V(r)$ *on* $(0, \infty)$ *satisfying the following statements:*

I. $V(r) = \tilde{V}(r)$ *for* $0 < r \leq b - \delta$.
II. $V(r) \in C^\infty(0, \infty)$ *and* $|V(r) - \tau^2| \leq \frac{h(r)}{1+r}$ *for* $r \geq b + \delta$.
III. $\sup_{b-\delta\leq r\leq b+\delta} |V(r)| \leq \tau^2 + 1 + \sup_{b-1\leq r\leq b} |\tilde{V}(r)|$.
IV. $-D^2 + V$ *has singular continuous spectrum on* $[\tau^2, \infty)$.

Let u be a solution of $-D^2 + \frac{\nu^2 - \frac{1}{4}}{r^2}$, $r < 1$, such that

$$u \in L^2(0, 1).$$

Recall that (by (15) and (16)) for $r < 1$,

$$u(r, \lambda) = \sqrt{r}J_\nu(\sqrt{\lambda}r).$$

Set $V(r) = \frac{\nu^2 - \frac{1}{4}}{r^2}$ for $r < 1$. Suppose we construct potentials $V(r)$ on $(0, x]$. We extend u to $(0, x]$ by solving

$$(-D^2 + V)u(r) = \lambda u(r) \tag{39}$$

for $0 \leq r \leq x$.

Let $\lambda = k^2$. For any $k > \tau$, let

$$\bar{k} = \sqrt{k^2 - \tau^2}.$$

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Rewrite equation (39) as

$$(-D^2 + V - \tau^2)u(r) = (\lambda - \tau^2)u(r) = \bar{k}^2 u(r). \tag{40}$$

It will be convenient to introduce the Prüfer variables R and θ, for equation (40), $R^2 = (u')^2 + \bar{k}^2 u^2$ and $\theta = \tan^{-1}(\bar{k}u/u')$ for $r \geq 1$. It is then a standard calculation that for $r \geq 1$,

$$(\log R(r, \bar{k})^2)' = \frac{1}{\bar{k}}(V(r) - \tau^2) \sin 2\theta(r, \bar{k}), \tag{41}$$

$$\theta(r, \bar{k})' = \bar{k} - \frac{1}{\bar{k}}(V(r) - \tau^2)(\sin \theta)^2. \tag{42}$$

Proof of Theorem 3.1. The proof of Theorem 3.1 closely follows the construction of [16, Theorem 1.1], so we skip the details. We point out several small modifications.

- Replace Lemma 2.1 in [16] with Lemma 2.4. Replace the Prüfer variables (2.2) and (2.3) in [16] with (41) and (42).
- I and III follow from Theorem 1.1 in [16].
- The potential constructed in [16] is not smooth. This issue can be addressed in the following way. In [16], Kiselev constructed the potential V piece by piece. We need to smooth the potential for the current piece first and then construct the next piece. II comes from the fact that we smooth the potential around $r = b$. □

Without loss of generality, assume $h(r)$ is positive, nondecreasing, $\lim_{r\to\infty} h(r) = \infty$ and

$$h(r) \leq 1 + r^{1/10}.$$

In the following b is a large positive constant, and δ is a small positive constant. We will need the following lemma.

Lemma 3.2 (Comparison theorem). *Suppose $f(r) \geq g(r)$ for $2 \leq r \leq r_0$. Let us consider two differential equations for $r \geq r_0$,*

$$f' + m(r)f^2(r) + \frac{Ae^{(n-1)r}}{\exp(\int_2^r 2(1 + f(x)e^{-(n-1)x})dx)} = h_1(r) \tag{43}$$

and

$$g' + m(r)g^2(r) + \frac{Ae^{(n-1)r}}{\exp(\int_2^r 2(1 + g(x)e^{-(n-1)x})dx)} = h_2(r), \tag{44}$$

where A is a nonnegative constant and $m(r) \geq 0$. Suppose $f(r_0) \geq g(r_0)$ and $h_1(r) \geq h_2(r)$ for all $r > r_0$. Then $f(r) \geq g(r)$ for all possible $r \geq r_0$.

Proof. Suppose $f(r) \geq g(r)$ for $r_0 < r < r_1$ and $f(r_1) = g(r_1)$. Since $f(r) \geq g(r)$ for $2 < r \leq r_0$, one has

$$\frac{Ae^{(n-1)r_1}}{\exp(\int_2^{r_1} 2(1 + f(x)e^{-(n-1)x})dx)} \leq \frac{Ae^{(n-1)r_1}}{\exp(\int_2^{r_1} 2(1 + g(x)e^{-(n-1)x})dx)}.$$

By (43), (44), and $h_1(r_1) \geq h_2(r_1)$, we have

$$f'(r_1) \geq g'(r_1).$$

It implies the lemma. □

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4. Proof of Theorem 1.2

We plan to use a rotationally symmetric metric to complete our construction. Our objective is to construct proper $f_1(r)$ so that Riemannian manifold

$$(M_n, g) = \left(\mathbb{R}^+ \times S^{n-1}(1) \cup \{O\}, dr^2 + f_1^2(r) g_{S^{n-1}(1)}\right)$$

satisfies Theorem 1.2. In the neighborhood of the origin, we will use the Euclidean metric. For $r \geq 2$. we will construct $f(r)$ so that

$$f_1(r) = \exp\left(\int_2^r (\sqrt{|K_0|} + f(x))dx\right) \tag{45}$$

will have the desired properties.

Define

$$K_{\text{rad}}(r) := -\frac{f_1''(r)}{f_1(r)}, \tag{46}$$

$$q(r) := \frac{(n-1)(n-3)}{4}\left(\frac{f_1'(r)}{f_1(r)}\right)^2 - \frac{(n-1)}{2}K(r) + \frac{\lambda_i}{f_1^2}. \tag{47}$$

Direct computation yields that for $r \geq 2$

$$K_{\text{rad}}(r) = -|K_0| - 2\sqrt{|K_0|}f(r) - f^2(r) - f'(r), \tag{48}$$

$$q(r) = \frac{(n-1)^2}{4}(\sqrt{|K_0|} + f(r))^2 + \frac{n-1}{2}f'(r) + \frac{\lambda_i}{f_1^2}. \tag{49}$$

Let

$$\tau = \frac{(n-1)}{2}\sqrt{|K_0|}. \tag{50}$$

In order to prove Theorem 1.2, we need to show that there exists f such that

$$q(r) = V(r) \tag{51}$$

and $K_{\text{rad}}(r)$ given by (48) satisfies our goal, where $V(r)$ is given by Theorem 3.1.

Without loss of generality, we assume $K_0 = -1$.

Proof of Theorem 1.2. For $r \leq 1$, let

$$\tilde{f}_1(r) = r. \tag{52}$$

For $r \in [2, b]$, let

$$\tilde{f}_1(r) = e^{2(r-2)}. \tag{53}$$

We extend $\tilde{f}_1(r)$ to $(0, b)$ so that $\tilde{f}_1(r) > 0$ and $\tilde{f}_1 \in C^\infty(0, b]$. Let $\tilde{V}(r)$ be given by (47) for $r \leq b$, namely,

$$\tilde{V}(r) = \frac{(n-1)(n-3)}{4}\left(\frac{\tilde{f}_1'}{\tilde{f}_1}\right)^2 + \frac{(n-1)}{2}\frac{\tilde{f}_1''}{\tilde{f}_1} + \frac{\lambda_i}{\tilde{f}_1^2}.$$

In particular, for $0 < r \leq 1$,

$$\tilde{V}(r) = \left(\frac{(n-1)(n-3)}{4} + \lambda_i\right)\frac{1}{r^2} = \frac{\nu^2 - \frac{1}{4}}{r^2}.$$

By Theorem 3.1, we obtain a potential $V(r)$. Now we are ready to define our metric. For $r \leq b - \delta$, let

$$f_1(r) = \tilde{f}_1(r).$$

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For $2 \leq r \leq b-\delta$, let $f(r)=1$ so that for $2 \leq r \leq b-\delta$,

$$f_1(r) = \exp(\int_2^r (1+f(x))dx).$$

Let us consider the following equation ($n \geq 2$):

$$\tau^2 + \frac{(n-1)^2}{2} f + \frac{(n-1)^2}{4} f^2 + \frac{n-1}{2} f' + \frac{\lambda_i}{\exp(\int_2^r 2(1+f(x))dx)} = V(r). \tag{54}$$

Since V is defined on $(0,\infty)$ and $f \equiv 1$ on $(2, b-\delta)$, let $f(r)$ solve (54) with initial condition $f(b-\delta)=1$. By choosing δ sufficiently small and III of Theorem 3.1, there is a unique solution $f(r)$ for $b-\delta \leq r \leq b+\delta$ such that $1/2 < f(r) < 2$ for $b-\delta \leq r \leq b+\delta$. Let $f(r)$ solve the equation (54) for $r \geq b+\delta$. By the implicit function theorem, there exists $\bar{\delta} > 0$ such that the solution $f(r)$ is well defined for $r \leq b+\delta+\bar{\delta}$. We claim (see Lemma 4.1 below) that there exists a unique solution $f(r)$ for all $r \geq b+\delta$ and, moreover,

$$|f(r)| + |f'(r)| = O\left(\frac{h(r)}{1+r}\right). \tag{55}$$

For $r > 2$, define

$$f_1(r) = \exp(\int_2^r (1+f(x))dx).$$

By our construction, for $r > 0$,

$$V(r) = \frac{(n-1)(n-3)}{4}\left(\frac{f_1'}{f_1}\right)^2 + \frac{n-1}{2}\frac{f_1''}{f_1} + \frac{\lambda_i}{f_1^2}. \tag{56}$$

By (48) and (55), one has the desired estimate on all terms in the definition of $K_{\text{rad}}(r)$, so we have

$$|K_{\text{rad}}(r) + 1| = O\left(\frac{h(r)}{1+r}\right).$$

By Theorem 3.1, $-D^2 + V$ has nonempty singular continuous spectrum. By (3) and (56), we have that $-\Delta$ also has nonempty singular continuous spectrum. It now remains to prove part (2) of Theorem 1.2. By Theorem 1.2 in [19] to prove

$$\sigma_{\text{ess}}(-\Delta) = \left[\frac{(n-1)^2}{4}, \infty\right),$$

it is enough to show $\lim_{r\to\infty} \Delta r = n-1$. Indeed, by (45), one has

$$\lim_{r\to\infty} \Delta r = \lim_{r\to\infty} (n-1)\frac{f_1'(r)}{f_1(r)} = (n-1) + \lim_{r\to\infty}(n-1)f(r) = n-1.$$

Thus

$$\sigma_{\text{ac}}(-\Delta) \subset \left[\frac{(n-1)^2}{4}, \infty\right).$$

By the fact that for $r > b+\delta$

$$\left|V(r) - \frac{(n-1)^2}{4}\right| \leq \frac{h(r)}{1+r},$$

one has (e.g., [2, 15])

$$\left[\frac{(n-1)^2}{4}, \infty\right) \subset \sigma_{\text{ac}}(-\Delta).$$

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Thus

$$\sigma_{\mathrm{ac}}(-\Delta) = \left[\frac{(n-1)^2}{4}, \infty\right)$$

which completes the proof. □

Lemma 4.1. *Suppose for* $2 \leq r \leq b-\delta$, $f(r) = 1$ *and for* $b-\delta \leq r \leq b+\delta$, $\frac{1}{2} \leq f(r) \leq 2$. *Suppose* $|V(r) - \tau^2| \leq \frac{h(r)}{1+r}$ *for* $r \geq b+\delta$. *Then there exists a unique solution* $f(r)$ *for all* $r \geq b+\delta$ *satisfying*

$$\tau^2 + \frac{(n-1)^2}{2} f + \frac{(n-1)^2}{4} f^2 + \frac{n-1}{2} f' + \frac{\lambda_i}{\exp(\int_2^r 2(1+f(x))dx)} = V(r), \tag{57}$$

and, moreover,

$$|f(r)| = O\left(\frac{h(r)}{1+r}\right), \tag{58}$$

and

$$|f'(r)| = O\left(\frac{h(r)}{1+r}\right). \tag{59}$$

Proof. Let $r_0 = b+\delta$. Let

$$f(r) = t(r) e^{-(n-1)r}, \tag{60}$$

and $t(r_0) = f(r_0)e^{(n-1)r_0} = t_0$. Then for $r \geq r_0$, $t(r)$ satisfies equation

$$t' + \frac{n-1}{2} t^2 e^{-(n-1)r} + \frac{2\lambda_i}{n-1} \frac{e^{(n-1)r}}{\exp(\int_2^r 2(1+t(x)e^{-(n-1)x})dx)} = g(r), \tag{61}$$

where $g(r) = \frac{2}{n-1}(V(r) - \tau^2)e^{(n-1)r}$. By the assumption, one has

$$|g(r)| \leq \frac{2}{n-1} \frac{h(r)}{1+r} e^{(n-1)r}. \tag{62}$$

By a simple computation, one has for large r_0,

$$\begin{aligned} \int_{r_0}^r \frac{h(x)}{1+x} e^{(n-1)x} dx &\leq \frac{h(r)}{1+r} e^{(n-1)r} \int_{r_0}^r \frac{1+r}{1+x} e^{(n-1)(x-r)} dx \\ &\leq \frac{3h(r)}{1+r} e^{(n-1)r}. \end{aligned} \tag{63}$$

By (61), one has

$$t(r) \leq t_0 + \int_{r_0}^r |g(x)| dx \leq t_0 + \frac{6h(r)}{1+r} e^{(n-1)r}. \tag{64}$$

Now we will use Lemma 3.2 to get the lower bound of $t(r)$. Let

$$\hat{t} = -10 \int_{r_0}^r \frac{h(x)}{1+x} e^{(n-1)x} dx.$$

Let $\hat{t}(r) = 0$ for $2 < r < r_0$. Direct computation yields that

$$\hat{t}' + \frac{n-1}{2} \hat{t}^2 e^{-(n-1)r} + \frac{2\lambda_i}{n-1} \frac{e^{(n-1)r}}{\exp(\int_2^r 2(1+\hat{t}(x)e^{-(n-1)x})dx)} = \hat{g}(r), \tag{65}$$

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where

$$\begin{aligned}\hat{g}(r) &= -10\frac{h(r)}{1+r}e^{(n-1)r} + O(1)\frac{e^{(n-1)r}}{\exp(\int_2^r 2(1+\hat{t}(x)e^{-(n-1)x})dx)} \\ &\quad -O(1)e^{-(n-1)r}\left(\int_{r_0}^r \frac{h(x)}{1+x}e^{(n-1)x}dx\right)^2 \\ &\leq -10\frac{h(r)}{1+r}e^{(n-1)r} + O(1)\frac{e^{(n-1)r}}{e^{2r}} - O(1)e^{(n-1)r}\frac{h^2(r)}{(1+r)^2} \\ &\leq -10\frac{h(r)}{1+r}e^{(n-1)r} + O(1)e^{(n-3)r} - O(1)e^{(n-1)r}\frac{h^2(r)}{(1+r)^2},\end{aligned}$$

where the first inequality holds by (63).

By (62) and choosing large r_0, one has $\hat{g}(r) \leq g(r)$ for $r \geq r_0$. By Lemma 3.2 and (61), one has

$$t(r) \geq \hat{t}(r) \geq -10\frac{h(r)}{1+r}e^{(n-1)r}. \tag{66}$$

By (64) and (66), we obtain that

$$|t(r)| \leq t(r_0) + 10\frac{h(r)}{1+r}e^{(n-1)r}. \tag{67}$$

It implies (58). (59) follows from (57) and (58). □

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