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ANDERSON LOCALIZATION FOR MULTI-FREQUENCY QUASI-PERIODIC OPERATORS ON $\mathbb{Z}^D$

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Abstract. We establish Anderson localization for general analytic k-frequency quasi-periodic operators on $\mathbb{Z}^d$ for *arbitrary* k, d.

1 Introduction and Main Results

The theory of quasiperiodic opetarors with analytic potentials has seen dramatic advances in the last 20 years, since the development of first non-perturbative methods for control of the Green's functions [Jit94, Jit99, BJ02, Bou05, BG20] that replaced earlier perturbation of eigenfunctions techniques. The most well-developed and remarkably rich theory concerns the case of one-dimensional one-frequency potentials, where powerful reducibility/dynamical techniques are particularly enhanced by the analyticity arguments. There are now non-perturbative results on both small and high coupling sides ([Bou05, MJ17, You18] and references therein), global theory [Avi15], and sharp arithmetic transitions and related universality (e.g. [AYZ17, JL18, JL18, JZ15]). However, if one increases either the dimension of the undelying torus (the number of frequencies) or, especially, the space dimension, the situation becomes significantly more complicated. First, non-perturbative results can be false [Bou05], so throwing away small measure sets of parameters where things actually do sometimes go bad, becomes a necessity. Even more importantly, one-dimensional (and therefore dynamical) techniques are not applicable in higher space dimension. The first multi-dimensional localization was obtained by perturbative (KAM) methods (with small measure set removal) by Chulaevsky-Dinaburg for long-range k-frequency operators on $\ell^2(\mathbb{Z}^d)$ with cos-type potential, for $k = 1$ and arbitrary d [CD93]. An a.e. (but still perturbative) result for long-range operators with cos potential, Diophantine frequency, $k = 1$ and any d was recently obtained as an application of general L^2 Aubry duality developed in [JK16] (see also [GYZ19] for a further enhancement). However, those results required cos-type potential, and moreover, neither perturbative nor Aubry duality techniques have been made to work to prove localization in the multi-frequency case, $k > 1$, even

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for $d = 1$. Bourgain-Goldstein-Schlag developed a way to apply some of the non-perturbative methods to the two dimensional case [BGS02], obtaining localization at high coupling for $k = d = 2$. This was extended by Bourgain to arbitrary $k = d$ [Bou07], where he developed a new powerful scheme that allowed to circumvent the arithmetic difficulties that restricted [BGS02] to $k = d = 2$. In this paper we extend Bourgain's result to the case of general k, d (in fact, an even significantly more general situation).

Let S be a Toeplitz (operator) matrix on $\ell^2(\mathbb{Z}^d)$ satisfying,

$$|S(n, n')| \leq e^{-\rho|n-n'|}, \quad \rho > 0, \tag{1.1}$$

where $|n| := \max_{1 \leq i \leq d} |n_i|$ for $n = (n_1, n_2, \dots, n_d) \in \mathbb{Z}^d$.

Let v be a real analytic function on $\mathbb{T}^b$, where $b = \sum_{i=1}^{d} b_i$ ($b_i \in \mathbb{N}$ for $1 \leq i \leq d$).

In this paper, we consider the following operators

$$H(x) = S + \lambda v(x + n\omega)\delta_{nn'}, \quad n, n' \in \mathbb{Z}^d, \tag{1.2}$$

where

$$x = (x_{11}, \dots, x_{b_1 1}, \dots, x_{1d}, \dots, x_{b_d d}) \in \mathbb{T}^b,$$
$$n\omega = (n_1\omega_{11}, \dots, n_1\omega_{b_1 1}, \dots, n_d\omega_{1d}, \dots, n_d\omega_{b_d d}).$$

Example 0. Taking $b_i = 1, i = 1, \dots, d$ and the nearest neighbor Laplacian S we obtain operators considered in [Bou07].

Example 1. $d = 2$, $b_1 = 2$, $b_2 = 1$. v is a function on $\mathbb{T}^3$. For $x = (x_1, x_2, x_3)$ and $\omega = (\omega_1, \omega_2, \omega_3)$, the operator becomes

$$H = S + \lambda v(x_1 + n_1\omega_1, x_2 + n_1\omega_2, x_3 + n_2\omega_3)\delta_{nn'}, \tag{1.3}$$

where $n = (n_1, n_2)$.

Example 2. $b = kd, b_i = k, i = 1, \dots, d,$ f is a function on $\mathbb{T}^k$, and

$$v(x_{11}, \dots, x_{k1}, \dots, x_{1d}, \dots, x_{kd}) = f(x_{11} + \cdots + x_{1d}, \cdots, x_{k1} + \cdots + x_{kd}).$$

Then the operator becomes

$$H(x) = S + \lambda f(x + nA)\delta_{nn'}, \tag{1.4}$$

where $x \in \mathbb{T}^k$, $n \in \mathbb{Z}^d$, and A is a d by k matrix of frequencies. This is the most general form of a d-dimensional quasiperiodic operator with a k-dimensional phase space. The Aubry dual family has the form

$$\tilde{H}(x) = F + \lambda s(x + An)\delta_{nn'}, \tag{1.5}$$

where $x \in \mathbb{T}^d$, $n \in \mathbb{Z}^k$, and F, S are Toeplitz operators with (n, n') terms given by the $n - n'$ Fourier coefficients of, correspondingly f, s. The standard Laplacian is therefore dual to the potential given by the sum of cosines, and the dual of a general analytic potential is a Toeplitz matrix as above.

Remark 1. When considering families (1.4) with A restricted to a linear submanifold of d by k matrices of frequencies, one needs to take b_i equal to the number of free variables in the ith row of A and adjust v accordingly. As such, the family considered in [Bou07] can of course also be recast in this language: it corresponds to A restricted to $diag(\omega_1, \dots, \omega_d)$.

We call $x \in \mathbb{T}^b$ the phase, $\omega \in \mathbb{T}^b$ the frequency and $\lambda \geq 0$ the coupling. Let

$$x^j := (x_{1j}, \dots, x_{b_j j}) \in \mathbb{T}^{b_j} \ (1 \leq j \leq d).$$

We assume v satisfies the following *non-degeneracy* condition: for any

$$(x^1, \dots, x^{j-1}, x^{j+1} \dots, x^d) \in \mathbb{T}^{b-b_j},$$

the function

$$\mathbb{T}^{b_j} \ni \theta \mapsto v(x^1, \dots, x^{j-1}, \theta, x^{j+1} \dots, x^d)$$

is nonconstant.

Denote by mes the Lebesgue measure. We say operator H satisfies Anderson localization if it has only pure point spectrum with exponentially decaying eigenfunctions.

Theorem 1.1. *Let $H(x)$ be given by (1.2) with v satisfying the non-degeneracy condition. Then for any $\delta > 0$, there is a $\lambda_0 = \lambda_0(\delta, v, \rho, b, d) > 0$ such that the following statement holds: for any $\lambda \geq \lambda_0$ and any $x \in \mathbb{T}^b$, there exists $\Omega = \Omega(x, \lambda v, \delta, \rho, b, d) \subset \mathbb{T}^b$ with $\mathrm{mes}(\mathbb{T}^b \setminus \Omega) \leq \delta$ such that for $\omega \in \Omega$, $H(x)$ satisfies Anderson localization.*

Remarks. (1) In particular, this holds for all operators (1.4) with arbitrary k, d and any non-constant analytic function f,[1] which is an important building block in the proof of absolutely continuous spectrum for operators (1.5) [BJP] and was the key initial motivation for our work.

(2) We note that our phase space dimension b satisfies $b \geq d$ since $b = \sum_{i=1}^d b_i$, $b_i \geq 1$. This is essential for our arguments. As shown in Example 2, general quasiperiodic operators always have $b \geq d$. However, operators (1.2) with $b < d$, for example $V_{n_1,n_2} = v(x + n_1\omega, x + n_2\omega)$, also appear naturally, e.g. in the study of interacting particles, and our proof does not apply in this setting. A localization result for a model with $b = 1, d = 2$ was recently obtained by Bourgain-Kachkovskiy [BK19].

[1] As in Remark 1, the non-degenracy condition on v leads to additional non-degeneracy conditions on f if the number of free variables in a certain row of the submanifold is bounded by 1. In particular, for A restricted to $diag(\omega_1, \dots, \omega_d)$, as in [Bou07], the required non-degeneracy condition is exactly as in [Bou07].

(3) Previous multidimensional/multifrequency localization results [BGS02, Bou07] were not only restricted to $k = d$, but also done only for the nearest neighbor Laplacian, i.e. $S(n, n') = \delta_{|n-n'|,1}$. The extension to general S as in (1.1) is motivated by the Aubry duality purposes in [BJP]. Treating general S instead of the standard Laplacian only adds small technical difficulties. Localization for long-range operators (general S) was previously obtained for $k = 1$ in [CD93] and, nonperturbatively, for $k = d = 1$ in [Bou05, BJ02].

The main scheme of our proof is definitely adapted from Bourgain [Bou07]. However, while our result is significantly more general and more technically complex, our argument can also be viewed as both a clarification and at the same time **streamlining** of [Bou07]. Indeed, our proof, while including more detail and hopefully increasing the readability, is only shorter than the corresponding part of Bourgain's. This is due to several important technical improvements that we add to Bourgain's scheme. One important highlight is that, in the process of deterministic multi-scale analysis proceeding from scale N_1 to N_2, a chain of scales between N_1 and N_2 has always been used in the past work, [Bou07, BGS02]. Here, instead of gluing "good" Green's function at multiple scales between N_1 and N_2 to establish the "goodness" of Green's function at scale N_2, we find a way to directly use the "good" Green's function at scale N_1 + subexponential bound of the norm to prove the "goodness" of Green's function at scale N_2.

Another issue we want to highlight is that the $k = d = 2$ analysis of [BGS02] required dealing with many different types of elementary regions, something that would be prohibitively difficult to carry out in higher dimensions. In dealing with higher dimensions in [Bou07] Bourgain significantly reduces the allowed elementary regions. This comes at the price of some complications in dealing with the lattice points at the boundary of the elementary regions, which Bourgain claims can be carried out, but provides no detail. We use the same (slightly corrected) type of restriction on the elementary regions but believe this issue is not entirely trivial and tackling it requires a certain modification of the procedure, which we provide in full detail.

Non-perturbative proofs of localization for $d > 1$ are in a sense a version of deterministic multi-scale analysis. The latter is a powerful method originally developed for random operators by Fröhlich and Spencer [FS83], that crucially relied on independence and Wegner's Lemma that is effectively dependent on rank-one perturbations. For the deterministic version, difficulties with lack of independence/rank one perturbations are circumvented by the semi-algebraic sets considerations and subharmonicity arguments [Bou05]. The non-perturbative proofs consist of two parts. First, one needs to obtain measure and complexity estimates for phases/frequencies with exponential off-diagonal decay and subexponential upper bounds for the matrix elements of the Green's function for box-restricted operators for a given energy. From this, localization follows through elimination of energy via an argument involving complexity bounds on semi-algebraic sets. The second part is by now rather standard and follows the reasonably short argument in [Bou07] essentially verbatim. In fact, it is the first part that presents the main difficulty associated with higher dimensions. Thus we focus only on the first, single energy, part here. This is also where the key difficulty in extending [BGS02] and the key difference between [Bou07] and [BGS02] lies. One needs to guarantee a sublinear upper bound on the number of times the ergodic trajectory hits certain forbidden regions of given measure/algebraic complexity, without further detail on the structure of those fordbidden regions. A key argument in [BGS02] is a Lemma that does guarantee it for $k = b = 2$ under an explicit arithmetic condition on the frequencies. Roughly, it means that too many points on the trajectory of rotation close to an algebraic curve of a bounded degree would force it to oscillate more than the degree allows. However, this statement is not extendable to $d \geq 3$. In [Bou07] Bourgain instead developed a way to restrict to suitable frequencies already for the first step, which turned out to be a very robust approach that we also develop here. Besides the elimination of energy argument, we do not include detailed proofs of two further statements very similar to those in [Bou07], and with proofs presented there in a very clear way. The proofs that are similar to Bourgain's that we do present either have certain novelty or contain important technical clarifications.

In Section 2 we introduce the main concepts and also list the above mentioned results for which we do not present detailed proofs. One such concept is "property P at scale N"—essentially, the single energy statement one wants to establish for all large scales, that allows to streamline certain formulations. Section 3 is devoted to the main multi-scale argument: property P at scales N, N^c implies property P at an interval of subexponentially large scales, Theorem 3.7. In Section 4 we take care of the initial scale and give a very short argument to obtain the final single energy estimate, Theorem 4.1, from Theorem 3.7. In the appendix we prove a several variables matrix-valued Cartan estimate (Lemma 3.5 used in the proof of Theorem 3.7), that follows Bourgain's one-variable argument in [Bou05] but also uses high-dimensional Cartan sets estimates of [GS08].

2 Preparations

2.1 Notation. For any $x \in \mathbb{R}^{d_1}$ and $X \subset \mathbb{R}^{d_1+d_2}$, denote the x-section of X by

$$X(x) := \{y \in \mathbb{R}^{d_2} : (x, y) \in X\}.$$

Let $\tilde{b} = \max_i b_i$. For any $x \in \mathbb{T}^b$ and $1 \leq j \leq d$, let $x_j^{\neg} = (x^1, \dots, x^{j-1}, x^{j+1} \dots, x^d) \in \mathbb{T}^{b-b_j}$. For $x = (x_1, x_2, \dots, x_l), y = (y_1, y_2, \dots, y_l) \in \mathbb{R}^l$, let $|x - y| = \max_i |x_i - y_i|$.

For $\Lambda_1, \Lambda \subset \mathbb{Z}^d$, we introduce

$$\operatorname{diam}(\Lambda) = \sup_{n,n' \in \Lambda} |n - n'|, \ \operatorname{dist}(m, \Lambda) = \inf_{n \in \Lambda} |m - n| \ (m \in \mathbb{R}^d),$$

and $\operatorname{dist}(\Lambda_1, \Lambda) = \inf_{n \in \Lambda_1} \operatorname{dist}(n, \Lambda)$.

We also use $\|\cdot\|$ as ℓ^2 norm of the matrix. For convenience, in the following, we study operator $\lambda^{-1}H(x)$. We always assume $\lambda > 1$. Since the spectra of $\lambda^{-1}H(x)$ are bounded by $C(S, v)$, we can further assume E is bounded.

2.2 Green's functions and elementary regions. For $\Lambda \subset \mathbb{Z}^d$, let R_Λ be the restriction operator, i.e., $(R_\Lambda\xi)(n) = \xi(n)$ for $n \in \Lambda$, and $(R_\Lambda\xi)(n) = 0$ for $n \notin \Lambda$. Denote by $H_\Lambda = R_\Lambda H R_\Lambda$ and the Green's functions

$$G_\Lambda(E; x) = (R_\Lambda(\lambda^{-1}H - E + i0)R_\Lambda)^{-1}.$$

We will also write G_Λ when there is no ambiguity. Clearly,

$$G_{n+\Lambda}(x) = G_\Lambda(x + n\omega). \tag{2.1}$$

We denote by Q_N an elementary region of size N centered at 0, which is one of the following regions,

$$Q_N = [-N, N]^d$$

or

$$Q_N = [-N, N]^d \setminus \{n \in \mathbb{Z}^d : \ n_i \in \varsigma_i, \quad 1 \le i \le d\},$$

where for $i = 1, 2, \ldots, d$, $\varsigma_i \in \{\{n < 0\}, \{n > 0\}, \emptyset\}$ and at least two ς_i are not $\emptyset$.

Denote by $\mathcal{E}_N^0$ the set of all elementary regions of size N centered at 0. Let $\mathcal{E}_N$ be the set of all translates of elementary regions, namely,

$$\mathcal{E}_N := \{n + Q_N\}_{n \in \mathbb{Z}^d, Q_N \in \mathcal{E}_N^0}.$$

2.3 Semi-algebraic sets.

Definition 2.1 (Chapter 9, [Bou05]). *A set $\mathcal{S} \subset \mathbb{R}^n$ is called a semi-algebraic set if it is a finite union of sets defined by a finite number of polynomial equalities and inequalities. More precisely, let $\{P_1, \ldots, P_s\} \subset \mathbb{R}[x_1, \ldots, x_n]$ be a family of real polynomials whose degrees are bounded by d. A (closed) semi-algebraic set $\mathcal{S}$ is given by an expression*

$$\mathcal{S} = \bigcup_j \bigcap_{\ell \in \mathcal{L}_j} \{x \in \mathbb{R}^n : \ P_\ell(x) \varsigma_{j\ell} 0\}, \tag{2.2}$$

where $\mathcal{L}_j \subset \{1, \ldots, s\}$ and $\varsigma_{j\ell} \in \{\ge, \le, =\}$. Then we say that $\mathcal{S}$ has degree at most sd. In fact, the degree of $\mathcal{S}$ which is denoted by $\deg(\mathcal{S})$, means the smallest sd over all representations as in (2.2).

In [Bou07], Bourgain proved a result for eliminating several variables.

Lemma 2.2 (Lemma 1.18, [Bou07]). *Let $\mathcal{S} \subset [0,1]^{d+r}$ be a semi-algebraic set of degree B and such that*

$$\mathrm{mes}(\mathcal{S}(y)) < \eta \text{ for } \forall\ y \in [0,1]^r.$$

Then the set

$$\left\{ (x_1, \ldots, x_{2^r}) \in [0,1]^{d2^r} : \bigcap_{1\le i\le 2^r} \mathcal{S}(x_i) \neq \emptyset \right\}$$

is semi-algebraic of degree at most B^C and measure at most

$$B^C \eta^{d^{-r}2^{-r(r-1)/2}},$$

where $C = C(d,r) > 0$.

Another important fact is the following decomposition Lemma for semi-algebraic sets in the product spaces.

Lemma 2.3 ([Bou07, Bou05]). *Let $\mathcal{S} \subset [0,1]^{d=d_1+d_2}$ be a semi-algebraic set of degree $\deg(\mathcal{S}) = B$ and $\mathrm{mes}_d(\mathcal{S}) \le \eta$, where*

$$\log B \ll \log \frac{1}{\eta}, \tag{2.3}$$

with

$$\eta^{\frac{1}{d}} \le \epsilon.$$

Then there is a decomposition of $\mathcal{S}$ as

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$$

such that the projection of $\mathcal{S}_1$ on $[0,1]^{d_1}$ has small measure

$$\mathrm{mes}_{d_1}(\mathrm{Proj}_{[0,1]^{d_1}} \mathcal{S}_1) \le B^{C(d)}\epsilon,$$

and $\mathcal{S}_2$ has the transversality property

$$\mathrm{mes}_{d_2}(\mathcal{L} \cap \mathcal{S}_2) \le B^{C(d)}\epsilon^{-1}\eta^{\frac{1}{d}},$$

where $\mathcal{L}$ is any d_2-dimensional hyperplane in $[0,1]^d$ s.t.,

$$\max_{1\le j\le d_1} |\mathrm{Proj}_{\mathcal{L}}(e_j)| < \epsilon,$$

where we denote by $e_1, \ldots, e_{d_1}$ the coordinate vectors in $\mathbb{R}^{d_1}$.

We then have

Lemma 2.4. *Suppose that* $\omega^i \in \mathbb{R}^{l_i}$ $(i = 1, 2, 3, \ldots, r)$ *and* $l = \sum_{i=1}^r l_i$. *Let* $\mathcal{S} \subset [0,1]^{lJ}$ *be a semi-algebraic set of degree* B *and such that*

$$\mathrm{mes}(\mathcal{S}) < \eta.$$

For $\omega = (\omega^1, \ldots, \omega^r) \in [0,1]^l$ *and* $n = (n_1, n_2, \ldots, n_r) \in \mathbb{Z}^r$, *define*

$$n\omega = (n_1\omega^1, n_2\omega^2, \ldots, n_r\omega^r).$$

Let $\mathcal{N}^1, \ldots, \mathcal{N}^{J-1} \subset \mathbb{Z}^r$ *be finite sets with the following property*

$$\min_{1\le s\le r} |n_s| > (B \max_{1\le s\le r} |m_s|)^C,$$

where $n \in \mathcal{N}^i, m \in \mathcal{N}^{i-1}$ $(2 \le i \le J-1)$, *where* $C = C(J, l)$. *Assume also*

$$\max_{n\in\mathcal{N}^{J-1}} |n|^C < \frac{1}{\eta}. \tag{2.4}$$

Then

$$\mathrm{mes}(\{\omega \in [0,1]^l : \ \exists\, n^{(i)} \ \in \mathcal{N}^i \ s.t.,\ (\omega, n^{(1)}\omega, \ldots, n^{(J-1)}\omega) \mod \mathbb{Z}^{lJ} \in \mathcal{S}\}) \le B^C\delta,$$

where

$$\delta^{-1} = \min_{n\in\mathcal{N}^1} \min_{1\le s\le r} |n_s|.$$

Proof. The proof follows from Lemmas 2.2 and 2.3 just as the proof of Lemma 1.20 in [Bou07] follows from the corresponding Lemma 1.18 and property (1.5) of semi-algebraic sets in [Bou07]. □

Definition 2.5. *We say* (E, x) *is* $(\bar\rho, N)$ *good, if for any* $Q_N \in \mathcal{E}_N^0$,

$$\|G_{Q_N}(E; x)\| \le e^{\sqrt{N}}, \tag{2.5}$$

$$|G_{Q_N}(E; x)(n, n')| \le e^{-\bar\rho|n-n'|} \text{ for } |n - n'| \ge \frac{N}{10}. \tag{2.6}$$

Definition 2.6. *We say Green's function satisfies property* P *with parameters* $(\gamma, \bar\rho)$ *at size* N *if there is a semi-algebraic set* $\Omega_N = \Omega_N(\lambda v, \rho, b, d) \subset \mathbb{T}^b$ *with* $\deg(\Omega_N) \le N^{4d}$ *such that the following statement is true: for any* $\omega \in \Omega_N$ *and* $E \in \mathbb{R}$, *there exists a set* $X_N = X_N(\lambda v, \rho, b, d, \omega, E) \subset \mathbb{T}^b$ *such that*

$$\sup_{1\le j\le d, x_j^{\neg} \in \mathbb{T}^{b-b_j}} \mathrm{mes}(X_N(x_j^{\neg})) \le e^{-N^\gamma}, \tag{2.7}$$

and for any x *not in* X_N, (E, x) *is* $(\bar\rho, N)$ *good.*

Theorem 2.7. *There exist small positive constants $c_3 < c_4 < 1$, where c_3 and c_4 depend on b, d such that the following statements are true. Let $c_1 = \frac{c_3}{4b}$ and $c_2 = c_1^2/2$. Fix a large number N_1. Let $N_2 = N_1^{2/c_1}$ and $N_3 = e^{N_2^{c_2}}$. Suppose the Green's functions satisfy property P at size N_1 with parameters $(c_1, \bar{\rho})$, and corresponding semi-algebraic sets Ω_{N_1}. Then there exists a semi-algebraic set $\Omega_3 \subset \Omega_{N_1}$ with $\deg(\Omega_3) \leq N_3^{4d}$ and $\mathrm{mes}((\Omega_{N_1} \backslash \Omega_3) \leq N_3{}^{-c_3}$ such that, if $\omega \in \Omega_3$, then for any $E \in \mathbb{R}$ and $x \in \mathbb{T}^b$, there exists $N_3^{c_3} < N < N_3^{c_4}$ such that, for all $k \in \Lambda \backslash \bar{\Lambda}$, $x + k\omega \mod \mathbb{Z}^b \notin X_{N_1}$, where*

$$\Lambda = [-N, N]^d, \bar{\Lambda} = [-N^{\frac{1}{10d}}, N^{\frac{1}{10d}}]^d.$$

Proof. The proof is based on Lemmas 2.2 and 2.4. For details, we refer the reader to the proof of the Claim in [Bou07, p.694]. To make it easier to check the corresponding relation between Theorem 2.7 and Claim in [Bou07], we present the alignment of our notations with these of [Bou07]. Let $X(B)$ denote the notation X used in [Bou07].

(1) $\tilde{b}(B) = 1$ since $b_i(B) = 1$, $i = 1, 2, \ldots, d$.
(2) $c_1 = c_1(B)$, $c_2 = c_2(B)$, $c_3 = c_5(B)$ and $c_4 = c_4(B)$. The formula before (2.8) in [Bou07] gives the relation between c_1 and c_2. The relation between c_1 and c_3 is presented at the end of Section 2 in [Bou07].
(3) $N_1 = N_2(B)$, $N_3 = \bar{N}(B)$ and $N_3^{c_3} = \bar{\bar{N}}(B)$. See (2.8), (2.11) and (2.24) in [Bou07] for the corresponding relations.
(4) $\Omega_3 = \Omega_{\bar{N}}(B)$. See (2.25) in [Bou07]. □

3 Resolvent Identities and Cartan's Lemma

Let $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^d$ and $\Lambda_1 \cap \Lambda_2 = \emptyset$. Let $\Lambda = \Lambda_1 \cup \Lambda_2$. Suppose that $R_\Lambda(\lambda^{-1}H(x) - E)R_\Lambda$ and $R_{\Lambda_i}(\lambda^{-1}H(x) - E)R_{\Lambda_i}$, $i = 1, 2$ are invertible. Then

$$G_\Lambda = G_{\Lambda_1} + G_{\Lambda_2} - \lambda^{-1}(G_{\Lambda_1} + G_{\Lambda_2})(H_\Lambda - H_{\Lambda_1} - H_{\Lambda_2})G_\Lambda.$$

If $m \in \Lambda_1$ and $n \in \Lambda$, we have

$$|G_\Lambda(m,n)| \leq |G_{\Lambda_1}(m,n)|\chi_{\Lambda_1}(n) + \lambda^{-1} \sum_{n' \in \Lambda_1, n'' \in \Lambda_2} e^{-\rho|n'-n''|}|G_{\Lambda_1}(m,n')||G_\Lambda(n'',n)|. \tag{3.1}$$

We remind

Lemma 3.1 (Schur test). *Suppose $A = A_{ij}$ is a symmetric matrix. Then*

$$\|A\| \leq \sup_i \sum_j |A_{ij}|.$$

We now prove

Lemma 3.2. *Let $M_0 \geq (\log N)^2, \bar{\rho} \in [\frac{\rho}{2}, \rho]$ and $M_1 \leq N$. Suppose $\Lambda \subset \mathbb{Z}^d$ is connected and $\operatorname{diam}(\Lambda) \leq 2N+1$. Suppose that for any $n \in \Lambda$, there exists some $W = W(n) \in \mathcal{E}_M$ with $M_0 \leq M \leq M_1$ such that $n \in W \subset \Lambda$, $\operatorname{dist}(n, \Lambda \backslash W) \geq \frac{M}{2}$ and*

$$\|G_{W(n)}(E;x)\| \leq 2e^{\sqrt{M}}, \tag{3.2}$$

$$|G_{W(n)}(E;x)(n,n')| \leq 2e^{-\bar{\rho}|n-n'|} \text{ for } |n-n'| \geq \frac{M}{10}. \tag{3.3}$$

We assume further that N is large enough so that

$$\sup_{M_0 \leq M \leq M_1} 2\lambda^{-1} e^{\sqrt{M}} (2M+1)^d e^{-\frac{3\rho}{20}M} \sum_{j=0}^{\infty} (2j+1)^d e^{-\frac{\rho}{2}j} \leq \frac{1}{2}. \tag{3.4}$$

Then

$$\|G_\Lambda(E;x)\| \leq 4(2M_1+1)^d e^{\sqrt{M_1}}.$$

Proof. For simplicity, we drop the dependence on E and x. Under the assumption of (3.4), it is easy to check that for all $M_0 \leq M \leq M_1$,

$$2\lambda^{-1}(2M+1)^d e^{\sqrt{M}+\frac{\rho}{10}M} \sum_{\substack{n_2 \in \Lambda \\ |n_2-n| \geq \frac{M}{2}}} e^{-\frac{\rho}{2}|n-n_2|} \leq \frac{1}{2}. \tag{3.5}$$

By (3.2) and (3.3), one has

$$|G_{W(n)}(n,n')| \leq 2e^{\sqrt{M}+\frac{\bar{\rho}}{10}M} e^{-\bar{\rho}|n-n'|}. \tag{3.6}$$

For each $n \in \Lambda$, applying (3.1) with $\Lambda_1 = W(n)$, one has

$$|G_\Lambda(n,n')| \leq |G_{W(n)}(n,n')| \chi_{W(n)}(n') + \lambda^{-1} \sum_{\substack{n_1 \in W(n) \\ n_2 \in \Lambda \backslash W(n)}} e^{-\rho|n_1-n_2|} |G_{W(n)}(n,n_1)||G_\Lambda(n_2,n')|. \tag{3.7}$$

By (3.6) and the fact that $|W(n)| \leq (2M+1)^d$, one has

$$\begin{aligned} |G_\Lambda(n,n')| &\leq |G_{W(n)}(n,n')| \chi_{W(n)}(n') + 2\lambda^{-1} \\ &\quad \times \sum_{\substack{n_1 \in W(n) \\ n_2 \in \Lambda \backslash W(n)}} e^{\sqrt{M}+\frac{\bar{\rho}}{10}M} e^{-\bar{\rho}|n-n_1|} e^{-\rho|n_1-n_2|} |G_\Lambda(n_2,n')| \\ &\leq |G_{W(n)}(n,n')| \chi_{W(n)}(n') + 2\lambda^{-1}(2M+1)^d e^{\sqrt{M}+\frac{\rho}{10}M} \\ &\quad \times \sum_{n_2 \in \Lambda \backslash W(n)} e^{-\frac{\rho}{2}|n-n_2|} |G_\Lambda(n_2,n')| \\ &\leq |G_{W(n)}(n,n')| \chi_{W(n)}(n') + 2\lambda^{-1}(2M+1)^d e^{\sqrt{M}+\frac{\rho}{10}M} \end{aligned}$$

$$\times \sum_{\substack{n_2\in\Lambda\\ |n_2-n|\geq \frac{M}{2}}} e^{-\frac{\rho}{2}|n-n_2|}|G_\Lambda(n_2,n')|, \tag{3.8}$$

where the last inequality holds by the assumption $\mathrm{dist}(n,\Lambda\backslash W(n))\geq \frac{M}{2}$.

Summing over $n'\in\Lambda$ in (3.8) and noticing (3.5) yields

$$\sup_{n\in\Lambda}\sum_{n'\in\Lambda}|G_\Lambda(n,n')|\leq 2(2M_1+1)^d e^{\sqrt{M_1}}+\frac{1}{2}\sup_{n_2\in\Lambda}\sum_{n'\in\Lambda}|G_\Lambda(n_2,n')|. \tag{3.9}$$

Now the lemma follows from Lemma 3.1. □

Theorem 3.3. *Assume* $\Lambda\subset\mathbb{Z}^d$ *is connected and* $\mathrm{diam}(\Lambda)\leq 2N+1$. *Assume* $\mathrm{diam}(\Lambda_1)\leq N^{\frac{1}{2d}}$. *Let* $M_0\geq(\log N)^2, \bar\rho\in[\frac{\rho}{2},\frac{4\rho}{5}]$. *Suppose that for any* $n\in\Lambda\backslash\Lambda_1$, *there exists some* $W=W(n)\in\mathcal{E}_M$ *with* $M_0\leq M\leq\sqrt{N}$ *such that* $n\in W$, $\mathrm{dist}(n,\Lambda\backslash\Lambda_1\backslash W)\geq\frac{M}{2}$, $W\subset\Lambda\backslash\Lambda_1$ *and*

$$\|G_W(E;x)\|\leq e^{\sqrt{M}},$$
$$|G_W(E;x)(n,n')|\leq e^{-\bar\rho|n-n'|} \text{ for } |n-n'|\geq\frac{M}{10}.$$

Suppose that

$$\|G_\Lambda(E;x)\|\leq e^{\sqrt{N}}.$$

Then

$$|G_\Lambda(E;x)(n,n')|\leq e^{-(\bar\rho-\frac{O(1)}{M_0^{1/2}})|n-n'|} \text{ for } |n-n'|\geq\frac{N}{10}.$$

Proof. As usual, we drop the dependence on E and x. Suppose $|n-n'|\geq N^{\frac{1}{d}}+1$. Obviously, one of n and n' is not in Λ_1. By the self-adjointness of Green's functions, we can assume $n\notin\Lambda_1$.

Applying (3.1) with $\Lambda_1=W=W(n)$, one has

$$|G_\Lambda(n,n')|\leq\lambda^{-1}\sum_{\substack{n_1\in W\\ n_2\in\Lambda\backslash W}} e^{-\rho|n_1-n_2|}|G_W(n,n_1)||G_\Lambda(n_2,n')|. \tag{3.10}$$

It implies (since $\lambda>1$)

$$\begin{aligned}|G_\Lambda(n,n')|\leq&\sum_{\substack{n_1\in W,|n_1-n|\leq\frac{M}{10}-1\\ n_2\in\Lambda\backslash W}} e^{-\rho|n_1-n_2|}|G_W(n,n_1)||G_\Lambda(n_2,n')|\\&+\sum_{\substack{n_1\in W,|n_1-n|\geq\frac{M}{10}\\ n_2\in\Lambda\backslash W}} e^{-\rho|n_1-n_2|}|G_W(n,n_1)||G_\Lambda(n_2,n')|\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{\substack{n_1\in W,|n_1-n|\le \frac{M}{10}-1\\ n_2\in\Lambda\setminus W}} e^{\sqrt{M}}e^{-\rho|n_1-n_2|}|G_\Lambda(n_2,n')|\\
&\qquad + \sum_{\substack{n_1\in W,|n_1-n|\ge \frac{M}{10}\\ n_2\in\Lambda\setminus W}} e^{-\rho|n_1-n_2|}e^{-\bar\rho|n-n_1|}|G_\Lambda(n_2,n')|\\
&\le \sum_{\substack{n_1\in W,|n_1-n|\le \frac{M}{10}-1\\ n_2\in\Lambda\setminus W}} e^{\sqrt{M}}e^{-\bar\rho|n-n_2|}|G_\Lambda(n_2,n')|\\
&\qquad + \sum_{\substack{n_1\in W,|n_1-n|\ge \frac{M}{10}\\ n_2\in\Lambda\setminus W}} e^{-\bar\rho|n-n_2|}|G_\Lambda(n_2,n')|\\
&\le (2N+1)^{2d}\sup_{n_2\in\Lambda\setminus W} e^{-(\bar\rho-\frac{O(1)}{\sqrt{M_0}})|n-n_2|}|G_\Lambda(n_2,n')|, \qquad (3.11)
\end{aligned}
$$

where the third inequality holds because of $\bar\rho\le\frac{4}{5}\rho$ and $|n-n_2|\ge\frac{M}{2}$.

Iterating (3.11) until $|n_2-n'|\le N^{\frac12}$ (but at most $\frac{2|n-n'|}{M_0}$ times), we have $|n-n'|\ge\frac{N}{10}$,

$$
\begin{aligned}
|G_\Lambda(n,n')| &\le (2N+1)^{\frac{O(|n-n'|)}{M_0}} e^{-(\bar\rho-\frac{O(1)}{\sqrt{M_0}})(|n-n'|-N^{\frac12})}e^{\sqrt{N}}\\
&\le e^{-(\bar\rho-\frac{O(1)}{M_0^{1/2}})|n-n'|}. \qquad \square
\end{aligned}
$$

Remark 3.4. In Lemma 3.2 and Theorem 3.3, the region Λ is only assumed to be connected, and is not necessarily an elementary region.

Lemma 3.5 (Several variables matrix-valued Cartan estimate). *Let $T(x)$ be a self-adjoint $N\times N$ matrix function of a parameter $x\in[-\delta,\delta]^J$ ($J\in\mathbb{Z}^+$) satisfying the following conditions:*

(i) *$T(x)$ is real analytic in $x\in[-\delta,\delta]^J$ and has a holomorphic extension to*

$$
\mathcal{D}_{\delta,\delta}=\left\{x=(x_i)_{1\le i\le J}\in\mathbb{C}^J:\sup_{1\le i\le J}|\Re x_i|\le\delta,\quad \sup_{1\le i\le J}|\Im x_i|\le\delta\right\}
$$

satisfying

$$
\sup_{x\in\mathcal{D}_{\delta,\delta}}\|T(x)\|\le B_1,\quad B_1\ge 1. \qquad (3.12)
$$

(ii) *For all $x\in[-\delta,\delta]^J$, there is subset $V\subset[1,N]$ with*

$$
|V|\le M,
$$

and

$$
\|(R_{[1,N]\setminus V}T(x)R_{[1,N]\setminus V})^{-1}\|\le B_2,\quad B_2\ge 1. \qquad (3.13)
$$

(iii)

$$\text{mes}\{x \in [-\delta, \delta]^J : \ \|T^{-1}(x)\| \geq B_3\} \leq 10^{-3J} J^{-J} \delta^J (1+B_1)^{-J}(1+B_2)^{-J}. \tag{3.14}$$

Let

$$0 < \epsilon \leq (1+B_1+B_2)^{-10M}. \tag{3.15}$$

Then

$$\text{mes}\left\{x \in [-\delta/2, \delta/2]^J : \ \|T^{-1}(x)\| \geq \epsilon^{-1}\right\} \leq C\delta^J e^{-c\left(\frac{\log \epsilon^{-1}}{M \log(B_2+B_3)}\right)^{1/J}}, \tag{3.16}$$

where $C = C(J, B_1), c = c(J, B_1) > 0$.

Proof. The proof is similar to that of the case $J = 1$ in Chapter 14 of [Bou05] (see also Remark 3 there). We use the higher dimensional Cartan sets techniques of [GS08]. For convenience, we give the details in the Appendix. □

Theorem 3.6. *Under the assumptions of Theorem 2.7, let* $\omega \in \Omega_{N_2} \cap \Omega_{N_1}$. *We assume for some* $x = (x^j, x_j^{\neg}) \in \mathbb{T}^b$, *there exist* $N \in [\frac{1}{4}N_3^{c_3}, N_3^{c_4}]$ *and* $\bar{\Lambda} \subset \Lambda \in \mathcal{E}_N$ *with* $\text{diam}(\bar{\Lambda}) \leq 10N^{\frac{1}{10d}}$ *such that, for any* $k \in \Lambda\backslash\bar{\Lambda}$, *there exists some* $k \in W \in \mathcal{E}_{N_1}$,$W \subset \Lambda\backslash\bar{\Lambda}$ *such that* $\text{dist}(k, \Lambda\backslash\bar{\Lambda}\backslash W) \geq \frac{N_1}{2}$, *and* $x + k\omega \mod \mathbb{Z}^b \notin X_{N_1}$. *Let*

$$Y = \{y \in \mathbb{R}^{b_j} : |y - x^j| \leq e^{-\rho N_1}, \|G_\Lambda(E; (y, x_j^{\neg}))\| \geq e^{\sqrt{N}}\}.$$

Then

$$\text{mes}(Y) \leq e^{-N^{1/3b_j}}. \tag{3.17}$$

Proof. Without loss of generality, we assume $j = 1$. Fix $x^1 \in \mathbb{T}^{b_1}$ and $x_1^{\neg} \in \mathbb{T}^{b-b_1}$.

Let $\mathcal{D}$ be the $e^{-\rho N_1}$ neighbourhood of x^1 in the complex plane, i,e.,

$$\mathcal{D} = \{z \in \mathbb{C}^{b_1} : \ |\Im z| \leq e^{-\rho N_1}, |\Re z - x^1| \leq e^{-\rho N_1}\}.$$

By the assumption of Theorem 3.6, one has for all $k \in \Lambda\backslash\bar{\Lambda}$ and $Q_{N_1} \in \mathcal{E}^0_{N_1}$,

$$\|G_{Q_{N_1}}(E; x + k\omega)\| \leq e^{\sqrt{N_1}}, \tag{3.18}$$

$$|G_{Q_{N_1}}(E; x + k\omega)(n, n')| \leq e^{-\bar{\rho}|n-n'|} \text{ for } |n - n'| \geq \frac{N_1}{10}. \tag{3.19}$$

By standard perturbation arguments[2], (3.18) and (3.19), we have for any $y \in \mathcal{D}$, $Q_{N_1} \in \mathcal{E}^0_{N_1}$, and $k \in \Lambda\backslash\bar{\Lambda}$,

$$\|G_{Q_{N_1}}(E; (x^1 + y, x_1^{\neg}) + k\omega)\| \leq 2e^{\sqrt{N_1}}, \tag{3.20}$$

[2] See e.g. the proof of Theorem 4.3.

$$|G_{Q_{N_1}}(E;(x^1+y,x_1^{\neg})+k\omega)(n,n')| \le 2e^{-\bar{\rho}|n-n'|} \text{ for } |n-n'| \ge \frac{N_1}{10}. \quad (3.21)$$

Substituting Λ with $\Lambda\backslash\bar{\Lambda}$ in Lemma 3.2, one has for any $y \in \mathcal{D}$,

$$\|G_{\Lambda\backslash\bar{\Lambda}}(E;(x^1+y,x_1^{\neg}))\| \le e^{2\sqrt{N_1}}. \quad (3.22)$$

We want to use Lemma 3.5. For this purpose, let

$$T(y) = \lambda^{-1}H_\Lambda((x^1+y,x_1^{\neg})) - E, J = b_1, \delta = e^{-\rho N_1}.$$

Now we are in the position to check the assumptions of Lemma 3.5. Obviously, $B_1 = O(1)$ since $\lambda > 1$ and E is bounded.

Let $V = \bar{\Lambda}$. By (3.22), one has

$$M = |\bar{\Lambda}| \le 30^d N^{1/10}, \quad B_2 = e^{2\sqrt{N_1}}. \quad (3.23)$$

By the fact that the Green's functions satisfy property P and (2.7), one has that both (2.5) and (2.6) hold at scale N_2 for all y except a set of $y \in \mathbb{T}^{b_1}$ with measure less than $e^{-N_2^{c_1}}$. It implies both (2.5) and (2.6) holds at scale N_2 for all $x+k\omega$ with $|k| \le N_3$ except a set of measure less than $(2N_3+1)^d e^{-N_2^{c_1}}$.

Applying Lemma 3.2 with $M_0 = M_1 = N_2$ and (2.1), one has

$$\|T^{-1}(y)\| \le 4(2N_2+1)^d e^{\sqrt{N_2}} \le 4e^{2\sqrt{N_2}} =: B_3,$$

except on a set of $y \in \mathbb{T}^{b_1}$ with measure less than $(2N_3+1)^d e^{-N_2^{c_1}}$.

Since $N_2 = N_1^{\frac{2}{c_1}}$, direct computation shows that

$$10^{-3b_1} b_1^{-b_1} \delta_1^{b_1} (1+B_1)^{-b_1} (1+B_2)^{-b_1} \ge e^{-N_2^{c_1}/2}.$$

This verifies (iii) in Lemma 3.5.

For $\epsilon = e^{-\sqrt{N}}$, by (3.23), one has

$$\epsilon < (1+B_1+B_2)^{-10M}.$$

By (3.16) of Lemma 3.5,

$$\text{mes}(Y) \le Ce^{-c\left(\frac{\sqrt{N}}{N_2 N^{1/10}}\right)^{1/b_1}} \le e^{-N^{1/3b_1}}. \quad (3.24)$$

□

Theorem 3.7. *Let $c_1, c_2, c_3, c_4, N_1, N_2, N_3, \Omega_3$ be given by Theorem 2.7, so in particular, Green's functions satisfy property P at N_1, N_2 with parameters $(c_1, \bar{\rho})$. Then for all $N_3 \le N \le N_3^2$, Green's functions satisfy property P at size N with parameters $(c_1, \bar{\rho} - \frac{O(1)}{N_1^{1/2}})$ and $\Omega_N = \Omega_3 \cap \Omega_{N_2}$, where $O(1)$ only depends on d.*

Proof. We fix $N \in [N_3, N_3^2]$ and $Q_N \in \mathcal{E}_N^0$. Let $\omega \in \Omega_{N_3}$.

For any $n \in Q_N$, replacing x with $x+n\omega$ in Theorem 2.7, there exits $N_3^{c_3} < \bar{N} < N_3^{c_4}$ such that, for all $k \in (n+\Lambda)\backslash(n+\bar{\Lambda})$, $x+k\omega \mod \mathbb{Z}^b \notin X_{N_1}$, where

$$\Lambda = [-\bar{N}, \bar{N}]^d, \bar{\Lambda} = [-\bar{N}^{\frac{1}{10d}}, \bar{N}^{\frac{1}{10d}}]^d, \tag{3.25}$$

and $n+\Lambda$, $n+\bar{\Lambda}$ are the shift of Λ and $\bar{\Lambda}$ by n.

We are going to possibly shrink the $n+\Lambda$ a little bit so that it is in Q_N. More precisely, we claim that for any $n \in Q_N$, there exist

$$\frac{1}{4}N_3^{c_3} \leq \tilde{N} \leq N_3^{c_4}, \tag{3.26}$$

$\Lambda_{\text{new}} \in \mathcal{E}_{\tilde{N}}$ and $\bar{\Lambda}_{\text{new}}$, such that

$$\Lambda_{\text{new}} \subseteq \Lambda, \quad \bar{\Lambda} \subseteq \bar{\Lambda}_{\text{new}}, \tag{3.27}$$

$$n \in \Lambda_{\text{new}} \subset Q_N, \quad \text{dist}(n, Q_N\backslash\Lambda_{\text{new}}) \geq \frac{\tilde{N}}{2} \tag{3.28}$$

and

$$\text{Diam}(\bar{\Lambda}_{\text{new}}) \leq 4\tilde{N}^{\frac{1}{10d}}. \tag{3.29}$$

Also for any $k \in \Lambda_{\text{new}}\backslash\bar{\Lambda}_{\text{new}}$, there exists some $\mathcal{E}_{N_1} \ni W \subset \Lambda_{\text{new}}\backslash\bar{\Lambda}_{\text{new}}$ such that

$$\text{dist}(k, \Lambda_{\text{new}}\backslash\bar{\Lambda}_{\text{new}}\backslash W) \geq \frac{N_1}{2}. \tag{3.30}$$

We split the proof into three cases.

Case 1: $n+\Lambda \subset Q_N$. In this case, let $\Lambda_{\text{new}} = n+\Lambda$ and $\bar{\Lambda}_{\text{new}} = n+\bar{\Lambda}$. See Case 1 of Figure 1.

Case 2: $(n+\Lambda)\cap(\mathbb{Z}^d\backslash Q_N)$ is non-empty and $\text{dist}(n+\bar{\Lambda}, \partial Q_N) \geq 2N_1$. See Case 2 of Figure 1. In this case, let $\bar{\Lambda}_{\text{new}} = n+\bar{\Lambda}$ (the black square). By shrinking $n+\Lambda$ a little bit, we can obtain proper $\Lambda_{\text{new}} \subset (n+\Lambda)\cap Q_N$ satisfying (3.28). In this case, $\Lambda_{\text{new}} =$ yellow part+black part. Since $\text{dist}(n+\bar{\Lambda}, \partial Q_N) \geq 2N_1$, we can also guarantee (3.30) holds.

Case 3: $(n+\Lambda)\cap(\mathbb{Z}^d\backslash Q_N)$ is non-empty and $\text{dist}(n+\bar{\Lambda}, \partial Q_N) \leq 2N_1$. In this case, making $(n+\bar{\Lambda})\cap Q_N$ possibly larger, we obtain $\bar{\Lambda}_{\text{new}} \subset Q_N$. We can also make sure for any $k \in Q_N\backslash\bar{\Lambda}_{\text{new}}$, there exists some $W \in \mathcal{E}_{N_1} \subset Q_N\backslash\bar{\Lambda}_{\text{new}}$

$$\text{dist}(k, Q_N\backslash\bar{\Lambda}_{\text{new}}\backslash W) \geq \frac{N_1}{2}. \tag{3.31}$$

See Figure 2. For B, $\bar{\Lambda}_{\text{new}} = (n+\bar{\Lambda})\cap Q_N$ (the black part). For A and C, $(n+\bar{\Lambda})\cap Q_N$ is the black part, and $\bar{\Lambda}_{\text{new}}$ is union of the black part and the gray part. Shrinking $n+\Lambda$, we can obtain proper Λ_{new} satisfying (3.28). For A snd C, $\Lambda_{\text{new}} =$ yellow part+ black part+gray part. For B, $\Lambda_{\text{new}} =$ yellow part + black part. This implies (3.30) by (3.31).

Fix $x_j^{\neg}$. Divide $\mathbb{T}^{b_j}$ into $e^{2b_j\rho N_1}$ cubes of size $e^{-\rho N_1}$.

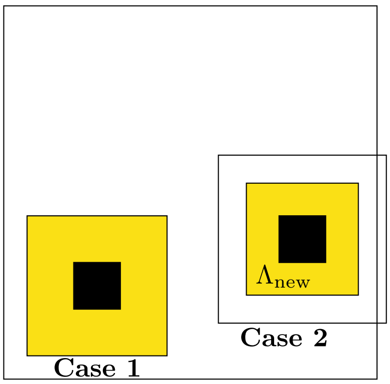

Figure 1: Λ_{new} is far away from the boundary.

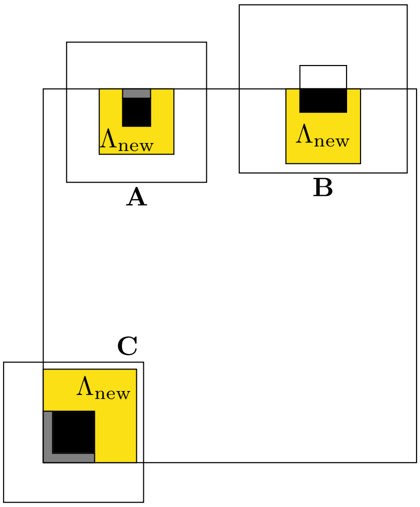

Figure 2: Case 3.

Applying Theorem 3.6 in each cube, ((3.27), (3.29) and (3.30) ensure we can use Theorem 3.6), there exists a set $Y_{\tilde{N}}(x_j^{\neg})$ such that

$$\mathrm{mes}(Y_{\tilde{N}}(x_j^{\neg})) \leq e^{2b_j\rho N_1} e^{-\tilde{N}^{\frac{1}{3b_j}}}, \tag{3.32}$$

and for $x = (x^j, x_j^{\neg})$ with $x^j \notin Y_{\tilde{N}}(x_j^{\neg})$,

$$\|G_{\Lambda_{\mathrm{new}}}(E; (x^j, x_j^{\neg}))\| \leq e^{\sqrt{\tilde{N}}}. \tag{3.33}$$

Setting $M_0 = N_1$, $\Lambda = \Lambda_{\mathrm{new}}$ and $\Lambda_1 = \bar{\Lambda}_{\mathrm{new}}$ in Theorem 3.3 ((2.5), (2.6), (3.27), (3.29), (3.30) and (3.33) ensure we can use Theorem 3.3), we have for such x,

$$|G_{\Lambda_{\mathrm{new}}}(E; x)(n, n')| \leq e^{-(\bar{\rho} - \frac{O(1)}{N_1^{1/2}})|n-n'|} \text{ for } |n - n'| \geq \frac{\tilde{N}}{10}. \tag{3.34}$$

Let

$$B_N(x_j^{\neg}) = \bigcup_{\frac{1}{4}N_3^{c_3} \le \tilde{N} \le N_3^{c_4}} Y_{\tilde{N}}(x_j^{\neg}). \tag{3.35}$$

By (3.32), (3.35) and since $c_1 = c_3/4\tilde{b}$, one has for any j and $x_j^{\neg} \in \mathbb{T}^{b-b_j}$,

$$\mathrm{mes}(B_N(x_j^{\neg})) \le e^{-N^{c_1}}. \tag{3.36}$$

Suppose $x^j \notin B_N(x_j^{\neg})$. Applying $\Lambda = Q_N$, $M_0 = \frac{1}{4}N_3^{c_3}$ and $M_1 = N_3^{c_4}$ in Lemma 3.2 since $N \in [N_3, N_3^2]$ ((3.26), (3.28), (3.33) and (3.34) ensure the assumption of Lemma 3.2), one has

$$\|G_{Q_N}(E;x)\| \le 4(2N_3^{c_4}+1)^d e^{\sqrt{N_3^{c_4}}} \le e^{\sqrt{N}}. \tag{3.37}$$

Applying $\Lambda = Q_N \in \mathcal{E}_N^0$, $M_0 = \frac{1}{4}N_3^{c_3}$ and $\Lambda_1 = \emptyset$ in Theorem 3.3, by (3.33), (3.34) and (3.37), we have

$$|G_{Q_N}(E;x)(n,n')| \le e^{-(\bar{\rho}-\frac{O(1)}{N_1^{1/2}})|n-n'|} \text{ for } |n-n'| \ge \frac{N}{10}. \tag{3.38}$$

Let

$$X_N = \{x \in \mathbb{T}^b : (E,x) \text{ is not } (\bar{\rho} - \frac{O(1)}{N_1^{1/2}}, N) \text{ good }\}, \quad \Omega_N = \Omega_3 \cap \Omega_{N_2}.$$

The theorem follows from (3.38), (3.37) and (3.36). □

4 Large Deviation Theorem for Green's Functions and Proof of Theorem 1.1

The main result of this section is the following large deviation theorem (LDT) for Green's functions.

Theorem 4.1 (LDT). *There exist constants $\gamma = \gamma(b,d) \in (0,1)$, $N_0 = N_0(v,\rho,b,d)$ and $\lambda_0 = \lambda_0(v,\rho,b,d)$, such that for all $N \ge N_0$ and $\lambda \ge \lambda_0$, the Green's functions satisfy property P with parameters $(\gamma, \frac{\rho}{2})$ at size N, and the corresponding semi-algebraic set Ω_N satisfying*

$$\mathrm{mes}(\mathbb{T}^b \backslash \cap_{N \ge N_0} \Omega_N) \to 0,$$

as $\lambda \to \infty$.

Compactness arguments and Theorem 8 in [PSS99] immediately imply

Lemma 4.2 (Łojasiewicz type Lemma). *For $E \in \mathbb{R}, \delta > 0$, define*

$$X := \{x \in \mathbb{T}^b : \ |v(x) - E| < \delta\}.$$

Then there are constants $C(v), a(v) > 0$ such that

$$\sup_{1 \leq j \leq d, x_j^{\neg} \in \mathbb{T}^{b-b_j}} \operatorname{mes}(X(x_j^{\neg})) \leq C(v)\delta^{a(v)}. \tag{4.1}$$

Theorem 4.3. *Let X be as in Lemma 4.2 and*

$$X_N := \bigcup_{|n| \leq N} \left\{x : \ x + n\omega \mod \mathbb{Z}^b \in X\right\}. \tag{4.2}$$

Then we have

$$\sup_{1 \leq j \leq d, x_j^{\neg} \in \mathbb{T}^{b-b_j}} \operatorname{mes}(X_N(x_j^{\neg})) \leq C(v)(2N+1)^d \delta^{a(v)}. \tag{4.3}$$

Moreover, if

$$\lambda \geq 2\delta^{-1}(2N+1)^d, \tag{4.4}$$

then for any $x \notin X_N, \ \omega \in \mathbb{T}^b$, we have for $Q_N \in \mathcal{E}_N^0$,

$$\|G_{Q_N}(E; x)\| \leq 2\delta^{-1}, \tag{4.5}$$

$$|G_{Q_N}(E; x)(n, n')| \leq 2\delta^{-1} e^{-\rho|n-n'|}. \tag{4.6}$$

Proof. The bound (4.3) follows from Lemma 4.2 immediately.

Let $x \notin X_N$ and fix $Q_N \in \mathcal{E}_N^0$. Let $A = \lambda^{-1} R_{Q_N} S R_{Q_N}$, with the kinetic term S being given by (1.1). Let B be diagonal part of the restriction of $\lambda^{-1}H - E$ on Q_N, namely,

$$B = R_{Q_N}(v(x + n\omega)\delta_{nn'} - E\delta_{nn'})R_{Q_N}.$$

By (4.2), one has

$$\min_{n \in Q_N} |v(x + n\omega) - E| \geq \delta.$$

It leads to

$$\|B^{-1}\| \leq \delta^{-1}. \tag{4.7}$$

Since $|S(n, n')| \leq e^{-\rho|n-n'|}$ for all n, n', by Lemma 3.1 again, one has for $N \geq N(\rho, d)$,

$$\|A\| \leq \lambda^{-1} \sup_{n \in Q_N} \sum_{n' \in Q_N} e^{-\rho|n-n'|} \leq \lambda^{-1}(2N+1)^d. \tag{4.8}$$

By (4.4),

$$\|AB^{-1}\| \leq \frac{1}{2}.$$

Combining with (4.7) and (4.8), we have the following Neumann series expansion

$$G_{Q_N} = B^{-1} \sum_{s \geq 0} (-AB^{-1})^s. \tag{4.9}$$

Thus one has

$$\|G_{Q_N}\| \leq ||B^{-1}|| \frac{1}{1 - ||AB^{-1}||} \leq 2\delta^{-1}. \tag{4.10}$$

It implies (4.5). In particular, (4.6) is also true for $n = n'$.

For $n \neq n'$, by (1.1), (4.9) and the fact that B is diagonal, we have

$$\begin{aligned} |G_{Q_N}(n, n')| &\leq \|B^{-1}\| \sum_{\substack{s \geq 1 \\ |k_i| \leq N}} \lambda^{-s} \delta^{-s} e^{-\rho|n-k_1| - \rho|k_1 - k_2| - \cdots |k_{s-1} - n'|} \\ &\leq \delta^{-1} e^{-\rho|n-n'|} \sum_{s \geq 1} (2N+1)^{sd} \lambda^{-s} \delta^{-s} \\ &\leq 2\delta^{-1} e^{-\rho|n-n'|}, \end{aligned}$$

where the last inequality holds by (4.4). □

Proof of Theorem 4.1. Denote the relation between N_1 and N_3 in Theorem 2.7 by f, i.e., $f(x) = e^{x^{c_1}}$. Let $N_0 = N_0(v, \rho, b, d)$ be sufficiently large. Denote by $f^{(n)}(x)$ the nth iteration of f, namely, $f^{(n)}(x) = f(f(f(\cdots x \cdots)))$. Let $g(x) = f^2(x)$. Clearly, $g(x) \geq f(x+1)$ for large x.

By letting $\delta = \frac{1}{2} e^{-\bar{N}^{1/2}}$ and Theorem 4.3, since $c_1 < 1/2$, the Green's functions satisfy property P with parameters $(c_1, \frac{4\rho}{5})$ for $N_0 \leq N \leq \bar{N}$ and $\Omega_N = \mathbb{T}^b$ if $\lambda \geq 4e^{\bar{N}^{1/2}}(2\bar{N}+1)^d$.

Theorem 3.7 allows us to proceed from scales N, $N^{\frac{2}{c_1}}$ to scales $[f(N), g(N)]$. Since we want to cover all scales, our initial step will consist of property P at the interval of scales $[N_1, f(N_1)]$. For this reason, we need to take $N_1 = \log\log\lambda$.

Initial step: For large λ, the Green's functions satisfy property P with parameters (c_1, ρ_0) for all $N_0 \leq N \leq g(\log\log\lambda)$ and $\Omega_N = \mathbb{T}^b$, where $\rho_0 = \frac{4\rho}{5}$.

Let

$$\rho_i = \frac{4\rho}{5} - \sum_{j=1}^{i} \frac{O(1)}{f^{(j)}(\log\log\lambda)^{1/2}}. \tag{4.11}$$

Applying Theorem 3.7 to $N_1 = \log\log\lambda, \log\log\lambda+1, \log\log\lambda+2, \ldots, f(\log\log\lambda)$, the Green's functions satisfy property P with parameters (c_1, ρ_1) for all $g(\log\log\lambda) \leq N \leq g(f(\log\log\lambda))$ since $g(x) \geq f(x+1)$. Moreover,

$$\begin{aligned} \operatorname{mes}\left(\bigcap_{N=\log\log\lambda}^{f(\log\log\lambda)} \Omega_N\right) &\geq 1 - \sum_{N=\log\log\lambda}^{f(\log\log\lambda)} \frac{1}{f(N)^{c_3}} \\ &\geq 1 - \sum_{N=\log\log\lambda}^{f(\log\log\lambda)} \frac{1}{N^5}. \end{aligned} \tag{4.12}$$

Applying Theorem 3.7 to $N_1 = f(\log\log\lambda), f(\log\log\lambda)+1, f(\log\log\lambda)+2, \ldots,$ $f^{(2)}(\log\log\lambda)$, the Green's functions satisfy property P with parameters (c_1, ρ_2) for all $g(f(\log\log\lambda)) \leq N \leq g(f^{(2)}(\log\log\lambda))$. Moreover,

$$\operatorname{mes}\left(\bigcap_{N=f(\log\log\lambda)+1}^{f^{(2)}(\log\log\lambda)} \Omega_N\right) \geq 1 - \sum_{N=f(\log\log\lambda)+1}^{f^{(2)}(\log\log\lambda)} \frac{1}{N^5}.$$

By induction, we have the Green's functions satisfy property P with parameters (c_1, ρ_i) for $g(f^{(i-1)}(\log\log\lambda)) \leq N \leq g(f^{(i)}(\log\log\lambda))$, $i = 1, 2, \ldots$. Moreover,

$$\operatorname{mes}\left(\bigcap_{N=f^{(i-1)}(\log\log\lambda)+1}^{f^{(i)}(\log\log\lambda)} \Omega_N\right) \geq 1 - \sum_{N=f^{(i-1)}(\log\log\lambda)+1}^{f^{(i)}(\log\log\lambda)} \frac{1}{N^5}. \tag{4.13}$$

Now Theorem 4.1 follows from (4.11) and (4.13). □

Proof of Theorem 1.1. With Theorem 4.1 at hand, the proof Theorem 1.1 is rather standard. We refer the readers to [Bou05, Section 3] or [BGS02, Section 6] for details. □

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Appendix A

In the following, we will prove the several variables matrix-valued Cartan estimate, i.e., Lemma 3.5. The proof is similar to that in [Bou05, Bou02]. Before going to the details, we recall some useful lemmas. The first result is the standard Schur's complement theorem. For convenience, we include a proof here.

Lemma A.1. *Let T be the matrix*

$$T = \begin{pmatrix} T_1 & T_2 \\ T_2^t & T_3 \end{pmatrix},$$

where T_1 is an invertible $n \times n$ matrix , T_2 is an $n \times k$ matrix and T_3 is a $k \times k$ matrix. Let

$$S = T_3 - T_2^t T_1^{-1} T_2.$$

Then T is invertible if and only if S is invertible, and

$$\|S^{-1}\| \le \|T^{-1}\| \le C(1 + \|T_1^{-1}\|)^2(1 + \|S^{-1}\|), \tag{A.1}$$

where C depends only on $\|T_2\|$.

Proof. It is easy to check that

$$T = \begin{pmatrix} T_1 & T_2 \\ T_2^t & T_3 \end{pmatrix} = \begin{pmatrix} I & 0 \\ T_2^t T_1^{-1} & I \end{pmatrix} \begin{pmatrix} I & T_2 \\ 0 & S \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ 0 & I \end{pmatrix}. \tag{A.2}$$

It implies T is invertible if and only if S is invertible and also the second inequality of (A.1). By (A.2), one has

$$\begin{aligned} T^{-1} &= \begin{pmatrix} T_1 & 0 \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} I & T_2 \\ 0 & S \end{pmatrix}^{-1} \begin{pmatrix} I & 0 \\ T_2^t T_1^{-1} & I \end{pmatrix}^{-1} \\ &= \begin{pmatrix} T_1^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & -T_2 S^{-1} \\ 0 & S^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -T_2^t T_1^{-1} & I \end{pmatrix} \\ &= \begin{pmatrix} \star & \star \\ \star & S^{-1} \end{pmatrix}. \end{aligned}$$

implying the first inequality of (A.1). □

We then introduce the higher dimensional Cartan sets Lemma of Goldstein-Schlag [GS08]. We denote by $\mathcal{D}(z, r)$ the standard disk on $\mathbb{C}$ of center z and radius $r > 0$.

Lemma A.2. [GS08, Lemma 2.15] *Let $f(z_1, \ldots, z_J)$ be an analytic function defined in a ploydisk $\mathcal{P} = \prod_{1 \le i \le J} \mathcal{D}(z_{i,0}, 1/2)$ and $\phi = \log|f|$. Let $\sup_{\underline{z} \in \mathcal{P}} \phi(\underline{z}) \le M, m \le \phi(\underline{z}_0)$, $\underline{z}_0 = (z_{1,0}, \ldots, z_{J,0})$. Given $F \gg 1$, there exists a set $\mathcal{B} \subset \mathcal{P}$ such that*

$$\phi(\underline{z}) > M - C(J)F(M - m), \text{ for } \forall\ \underline{z} \in \prod_{1 \le i \le J} \mathcal{D}(z_{i,0}, 1/4) \setminus \mathcal{B}, \tag{A.3}$$

and

$$\operatorname{mes}(\mathcal{B} \cap \mathbb{R}^J) \le C(J) e^{-F^{1/J}}. \tag{A.4}$$

Proof of Lemma 3.5. The proof is similar to that of Proposition 14.1 in [Bou05] in case $J = 1$ and Lemma 1.43 in [Bou02] without explicit bounds. In the following proof, $C = C(B_1, J)$ and $c = c(B_1, J)$.
Let

$$\mu = 10^{-2} J^{-1} \delta (1 + B_1)^{-1} (1 + B_2)^{-1}.$$

Fix

$$x_0 \in [-\delta/2, \delta/2]^J$$

and consider $T(z)$ with $|z - x_0| = \sup_{1 \le i \le J} |z_i - x_{0,i}| < \mu$. Thanks to Cauchy's estimate and (3.12), one obtains for $|z - x_0| < \mu$,

$$\|\partial_{z_i} T(z)\| \le \frac{4B_1}{\delta}, \quad i = 1, 2, \ldots, J,$$

which implies

$$\|T(z) - T(x_0)\| \le \frac{4JB_1\mu}{\delta} \le 25^{-1}(1 + B_2)^{-1}.$$

From the assumption (ii) of Lemma 3.5, we can find $V = V(x_0)$ so that $|V| \le M$ and (3.13) is satisfied. Denote by $V^c = [1, N] \backslash V$. Thus using the standard Neumann series argument and (3.13), one has

$$\|(R_{V^c} T(z) R_{V^c})^{-1}\| \le 2B_2 \text{ for } |z - x_0| < \mu. \tag{A.5}$$

We define for $|z - x_0| < \mu$ the analytic self-adjoint function

$$S(z) = R_V T(z) R_V - R_V T(z) R_{V^c} (R_{V^c} T(z) R_{V^c})^{-1} R_{V^c} T(z) R_V. \tag{A.6}$$

Then by (A.5) and (A.6), we have

$$\|S(z)\| \le 3B_1^2 B_2. \tag{A.7}$$

Recalling Lemma A.1, if $S(z)$ is invertible, so is $T(z)$ and by (A.1),

$$\|S^{-1}(z)\| \le C\|T^{-1}(z)\| \le CB_2^2(1 + \|S^{-1}(z)\|). \tag{A.8}$$

For $x \in \mathbb{R}^J$, one has

$$||S(x)||^M \ge |\det S(x)| = \prod_{\lambda \in \sigma(S(x))} |\lambda| \ge \|S^{-1}(x)\|^{-M}. \tag{A.9}$$

By (A.7), one has

$$\|S^{-1}(x)\| \le \frac{\|S(x)\|^{M-1}}{|\det S(x)|} \le \frac{(3B_1^2 B_2)^M}{|\det S(x)|}. \tag{A.10}$$

Let

$$\phi(z) = \log|\det S(x_0 + \mu z)|, \ \ |z| < 1.$$

Then by (A.9) and (A.7),

$$\sup_{|z|<1} \phi(z) \leq CM \log B_2. \tag{A.11}$$

By (3.14) and the definition of μ, there is some x_1 with $|x_0 - x_1| < \mu/10$ such that

$$\|T^{-1}(x_1)\| \leq B_3. \tag{A.12}$$

Hence by (A.8), $\|S^{-1}(x_1)\| \leq CB_3$, and from (A.9),

$$\phi(a) \geq -CM \log B_3, \tag{A.13}$$

where $a = \frac{x_1 - x_0}{\mu}$, so $|a| < 1/10$. Let

$$\mathcal{P} = \prod_{1 \leq i \leq J} \mathcal{D}(a_i, 1/2).$$

Then one has

$$\sup_{z \in \mathcal{P}} \phi(z) \leq CM \log B_2, \phi(a) \geq -CM \log B_3.$$

Applying Lemma A.2 and recalling (A.3), (A.4), for any $F \gg 1$, there is some set $\mathcal{B} \subset \prod_{1 \leq i \leq J} \mathcal{D}(a_i, 1/4)$ with

$$\phi(z) \geq -CFM \log(B_2 + B_3) \text{ for } z \in \prod_{1 \leq i \leq J} \mathcal{D}(a_i, 1/4) \setminus \mathcal{B}, \tag{A.14}$$

and

$$\text{mes}(\mathcal{B} \cap \mathbb{R}^J) \leq Ce^{-F^{1/J}}. \tag{A.15}$$

For $0 < \epsilon < 1$, let

$$F = \frac{-c \log \epsilon}{M \log(B_2 + B_3)}.$$

Then by (A.14) and (A.15),

$$\begin{aligned} &\text{mes}\left\{x \in \mathbb{R}^J : \ |x - x_1| < \mu/4 \text{ and } |\det S(x)| \leq \epsilon\right\} \\ &\quad = \mu^J \text{mes}\left\{x \in \mathbb{R}^J : \ |x - a| < 1/4 \text{ and } \phi(x) \leq \log \epsilon\right\} \\ &\quad \leq C\mu^J e^{-F^{1/J}}. \end{aligned}$$

Since $|x_0 - x_1| < \mu/10$, we have

$$\operatorname{mes}\left\{x \in \mathbb{R}^J : \; |x - x_0| < \mu/8 \text{ and } |\det(S(x))| \le \epsilon\right\} \le C\mu^J e^{-c\left(\frac{\log \epsilon^{-1}}{M \log(B_2+B_3)}\right)^{1/J}}. \tag{A.16}$$

Recalling (A.8), (A.10) and (3.15), one has for $|x - x_0| < \mu/8$ and $|\det S(x)| \ge \epsilon$,

$$\|T^{-1}(x)\| \le C(1 + B_2^2)(1 + \epsilon^{-1}(3B_1^2 B_2)^M) \le C\epsilon^{-2}. \tag{A.17}$$

Covering $[-\frac{\delta}{2}, \frac{\delta}{2}]^J$ by cubes of side $\mu/4$, and combining (A.16) and (A.17), one has

$$\operatorname{mes}\left\{x \in [-\delta/2, \delta/2]^J : \; \|T^{-1}(x)\| \ge \epsilon^{-2}\right\} \le C\delta^J e^{-c\left(\frac{\log \epsilon^{-1}}{M \log(B_2+B_3)}\right)^{1/J}}. \qquad \square$$

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