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Almost Mathieu operators with completely resonant phases

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Abstract. Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and $\beta(\alpha) = \limsup_{n\to\infty}(\ln q_{n+1})/q_n < \infty$, where p_n/q_n is the continued fraction approximation to α. Let $(H_{\lambda,\alpha,\theta}u)(n) = u(n+1) + u(n-1) + 2\lambda \cos 2\pi(\theta + n\alpha)u(n)$ be the almost Mathieu operator on $\ell^2(\mathbb{Z})$, where $\lambda, \theta \in \mathbb{R}$. Avila and Jitomirskaya [The ten Martini problem. *Ann. of Math.* (2), **170**(1) (2009), 303–342] conjectured that, for $2\theta \in \alpha\mathbb{Z} + \mathbb{Z}$, $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $|\lambda| > e^{2\beta(\alpha)}$. In this paper, we develop a method to treat simultaneous frequency and phase resonances and obtain that, for $2\theta \in \alpha\mathbb{Z} + \mathbb{Z}$, $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $|\lambda| > e^{3\beta(\alpha)}$.

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1. *Introduction*

The almost Mathieu operator (AMO) is the (discrete) quasi-periodic Schrödinger operator on $\ell^2(\mathbb{Z})$ given by

$$(H_{\lambda,\alpha,\theta}u)(n) = u(n+1) + u(n-1) + 2\lambda \cos 2\pi(\theta + n\alpha)u(n),$$

where λ is the coupling, α is the frequency, and θ is the phase.

The AMO is the most studied quasi-periodic Schrödinger operator, arising naturally as a physical model. We refer the reader to [**34**, **40**] and the references therein for physical background. Most recently, there are a lot of interesting topics related to AMOs, e.g. [**4**, **27**, **28**, **31**, **33**, **35**, **45**].

We say that the phase $\theta \in \mathbb{R}$ is completely resonant with respect to frequency α if $2\theta \in \alpha\mathbb{Z} + \mathbb{Z}$. In this paper, we always assume that $\alpha \in \mathbb{R}\backslash\mathbb{Q}$.

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Conjecture 1. Avila and Jitomirskaya [**1**, **2**] assert that, for $2\theta \in \alpha\mathbb{Z} + \mathbb{Z}$, $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $|\lambda| > e^{2\beta}$, where

$$\beta = \beta(\alpha) = \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n}$$

and p_n/q_n is the continued fraction approximation to α.

Completely resonant phases of quasi-periodic operators correspond to the rational rotation numbers with respect to frequency in the Aubry dual model. We refer the reader to [**13**, **18**, **29**] for the Aubry duality. The (quantitative) reducibility of Schrödinger cocycles with rational rotation numbers is related to many topics in quasi-periodic operators. For example, it is a good approach to show that all the spectral gaps G_m labeled by the gap labeling theorem (the rotation number ρ on gap G_m satisfies $2\rho = m\alpha \mod \mathbb{Z}$) [**7**, **32**] are open (named after the dry ten Martini problem for the AMO). The dry ten Martini problem is stronger than the ten Martini problem (the latter one was finally solved by Avila and Jitomirskaya [**2**]). We should mention that the dry ten Martini problem is still open for all parameters. The non-critical coupling case has been solved by Avila-You-Zhou [**5**]. It is also related to the Hölder continuity of Lyapunov exponents, rotation numbers and the integrated density of states.

The reducibility of the Schrödinger cocycles with rational rotation numbers was first established by Moser and Pöschel [**41**], who modified the proof of reducibility of cocycles with Diophantine rotation numbers [**14**]. See [**15**, **19**] for more precise results. It was first realized by Puig [**42**, **43**] that localization at completely resonant phases leads to reducibility for Schrödinger cocycles with rational rotation numbers for the dual model. The argument was significantly developed in [**3**, **20**, **36**, **39**].

For completely resonant phases, Jitomirskaya–Koslover–Schulteis [**25**] proved localization for $\alpha \in DC$ via a simple modification of the proof in [**23**]. We say $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ satisfies the Diophantine condition (DC) if there exist $\tau > 1$, $\kappa > 0$ such that

$$\|k\alpha\| \geq \kappa|k|^{-\tau} \quad \text{for any } k \in \mathbb{Z} \setminus \{0\},$$

where $\|x\| = \text{dist}(x, \mathbb{Z})$. Their result can be extended to α with $\beta(\alpha) = 0$ without any difficulty. In order to avoid too many concepts, if $\beta(\alpha) = 0$, we call α Diophantine. To the contrary, if $\beta(\alpha) > 0$, we call α Liouville.

Recently, there have been several remarkable sharp arithmetic transition results for all parameters. In particular, phase transitions happen in a positive Lyapunov exponent regime for Liouville frequencies [**6**, **21**, **24**, **26**, **27**]. Later, a universal (reflective) hierarchical structure of eigenfunctions was established in the localization regime [**27**, **28**] with an arithmetic condition on θ. However, all the sharp results aforementioned excluded the completely resonant phases. The purpose of this paper is to consider the missing part.

We prove Conjecture 1 for $|\lambda| > e^{3\beta}$.

THEOREM 1.1. *Suppose frequency $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ satisfies $\beta(\alpha) < \infty$. Then the AMO $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $2\theta \in \alpha\mathbb{Z} + \mathbb{Z}$ and $|\lambda| > e^{3\beta(\alpha)}$. Moreover, if ϕ is an eigenfunction, that is, $H_{\lambda,\alpha,\theta}\phi = E\phi$, then*

$$\limsup_{k\to\infty} \frac{\ln(\phi^2(k) + \phi^2(k-1))}{2|k|} \leq -(\ln\lambda - 3\beta).$$

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Remark 1.2. For α with $\beta(\alpha) = +\infty$, $H_{\lambda,\alpha,\theta}$ has a purely singular continuous spectrum [**17**, **44**] if $|\lambda| > 1$.

Now we will discuss the history of Conjecture 1 and also our approach to the proof of Theorem 1.1. We state another related conjecture first. Define

$$\delta(\alpha, \theta) = \limsup_{n\to\infty} \frac{-\ln \|2\theta + n\alpha\|}{|n|}.$$

Conjecture 2. Jitomirskaya [**22**] conjectured the following.

(2a) (Diophantine phase) $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $|\lambda| > e^{\beta(\alpha)}$ and $\delta(\alpha, \theta) = 0$, and $H_{\lambda,\alpha,\theta}$ has a purely singular continuous spectrum for all θ if $1 < |\lambda| < e^{\beta(\alpha)}$.

(2b) (Diophantine frequency) Suppose $\beta(\alpha) = 0$. $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $|\lambda| > e^{\delta(\alpha,\theta)}$ and has a purely singular continuous spectrum if $1 < |\lambda| < e^{\delta(\alpha,\theta)}$.

Notice that $\beta(\alpha) = 0$ for almost every α, and $\delta(\alpha, \theta) = 0$ for almost every θ and fixed α.

The case $\beta(\alpha) = 0$ and $\delta(\alpha, \theta) = 0$ of Conjecture 2 was solved by Jitomirskaya in her pioneering paper [**23**]. Avila and Jitomirskaya [**2**] proved the localization part for Diophantine phases in the regime $|\lambda| > e^{(16/9)\beta}$, which was a key step in solving the ten Martini problem. Liu and Yuan followed their proof and extended the result to $|\lambda| > e^{(3/2)\beta}$ [**37**]. Liu and Yuan [**38**] further developed Avila–Jitomirskaya's technique in [**2**] and verified Conjecture 1 in the regime $|\lambda| > e^{7\beta}$. Here, $\frac{3}{2}$ and 7 are the limits of the method in [**2**].

Recently, Avila–You–Zhou [**6**] proved the singular continuous spectrum part of Conjecture 2a, as well as the measure-theoretic version of Conjecture 2a: $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization for $|\lambda| > e^{\beta}$ and almost every θ. See also [**24**]. Diophantine frequency (Conjecture 2b) and the localization part of the Diophantine phase (Conjecture 2a) were proved by Jitomirskaya and Liu [**27**, **28**], who developed Avila–Jitomirskaya's scheme and found a better way of dealing with the phase and frequency resonances.

One of the ideas of [**27**, **28**] is that they treat the values of the generalized eigenfunction at resonant points as variables and obtain the localization via solving the equations of resonant points, not just by using block expansion and the exponential decay of the Green functions. We should mention that the Green's functions are not necessarily exponential decay in [**27**, **28**] and also in the present paper.

We want to explain the motivations for Conjectures 1 and 2, and also explain the new challenge for completely resonant phases. For the Diophantine frequency $\beta(\alpha) = 0$, the resonant points come from the phase resonances. Roughly speaking, if $\|2\theta + k\alpha\|$ is small, k is called a phase resonance. For the Diophantine phase $\delta(\alpha, \theta) = 0$, the resonant points come from the frequency resonances. Roughly speaking, if $\|k\alpha\|$ is small, k is called a frequency resonance. Phase resonances lead to reflective repetitions of potential [**30**] and frequency resonances lead to repetitions of potential [**17**, **44**]. Indeed, all known proofs of localization, for example [**10–12**, **16**], are based, in one way or another, on avoiding resonances and removing resonance-producing parameters. For AMOs and $|\lambda| > 1$, the Lyapunov exponent is $\ln |\lambda|$. Conjecture 2 says that the competition between

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the Anderson localization and the singular continuous spectrum is actually the competition between the Lyapunov exponent and the strength of the resonance. Conjecture 2a says that, without phase resonances, if the Lyapunov exponent beats the frequency resonance, then Anderson localization follows. Otherwise, $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum. Conjecture 2b says that without frequency resonances, if the Lyapunov exponent beats the phase resonance, then Anderson localization follows. Otherwise, $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum.

For completely resonant phases, $2\theta \in \alpha\mathbb{Z} + \mathbb{Z}$, $\delta(\alpha, \theta) = \beta(\alpha)$. Thus phase resonances and frequency resonances happen at the same time. Conjecture 1 says that if the Lyapunov exponent beats the frequency resonance plus the phase resonance, then the Anderson localization follows. This is the first challenge in our paper since we need to deal with frequency and phase resonances simultaneously. The second challenge is to avoid complete resonance. In dealing with Conjecture 1, the original arguments of Jitomirskaya [**23**] do not work directly since there is complete resonance. In [**25**], Jitomirskaya–Koslover–Schulteis found a trick to avoid complete resonance by shrinking the size of the interval around zero (we refer it as the 'shrinking scale' technique). Later, the shrinking scale technique was fully explored in [**3**, **20**, **38**, **39**]. It is natural to develop the shrinking scale technique and the localization arguments in [**27**, **28**] to treat our situation. Since we shrink the scale, there is one phase resonance *and* one frequency phase resonance in a half scale. It is different from the situation in Conjecture 2, where there is one phase resonance *or* one frequency resonance in one scale. Using the full strength of the localization proof of [**27**, **28**] to treat both phase resonances and frequency resonances, one can only obtain the Anderson localization for $|\lambda| > e^{4\beta}$ in Conjecture 1, where four is the non-trivial technical limit in such approach. We bring several new ingredients that go beyond the technique of [**3**, **20**, **25**, **27**, **28**, **38**, **39**] and allow us to improve the constant to three, thus going well beyond the previous technical limit. In particular, instead of using Lagrange interpolation uniformly, we treat Lagrange interpolation individually during the process of finding the points without 'small divisors'. This gives us significantly more varieties to construct Green functions. We believe our method has wider applicability to Anderson localization.

2. *Some notation and known facts*

It is well known that in order to prove Anderson localization of $H_{\lambda,\alpha,\theta}$, we only need to show the following statements [**8**]. Assume that ϕ is a generalized function, i.e.,

$$H\phi = E\phi \quad \text{and} \quad |\phi(k)| \leq 1 + |k| \quad \text{for some } E.$$

Then there exists some constant $c > 0$ such that

$$|\phi(k)| \leq Ce^{-c|k|} \quad \text{for all } k.$$

It suffices to consider α with $0 < \beta(\alpha) < \infty$. Without loss of generality, we assume that $\lambda > e^{3\beta}$, $\theta \in \{\alpha/2, \alpha/2 + \frac{1}{2}, 0, \frac{1}{2}\}$ (shift is a unitary operator). In order to avoid too many forms of notation, we still use $2\theta \in \alpha\mathbb{Z} + \mathbb{Z}$ to represent $\theta \in \{\alpha/2, \alpha/2 + \frac{1}{2}, 0, \frac{1}{2}\}$. We also assume that $E \in \Sigma_{\lambda,\alpha}$ (denote by $\Sigma_{\lambda,\alpha}$ the spectrum of operator $H_{\lambda,\alpha,\theta}$ since the spectrum does not depend on θ). For simplicity, we usually omit the dependence on parameters E, λ, α, θ.

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Given a generalized eigenfunction ϕ of $H_{\lambda,\alpha,\theta}$, without loss of generality, assume that $\phi(0) = 1$. Our objective is to show that there exists some specific $c > 0$ such that

$$|\phi(k)| \le e^{-c|k|} \quad \text{for } k \to \infty.$$

Let us denote

$$P_k(\theta) = \det(R_{[0,k-1]}(H_{\lambda,\alpha,\theta} - E)R_{[0,k-1]}).$$

It is easy to see that $P_k(\theta)$ is an even function of $\theta + \frac{1}{2}(k-1)\alpha$ and can be written as a polynomial of degree k in $\cos 2\pi(\theta + \frac{1}{2}(k-1)\alpha)$: i.e.,

$$P_k(\theta) = \sum_{j=0}^{k} c_j \cos^j 2\pi\left(\theta + \frac{1}{2}(k-1)\alpha\right) \triangleq Q_k\left(\cos 2\pi\left(\theta + \frac{1}{2}(k-1)\alpha\right)\right). \tag{1}$$

LEMMA 2.1. [**2**, p. 16] *The following inequality holds.*

$$\lim_{k\to\infty} \sup_{\theta\in\mathbb{R}} \frac{1}{k} \ln|P_k(\theta)| \le \ln\lambda.$$

By Cramer's rule (see [**9**, p. 15], for example) for given x_1 and $x_2 = x_1 + k - 1$, with $y \in I = [x_1, x_2] \subset \mathbb{Z}$,

$$|G_I(x_1, y)| = \left|\frac{P_{x_2-y}(\theta + (y+1)\alpha)}{P_k(\theta + x_1\alpha)}\right|, \tag{2}$$

$$|G_I(y, x_2)| = \left|\frac{P_{y-x_1}(\theta + x_1\alpha)}{P_k(\theta + x_1\alpha)}\right|. \tag{3}$$

By Lemma 2.1, the numerators in (2) and (3) can be bounded uniformly with respect to θ. Namely, for any $\varepsilon > 0$,

$$|P_n(\theta)| \le e^{(\ln\lambda+\varepsilon)n} \tag{4}$$

for large enough n.

Definition 2.2. Fix $t > 0$. A point $y \in \mathbb{Z}$ will be called (t, k) regular if there exists an interval $[x_1, x_2]$ containing y, where $x_2 = x_1 + k - 1$, such that

$$|G_{[x_1,x_2]}(y, x_i)| \le e^{-t|y-x_i|} \quad \text{and} \quad |y - x_i| \ge \tfrac{1}{7}k \quad \text{for } i = 1, 2.$$

It is easy to check that ([**9**, p. 61])

$$\phi(x) = -G_{[x_1,x_2]}(x_1, x)\phi(x_1 - 1) - G_{[x_1,x_2]}(x, x_2)\phi(x_2 + 1), \tag{5}$$

where $x \in I = [x_1, x_2] \subset \mathbb{Z}$.

Given a set $\{\theta_1, \ldots, \theta_{k+1}\}$, the Lagrange interpolation terms La_i, $i = 1, 2, \ldots, k+1$, are defined by

$$La_i = \ln \max_{x\in[-1,1]} \prod_{j=1, j\ne i}^{k+1} \frac{|x - \cos 2\pi\theta_j|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_j|}. \tag{6}$$

The following lemma is another form of Lemma 9.3 in [**2**].

LEMMA 2.3. *Given a set* $\{\theta_1, \ldots, \theta_{k+1}\}$, *there exists some* θ_i *in set* $\{\theta_1, \ldots, \theta_{k+1}\}$ *such that*

$$P_k\left(\theta_i - \frac{k-1}{2}\alpha\right) \ge \frac{e^{k\ln\lambda - La_i}}{k+1}.$$

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Proof. Otherwise, for all $i = 1, 2, \ldots, k+1$,

$$Q_k(\cos 2\pi\theta_i) = P_k\left(\theta_i - \frac{k-1}{2}\alpha\right) < \frac{e^{k\ln\lambda - La_i}}{k+1}.$$

By (1), we can write the polynomial $Q_k(x)$ in the Lagrange interpolation form at points $\cos 2\pi\theta_i$, $i = 1, 2, \ldots, k+1$. Thus

$$\begin{aligned}|Q_k(x)| &= \left|\sum_{i=1}^{k+1} Q_k(\cos 2\pi\theta_i)\frac{\prod_{j\neq i}(x - \cos 2\pi\theta_j)}{\prod_{j\neq i}(\cos 2\pi\theta_i - \cos 2\pi\theta_j)}\right| \\ &< (k+1)\frac{e^{k\ln\lambda - La_i}}{k+1}e^{La_i} = e^{k\ln\lambda}\end{aligned}$$

for all $x \in [-1, 1]$. By (1) again, $|P_k(x)| < e^{k\ln\lambda}$ for all $x \in \mathbb{R}$. However, by Herman's subharmonic function methods (see p.16 [**9**]), $\int_{\mathbb{R}/\mathbb{Z}} \ln|P_k(x)|dx \geq k\ln\lambda$. This is impossible. □

Fix a sufficiently small constant η, which will be determined later. Let $b_n = \eta q_n$. For any $y \neq 0$, we will distinguish between two cases:

(i) $\operatorname{dist}(y, q_n\mathbb{Z} + (q_n/2)\mathbb{Z}) \leq b_n$, called n-resonance; and

(ii) $\operatorname{dist}(y, q_n\mathbb{Z} + (q_n/2)\mathbb{Z}) > b_n$, called n-non-resonance.

THEOREM 2.4. [**38**] *Assume that $2\theta \in \alpha\mathbb{Z} + \mathbb{Z}$ and $\lambda > 1$.*

Suppose either:

(i) *$b_n \leq |y| < Cb_{n+1}$ for some $C > 1$ and y is n-non-resonant;*

or

(ii) *$|y| \leq Cq_n$ and* $\operatorname{dist}(y, q_n\mathbb{Z} + (q_n/2)\mathbb{Z}) > b_n$.

Let n_0 be the least positive integer such that $4q_{n-n_0} \leq \operatorname{dist}(y, q_n\mathbb{Z} + (q_n/2)\mathbb{Z}) - 2$. Let $s \in \mathbb{N}$ be the largest number such that $4sq_{n-n_0} \leq \operatorname{dist}(y, q_n\mathbb{Z} + (q_n/2)\mathbb{Z}) - 2$. Then, for any $\varepsilon > 0$ and sufficiently large n, y is $(\ln\lambda - \varepsilon, 6sq_{n-n_0} - 1)$ regular.

The proof of Theorem 2.4 builds on the ideas used in the proof of Lemma B.4 in [**27**], which is originally from [**2**]. However, it requires some modifications to avoid the completely resonant phases. Thus we give the proof in Appendix A.1.

The following lemma can be proved directly by block expansion and Theorem 2.4, which is similar to the proof of Lemma 4.1 in [**27**]. We also give the proof in the Appendix.

LEMMA 2.5. *Suppose $k \in [jq_n, (j+\frac{1}{2})q_n]$ or $k \in [(j+\frac{1}{2})q_n, (j+1)q_n]$ with $0 \leq |j| \leq C(b_{n+1}/q_n) + C$, and* $\operatorname{dist}(k, q_n\mathbb{Z} + (q_n/2)\mathbb{Z}) \geq 10\eta q_n$. *Let $d_t = |k - tq_n|$ for $t \in \{j, j+\frac{1}{2}, j+1\}$. Then, for sufficiently large n,*

$$\begin{aligned}|\phi(k)| \leq \max\{&r_j \exp\{-(\ln\lambda - \eta)(d_j - 3\eta q_n)\}, \\ &r_{j+1/2}\exp\{-(\ln\lambda - \eta)(d_{j+1/2} - 3\eta q_n)\}\},\end{aligned} \tag{7}$$

or

$$\begin{aligned}|\phi(k)| \leq \max\{&r_{j+1/2}\exp\{-(\ln\lambda - \eta)(d_{j+1/2} - 3\eta q_n)\}, \\ &r_{j+1}\exp\{-(\ln\lambda - \eta)(d_{j+1} - 3\eta q_n)\}\}.\end{aligned} \tag{8}$$

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3. *Proof of Theorem 1.1*

We always assume that n is large enough and that C is a large constant below. Denote by $\lfloor x\rfloor$ the largest integer less or equal to x.

Let

$$r_j = \sup_{|r|\le 10\eta} |\phi(jq_n + rq_n)|$$

and

$$r_{j+1/2} = \sup_{|r|\le 10\eta} \left|\phi\left(jq_n + \left\lfloor \frac{q_n}{2}\right\rfloor + rq_n\right)\right|.$$

We prove a crucial theorem first.

THEOREM 3.1. *Let* $|\ell| \le (b_{n+1}/q_n) + 3$. *Then, except for* r_0,

$$r_\ell \le \exp\{-(\ln\lambda - 3\beta - C\eta)|\ell|q_n\} \tag{9}$$

and

$$r_{\ell-1/2} \le \exp\{-(\ln\lambda - 3\beta - C\eta)|\ell - \tfrac{1}{2}|q_n\}. \tag{10}$$

LEMMA 3.2. *For any* $|j| \le 4(b_{n+1}/q_n) + 16$,

$$r_{j+1/2} \le \exp\{-\tfrac{1}{2}(\ln\lambda - 2\beta - C\eta)q_n\}\max\{r_j, r_{j+1}\}.$$

Proof. Take $\phi(jq_n + \lfloor q_n/2\rfloor + rq_n)$ with $|r| \le 10\eta$ into consideration. Without loss of generality, assume that $j \ge 0$. Let n_0 be the least positive integer such that

$$\frac{1}{\eta}q_{n-n_0} \le \left(\frac{1}{6} - 2\eta\right)q_n.$$

Let s be the largest positive integer such that $sq_{n-n_0} \le (\frac{1}{6} - 2\eta)q_n$. Then

$$s \ge \frac{1}{\eta}.$$

By the fact that $(s+1)q_{n-n_0} \ge (\frac{1}{6} - 2\eta)q_n$, one has

$$(\tfrac{1}{6} - 3\eta)q_n \le sq_{n-n_0} \le (\tfrac{1}{6} - 2\eta)q_n. \tag{11}$$

Set $I_1, I_2 \subset \mathbb{Z}$ as follows:

$$I_1 = [-2sq_{n-n_0}, -1];$$
$$I_2 = \left[jq_n + \left\lfloor\frac{q_n}{2}\right\rfloor - (s + \lfloor\eta s\rfloor)q_{n-n_0},\ jq_n + \left\lfloor\frac{q_n}{2}\right\rfloor + (s + \lfloor\eta s\rfloor)q_{n-n_0} - 1\right];$$

and let $\theta_m = \theta + m\alpha$ for $m \in I_1 \cup I_2$. The set $\{\theta_m\}_{m\in I_1\cup I_2}$ consists of $(4s + 2\lfloor\eta s\rfloor)q_{n-n_0}$ elements. Let $k = (4s + 2\lfloor\eta s\rfloor)q_{n-n_0} - 1$.

By modifying the proof of [**2**, Lemma 9.9] or [**38**, Lemma 4.1], we can prove the claim (Claim 1): for any $\varepsilon > 0$, $m \in I_1$, one has $La_m \le \varepsilon q_n$; and for any $m \in I_2$, one has $La_m \le q_n(\beta + \varepsilon)$. We also give the proof in the Appendix.

By Lemma 2.3, there exists some $j_0 \in I_1$ such that $P_k(\theta_{j_0} - ((k-1)/2)\alpha) \ge e^{k\ln\lambda - \varepsilon q_n}$, or some $j_0 \in I_2$ such that $P_k(\theta_{j_0} - ((k-1)/2)\alpha) \ge e^{k\ln\lambda - (\beta+\varepsilon)q_n}$.

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Suppose $j_0 \in I_1$, i.e., $P_k(\theta_{j_0} - ((k-1)/2)\alpha) \geq e^{k\ln\lambda - \varepsilon q_n}$. Let $I = [j_0 - 2sq_{n-n_0} - \lfloor s\eta \rfloor q_{n-n_0} + 1,\ j_0 + 2sq_{n-n_0} + \lfloor s\eta \rfloor q_{n-n_0} - 1] = [x_1, x_2]$. Denote $x_1' = x_1 - 1$ and $x_2' = x_2 + 1$.

By (2)–(4), it is easy to verify that

$$\begin{aligned} |G_I(0, x_i)| &\leq e^{(\ln\lambda+\varepsilon)(k-2-|x_i|) - k\ln\lambda + \varepsilon q_n} \\ &\leq e^{-|x_i|\ln\lambda + C\varepsilon q_n}. \end{aligned}$$

Using (5) and noticing that $|x_i| \geq (\eta s/2) q_{n-n_0}$, we obtain

$$|\phi(0)| \leq \sum_{i=1,2} e^{-(\eta s/2)q_{n-n_0}\ln\lambda + C\varepsilon q_n} |\phi(x_i')| < 1, \tag{12}$$

where the second inequality holds by (11). This is contradicted by the fact that $\phi(0) = 1$.

Thus there exists $j_0 \in I_2$ such that $P_k(\theta_{j_0} - ((k-1)/2)\alpha) \geq e^{k\ln\lambda - (\beta+\varepsilon)q_n}$. Let $I = [j_0 - 2sq_{n-n_0} - \lfloor s\eta \rfloor q_{n-n_0} + 1,\ j_0 + 2sq_{n-n_0} + \lfloor s\eta \rfloor q_{n-n_0} - 1] = [x_1, x_2]$. By (2), (3) and (4) again,

$$|G_I(p, x_i)| \leq e^{(\ln\lambda+\varepsilon)(k-2-|p-x_i|) - k\ln\lambda + \beta q_n + \varepsilon q_n}, \tag{13}$$

where $p = jq_n + \lfloor q_n/2 \rfloor + rq_n$. Using (5), we obtain

$$|\phi(p)| \leq \sum_{i=1,2} e^{(\beta + C\eta)q_n} |\phi(x_i')| e^{-|p-x_i|\ln\lambda}. \tag{14}$$

Let $d_{i,i_1,i_2} = |x_i - i_1 q_n - i_2(q_n/2)|$, where $i = 1, 2$, $i_1 \in \mathbb{Z}$ and $i_2 = 0, 1$. If $d_{i,i_1,i_2} \geq 10\eta q_n$, then we replace $\phi(x_i)$ in (14) with (7) (or (8)). If $d_{i,i_1,i_2} \leq 10\eta q_n$, then we replace $\phi(x_i')$ in (14) with $r_{i_1+(i_2/2)}$. Then

$$\begin{aligned} r_{j+1/2} \leq \max\{&\exp\{-\tfrac{1}{2}(\ln\lambda - 2\beta - C\eta)q_n\} r_j, \exp\{-\tfrac{1}{2}(\ln\lambda - 2\beta - C\eta)q_n\} r_{j+1}, \\ &\exp\{-2sq_{n-n_0}\ln\lambda + \beta q_n + C\eta q_n\} r_{j+1/2}\}. \end{aligned} \tag{15}$$

By (11),

$$\begin{aligned} -2sq_{n-n_0}\ln\lambda + \beta q_n + C\eta q_n &< \left(-\frac{\ln\lambda}{3} + \beta + C\eta\right) q_n \\ &< 0, \end{aligned}$$

for small η. This implies that

$$r_{j+1/2} \leq \exp\{-2(s\eta + s)q_{n-n_0}\ln\lambda + \beta q_n + C\eta q_n\} r_{j+1/2}$$

cannot happen.

Thus (15) becomes

$$r_{j+1/2} \leq \max\{\exp\{-\tfrac{1}{2}(\ln\lambda - 2\beta - C\eta)q_n\} r_j, \exp\{-\tfrac{1}{2}(\ln\lambda - 2\beta - C\eta)q_n\} r_{j+1}\}. \tag{16}$$

□

LEMMA 3.3. *For* $1 \leq |j| \leq 4(b_{n+1}/q_n) + 12$,

$$r_j \leq \max\left\{\max_{t\in O}\{\exp\{-(|t|\ln\lambda - \beta - C\eta)q_n\} r_{j+t}\}, \exp\{-(\ln\lambda - 3\beta - C\eta)q_n\} r_{\pm 1}\right\}, \tag{17}$$

where $O = \{\pm\frac{3}{2}, \pm\frac{1}{2}\}$.

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Proof. It suffices to estimate $\phi(jq_n + rq_n)$ with $|j| \geq 1$ and $|r| \leq 10\eta$. Without loss of generality, assume that $j \geq 1$. Let n_0 be the least positive integer such that

$$\frac{1}{\eta} q_{n-n_0} \leq \frac{q_n}{6} - 2.$$

Let s be the largest positive integer such that $sq_{n-n_0} \leq (q_n/6) - 2$. Then $s \geq 1/\eta$.

Set $J_1, J_2, J_3 \subset \mathbb{Z}$ as follows:

$$J_1 = [-2sq_{n-n_0}, -1];$$
$$J_2 = [jq_n - 3sq_{n-n_0}, jq_n - 2sq_{n-n_0} - 1] \cup [jq_n + 2sq_{n-n_0}, jq_n + 3sq_{n-n_0} - 1];$$
$$J_3 = [jq_n - 2sq_{n-n_0}, jq_n + 2sq_{n-n_0} - 1];$$

and let $\theta_m = \theta + m\alpha$ for $m \in J_1 \cup J_2 \cup J_3$. The set $\{\theta_m\}_{m \in J_1 \cup J_2 \cup J_3}$ consists of $8sq_{n-n_0}$ elements. By modifying the proof of [**2**, Lemma 9.9] or [**38**, Lemma 4.1] again, we can prove the claim (Claim 2) that, for any $m \in J_1 \cup J_3$ and any $\varepsilon > 0$, $La_m \leq 2(\beta + \varepsilon)q_n$, and for any $m \in J_2$, $La_m \leq (\beta + \varepsilon)q_n$. We also give the details of proof in the Appendix.

Applying Lemma 2.3, there exists some j_0 with $j_0 \in J_1 \cup J_3$ such that

$$P_{8sq_{n-n_0}-1}(\theta_{j_0} - (4sq_{n-n_0} - 1)\alpha) \geq e^{8sq_{n-n_0} \ln \lambda - 2\beta q_n - \varepsilon q_n},$$

or there exists some j_0 with $j_0 \in J_2$ such that

$$P_{8sq_{n-n_0}-1}(\theta_{j_0} - (4sq_{n-n_0} - 1)\alpha) \geq e^{8sq_{n-n_0} \ln \lambda - \beta q_n - \varepsilon q_n}.$$

If $j_0 \in J_2$, let $I = [j_0 - 4sq_{n-n_0} + 1, j_0 + 4sq_{n-n_0} - 1] = [x_1, x_2]$. Then

$$|G_I(jq_n + rq_n, x_i)| \leq e^{(\ln \lambda + \eta)(8sq_{n-n_0} - 2 - |jq_n + rq_n - x_i|) - 8sq_{n-n_0} \ln \lambda + \beta q_n + C\eta q_n}. \tag{18}$$

Using (5), we obtain

$$|\phi(jq_n + rq_n)| \leq \sum_{i=1,2} e^{(\beta + C\eta)q_n} |\phi(x_i')| e^{-|jq_n + rq_n - x_i| \ln \lambda}. \tag{19}$$

Recall that $d_{i,i_1,i_2} = |x_i - i_1 q_n - i_2 q_n/2|$, where $i = 1, 2$, $i_1 \in \mathbb{Z}$ and $i_2 = 0, 1$. If $d_{i,i_1,i_2} \geq 10\eta q_n$, then we replace $\phi(x_i')$ in (19) with (7) (or (8)). If $d_{i,i_1,i_2} \leq 10\eta q_n$, then we replace $\phi(x_i')$ in (19) with $r_{i_1 + (i_2/2)}$.

Then by (19),

$$r_j \leq \exp\{\beta q_n + C\eta q_n\} \max \left\{ \max_{t \in O} \{\exp\{-|t| q_n \ln \lambda\} r_{j+t}, \exp\{-2sq_{n-n_0} \ln \lambda\} r_j\} \right\},$$

where $O = \pm\frac{3}{2}, \pm 1, \pm\frac{1}{2}$.

Note that $sq_{n-n_0} \geq (1 - \eta)\frac{1}{6}q_n$ (using $(s+1)q_{n-n_0} > \frac{1}{6}q_n - 2$ and $s \geq 1/\eta$). Then

$$r_j \leq \exp\{\beta q_n + C\eta q_n\} \exp\{-2sq_{n-n_0} \ln \lambda\} r_j$$

cannot happen since $\ln \lambda > 3\beta$.

Thus

$$r_j \leq \max_{t \in O} \{\exp\{\beta q_n + C\eta q_n - |t| q_n \ln \lambda\} r_{j+t}\},$$

where $O = \pm\frac{3}{2}, \pm 1, \pm\frac{1}{2}$. This implies (17).

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If $j_0 \in J_3$, by the same arguments,

$$r_j \leq \max_{t\in\{\pm 1,\pm 1/2\}} \{\exp\{2\beta q_n + C\eta q_n - |t| q_n \ln\lambda\} r_{j+t}\}.$$

Using the estimate of $r_{j\pm 1/2}$ in Lemma 3.2,

$$r_j \leq \exp\{-(\ln\lambda - 3\beta - C\eta)q_n\} \max\{r_{j\pm 1}, r_j\}.$$

By the same reason,

$$r_j \leq \exp\{-(\ln\lambda - 3\beta - C\eta)q_n\} r_j$$

cannot happen. Thus

$$r_j \leq \exp\{-(\ln\lambda - 3\beta - C\eta)q_n\} r_{j\pm 1}. \tag{20}$$

This also implies (17).

If $j_0 \in J_1$, then (20) holds for $j = 0$, which will lead to $|\phi(0)| < 1$. This is impossible. □

3.1. *Proof of Theorem 3.1.* By Lemmas 3.2 and 3.3, for $1 \leq j \leq 2(b_{n+1}/q_n) + 4$,

$$r_{j-1/2} \leq \exp\{-\tfrac{1}{2}(\ln\lambda - 3\beta - C\eta)q_n\} \max\{r_{j-1}, r_j\} \tag{21}$$

and

$$r_j \leq \max_{t\in O}\{\exp\{-|t|(\ln\lambda - 3\beta - C\eta)q_n\} r_{j+t}\}, \tag{22}$$

where $O = \{\pm\frac{3}{2}, \pm 1, \pm\frac{1}{2}\}$. For $-b_{n+1}/q_n - 3 \leq j \leq -1$,

$$r_{j+1/2} \leq \exp\{-\tfrac{1}{2}(\ln\lambda - 3\beta - C\eta)q_n\} \max\{r_{j+1}, r_j\} \tag{23}$$

and

$$r_j \leq \max_{t\in O}\{\exp\{-|t|(\ln\lambda - 3\beta - C\eta)q_n\} r_{j+t}\}. \tag{24}$$

Suppose $\ell > 0$. By letting $j = \ell$ in (22) and (21) and iterating 2ℓ times or until $j \leq 1$, we obtain

$$r_\ell \leq (2\ell + 2)q_n \exp\{-(\ln\lambda - 3\beta - C\eta)\ell q_n\} \tag{25}$$

and

$$r_{\ell-1/2} \leq (2\ell + 2)q_n \exp\{-(\ln\lambda - 3\beta - C\eta)(\ell - \tfrac{1}{2})q_n\}. \tag{26}$$

Notice that we have used the fact that $|r_j| \leq (|j| + 2)q_n$ and $|r_{j-1/2}| \leq (|j - \frac{1}{2}| + 2)q_n$.

Suppose $\ell < 0$. By letting $j = \ell$ in (24) and (23) and iterating $2|\ell|$ times or until $j \geq -1$, we obtain

$$r_\ell \leq (2\ell + 2)q_n \exp\{-(\ln\lambda - 3\beta - C\eta)|\ell| q_n\} \tag{27}$$

and

$$r_{\ell+1/2} \leq (2\ell + 2)q_n \exp\{-(\ln\lambda - 3\beta - C\eta)|\ell + \tfrac{1}{2}| q_n\}. \tag{28}$$

Now Theorem 3.1 follows from (25)–(28).

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3.2. *Proof of Theorem 1.1.* Without loss of generality, we assume that $k > 0$. Let $\eta > 0$ be much smaller than $\ln\lambda - 3\beta$. For any k, let n be such that $b_n \le k < b_{n+1}$.

Case 1. $\mathrm{dist}(k, q_n\mathbb{Z} + (q_n/2)\mathbb{Z}) \le 10\eta q_n$.

In this case, applying Theorem 3.1,

$$|\phi(k)|, \quad |\phi(k-1)| \le \exp\{-(\ln\lambda - 3\beta - C\eta)|k|\}. \tag{29}$$

Case 2. $\mathrm{dist}(k, q_n\mathbb{Z} + (q_n/2)\mathbb{Z}) \ge 10\eta q_n$.

Let $0 \le j \le b_{n+1}/q_n$ such that $k \in [jq_n, (j+\frac{1}{2})q_n]$ or $k \in [(j+\frac{1}{2})q_n, (j+1)q_n]$. By Lemma 2.5 and Theorem 3.1, one also has

$$|\phi(k)|, \quad |\phi(k-1)| \le \exp\{-(\ln\lambda - 3\beta - C\eta)|k|\}. \tag{30}$$

By (29), (30) and letting $\eta \to 0$,

$$\limsup_{k\to\infty} \frac{\ln(\phi^2(k) + \phi^2(k-1))}{2|k|} \le -(\ln\lambda - 3\beta).$$

We have finished the proof.

A. *Appendix. Proof of Theorem 2.4, Claims 1 and 2*

Let p_n/q_n be the continued fraction approximation to α. Then

$$\text{for all } 1 \le k < q_{n+1}, \quad \mathrm{dist}(k\alpha, \mathbb{Z}) \ge |q_n\alpha - p_n| \tag{A.1}$$

and

$$\frac{1}{2q_{n+1}} \le |q_n\alpha - p_n| \le \frac{1}{q_{n+1}}. \tag{A.2}$$

LEMMA A.1. [**2**, Lemma 9.7] *Let* $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $x \in \mathbb{R}$ *and* $0 \le \ell_0 \le q_n - 1$ *be such that* $|\sin\pi(x + \ell_0\alpha)| = \inf_{0\le\ell\le q_n-1} |\sin\pi(x+\ell\alpha)|$. *Then, for some absolute constant* $C > 0$,

$$-C\ln q_n \le \sum_{\substack{\ell=0\\ \ell\ne\ell_0}}^{q_n-1} \ln|\sin\pi(x+\ell\alpha)| + (q_n-1)\ln 2 \le C\ln q_n. \tag{A.3}$$

A.1. *Proof of Theorem 2.4.* We only give the proof of Case 1: $b_n \le |y| < Cb_{n+1}$ is non-resonant.

By the definition of s and n_0, we have $4sq_{n-n_0} \le \mathrm{dist}(y, q_n\mathbb{Z}) - 2$ and $4q_{n-n_0+1} > \mathrm{dist}(y, q_n\mathbb{Z}) - 2$. This leads to $sq_{n-n_0} \le q_{n-n_0+1}$. Set $I_1, I_2 \subset \mathbb{Z}$ as follows:

$$I_1 = [-2sq_{n-n_0}, -1];$$
$$I_2 = [y - 2sq_{n-n_0}, y + 2sq_{n-n_0} - 1];$$

and let $\theta_j = \theta + j\alpha$ for $j \in I_1 \cup I_2$. The set $\{\theta_j\}_{j\in I_1\cup I_2}$ consists of $6sq_{n-n_0}$ elements.

Let $k = 6sq_{n-n_0} - 1$. We estimate La_i first. For this reason, let $x = \cos 2\pi a$ and take the logarithm in (6). Then

$$\ln \prod_{\substack{j\in I_1\cup I_2\\ j\ne i}} \frac{|\cos 2\pi a - \cos 2\pi\theta_j|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_j|}$$
$$= \sum_{\substack{j\in I_1\cup I_2\\ j\ne i}} \ln|\cos 2\pi a - \cos 2\pi\theta_j| - \sum_{\substack{j\in I_1\cup I_2\\ j\ne i}} \ln|\cos 2\pi\theta_i - \cos 2\pi\theta_j|.$$

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We start to estimate $\sum_{j\in I_1\cup I_2, j\neq i} \ln|\cos 2\pi a - \cos 2\pi\theta_j|$. Obviously,

$$\begin{aligned}
&\sum_{\substack{j\in I_1\cup I_2\\ j\neq i}} \ln|\cos 2\pi a - \cos 2\pi\theta_j| \\
&= \sum_{\substack{j\in I_1\cup I_2\\ j\neq i}} \ln|\sin\pi(a+\theta_j)| + \sum_{\substack{j\in I_1\cup I_2\\ j\neq i}} \ln|\sin\pi(a-\theta_j)| + (6sq_{n-n_0}-1)\ln 2 \\
&= \Sigma_+ + \Sigma_- + (6sq_{n-n_0}-1)\ln 2.
\end{aligned}$$

Both Σ_+ and Σ_- consist of $6s$ terms of the form of (A.3), plus 6s terms of the form

$$\ln \min_{j=0,1,\ldots,q_{n-n_0}} |\sin\pi(x+j\alpha)|,$$

minus $\ln|\sin\pi(a\pm\theta_i)|$. Thus, using (A.3) 6s times of Σ_+ and Σ_-, respectively, gives

$$\sum_{\substack{j\in I_1\cup I_2\\ j\neq i}} \ln|\cos 2\pi a - \cos 2\pi\theta_j| \leq -6sq_{n-n_0}\ln 2 + Cs\ln q_{n-n_0}. \tag{A.4}$$

Let $a=\theta_i$. We obtain

$$\begin{aligned}
&\sum_{\substack{j\in I_1\cup I_2\\ j\neq i}} \ln|\cos 2\pi \theta_i - \cos 2\pi\theta_j| \\
&= \sum_{\substack{j\in I_1\cup I_2\\ j\neq i}} \ln|\sin\pi(\theta_i+\theta_j)| + \sum_{\substack{j\in I_1\cup I_2\\ j\neq i}} \ln|\sin\pi(\theta_i-\theta_j)| + (6sq_{n-n_0}-1)\ln 2 \\
&= \Sigma_+ + \Sigma_- + (6sq_{n-n_0}-1)\ln 2,
\end{aligned} \tag{A.5}$$

where

$$\Sigma_+ = \sum_{\substack{j\in I_1\cup I_2\\ j\neq i}} \ln|\sin\pi(2\theta+(i+j)\alpha)|$$

and

$$\Sigma_- = \sum_{\substack{j\in I_1\cup I_2\\ j\neq i}} \ln|\sin\pi(i-j)\alpha|.$$

We will estimate Σ_+. Set $J_1=[-2s,-1]$ and $J_2=[0,4s-1]$, which are two adjacent disjoint intervals of length $2s$ and $4s$, respectively. Then $I_1\cup I_2$ can be represented as a disjoint union of segments B_j, $j\in J_1\cup J_2$, each of length q_{n-n_0}.

Applying (A.3) to each B_j, we obtain

$$\Sigma_+ \geq -6sq_{n-n_0}\ln 2 + \sum_{j\in J_1\cup J_2} \ln|\sin\pi\hat{\theta}_j| - Cs\ln q_{n-n_0} - \ln|\sin 2\pi(\theta+i\alpha)|, \tag{A.6}$$

where

$$|\sin\pi\hat{\theta}_j| = \min_{\ell\in B_j} |\sin\pi(2\theta+(\ell+i)\alpha)|. \tag{A.7}$$

By the construction of I_1 and I_2,

$$2\theta+(\ell+i)\alpha = \pm(mq_n\alpha + r_1\alpha) \mod \mathbb{Z} \tag{A.8}$$

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or

$$2\theta + (\ell + i)\alpha = \pm r_2\alpha \mod \mathbb{Z}, \tag{A.9}$$

where $0 \le m \le C(b_{n+1}/q_n)$ and $1 \le r_i < q_n$, $i = 1, 2$.

By (A.1) and (A.2), it follows that

$$\begin{aligned}\min_{\ell\in I_1\cup I_2} \ln|\sin\pi(2\theta + (\ell+i)\alpha)| &\ge C\ln\left(\|r_i\alpha\|_{\mathbb{R}/\mathbb{Z}} - \frac{\Delta_{n-1}}{2}\right)\\ &\ge C\ln\left(\Delta_{n-1} - \frac{\Delta_{n-1}}{2}\right)\\ &\ge \ln C\frac{\Delta_{n-1}}{2} \ge -C\ln q_n,\end{aligned} \tag{A.10}$$

since $\|mq_n\alpha\|_{\mathbb{R}/\mathbb{Z}} \le C(\eta q_{n+1}/q_n)\Delta_n \le (\Delta_{n-1}/2)$.

By the construction of I_1 and I_2, we also have

$$\min_{\substack{i\ne j\\ i,j\in I_1\cup I_2}} \ln|\sin\pi(j-i)\alpha)| \ge -C\ln q_n. \tag{A.11}$$

Next, we estimate $\sum_{j\in J_1}\ln|\sin\pi\hat{\theta}_j|$. Assume that $\hat{\theta}_{j+1} = \hat{\theta}_j + q_{n-n_0}\alpha$ for every $j, j+1 \in J_1$. In this case, for any $i, j \in J_1$ and $i \ne j$,

$$\|\hat{\theta}_i - \hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} \ge \|q_{n-n_0}\alpha\|_{\mathbb{R}/\mathbb{Z}}. \tag{A.12}$$

By the Stirling formula, (A.10) and (A.12),

$$\begin{aligned}\sum_{j\in J_1}\ln|\sin 2\pi\hat{\theta}_j| &> 2\sum_{j=1}^{s}\ln(j\Delta_{n-n_0}) - C\ln q_n\\ &> 2s\ln\frac{s}{q_{n-n_0+1}} - C\ln q_n - Cs.\end{aligned} \tag{A.13}$$

In the other cases, decompose J_1 into maximal intervals T_κ such that, for $j, j+1 \in T_\kappa$, we have $\hat{\theta}_{j+1} = \hat{\theta}_j + q_{n-n_0}\alpha$. Notice that the boundary points of an interval T_κ are either boundary points of J_1 or satisfy $\|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} + \Delta_{n-n_0} \ge \Delta_{n-n_0-1}/2$. This follows from the fact that if $0 < |z| < q_{n-n_0}$, then $\|\hat{\theta}_j + q_{n-n_0}\alpha\|_{\mathbb{R}/\mathbb{Z}} \le \|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} + \Delta_{n-n_0}$ and $\|\hat{\theta}_j + (z + q_{n-n_0})\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge \|z\alpha\|_{\mathbb{R}/\mathbb{Z}} - \|\hat{\theta}_j + q_{n-n_0}\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge \Delta_{n-n_0-1} - \|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} - \Delta_{n-n_0}$. Assuming that $T_\kappa \ne J_1$, there exists $j \in T_\kappa$ such that $\|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} \ge (\Delta_{n-n_0-1}/2) - \Delta_{n-n_0}$.

If T_κ contains some j with $\|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} < \Delta_{n-n_0-1}/10$, then

$$\begin{aligned}|T_\kappa| &\ge \frac{(\Delta_{n-n_0-1}/2) - \Delta_{n-n_0} - (\Delta_{n-n_0-1}/10)}{\Delta_{n-n_0}}\\ &\ge \frac{1}{4}\frac{\Delta_{n-n_0-1}}{\Delta_{n-n_0}} - 1 \ge \frac{s}{8} - 1,\end{aligned} \tag{A.14}$$

since $sq_{n-n_0} \le q_{n-n_0+1}$, where $|T_\kappa| = b - a + 1$ for $T_\kappa = [a, b]$. For such T_κ, a similar estimate to (A.13) gives

$$\begin{aligned}\sum_{j\in T_\kappa}\ln|\sin\pi\hat{\theta}_j| &\ge |T_\kappa|\ln\frac{|T_\kappa|}{q_{n-n_0+1}} - Cs - C\ln q_n\\ &\ge |T_\kappa|\ln\frac{s}{q_{n-n_0+1}} - Cs - C\ln q_n.\end{aligned} \tag{A.15}$$

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If T_κ does not contain any j with $\|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} < \Delta_{n-n_0-1}/10$, then, by (A.2),

$$\sum_{j\in T_\kappa} \ln|\sin\pi\hat{\theta}_j| \geq -|T_\kappa| \ln q_{n-n_0} - C|T_\kappa|$$
$$\geq |T_\kappa| \ln \frac{s}{q_{n-n_0+1}} - C|T_\kappa|. \tag{A.16}$$

By (A.15) and (A.16),

$$\sum_{j\in J_1} \ln|\sin\pi\hat{\theta}_j| \geq 2s \ln \frac{s}{q_{n-n_0+1}} - Cs - C\ln q_n. \tag{A.17}$$

Similarly,

$$\sum_{j\in J_2} \ln|\sin\pi\hat{\theta}_j| \geq 4s \ln \frac{s}{q_{n-n_0+1}} - Cs - C\ln q_n. \tag{A.18}$$

Putting (A.6), (A.17) and (A.18) together gives

$$\Sigma_+ > -6sq_{n-n_0}\ln 2 + 6s\ln\frac{s}{q_{n-n_0+1}} - Cs\ln q_{n-n_0} - C\ln q_n. \tag{A.19}$$

Now we start to estimate Σ_-.

By replacing (A.10) with (A.11), and following the proof of (A.19), we obtain

$$\Sigma_- > -6sq_{n-n_0}\ln 2 + 6s\ln\frac{s}{q_{n-n_0+1}} - Cs\ln q_{n-n_0} - C\ln q_n. \tag{A.20}$$

By (A.5), (A.19) and (A.20), we obtain

$$\sum_{\substack{j\in I_1\cup I_2\\ j\neq i}} \ln|\cos 2\pi\theta_i - \cos 2\pi\theta_j|$$
$$\geq -6sq_{n-n_0}\ln 2 + 6s\ln\frac{s}{q_{n-n_0+1}} - Cs\ln q_{n-n_0} - C\ln q_n. \tag{A.21}$$

By (A.4) and (A.21), for any $i \in I_1 \cup I_2$,

$$\prod_{\substack{j\in I_1\cup I_2\\ j\neq i}} \frac{|x - \cos 2\pi\theta_j|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_j|} \leq e^{6sq_{n-n_0}(-2\ln(s/q_{n-n_0+1})/q_{n-n_0}+\varepsilon)}.$$

Using the fact that $4(s+1)q_{n-n_0} > \eta q_n - 2$, one has, for any $i \in I_1 \cup I_2$,

$$\prod_{\substack{j\in I_1\cup I_2\\ j\neq i}} \frac{|x - \cos 2\pi\theta_j|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_j|} \leq e^{sq_{n-n_0}\varepsilon}. \tag{A.22}$$

This implies that $La_i \leq \varepsilon sq_{n-n_0}$ for any $i = 1, 2, \ldots, k+1$, where $k = 6sq_{n-n_0} - 1$.

Applying Lemma 2.3, there exists some j_0 with $j_0 \in I_1 \cup I_2$ such that

$$P_{k-1}\left(\theta_{j_0} - \frac{k-1}{2}\alpha\right) \geq e^{(\ln\lambda - \varepsilon)k}.$$

Firstly, we assume that $j_0 \in I_2$.

Set $I = [j_0 - 3sq_{n-n_0} + 1,\ j_0 + 3sq_{n-n_0} - 1] = [x_1, x_2]$. By (2), (3) and (4) again,

$$|G_I(y, x_i)| \leq \exp\{(\ln\lambda + \varepsilon)(6sq_{n-n_0} - 1 - |y - x_i|) - 6sq_{n-n_0}(\ln\lambda - \varepsilon)\}.$$

Notice that $|y - x_i| \geq sq_{n-n_0}$. We obtain

$$|G_I(y, x_i)| \leq \exp\{-(\ln\lambda - \varepsilon)|y - x_i|\}. \tag{A.23}$$

If $j_0 \in I_1$, we may let $y = 0$ in (A.23). By (5), we get

$$|\phi(0)| \leq 6sq_{n-n_0}\exp\{-(\ln\lambda - \varepsilon)sq_{n-n_0}\}.$$

This contradicts $\phi(0) = 1$. Thus $j_0 \in I_2$, and the theorem follows from (A.23).

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A.2. *Proof of Claim 1.* By the construction of I_1 and I_2 in Claim 1, (A.1) and (A.2), we have, for $i \in I_1$,

$$\min_{\ell \in I_1 \cup I_2} \ln |\sin\pi(2\theta + (\ell + i)\alpha)| \geq -C \ln q_n \tag{A.24}$$

and

$$\min_{\substack{i \neq j \\ j \in I_1 \cup I_2}} \ln |\sin\pi(j - i)\alpha)| \geq -C \ln q_n. \tag{A.25}$$

Replacing (A.10) with (A.24) and (A.11) with (A.25), and following the proof of (A.22), we can show that, for any $i \in I_1$,

$$\prod_{\substack{j \in I_1 \cup I_2 \\ j \neq i}} \frac{|x - \cos 2\pi\theta_j|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_j|} \leq e^{\varepsilon s q_{n-n_0}}.$$

This implies that, for $i \in I_1$, $La_i \leq \varepsilon q_n$. By the construction of I_1 and I_2 in Claim 1, (A.1) and (A.2) again, we have, for $i \in I_2$,

$$\min_{\ell \in I_1 \cup I_2} \ln |\sin\pi(2\theta + (\ell + i)\alpha)|_{\mathbb{R}/\mathbb{Z}} \geq -\beta q_n - C \ln q_n \tag{A.26}$$

and

$$\min_{\substack{i \neq j \\ j \in I_1 \cup I_2}} \ln |\sin\pi(j - i)\alpha)| \geq -C \ln q_n. \tag{A.27}$$

We should mention that, for each $i \in I_2$, there is exact one $j \in I_1 \cup I_2$ such that the lower bound of (A.26) can be achieved.

Replacing (A.10) with (A.26) and (A.11) with (A.27), and following the proof of (A.22), we can show that, for any $i \in I_1$,

$$\prod_{\substack{j \in I_1 \cup I_2 \\ j \neq i}} \frac{|x - \cos 2\pi\theta_j|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_j|} \leq e^{\varepsilon s q_{n-n_0} + \beta q_n}.$$

This implies that, for any $i \in I_2$, $La_i \leq q_n(\beta + \varepsilon)$.

A.3. *Proof of Claim 2.* Let $J_3^1 = [jq_n - 2sq_{n-n_0}, jq_n - 1]$ and

$$J_3^2 = [jq_n, +2sq_{n-n_0} - 1]$$

so that $J_3 = J_3^1 \cup J_3^2$. Let $I = J_1 \cup J_2 \cup J_2$.

Case 1. $i \in J_1 \cup J_3^1$. By the construction of J_1, J_2 and J_3 in Claim 2, and by (A.1), (A.2),

$$\min_{\ell \in I} \ln |\sin\pi(2\theta + (\ell + i)\alpha)| \geq -\beta q_n - C \ln q_n \tag{A.28}$$

and

$$\min_{\substack{i \neq j \\ j \in I}} \ln |\sin\pi(j - i)\alpha)| \geq -\beta q_n - C \ln q_n. \tag{A.29}$$

Moreover, there are exactly two ℓ, $j \in I$ such that the lower bound of (A.28) can be achieved for ℓ and the lower bound of (A.29) can be achieved for j.

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Case 2. $i \in J_1 \cup J_3^2$. By the same reason,

$$\min_{\ell \in I} \ln |\sin \pi (2\theta + (\ell + i)\alpha)| \geq -\beta q_n - C \ln q_n \tag{A.30}$$

and

$$\min_{\substack{i \neq j \\ j \in I}} \ln |\sin \pi (j - i)\alpha)| \geq -C \ln q_n. \tag{A.31}$$

Moreover, there are exactly two $\ell_1, \ell_2 \in I$ such that the lower bound of (A.30) can be achieved for both ℓ_1 and ℓ_2.

Case 3. $i \in J_2$. By the same reason,

$$\min_{\ell \in I} \ln |\sin \pi (2\theta + (\ell + i)\alpha)| \geq -\beta q_n - C \ln q_n \tag{A.32}$$

and

$$\min_{\substack{i \neq j \\ j \in I}} \ln |\sin \pi (j - i)\alpha)| \geq -C \ln q_n. \tag{A.33}$$

Moreover, there is exactly one $\ell \in I$ such that the lower bound of (A.32) can be achieved for ℓ.

Now following the proof of the Claim 1, we can prove Claim 2.

B. *Appendix. Proof of Lemma 2.5*

Without loss of generality, we assume that $k \in [jq_n, (j + \frac{1}{2})q_n]$ and $j \geq 0$. Let $d_j = k - jq_n$ and $d_{j+1/2} = (j + \frac{1}{2})q_n - k$.

For any $y \in [jq_n + \eta q_n, (j + \frac{1}{2})q_n - \eta q_n]$, by Theorem 2.4, y is regular with $\tau = \ln \lambda - \eta$. Therefore there exists an interval $I(y) = [x_1, x_2] \subset [jq_n, (j + \frac{1}{2})q_n]$ such that $y \in I(y)$,

$$\text{dist}(y, \partial I(y)) \geq \frac{1}{7}|I(y)| \geq \frac{q_{n-n_0}}{2} \tag{B.1}$$

and

$$|G_{I(y)}(y, x_i)| \leq e^{-(\ln \lambda - \eta)|y - x_i|}, \quad i = 1, 2, \tag{B.2}$$

where $\partial I(y)$ is the boundary of the interval $I(y)$, i.e.,$\{x_1, x_2\}$, and $|I(y)|$ is the size of $I(y) \cap \mathbb{Z}$, i.e., $|I(y)| = x_2 - x_1 + 1$. For $z \in \partial I(y)$, let z' be the neighbor of z, (i.e., $|z - z'| = 1$) not belonging to $I(y)$.

If $x_2 + 1 \leq (j + \frac{1}{2})q_n - \eta q_n$ or $x_1 - 1 \geq jq_n + \eta q_n$, we can expand $\phi(x_2 + 1)$ or $\phi(x_1 - 1)$ using (5). We can continue this process until we arrive at z such that $z + 1 > (j + \frac{1}{2})q_n - \eta q_n$ or $z - 1 < jq_n + \eta q_n$, or until the iterating number reaches $\lfloor 4q_n/q_{n-n_0} \rfloor$. Thus, by (5),

$$\phi(k) = \sum_{s; z_{i+1} \in \partial I(z_i')} G_{I(k)}(k, z_1) G_{I(z_1')}(z_1', z_2) \cdots G_{I(z_s')}(z_s', z_{s+1}) \phi(z_{s+1}'), \tag{B.3}$$

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where in each term of the summation one has $jq_n + \eta q_n + 1 \le z_i \le (j + \frac{1}{2})q_n - \eta q_n - 1$, $i = 1, \ldots, s$, and either $z_{s+1} \notin [jq_n + \eta q_n + 1, (j + \frac{1}{2})q_n - \eta q_n - 1]$, $s + 1 < \lfloor 4q_n/q_{n-n_0} \rfloor$ or $s + 1 = \lfloor 4q_n/q_{n-n_0} \rfloor$. We should mention that $z_{s+1} \in [jq_n, (j + \frac{1}{2})q_n]$.

If $z_{s+1} \in [jq_n, jq_n + \eta q_n]$, $s + 1 < \lfloor 4q_n/q_{n-n_0} \rfloor$, this implies that

$$|\phi(z'_{s+1})| \le r_j.$$

By (B.2),

$$\begin{aligned} &|G_{I(k)}(k, z_1)G_{I(z'_1)}(z'_1, z_2) \cdots G_{I(z'_s)}(z'_s, z_{s+1})\phi(z'_{s+1})| \\ &\quad \le r_j = e^{-(\ln \lambda - \eta)(|k - z_1| + \sum_{i=1}^{s} |z'_i - z_{i+1}|)} \\ &\quad \le r_j e^{-(\ln \lambda - \eta)(|k - z_{s+1}| - (s+1))} \\ &\quad \le r_j e^{-(\ln \lambda - \eta)(d_j - 2\eta q_n - 4 - (4q_n/q_{n-n_0}))}. \end{aligned} \tag{B.4}$$

If $z_{s+1} \in [(j + \frac{1}{2})q_n - \eta q_n, (j + \frac{1}{2})q_n]$, $s + 1 < \lfloor 4q_n/q_{n-n_0} \rfloor$, then, by the same arguments,

$$\begin{aligned} &|G_{I(k)}(k, z_1)G_{I(z'_1)}(z'_1, z_2) \cdots G_{I(z'_s)}(z'_s, z_{s+1})\phi(z'_{s+1})| \\ &\quad \le r_{j+1/2} e^{-(\ln \lambda - \eta)(d_{j+1/2} - 2\eta q_n - 4 - (4q_n/q_{n-n_0}))}. \end{aligned} \tag{B.5}$$

If $s + 1 = \lfloor 4q_n/q_{n-n_0} \rfloor$, using (B.1) and (B.2), we obtain

$$\begin{aligned} &|G_{I(k)}(k, z_1)G_{I(z'_1)}(z'_1, z_2) \cdots G_{I(z'_s)}(z'_s, z_{s+1})\phi(z'_{s+1})| \\ &\quad \le e^{-(\ln \lambda - \eta)(1/2)q_{n-n_0} \lfloor 4q_n/q_{n-n_0} \rfloor} |\phi(z'_{s+1})|. \end{aligned} \tag{B.6}$$

Notice that the total number of terms in (B.3) is at most $2^{\lfloor 4q_n/q_{n-n_0} \rfloor}$ and $d_j, d_{j+1/2} \ge 10\eta q_n$. By (B.4)–(B.6),

$$\begin{aligned} |\phi(k)| \le \max \Big\{ & r_j e^{-(\ln \lambda - \eta)(d_j - 3\eta q_n)}, r_{j+1/2} e^{-(\ln \lambda - \eta)(d_{j+1/2} - 3\eta q_n)}, \\ & e^{-(\ln \lambda - \eta)q_n} \max_{p \in [jq_n, (j+1/2)q_n]} |\phi(p)| \Big\}. \end{aligned} \tag{B.7}$$

Now we will show that, for any $p \in [jq_n, (j + \frac{1}{2})q_n]$, one has $|\phi(p)| \le \max\{r_j, r_{j+1/2}\}$. Then (B.7) implies Lemma 2.5. Otherwise, by the definition of r_j, if $|\phi(p')|$ is the largest one of $|\varphi(z)|$, $z \in [jq_n + 10\eta q_n + 1, (j + \frac{1}{2})q_n - 10\eta q_n - 1]$, then $|\phi(p')| > \max\{r_j, r_{j+1/2}\}$. Applying (B.7) to $\phi(p')$ and noticing that $\operatorname{dist}(p', q_n\mathbb{Z}) \ge 10\eta q_n$, we get

$$|\phi(p')| \le e^{-7(\ln \lambda - \eta)\eta q_n} \max\{r_j, r_{j+1/2}, |\phi(p')|\}.$$

This is impossible because $|\phi(p')| > \max\{r_j, r_{j+1/2}\}$.

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REFERENCES

[**1**] A. Avila and S. Jitomirskaya. Solving the ten Martini problem. *Mathematical Physics of Quantum Mechanics (Lecture Notes in Physics, 690)*. Springer, Berlin, 2006, pp. 5–16.

[**2**] A. Avila and S. Jitomirskaya. The ten Martini problem. *Ann. of Math.* (2) **170**(1) (2009), 303–342.

[**3**] A. Avila and S. Jitomirskaya. Almost localization and almost reducibility. *J. Eur. Math. Soc.* **12**(1) (2010), 93–131.

[**4**] A. Avila, S. Jitomirskaya and Q. Zhou. Second phase transition line. *Math. Ann.* **370**(1–2) (2018), 271–285.

[**5**] A. Avila, J. You and Q. Zhou. Dry ten Martini problem in non-critical case. *Preprint*.

[**6**] A. Avila, J. You and Q. Zhou. Sharp phase transitions for the almost Mathieu operator. *Duke Math. J.* **166**(14) (2017), 2697–2718.

[**7**] J. Bellissard, R. Lima and D. Testard. Almost periodic Schrödinger operators. *Mathematics+ Physics (Lectures on Recent Results, 1)*. World Scientific, Singapore, 1985, pp. 1–64.

[**8**] J. M. Berezanskii. *Expansions in Eigenfunctions of Self-Adjoint Operators (Translations of Mathematical Monographs, 17)*. American Mathematical Society, Providence, RI, 1968.

[**9**] J. Bourgain. *Green's Function Estimates for Lattice Schrödinger Operators and Applications (Annals of Mathematics Studies, 158)*. Princeton University Press, Princeton, NJ, 2005.

[**10**] J. Bourgain and M. Goldstein. On nonperturbative localization with quasi-periodic potential. *Ann. Math.* **152**(3) (2000), 835–879.

[**11**] J. Bourgain, M. Goldstein and W. Schlag. Anderson localization for Schrödinger operators on $\mathbf{Z}^2$ with quasi-periodic potential. *Acta Math.* **188**(1) (2002), 41–86.

[**12**] J. Bourgain and S. Jitomirskaya. Absolutely continuous spectrum for 1D quasiperiodic operators. *Invent. Math.* **148**(3) (2002), 453–463.

[**13**] V. Chulaevsky and F. Delyon. Purely absolutely continuous spectrum for almost Mathieu operators. *J. Statist. Phys.* **55**(5–6) (1989), 1279–1284.

[**14**] E. Dinaburg and Y. G. Sinai. The one-dimensional Schrödinger equation with a quasiperiodic potential. *Funct. Anal. Appl.* **9**(4) (1975), 279–289.

[**15**] L. H. Eliasson. Floquet solutions for the 1-dimensional quasi-periodic Schrödinger equation. *Commun. Math. Phys.* **146**(3) (1992), 447–482.

[**16**] J. Fröhlich, T. Spencer and P. Wittwer. Localization for a class of one dimensional quasi-periodic Schrödinger operators. *Commun. Math. Phys.* **132**(1) (1990), 5–25.

[**17**] A. Y. Gordon. The point spectrum of the one-dimensional Schrödinger operator. *Uspekhi Mat. Nauk* **31**(4) (1976), 257–258.

[**18**] A. Y. Gordon, S. Jitomirskaya, Y. Last and B. Simon. Duality and singular continuous spectrum in the almost Mathieu equation. *Acta Math.* **178**(2) (1997), 169–183.

[**19**] S. Hadj Amor. Hölder continuity of the rotation number for quasi-periodic co-cycles in $\mathrm{SL}(2, \mathbb{R})$. *Commun. Math. Phys.* **287**(2) (2009), 565–588.

[**20**] R. Han. Dry ten Martini problem for the non-self-dual extended Harper's model. *Trans. Amer. Math. Soc.* **370**(1) (2018), 197–217.

[**21**] R. Han and S. Jitomirskaya. Full measure reducibility and localization for quasiperiodic Jacobi operators: a topological criterion. *Adv. Math.* **319** (2017), 224–250.

[**22**] S. Jitomirskaya. Almost everything about the almost Mathieu operator. II. *XIth International Congress of Mathematical Physics (Paris, 1994)*. International Press, Cambridge, MA, 1995, pp. 373–382.

[**23**] S. Jitomirskaya. Metal-insulator transition for the almost Mathieu operator. *Ann. Math.* **150**(3) (1999), 1159–1175.

[**24**] S. Jitomirskaya and I. Kachkovskiy. L^2-reducibility and localization for quasiperiodic operators. *Math. Res. Lett.* **23**(2) (2016), 431–444.

[**25**] S. Jitomirskaya, D. A. Koslover and M. S. Schulteis. Localization for a family of one-dimensional quasiperiodic operators of magnetic origin. *Ann. Henri Poincaré* **6**(1) (2005), 103–124.

[**26**] S. Jitomirskaya and W. Liu. Arithmetic spectral transitions for the Maryland model. *Commun. Pure Appl. Math.* **70**(6) (2017), 1025–1051.

[**27**] S. Jitomirskaya and W. Liu. Universal hierarchical structure of quasiperiodic eigenfunctions. *Ann. of Math.* (2) **187**(3) (2018), 721–776.

[**28**] S. Jitomirskaya and W. Liu. Universal reflective-hierarchical structure of quasiperiodic eigenfunctions and sharp spectral transition in phase. *Preprint*, 2018, arXiv:1802.00781.

[**29**] S. Jitomirskaya and C. A. Marx. Analytic quasi-periodic Schrödinger operators and rational frequency approximants. *Geom. Funct. Anal.* **22**(5) (2012), 1407–1443.

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[**30**] S. Jitomirskaya and B. Simon. Operators with singular continuous spectrum. III. Almost periodic Schrödinger operators. *Commun. Math. Phys.* **165**(1) (1994), 201–205.

[**31**] S. Jitomirskaya and S. Zhang. Quantitative continuity of singular continuous spectral measures and arithmetic criteria for quasiperiodic Schrödinger operators. *Preprint*, 2015, arXiv:1510.07086.

[**32**] R. Johnson and J. Moser. The rotation number for almost periodic potentials. *Commun. Math. Phys.* **84**(3) (1982), 403–438.

[**33**] I. Krasovsky. Central spectral gaps of the almost Mathieu operator. *Commun. Math. Phys.* **351**(1) (2017), 419–439.

[**34**] Y. Last. Spectral theory of Sturm–Liouville operators on infinite intervals: a review of recent developments. *Sturm–Liouville Theory*. Birkhäuser, Basel, 2005, pp. 99–120.

[**35**] Y. Last and M. Shamis. Zero Hausdorff dimension spectrum for the almost Mathieu operator. *Commun. Math. Phys.* **348**(3) (2016), 729–750.

[**36**] W. Liu and Y. Shi. Upper bounds on the spectral gaps of quasi-periodic Schrödinger operators with Liouville frequencies. *J. Spectr. Theory*, to appear.

[**37**] W. Liu and X. Yuan. Anderson localization for the almost Mathieu operator in the exponential regime. *J. Spectr. Theory* **5**(1) (2015), 89–112.

[**38**] W. Liu and X. Yuan. Anderson localization for the completely resonant phases. *J. Funct. Anal.* **268**(3) (2015), 732–747.

[**39**] W. Liu and X. Yuan. Spectral gaps of almost Mathieu operators in the exponential regime. *J. Fractal Geom.* **2**(1) (2015), 1–51.

[**40**] C. A. Marx and S. Jitomirskaya. Dynamics and spectral theory of quasi-periodic Schrödinger-type operators. *Ergod. Th. & Dynam. Sys.* **37**(8) (2017), 2353–2393.

[**41**] J. Moser and J. Pöschel. An extension of a result by Dinaburg and Sinai on quasi-periodic potentials. *Comment. Math. Helv.* **59**(1) (1984), 39–85.

[**42**] J. Puig. Cantor spectrum for the almost Mathieu operator. *Commun. Math. Phys.* **244**(2) (2004), 297–309.

[**43**] J. Puig. A nonperturbative Eliasson's reducibility theorem. *Nonlinearity* **19**(2) (2006), 355–376.

[**44**] B. Simon. Almost periodic Schrödinger operators: a review. *Adv. Appl. Math.* **3**(4) (1982), 463–490.

[**45**] F. Yang. Spectral transition line for the extended Harper's model in the positive Lyapunov exponent regime. *J. Funct. Anal.* **275**(3) (2018), 712–734.

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