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Growth of the Eigensolutions of Laplacians on Riemannian Manifolds I: Construction of Energy Function

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In this paper, we consider the eigensolutions of $-\Delta u + Vu = \lambda u$, where Δ is the Laplacian on a non-compact complete Riemannian manifold. We develop Kato's methods on manifold and establish the growth of the eigensolutions as r goes to infinity based on the asymptotical behaviors of Δr and $V(x)$, where $r = r(x)$ is the distance function on the manifold. As applications, we prove several criteria of absence of eigenvalues of Laplacian, including a new proof of the absence of eigenvalues embedded into the essential spectra of free Laplacian if the radial curvature of the manifold satisfies $K_{\mathrm{rad}}(r) = -1 + \frac{o(1)}{r}$.

1 Introduction and Main Results

Let (M, g) be a connected n-dimensional non-compact complete Riemannian manifold ($n \geq 2$). The Laplace–Beltrami operator on (M, g) is essentially self-adjoint on $C_0^\infty(M)$. We denote the self-adjoint extension by Δ (the Laplacian).

Assume there exists $U \subset M$ such that $M \backslash U$ is connected and the induced outward normal exponential map $\exp_{\partial U}^{\perp} : N^+(\partial U) \to M - \overline{U}$ is a diffeomorphism, where $N^+(\partial U) = \{v \in T(\partial U) \mid v \text{ is outward normal to } \partial U\}$. As in [17, 19], let r be the distance function from ∂U defined on $M - \overline{U}$.

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We are interested in the spectral theory of Δ and asymptotic behavior of the eigensolutions of

$$-\Delta u + Vu = \lambda u \tag{1}$$

as $r(x)$ goes to infinity.

For Euclidean space $\mathbb{R}^n$, that is $M = \mathbb{R}^n$, there are rich results about spectral theory of $-\Delta + V$ with decaying potential V. A typical application of Weyl's theorem states that the essential spectrum $\sigma_{\text{ess}}(-\Delta + V) = [0, \infty)$ if $\limsup |V(x)| = 0$. It is interesting to investigate if there exists eigenvalue embedded into the essential spectrum. Kato [13] addressed this problem and showed that there is no eigenvalue $\lambda > a^2$ if $\limsup |x||V(x)| = a$. This implies there is no eigenvalue embedded into the essential spectrum if $V(x) = \frac{o(1)}{1+|x|}$. There is an alternative proof similar to Kato's by Vakulenko [31]. By Neuman–Wigner type functions [32], $V(x) = \frac{o(1)}{1+|x|}$ is optimal so that $V(x) = \frac{o(1)}{1+|x|}$ is a spectral transition for eigenvalue embedded into the essential spectrum. For more examples about (finite or dense) eigenvalues embedded into essential spectrum, see [15, 23, 29]. Under stronger assumption on the perturbation, for example $V(x) = \frac{O(1)}{(1+|x|)^{1+\delta}}$ for some $\delta > 0$ or $V \in L^p(\mathbb{R}^n)$ for proper $p > 0$, the limiting absorption principle holds, originally from Agmon's theory [2, 25]. Thus operator $-\Delta + V$ has no singular continuous spectrum. See the survey paper [27] for more details. For one-dimensional case, there are more results. For example, $V(x) = \frac{O(1)}{1+|x|}$ is a spectral transition for singular continuous spectrum embedded into the essential spectrum [3, 14]. Agmon [1] and Simon [28], using Kato's methods, independently obtained the quantitative bounds for a class of potentials $V(x) = V_1(x) + V_2(x)$, where $\limsup |x||V_1(x)| < \infty$, $\limsup V_2(x) = 0$, and $\limsup |x||\frac{\partial V_2}{\partial r}| < \infty$ ($\frac{\partial V_2}{\partial r}$ is the derivative with radial direction). We refer the readers to Simon's review [30] for the full details on Kato's method, its applications, and related topics.

There are a series of Kumura's and Donnelly papers [4, 5, 16, 17, 19] studying the eigenvalues embedded into essential spectrum of the Laplacian on manifolds. See Donnelly review [6]. The results are based on construction of energy functions. However they do not fully use Kato's method. As a result, they need some geometric condition that we have shown to be unnecessary.

Our goal is to develop Kato's method (also Agmon's and Simon's generalizations) on manifolds. This is the first of our series papers, which in particular implies sharp bounds for asymptotically hyperbolic manifolds. Let us consider the eigensolution $-\Delta u + Vu = \lambda u$. We show that under some weak convexity assumption on a manifold, asymptotical behaviors of Δr and potentials V can determine whether there is an

eigenvalue embedded into essential spectrum, where Δr is the Laplacian of distance function $r(x)$. Δr comes from geometry and V comes from the Schödinger operator on Euclidean space. The interesting thing is that Δr is the only term involved in the geometry, completely determining the threshold for embedded eigenvalue. For example, even to obtain our Corollary 1.2, Kumura [17] needs an extra condition on Ricci curvature of a manifold.

The original idea of Kato [13] to study the growth of eigensolutions on Euclidean space contains four steps: construct energy function for eigen-equation; prove the monotonicity of energy function with respect to r ($r = |x|$ in the Schrödinger case); set up the positivity of initial energy; and obtain the growth of eigensolution. The first challenge is to construct the energy function since the simple sum of potential energy and kinetic energy does not work well even for Euclidean space in higher dimensions. We will give the general construction of energy functions on manifolds, which can be adapted to various situations easily. During the process, we borrow some derivative estimates from [17] to set up the monotonicity of energy function (see §2). However, we improve the previous arguments significantly in several aspects, including the subtle geometry analysis. Another main novelty here is that we give universal method to treat all cases of manifolds and potentials. Here, we just fix some indices to make energy functions work for this paper. We believe our method has a wider applicability. For example, in the following [22], we give a new way to verify the positivity of initial energy, which works well for the asymptotically flat manifolds.

The results of [8, 16] show that $\sigma_{\mathrm{ess}}(-\Delta) = [\frac{a^2}{4}, \infty)$ if $\lim \Delta r = a$.

In order to describe our results, some notations are necessary. Let g be the metric and ∇ be the covariant derivative. Denote Hessian of r by ∇dr. For simplicity, let dx be volume form (or restriction on sphere if necessary). Sometimes we also use $|\cdot|$ as the norm of vector. All the functions f on the manifolds in this paper depend on x. For simplicity, we ignore the dependence on sphere and use $f(r)$ instead of $f(x)$.

Let u be a nonzero real solution (Actually, all the results in the paper hold for complex solution u. We give up it here because it is not our main goal.) of eigenequation (1) and define

$$M(r) = M(r; u) = \left(\int_{|r(x)|=r} |u(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}},$$

$$N(r) = N(r; u) = \left(\int_{|r(x)|=r} \left| \frac{\partial u}{\partial r} \right|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}.$$

Our first main result is

Theorem 1.1. [**Basic version**] Let the potential $V(r) = V_1(r) + V_2(r)$. Assume

$$\limsup_{r\to\infty} |rV_1(r)| \le a_1, \limsup_{r\to\infty} |V_2(r)| = 0, \limsup_{r\to\infty} r\left|\frac{\partial V_2}{\partial r}\right| \le a_2,$$

for some nonnegative constants a_1, a_2.

Assume

$$\liminf_{r\to\infty} [r\nabla \mathrm{d}r - a_3\hat{g}] \ge 0,$$

for some $a_3 > 0$, where $\hat{g} = g - \mathrm{d}r \otimes \mathrm{d}r$, and

$$\limsup_{r\to\infty} r\left|\Delta r - a_4 - \frac{a_5}{r}\right| \le \delta,$$

for some nonnegative constants a_4, a_5, δ.

Suppose

$$\mu > \delta, 2a_3 > \mu + \delta, a_3 > 1 + \delta, \tag{2}$$

and

$$\lambda > \max\left\{\frac{a_4^2}{4} + \frac{a_2}{\mu-\delta} + \frac{1}{4}\frac{(2a_1+\delta a_4)^2}{\mu^2-\delta^2}, \frac{a_4^2}{4} + \frac{a_2}{2(a_3-\delta)}\right\}. \tag{3}$$

Then we have

$$\liminf_{r\to\infty} r^{\mu}\left[M(r)^2 + N(r)^2\right] = \infty.$$

Based on Theorem 1.1 (By the fact that $\Delta + V$ is essentially self-adjoint, we have $\nabla u \in L^2(M)$ if the eigensolution $u \in L^2(M)$.), we have several immediate corollaries.

Corollary 1.2. Let the potential $V(r) = V_1(r) + V_2(r)$. Suppose

$$|V_1(r)| = \frac{o(1)}{r}, |V_2(r)| = o(1), \left|\frac{\partial V_2}{\partial r}\right| = \frac{o(1)}{r},$$

as r goes to infinity.

Suppose

$$\liminf_{r\to\infty} [r\nabla \mathrm{d}r - (1+\epsilon))\hat{g}] \ge 0,$$

for some $\epsilon > 0$, and

$$\Delta r = a + \frac{b}{r} + \frac{o(1)}{r}.$$

Then for any $\lambda > \frac{a^2}{4}$ and $\mu > 0$, we have

$$\liminf_{r\to\infty} r^{\mu}\left[M(r)^2 + N(r)^2\right] = \infty.$$

In particular, $-\Delta$ admits no eigenvalue larger than $\frac{a^2}{4}$.

Corollary 1.3. Suppose

$$\limsup_{r\to\infty} r|\nabla \mathrm{d}r - \hat{g}| \le A,$$

and $(n-1)A < 1$. Then $-\Delta$ does not have eigenvalue larger than $\frac{(n-1)^2}{4} + \frac{(n-1)^4A^2}{4(1-(n-1)^2A^2)}$.

Remark 1.4. By some direct modifications, a similar result can also be obtained under the assumption

$$-\frac{B}{r} \le \nabla \mathrm{d}r - \hat{g} \le \frac{A}{r},$$

as r goes to infinity. Thus the corollary improves Theorem 1.1 in [17] by removing extra assumption on the Ricci curvature.

Corollary 1.5. Suppose there exists a $r_0 > 0$ such that

$$\nabla \mathrm{d}r \ge 0$$

for $r = r_0$, and

$$-1 - \frac{2A}{r} \le K_{\mathrm{rad}}(r) \le -1 + \frac{2A}{r} < 0,$$

for $r \ge r_0$, where $K_{\mathrm{rad}}(r)$ is the radial curvature (In Geometry, radial curvature is the sectional curvature with one fixed direction $\frac{\partial}{\partial r}$. We refer the reader to [10] for formal definition and applications.). Suppose

$$(n-1)A < 1.$$

Then $-\Delta$ does not have eigenvalues larger than $\frac{(n-1)^2}{4} + \frac{(n-1)^4A^2}{4(1-(n-1)^2A^2)}$. In particular, $-\Delta$ does not have eigenvalue larger than $\frac{(n-1)^2}{4}$ if $K_{\mathrm{rad}}(r) + 1 = \frac{o(1)}{r}$.

Just we mentioned in the introduction, in Theorem 1.1 Δr is from the geometry and V is the potential from Schödinger operator on Euclidean space. For the potential part, we develop Agmon–Simon's generalization on manifolds. For the geometric part,

we just develop Kato's method so that there is no derivative of Δr involved in. Our next two theorems are to develop Agmon–Simon's generalization on geometric part of manifolds. Namely, if we know the information of $\frac{\partial \Delta r}{\partial r}$ (or gradient of Δr), we can get similar results to Theorem 1.1.

Theorem 1.6. [**Gradient version**] Let the potential $V(r) = V_1(r) + V_2(r)$. Suppose

$$\limsup_{r\to\infty} |rV_1(r)| \le a_1, \limsup_{r\to\infty} |V_2(r)| = 0, \limsup_{r\to\infty} r\left|\frac{\partial V_2}{\partial r}\right| \le a_2,$$

for some nonnegative constants a_1, a_2.

Suppose

$$\liminf_{r\to\infty}[r\nabla \mathrm{d}r - a_3\hat{g}] \ge 0,$$

for some $a_3 > 1$. Suppose

$$\Delta r = a_4 + \frac{a_5}{r} + \frac{\bar{\delta}(r)}{r}$$

and

$$\limsup_{r\to\infty}\left|\frac{\partial\bar{\delta}(r)}{\partial r}\right| \le \delta_1, \limsup_{r\to\infty}\left|\left(\nabla - \frac{\partial}{\partial r}\right)\bar{\delta}(r)\right| \le \delta_2, \limsup_{r\to\infty}|\bar{\delta}(r)| \le \delta,$$

for some nonnegative constants $a_4, a_5, \delta_1, \delta_2, \delta$.

Suppose

$$2a_3 > \mu,$$
$$\lambda > \frac{a_4^2}{4} + \frac{1}{\mu}\left[a_2 + \frac{(2a_1+\delta_1)^2}{4\mu} + \frac{\delta_2^2}{8a_3 - 4\mu} + \frac{a_4\delta_1}{2}\right], \tag{4}$$

and

$$\lambda > \frac{a_4^2}{4} + \min_{2\le s_0\le 2a_3}\left\{\frac{a_2}{s_0} + \frac{a_4\delta_1}{2s_0} + \frac{\delta_2^2}{(8a_3 - 4s_0)s_0}\right\}. \tag{5}$$

Then we have

$$\liminf_{r\to\infty} r^{\mu}\left[M(r)^2 + N(r)^2\right] = \infty.$$

Remark 1.7.

- The bounds on the right of (4) and (5) depend on δ_1, not δ.
- We can also obtain some interesting corollaries like Corollaries 1.2 and 1.3.

Theorem 1.8. [**Mixed version**] Let the potential $V(r) = V_1(r) + V_2(r)$. Suppose

$$\limsup_{r\to\infty} |rV_1(r)| \le a_1, \limsup_{r\to\infty} |V_2(r)| = 0, \limsup_{r\to\infty} r\left|\frac{\partial V_2}{\partial r}\right| \le a_2,$$

for nonnegative constants a_1, a_2.

Suppose

$$\liminf_{r\to\infty}[r\nabla \mathrm{d}r - a_3\hat{g}] \ge 0,$$

for some $a_3 > 1$. Suppose

$$\Delta r = a_4 + \frac{a_5}{r} + \frac{\bar{\delta}(r)}{r}$$

and

$$\limsup_{r\to\infty}\left|\frac{\partial\bar{\delta}(r)}{\partial r}\right| \le \delta_1, \limsup_{r\to\infty} |\bar{\delta}(r)| \le \delta,$$

for some nonnegative constants $a_4, a_5, \delta_1, \delta$.

Suppose

$$\mu > \delta, 2a_3 > \mu + \delta, a_3 > 1 + \delta, \tag{6}$$

and

$$\lambda > \max\left\{\frac{a_4^2}{4} + \frac{a_2}{\mu-\delta} + \frac{a_4\delta_1}{2(\mu-\delta)} + \frac{a_1^2}{\mu^2-\delta^2}, \frac{a_4^2}{4} + \frac{2a_2 + a_4\delta_1}{4(a_3-\delta)}\right\}. \tag{7}$$

Then we have

$$\liminf_{r\to\infty} r^{\mu}\left[M(r)^2 + N(r)^2\right] = \infty.$$

Corollary 1.9. Suppose there exists a $r_0 > 0$ such that

$$\nabla \mathrm{d}r \ge 0$$

for $r = r_0$, and

$$-1 - \frac{2A}{r} \le K_{\mathrm{rad}}(r) \le -1 + \frac{2A}{r} < 0,$$

for $r \ge r_0$, where $K_{\mathrm{rad}}(r)$ is the radial curvature. Suppose

$$(n-1)A < 1.$$

Then $-\Delta$ does not have eigenvalue larger than $\frac{(n-1)^2}{4} + \frac{2(n-1)^2A}{1-(n-1)A}$.

Remark 1.10. The lower bound $\frac{(n-1)^2}{4} + \frac{2(n-1)^2 A}{1-(n-1)A}$ is exactly the bound given in [17].

We should remind that Corollaries 1.5 and 1.9 gave different bounds, and n and A will decide which one is better. However by the combination of energy functions in Theorems 1.6 and 1.8, we can get a universal bound, which is better than that in Corollaries 1.5 and 1.9.

Theorem 1.11. Suppose there exists a $r_0 > 0$ such that

$$\nabla \mathrm{d}r \geq 0$$

for $r = r_0$, and

$$-1 - \frac{2A}{r} \leq K_{\mathrm{rad}}(r) \leq -1 + \frac{2A}{r} < 0,$$

for $r \geq r_0$, where $K_{\mathrm{rad}}(r)$ is the radial curvature. Suppose

$$(n-1)A < 1$$

and

$$\lambda > \frac{(n-1)^2}{4} + \min_{\sigma \in [0,1]} \left\{ \sigma^2 \frac{(n-1)^4 A^2}{4(1-(n-1)^2 A^2)} + (1-\sigma) \frac{2(n-1)^2 A}{1-(n-1)A} \right\},$$

then λ cannot be an eigenvalue of the free Laplacian.

Remark 1.12. Actually, by the combination of energy functions in Theorems 1.6 and 1.8, we can set up a generally stronger Theorem with more generality. We do not want to explore the general case here, we just give a better bound than that in Corollaries 1.5 and 1.9.

We want to see more about Corollaries 1.5 and 1.9 and Theorem 1.11. Radial curvature $K_{\mathrm{rad}}(r)$ is a feature of hyperbolic manifold and flat manifold.

For the asymptotically hyperbolic case, the sharp transition is given by Kumura [17] by studying the eigensolutions directly. He excludes eigenvalues greater than $\frac{(n-1)^2}{4}$ under the assumption that $K_{\mathrm{rad}}(r) + 1 = o(r^{-1})$, and also constructs a manifold with the radial curvature $K_{\mathrm{rad}}(r) + 1 = O(r^{-1})$ and with an eigenvalue $\frac{(n-1)^2}{4} + 1$ embedding into its essential spectrum $[\frac{(n-1)^2}{4}, \infty)$. Before that some partial results on the absence of eigenvalues were obtained in papers [4, 24].

For the asymptotically flat case, several authors [5, 7, 9, 19] showed the absence of positive eigenvalues of free Laplacian under some assumptions on the radial curvature. Roughly speaking, they assume $|K_{\text{rad}}(r)| \leq \frac{\delta}{1+r^2}$ for small δ. See Donnelly's review paper [6] for more results.

Jitomirskaya and Liu also constructed examples which show that dense eigenvalues and singular continuous spectrum can embed into essential spectrum of Laplacian in both cases [11, 12]. We mentioned that Kumura also studied other related topics in papers [18, 20, 21]. There are also other topics about asymptotically flat and asymptotically hyperbolic manifolds. See [26] and the references therein.

The rest of the paper is organized as follows: in Section 2, we present some basic knowledge. In §3, we will give the general construction of energy functions. In Section 4, by fixing some indices in the energy functions, we prove Theorem 1.1 and Corollaries 1.2, 1.3, and 1.5. In Section 5, by adapting the energy functions, we prove Theorem 1.6. In Section 6, by adapting the energy functions, we prove Theorems 1.8 and 1.11 and Corollary 1.9. Our proof is self-contained except the unique continuation theorem and some basic geometry results (Lemma 4.5).

2 Preliminaries and Derivative Lemma

Let $S_t = \{x \in M : r(x) = t\}$, $\omega \in S_r$, and $x \in M$. Thus (r, ω) is a local coordinate system for M (ω depends on r. We ignore the dependence for simplicity.). Let $< \cdot, \cdot >$ be the metric on the Riemannian manifold.

Choose a function $\rho(r)$, which will be specified later. Let $\hat{L} = e^{\rho} L e^{-\rho}$, where $L = -\Delta + V$. Then

$$\begin{array}{ccc} L^2(M, dg) & \xrightarrow{-\Delta+V} & L^2(M, dg) \\ {\scriptstyle e^{\rho}}\downarrow & & \downarrow{\scriptstyle e^{\rho}} \\ L^2(M, e^{-2\rho} dg) & \xrightarrow[\hat{L}]{} & L^2(M, e^{-2\rho} dg) \end{array}.$$

Let

$$v = e^{\rho} u.$$

Then, one has

$$\nabla u = -\rho' e^{-\rho} v \nabla r + e^{-\rho} \nabla v,$$

and

$$\Delta u = \operatorname{div} \nabla u = e^{-\rho} \Delta v - 2\rho' e^{-\rho} \frac{\partial v}{\partial r} + \left(\rho'^2 - \rho'' - \rho' \Delta r\right) e^{-\rho} v.$$

So the eigen-equation (1) becomes

$$-\Delta v + 2\rho' \frac{\partial v}{\partial r} + (V + V_0)v = \lambda v, \tag{8}$$

where

$$V_0 = \rho' \Delta r + \rho'' - \rho'^2. \tag{9}$$

Lemma 2.1. [17] Let X be a vector field. Then

$$\frac{\partial}{\partial r} \int_{S_r} < X, \nabla r > e^{-2\rho} \, \mathrm{d}x = \int_{S_r} (\mathrm{div} X - 2\rho' < X, \nabla r >) e^{-2\rho} \, \mathrm{d}x. \tag{10}$$

Proof. First, one has

$$\mathrm{div}\left(Xe^{-2\rho}\right) = e^{-2\rho} \mathrm{div}(X) - 2\rho' e^{-2\rho} < X, \nabla r > .$$

Integration by part, we get

$$\int_{S_{t_2}} < X, \nabla r > e^{-2\rho} \, \mathrm{d}x - \int_{S_{t_1}} < X, \nabla r > e^{-2\rho} \, \mathrm{d}x = \int_{t_1 \le |r(x)| \le t_2} (\mathrm{div} X - 2\rho' < X, \nabla r >) e^{-2\rho} \, \mathrm{d}x,$$

which implies (10). ■

Lemma 2.2. [17]

$$\frac{\partial}{\partial r} \int_{S_r} f e^{-2\rho} \, \mathrm{d}x = \int_{S_r} \left[\frac{\partial f}{\partial r} + f(\Delta r - 2\rho') \right] e^{-2\rho} \, \mathrm{d}x.$$

Proof. Let $X = f \nabla r$. By direct computation, one has

$$\mathrm{div} X = \mathrm{div}(f \nabla r) = \frac{\partial f}{\partial r} + f \Delta r.$$

Putting X into Lemma 2.1, Lemma 2.2 follows. ■

Now we always assume u is a nonzero solution of $-\Delta u + Vu = \lambda u$, where $V = V_1 + V_2$. Let

$$v_m = r^m v$$

with $m \geq 0$. By (8), we get the equation of v_m,

$$\Delta v_m - \left(\frac{2m}{r} + 2\rho'\right)\frac{\partial v_m}{\partial r} + \left(\frac{m(m+1)}{r^2} + \frac{m}{r}(2\rho' - \Delta r) - V_0 - V_1 - V_2 + \lambda\right) v_m = 0. \quad (11)$$

3 Construction of the Energy Functions

In this section, we will give the general construction of energy functions and derive the formulas for their derivatives.

Let A_r be the Laplacian on sphere r. Using $\Delta u = \frac{\partial^2 u}{\partial r^2} + \Delta r \frac{\partial u}{\partial r} + A_r u$, (11) becomes

$$\frac{\partial^2 v_m}{\partial r^2} - \left(-\Delta r + \frac{2m}{r} + 2\rho'\right)\frac{\partial v_m}{\partial r} + A_r v_m + \left(\frac{m(m+1)}{r^2} + \frac{m}{r}(2\rho' - \Delta r) - V_0 - V_1 - V_2 + \lambda\right) v_m = 0. \quad (12)$$

Let us mention our intuition to construct energy functions. We view (12) as one-dimensional Schrödinger operator (r is the variable). (12) is not the normal form--$D^2 + q$ since $-\Delta r + \frac{2m}{r} + 2\rho'$ is not 0. So the first step we need to do is to choose ρ such that $-\Delta r + \frac{2m}{r} + 2\rho'$ is smaller than $\frac{O(1)}{r}$. The energy function for equation $u'' + qu = 0$ is $\frac{1}{2}|u'|^2 + \frac{1}{2}qu^2$. Similarly, the usual energy functions of (12) are taken with the form as $\frac{1}{2}\left|\frac{\partial v_m}{\partial r}\right|^2 + \frac{1}{2}qv_m^2$ with average on the sphere. Since we cannot make $-\Delta r + \frac{2m}{r} + 2\rho'$ zero, extra term $q_1 \frac{\partial v_m}{\partial r} v_m$ should be added into the energy functions. By the fact

$$\int_{S_r}\left[\frac{1}{2}\left|\frac{\partial v_m}{\partial r}\right|^2 + \frac{1}{2}(A_r v_m, v_m)\right] \mathrm{d}x = \int_{S_r}\left[\left|\frac{\partial v_m}{\partial r}\right|^2 - \frac{1}{2}|\nabla v_m|^2\right] \mathrm{d}x,$$

it is natural to construct

$$\begin{aligned} F(m, r, t, s) &= r^s \int_{S_r} \frac{1}{2}\left[q_1 \frac{\partial v_m}{\partial r} v_m + \left(\frac{m(m+1)}{r^2} - \frac{t}{r} + q_2 + \lambda\right) v_m^2\right] e^{-2\rho}\, \mathrm{d}x \\ &\quad + r^s \int_{S_r}\left[\left|\frac{\partial v_m}{\partial r}\right|^2 - \frac{1}{2}|\nabla v_m|^2\right] e^{-2\rho}\, \mathrm{d}x \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}, \end{aligned}$$

where

$$\begin{aligned}\mathrm{I} &= r^s \int_{S_r} \left[\left| \frac{\partial v_m}{\partial r} \right|^2 - \frac{1}{2} |\nabla v_m|^2 \right] e^{-2\rho} \, \mathrm{d}x \\ &= \frac{1}{2} r^s \int_{S_r} \left[\left| \frac{\partial v_m}{\partial r} \right|^2 - |\nabla_\omega v_m|^2 \right] e^{-2\rho} \, \mathrm{d}x,\end{aligned}$$

and

$$\mathrm{II} = \frac{1}{2} r^s \int_{S_r} \left[\frac{m(m+1)}{r^2} - \frac{t}{r} + q_2 + \lambda \right] v_m^2 e^{-2\rho} \, \mathrm{d}x,$$

and

$$\mathrm{III} = \frac{1}{2} r^s \int_{S_r} \left[q_1 \frac{\partial v_m}{\partial r} v_m \right] e^{-2\rho} \, \mathrm{d}x.$$

We begin with the derivation of $\frac{\partial}{\partial r}\mathrm{I}$.

By Lemma 2.2, one has

$$\begin{aligned}\frac{\partial}{\partial r}\mathrm{I} = {} & s r^{s-1} \int_{S_r} \left[\frac{1}{2} \left| \frac{\partial v_m}{\partial r} \right|^2 - \frac{1}{2} |\nabla_\omega v_m|^2 \right] e^{-2\rho} \, \mathrm{d}x \\ & + r^s \int_{S_r} \left[\frac{\partial v_m}{\partial r} \frac{\partial^2 v_m}{\partial r^2} - \frac{1}{2} \frac{\partial}{\partial r} < \nabla_\omega v_m, \nabla_\omega v_m > \right] e^{-2\rho} \, \mathrm{d}x \\ & + r^s \int_{S_r} \frac{1}{2} (\Delta r - 2\rho') \left[\left| \frac{\partial v_m}{\partial r} \right|^2 - |\nabla_\omega v_m|^2 \right] e^{-2\rho} \, \mathrm{d}x.\end{aligned}$$

Using $\Delta v_m = \frac{\partial^2 v_m}{\partial r^2} + \Delta r \frac{\partial v_m}{\partial r} + \Delta_\omega v_m$, we get

$$\begin{aligned}\frac{\partial}{\partial r}\mathrm{I} = {} & \int_{S_r} \left[\frac{s}{2} r^{s-1} \left| \frac{\partial v_m}{\partial r} \right|^2 - 2 r^s \rho' \left| \frac{\partial v_m}{\partial r} \right|^2 + r^s \frac{\partial v_m}{\partial r} \Delta v_m + \frac{r^s}{2} (2\rho' - \Delta r) \left| \frac{\partial v_m}{\partial r} \right|^2 \right] e^{-2\rho} \, \mathrm{d}x \\ & + r^s \int_{S_r} \left[\Big(- \frac{s}{2r} + \frac{1}{2} (2\rho' - \Delta r) \Big) \hat{g}(\nabla v_m, \nabla v_m) \right] e^{-2\rho} \, \mathrm{d}x \\ & + r^s \int_{S_r} \left[\left\langle \nabla_\omega \frac{\partial v_m}{\partial r}, \nabla_\omega v_m \right\rangle - \frac{1}{2} \frac{\partial}{\partial r} < \nabla_\omega v_m, \nabla_\omega v_m > \right] e^{-2\rho} \, \mathrm{d}x.\end{aligned}$$

By some basic computation, one has

$$\begin{aligned}\left\langle \nabla_\omega \frac{\partial v_m}{\partial r}, \nabla_\omega v_m \right\rangle - \frac{1}{2} \frac{\partial}{\partial r} < \nabla_\omega v_m, \nabla_\omega v_m > &= \left\langle \nabla_\omega \frac{\partial v_m}{\partial r}, \nabla_\omega v_m \right\rangle - \left\langle \nabla_{\frac{\partial}{\partial r}} \nabla_\omega v_m, \nabla_\omega v_m \right\rangle \\ &= (\nabla dr)(\nabla_\omega v_m, \nabla_\omega v_m).\end{aligned}$$

Finally we get

$$\frac{\partial}{\partial r}\mathrm{I} = \int_{S_r}\left[\frac{s}{2}r^{s-1}\left|\frac{\partial v_m}{\partial r}\right|^2 - 2r^s\rho'\left|\frac{\partial v_m}{\partial r}\right|^2 + r^s\frac{\partial v_m}{\partial r}\Delta v_m + \frac{r^s}{2}(2\rho' - \Delta r)\left|\frac{\partial v_m}{\partial r}\right|^2\right]e^{-2\rho}\,\mathrm{d}x$$
$$+ r^s\int_{S_r}\left[\left(\nabla \mathrm{d}r + \left(-\frac{s}{2r} + \frac{1}{2}(2\rho' - \Delta r)\right)\hat{g}\right)(\nabla v_m, \nabla v_m)\right]e^{-2\rho}\,\mathrm{d}x. \tag{13}$$

Now we are in the position to obtain $\frac{\partial}{\partial r}\mathrm{II}$.

By Lemma 2.2 again, one has

$$\frac{\partial}{\partial r}\mathrm{II} = \int_{S_r}\left[\frac{\partial}{\partial r}\frac{r^s}{2}\left(\frac{m(m+1)}{r^2} - \frac{t}{r} + q_2 + \lambda\right)v_m^2\right]e^{-2\rho}\,\mathrm{d}x$$
$$+ \int_{S_r}(\Delta r - 2\rho')\frac{r^s}{2}\left[\frac{m(m+1)}{r^2} - \frac{t}{r} + q_2 + \lambda\right]v_m^2 e^{-2\rho}\,\mathrm{d}x$$
$$= \int_{S_r}\left[\frac{s-2}{2}r^{s-3}m(m+1) - \frac{s-1}{2}tr^{s-2} + \frac{r^s}{2}\frac{\partial q_2}{\partial r} + \frac{s}{2}r^{s-1}q_2 + \lambda\frac{s}{2}r^{s-1}\right]v_m^2 e^{-2\rho}\,\mathrm{d}x$$
$$+ r^s\int_{S_r}\left[\frac{m(m+1)}{r^2} - \frac{t}{r} + q_2 + \lambda\right]v_m\frac{\partial v_m}{\partial r}e^{-2\rho}\,\mathrm{d}x$$
$$+ \frac{r^s}{2}\int_{S_r}(\Delta r - 2\rho')\left[\frac{m(m+1)}{r^2} - \frac{t}{r} + q_2 + \lambda\right]v_m^2 e^{-2\rho}\,\mathrm{d}x. \tag{14}$$

Similarly, by Lemma 2.2 again, we have

$$\frac{\partial}{\partial r}\mathrm{III} = \int_{S_r}\left[\frac{1}{2}q_1 r^s v_m\frac{\partial v_m}{\partial r}(\Delta r - 2\rho') + \frac{\partial}{\partial r}\left(r^s\frac{1}{2}q_1\frac{\partial v_m}{\partial r}v_m\right)\right]e^{-2\rho}\,\mathrm{d}x$$
$$= \int_{S_r}\left[\frac{r^s}{2}q_1(\Delta r - 2\rho') + \frac{1}{2}sr^{s-1}q_1 + \frac{1}{2}r^s\frac{\partial q_1}{\partial r}\right]\frac{\partial v_m}{\partial r}v_m\Bigg]e^{-2\rho}\,\mathrm{d}x$$
$$+ \int_{S_r}\left[\frac{1}{2}r^s q_1\left|\frac{\partial v_m}{\partial r}\right|^2 + \frac{1}{2}r^s q_1 v_m\frac{\partial^2 v_m}{\partial^2 r}\right]e^{-2\rho}\,\mathrm{d}x$$
$$= \int_{S_r}\left[\frac{r^s}{2}q_1(\Delta r - 2\rho') + \frac{s}{2}r^{s-1}q_1 + \frac{r^s}{2}\frac{\partial q_1}{\partial r}\right]\frac{\partial v_m}{\partial r}v_m e^{-2\rho}\,\mathrm{d}x$$
$$+ \int_{S_r}\left[\frac{r^s}{2}q_1\left|\frac{\partial v_m}{\partial r}\right|^2 + \frac{r^s}{2}q_1 v_m\left(\Delta v_m - \Delta r\frac{\partial v_m}{\partial r} - \Delta_\omega v_m\right)\right]e^{-2\rho}\,\mathrm{d}x. \tag{15}$$

Putting (13), (14), and (15) together and using (11), we conclude that

$$\begin{aligned}\frac{\partial F(m,r,t,s)}{\partial r} &= \frac{\partial \mathrm{I}}{\partial r}+\frac{\partial \mathrm{II}}{\partial r}+\frac{\partial \mathrm{III}}{\partial r} \qquad (16)\\ &= \int_{S_r}\left[r^s\left((\nabla dr)-\left(\frac{s}{2r}-\frac{1}{2}(2\rho'-\Delta r)\right)\hat{g}\right)(\nabla v_m,\nabla v_m)+\frac{1}{2}q_1 r^s v_m(-\Delta_\omega v_m)\right]e^{-2\rho}\,\mathrm{d}x\\ &\quad+\int_{S_r}\left[2mr^{s-1}+\frac{r^s}{2}(2\rho'-\Delta r)+\frac{1}{2}q_1 r^s+\frac{s}{2}r^{s-1}\right]\left|\frac{\partial v_m}{\partial r}\right|^2 e^{-2\rho}\,\mathrm{d}x\\ &\quad+\int_{S_r}\left[r^s\left(V_0+V_1+V_2+q_2-\frac{t}{r}\right)+r^{s-1}m(\Delta r-2\rho')\right]\frac{\partial v_m}{\partial r}v_m e^{-2\rho}\,\mathrm{d}x\\ &\quad+\int_{S_r}\left[\frac{1}{2}sr^{s-1}q_1+mr^{s-1}q_1+\frac{1}{2}r^s\frac{\partial q_1}{\partial r}\right]\frac{\partial v_m}{\partial r}v_m e^{-2\rho}\,\mathrm{d}x\\ &\quad+\int_{S_r}\left[\frac{s-2}{2}r^{s-3}m(m+1)-\frac{s-1}{2}tr^{s-2}+\frac{r^s}{2}\frac{\partial q_2}{\partial r}+\frac{s}{2}r^{s-1}q_2+\lambda\frac{s}{2}r^{s-1}\right]v_m^2 e^{-2\rho}\,\mathrm{d}x\\ &\quad+\int_{S_r}\frac{r^s}{2}(\Delta r-2\rho')\left[\frac{m(m+1)}{r^2}-\frac{t}{r}+q_2+\lambda\right]v_m^2 e^{-2\rho}\,\mathrm{d}x\\ &\quad+\int_{S_r}-\frac{r^s}{2}q_1\left[\frac{m(m+1)}{r^2}+\frac{m}{r}\left(2\rho'-\Delta r\right)-V_0-V_1-V_2+\lambda\right]v_m^2 e^{-2\rho}\,\mathrm{d}x.\end{aligned}$$

4 Proof of Theorem 1.1 and Some Corollaries

Let

$$\bar{\delta}(r)=r\left(\Delta r-a_4-\frac{a_5}{r}\right)$$

$$\bar{a}_1(r)=rV_1(r)$$

$$\bar{a}_2(r)=r\frac{\partial V_2}{\partial r}.$$

By the assumptions of Theorem 1.1,

$$\limsup_{r\to\infty}|\bar{\delta}(r)|\le\delta,\limsup_{r\to\infty}|\bar{a}_1(r)|\le a_1,\limsup_{r\to\infty}|\bar{a}_2(r)|\le a_2.$$

Let $0<t<1$ be small enough, $2\rho'=a_4+\frac{a_5}{r}$ and $q_1=0$. Direct computation of (9) implies that

$$V_0=\frac{a_4^2}{4}+\frac{a_4a_5}{2r}+\frac{a_4\bar{\delta}}{2r}+\frac{O(1)}{r^2}.$$

We should mention that $O(1)$ and $o(1)$ only depend on constants in the assumptions of Theorem 1.1, not depend on m, t.

Let $q_2 = -\frac{a_4^2}{4} - \frac{a_4 a_5}{2r} - V_2$. By (16), we have

$$\frac{\partial F(m,r,t,s)}{\partial r} = r^{s-1}\int_{S_r}\left[\left(r(\nabla dr) - \left(\frac{s}{2}+\frac{1}{2}\bar{\delta}\right)\hat{g}\right)(\nabla v_m, \nabla v_m)\right]e^{-2\rho}\,dx \tag{17}$$

$$+ r^{s-1}\int_{S_r}\left[2m - \frac{\bar{\delta}}{2} + \frac{s}{2}\right]\left|\frac{\partial v_m}{\partial r}\right|^2 e^{-2\rho}\,dx \tag{18}$$

$$+ r^{s-1}\int_{S_r}\left[\bar{a}_1 + \frac{\bar{\delta}a_4}{2} - t + \bar{\delta}\frac{m}{r} + o(1)\right]\frac{\partial v_m}{\partial r}v_m e^{-2\rho}\,dx \tag{19}$$

$$+ r^{s-1}\int_{S_r}\left[\left(\lambda - \frac{a_4^2}{4}\right)\left(\frac{s}{2}+\frac{\bar{\delta}}{2}\right) - \frac{\bar{a}_2}{2} + o(1)\right]v_m^2 e^{-2\rho}\,dx \tag{20}$$

$$+ r^{s-1}\int_{S_r}\frac{m(m+1)}{r^2}\left[\frac{s-2}{2}+\frac{\bar{\delta}}{2}\right]v_m^2 e^{-2\rho}\,dx. \tag{21}$$

Theorem 4.1. Under the assumptions of Theorem 1.1, there exist $s_0, R_0, m_0 > 0$ such that for $m \geq m_0$ and $r \geq R_0$,

$$\frac{\partial F(m,r,t,s_0)}{\partial r} > 0.$$

Proof. Let s_0 be such that $s_0 < 2a_3 - \delta$ and sufficiently close to $2a_3 - \delta$. By the assumption $a_3 > 1 + \delta$ (see (2)), one has

$$r(\nabla dr) - \left(\frac{s_0}{2}+\frac{1}{2}\bar{\delta}\right)\hat{g} \geq 0,$$

for large r, which implies

$$(17) > 0. \tag{22}$$

By assumption $a_3 > 1 + \delta$ (see (2)) again and $\lambda > \frac{a_4^2}{4} + \frac{a_2}{2(a_3-\delta)}$ (see (3)), one has

$$\left(\lambda - \frac{a_4^2}{4}\right)\left(\frac{s_0}{2}+\frac{\bar{\delta}}{2}\right) - \frac{\bar{a}_2}{2} > 0,$$

and

$$\frac{s_0 - 2}{2} + \frac{\bar{\delta}}{2} > 0.$$

By Cauchy–Schwartz inequality, one has

$$\left[\frac{m(m+1)}{r^2}\left(\frac{s_0-2}{2}+\frac{\bar{\delta}}{2}\right)+\left(\lambda-\frac{a_4^2}{4}\right)\left(\frac{s_0}{2}+\frac{\bar{\delta}}{2}\right)-\frac{\bar{a}_2}{2}\right]v_m^2+\left[2m-\frac{\bar{\delta}}{2}+\frac{s_0}{2}\right]\left|\frac{\partial v_m}{\partial r}\right|^2 > \left|\left[\frac{\bar{a}_1}{2}+\frac{\bar{\delta}a_4}{2}-t+\bar{\delta}\frac{m}{r}\right]\frac{\partial v_m}{\partial r}v_m\right|$$

for large m and r. Thus, one has

$$|(19)| < (18)+(20)+(21) \tag{23}$$

for large m and r.

By (22) and (23), we obtain Theorem 4.1. ■

Theorem 4.2. Let $s < \mu$ and s be sufficiently close to μ. Then under the conditions of Theorem 1.1, we have

$$\frac{\partial F(0,r,0,s)}{\partial r} > 0,$$

for large r.

Proof. Let $m = 0$, $t = 0$ in (17)–(21), one has

$$\begin{aligned}\frac{\partial F(0,r,0,s)}{\partial r} &= \int_{S_r} r^{s-1}\left[r(\nabla dr)-\left(\frac{s}{2}+\frac{1}{2}\bar{\delta}\right)\hat{g})(\nabla v,\nabla v)\right]e^{-2\rho}\,\mathrm{d}x\\ &+ r^{s-1}\int_{S_r}\left[-\frac{\bar{\delta}}{2}+\frac{s}{2}\right]\left|\frac{\partial v}{\partial r}\right|^2 e^{-2\rho}\,\mathrm{d}x\\ &+ r^{s-1}\int_{S_r}\left[\bar{a}_1+\frac{\bar{\delta}a_4}{2}+o(1)\right]\frac{\partial v}{\partial r}ve^{-2\rho}\,\mathrm{d}x\\ &+ r^{s-1}\int_{S_r}\left[\left(\lambda-\frac{a_4^2}{4}\right)\left(\frac{s}{2}+\frac{\bar{\delta}}{2}\right)-\frac{\bar{a}_2}{2}+o(1)\right]v^2e^{-2\rho}\,\mathrm{d}x.\end{aligned}$$

We will show that for large r,

$$\frac{\partial F(0,r,0,s)}{\partial r} > 0.$$

By Cauchy–Schwartz inequality, it suffices to prove

$$4\left[\left(\lambda-\frac{a_4^2}{4}\right)\left(\frac{s}{2}+\frac{\bar{\delta}}{2}\right)-\frac{\bar{a}_2}{2}\right]\left[-\frac{\bar{\delta}}{2}+\frac{s}{2}\right] > \left|\bar{a}_1+\frac{\bar{\delta}}{2}a_4\right|^2. \tag{24}$$

Solving inequality (24), we get

$$\lambda > \frac{a_4^2}{4} + \frac{a_2}{s+\bar{\delta}} + \frac{1}{4}\frac{(2\bar{a}_1 + \bar{\delta}a_4)^2}{s^2 - \bar{\delta}^2}. \tag{25}$$

It is clear that (25) holds if

$$\lambda > \frac{a_4^2}{4} + \frac{a_2}{s-\delta} + \frac{1}{4}\frac{(2a_1 + \delta a_4)^2}{s^2 - \delta^2}, \tag{26}$$

which follows from the assumption (3) and the fact s is close to μ. ■

Theorem 4.3. There exist $m_0 \geq 0$ and $R_0 > 0$ such that

$$F(m_0, r, t, 0) > 0$$

for all $r \geq R_0$.

Proof. It is easy to check that

$$\begin{aligned} F(m, r, t, 0) &= \int_{S_r} \left[\frac{1}{2}\left(\frac{m(m+1)}{r^2} - \frac{t}{r} + q_2 + \lambda \right) v_m^2 + \left(\left| \frac{\partial v_m}{\partial r} \right|^2 - \frac{1}{2} \left| \nabla v_m \right|^2 \right) \right] e^{-2\rho} \, dx \\ &= \frac{r^{2m}}{2} \int_{S_r} \left[\left(\frac{2m^2 + m}{r^2} - \frac{t}{r} + q_2 + \lambda \right) v^2 + \left| \frac{\partial v}{\partial r} \right|^2 + 2\frac{m}{r}\frac{\partial v}{\partial r} v - |\nabla_\omega v|^2 \right] e^{-2\rho} \, dx. \end{aligned}$$

By unique continuation theorem and fact that u is nonzero, there exists large enough R_0 such that $\int_{S_{R_0}} v^2 e^{-2\rho} \, dx \neq 0$. Let m_0 be large enough so that

$$F(m_0, R_0, t, 0) > 0.$$

By Theorem 4.1, we get

$$F(m_0, r, t, 0) > 0,$$

for all $r > R_0$. ■

Theorem 4.4. Assume that $\int_{S_r} v^2 e^{-2\rho} \, dx$ is not monotone increasing (with respect to r) in any semi-infinite interval $r \geq R$. Then there exists a sequence r_n goes to infinity such that

$$F(0, 0, r_n, 0) > 0.$$

Proof. By the assumption, there exists a sequence r_n goes to infinity and such that

$$\frac{\partial}{\partial r}\int_{S_r} v^2 e^{-2\rho}\,\mathrm{d}x < 0$$

for $r = r_n$. By Lemma 2.2, one has

$$\int_{S_r}\left[2v\frac{\partial v}{\partial r} + (\Delta r - 2\rho')v^2\right]e^{-2\rho}\,\mathrm{d}x < 0 \tag{27}$$

for $r = r_n$. By some direct computation, we have

$$\begin{aligned}
F(m_0, r, t, 0) &= \int_{S_r}\left[\frac{1}{2}\left(\frac{m_0(m_0+1)}{r^2} - \frac{t}{r} + q_2 + \lambda\right)v_{m_0}^2 + \left(\left|\frac{\partial v_{m_0}}{\partial r}\right|^2 - \frac{1}{2}|\nabla v_{m_0}|^2\right)\right]e^{-2\rho}\,\mathrm{d}x \\
&= \frac{r^{2m_0}}{2}\int_{S_r}\left[\left(\frac{m_0(m_0+1)}{r^2} - \frac{t}{r} + q_2 + \lambda\right)v^2 + \left|\frac{\partial v}{\partial r} + \frac{m_0}{r}v\right|^2 - |\nabla_\omega v|^2\right]e^{-2\rho}\,\mathrm{d}x \\
&= \frac{r^{2m_0}}{2}\int_{S_r}\left[(q_2+\lambda)v^2 + \left|\frac{\partial v}{\partial r}\right|^2 - |\nabla_\omega v|^2\right]e^{-2\rho}\,\mathrm{d}x \\
&\quad + \frac{r^{2m_0}}{2}\int_{S_r}\left[\left(\frac{2m_0^2+m_0}{r^2} - \frac{t}{r}\right)v^2 + 2\frac{m_0}{r}\frac{\partial v}{\partial r}v\right]e^{-2\rho}\,\mathrm{d}x \\
&= \frac{r^{2m_0}}{2}\int_{S_r}\left[(q_2+\lambda)v^2 + \left|\frac{\partial v}{\partial r}\right|^2 - |\nabla_\omega v|^2\right]e^{-2\rho}\,\mathrm{d}x \\
&\quad + \frac{m_0 r^{2m_0}}{2r}\int_{S_r}\left[2v\frac{\partial v}{\partial r} + (\Delta r - 2\rho')v^2\right]e^{-2\rho}\,\mathrm{d}x \\
&\quad + \frac{r^{2m_0}}{2}\int_{S_r}\left[\frac{2m_0^2+m_0}{r^2} - \frac{t}{r} - \frac{m_0}{r}(\Delta r - 2\rho')\right]v^2 e^{-2\rho}\,\mathrm{d}x \\
&= r^{2m_0}F(0,0,r,0) \\
&\quad + \frac{m_0 r^{2m_0}}{2r}\int_{S_r}\left[2v\frac{\partial v}{\partial r} + (\Delta r - 2\rho')v^2\right]e^{-2\rho}\,\mathrm{d}x \\
&\quad + \frac{r^{2m_0}}{2}\int_{S_r}\left[\frac{2m_0^2+m_0}{r^2} - \frac{t}{r} - \frac{m_0}{r}(\Delta r - 2\rho')\right]v^2 e^{-2\rho}\,\mathrm{d}x.
\end{aligned} \tag{28}$$

For large r, one has

$$\frac{r^{2m_0}}{2}\int_{S_R}\left[\frac{2m_0^2+m_0}{r^2} - \frac{t}{r} - \frac{m_0}{r}(\Delta r - 2\rho')\right]v^2 e^{-2\rho}\,\mathrm{d}x < 0,$$

since $t > 0$.

Combing with (27) and (28), one has

$$F(m_0, r_n, t, 0) < F(0, r_n, 0, 0).$$

By Theorem 4.3, we have $F(m_0, r_n, t, 0) > 0$. Thus we get

$$F(0, 0, r_n, 0) > 0.$$

■

Proof of Theorem 1.1.

Proof. It suffices to assume $\int_{S_r} v^2 e^{-2\rho}\, dx$ is not monotone increasing (with respect to r) in any semi-infinite interval $r \geq R$.

By Theorems 4.2 and 4.4, there exists $\gamma > 0$ such that

$$F(0, r, 0, s) \geq \gamma,$$

for large r. Thus

$$r^s \int_{S_r} \left(v^2 + \left| \frac{\partial v}{\partial r} \right|^2 \right) e^{-2\rho}\, dx \geq \gamma_0$$

for some $\gamma_0 > 0$.

By the fact $v = e^{\rho} u$, we get that

$$\liminf_{r \to \infty} r^s \left[M(r)^2 + N(r)^2 \right] > 0.$$

Recalling that $s < \mu$, we have

$$\liminf_{r \to \infty} r^{\mu} \left[M(r)^2 + N(r)^2 \right] = \infty.$$

■

Proof of Corollary 1.2.

Proof. The Corollary 1.2 follows from Theorem 1.1 directly. ■

Proof of Corollary 1.3.

Proof. The Corollary 1.3 follows from Theorem 1.1 and the fact that Δr is the trace of ∇dr. ■

Before we finish the proof of Corollary 1.5, a lemma is necessary.

Lemma 4.5. Suppose there exists a $r_0 > 0$ such that

$$\nabla \mathrm{d}r \geq 0$$

for $r = r_0$, and

$$-1 - \frac{2A}{r} \leq K_{\mathrm{rad}}(r) \leq -1 + \frac{2A}{r} < 0,$$

for $r \geq r_0$, where $K_{\mathrm{rad}}(r)$ is the radial curvature.

Then

$$|\nabla \mathrm{d}r - \hat{g}| \leq \frac{A + o(1)}{r}, \tag{29}$$

and

$$\left|\frac{\partial \Delta r}{\partial r}\right| \leq \frac{4(n-1)A + o(1)}{r}. \tag{30}$$

Proof. (29) and (30) can be proved by comparison theorem and Weitzenböck formula. See [17] for details. ■

Proof of Corollary 1.5.

Proof. Under the curvature condition of Corollary 1.5 and Lemma 4.5, one has

$$\limsup_{r\to\infty} |\nabla \mathrm{d}r - \hat{g}| \leq \frac{A}{r}.$$

Now Corollary 1.5 follows from Corollary 1.3 . ■

5 Proof of Theorem 1.6

Let $q_1 = \Delta r - 2\rho'$, $q_2 = -\frac{a_4^2}{4} - \frac{a_4\bar{\delta}}{2r} - \frac{a_4 a_5}{2r} - V_2$ and the others as the same in §4. By (16) and integration by part, one has

$$\begin{aligned}\frac{\partial F(m,r,t,s)}{\partial r} &= r^{s-1}\int_{S_r}\left[\left(r(\nabla \mathrm{d}r) - \frac{s}{2}\hat{g}\right)(\nabla v_m, \nabla v_m) + \frac{1}{2}r^s v_m < \nabla_\omega q_1, \nabla_\omega v_m >\right] e^{-2\rho}\,\mathrm{d}x\\ &+ r^{s-1}\int_{S_r}\left[2m + \frac{s}{2}\right]\left|\frac{\partial v_m}{\partial r}\right|^2 e^{-2\rho}\,\mathrm{d}x\\ &+ r^{s-1}\int_{S_r} r\left[q_2 - \frac{t}{r} + V_0 + V_1 + V_2\right]\frac{\partial v_m}{\partial r} v_m e^{-2\rho}\,\mathrm{d}x\\ &+ r^{s-1}\int_{S_r}\left[\frac{s}{2}q_1 + \frac{1}{2}r\frac{\partial q_1}{\partial r} + 2mq_1\right]\frac{\partial v_m}{\partial r} v_m e^{-2\rho}\,\mathrm{d}x\\ &+ \int_{S_r}\left[\frac{s-2}{2}r^{s-3}m(m+1) - \frac{s-1}{2}tr^{s-2} + \frac{r^s}{2}\frac{\partial q_2}{\partial r} + \frac{s}{2}r^{s-1}q_2 + \lambda\frac{s}{2}r^{s-1}\right] v_m^2 e^{-2\rho}\,\mathrm{d}x\\ &+ \int_{S_r}\frac{1}{2}r^s q_1\left[-\frac{t}{r} + \frac{m}{r}q_1 + q_2 + V_0 + V_1 + V_2\right] v_m^2 e^{-2\rho}\,\mathrm{d}x.\end{aligned}$$

Let

$$\frac{\partial \bar{\delta}(r)}{\partial r} = \bar{\delta}_1, \left(\nabla - \frac{\partial}{\partial r}\right)\bar{\delta}(r) = \bar{\delta}_2.$$

By the assumptions of Theorem 1.6, we get

$$\begin{aligned}\frac{\partial F(m,r,t,s)}{\partial r} &= r^{s-1}\int_{S_r}\left[\left(r(\nabla \mathrm{d}r) - \left(\frac{s}{2}\right)\hat{g}\right)(\nabla v_m, \nabla v_m) + \frac{1}{2}r^{s-1} v_m < \bar{\delta}_2, \nabla_\omega v_m >\right] e^{-2\rho}\,\mathrm{d}x\\ &+ r^{s-1}\int_{S_r}\left[2m + \frac{s}{2}\right]\left|\frac{\partial v_m}{\partial r}\right|^2 e^{-2\rho}\,\mathrm{d}x\\ &+ r^{s-1}\int_{S_r}\left[-t + \bar{a}_1 + \frac{1}{2}\bar{\delta}_1 + 2\bar{\delta}\frac{m}{r} + o(1)\right]\frac{\partial v_m}{\partial r} v_m e^{-2\rho}\,\mathrm{d}x\\ &+ r^{s-1}\int_{S_r}\left[-\frac{a_4\bar{\delta}_1}{4} - \frac{s}{8}a_4^2 + \frac{s}{2}\lambda - \frac{\bar{a}_2}{2} + o(1)\right] v_m^2 e^{-2\rho}\,\mathrm{d}x\\ &+ r^{s-1}\int_{S_r}\left[\frac{s-2}{2}\frac{m(m+1)}{r^2} + \frac{\bar{\delta}^2}{2}\frac{m}{r^2}\right] v_m^2 e^{-2\rho}\,\mathrm{d}x. \qquad (31)\end{aligned}$$

Theorem 5.1. Under the conditions of Theorem 1.6, there exist $s_0, R_0, m_0 > 0$ such that for $m \geq m_0$ and $r \geq R_0$,

$$\frac{\partial F(m,r,t,s_0)}{\partial r} > 0.$$

Proof. Let $2 < s_0 < 2a_3$, which will be specified later. One has

$$\left|\frac{1}{2}v_m < \bar{\delta}_2, \nabla_\omega v_m >\right| \leq \frac{v_m^2}{\sigma} + \left(a_3 - \frac{s_0}{2} + \epsilon(\sigma)\right)|\nabla_\omega v_m|^2,$$

where σ is any constant such that

$$\sigma < \frac{16a_3 - 8s_0}{|\delta_2|^2},$$

and $\epsilon(\sigma) > 0$ is small. Let σ be close to $\frac{16a_3-8s_0}{|\delta_2|^2}$. By assumption (5), there exists s_0 with $2 < s_0 < 2a_3$ such that

$$-\frac{a_4\bar{\delta}_1}{4} - \frac{s_0}{8}a_4^2 + \frac{s_0}{2}\lambda - \frac{\bar{a}_2}{2} - \frac{1}{\sigma} > 0.$$

By Cauchy–Schwartz inequality, we have

$$\begin{aligned} &4\left[\frac{s_0-2}{2}\frac{m(m+1)}{r^2} + \frac{\bar{\delta}^2}{2}\frac{m}{r^2} - \frac{a_4\bar{\delta}_1}{4} - \frac{s_0}{8}a_4^2 + \frac{s_0}{2}\lambda - \frac{\bar{a}_2}{2} - \frac{1}{\sigma}\right]\left[2m + \frac{s_0}{2}\right] \\ &\quad > \left[-t + \bar{a}_1 + \frac{1}{2}\bar{\delta}_1 + 2\bar{\delta}\frac{m}{r}\right]^2 \end{aligned} \tag{32}$$

for large m and large r. Putting all the estimates together, we obtain

$$\frac{\partial F(m, r, t, s_0)}{\partial r} > 0$$

for large m and r. ■

Theorem 5.2. Under the assumptions of Theorem 1.6, we have

$$\frac{\partial F(0, r, 0, s)}{\partial r} > 0,$$

for large r.

Proof. Let $m, t = 0$ in (31), one has

$$\begin{aligned}\frac{\partial F(0,r,0,s)}{\partial r} &= r^{s-1}\int_{S_r}\left[\left(r(\nabla \mathrm{d}r) - \frac{s}{2}\hat{g}\right)(\nabla v, \nabla v) + \frac{1}{2}r^{s-1}v < \bar{\delta}_2, \nabla_\omega v >\right] e^{-2\rho}\,\mathrm{d}x \\ &+ r^{s-1}\int_{S_r}\left[\frac{s}{2}\right]\left|\frac{\partial v}{\partial r}\right|^2 e^{-2\rho}\,\mathrm{d}x \\ &+ r^{s-1}\int_{S_r}\left[\bar{a}_1 + \frac{1}{2}\bar{\delta}_1 + o(1)\right]\frac{\partial v}{\partial r}ve^{-2\rho}\,\mathrm{d}x \\ &+ r^{s-1}\int_{S_r}\left[-\frac{a_4\bar{\delta}_1}{4} - \frac{s}{8}a_4^2 + \frac{s}{2}\lambda - \frac{\bar{a}_2}{2} + o(1)\right]v^2e^{-2\rho}\,\mathrm{d}x.\end{aligned}$$

By the proof of (32), it suffices to show that

$$4\left[-\frac{a_4\bar{\delta}_1}{4} - \frac{s}{8}a_4^2 + \frac{s}{2}\lambda - \frac{\bar{a}_2}{2} - \frac{1}{\sigma}\right]\left[\frac{s}{2}\right] > \left[\bar{a}_1 + \frac{1}{2}\bar{\delta}_1\right]^2.$$

This holds by assumption (4). ■

Proof of Theorem 1.6.

Proof. The proof follows from the proof of Theorem 1.1. We only need to replace Theorems 4.1 and 4.2, with Theorems 5.1 and 5.2. ■

6 Proof of Theorems 1.8 and 1.11 and Corollary 1.9

Let $q_1 = 0$, $q_2 = -\frac{a_4^2}{4} - \frac{a_4\bar{\delta}}{2r} - \frac{a_4a_5}{2r} - V_2$ and the others as before. By (16), we have

$$\frac{\partial F(m,r,t,s)}{\partial r} = r^{s-1}\int_{S_r}\left[r(\nabla \mathrm{d}r) - \left(\frac{s}{2} + \frac{1}{2}\bar{\delta}\right)\hat{g})(\nabla v_m, \nabla v_m)\right]e^{-2\rho}\,\mathrm{d}x \tag{33}$$

$$+ r^{s-1}\int_{S_r}\left[2m - \frac{\bar{\delta}}{2} + \frac{s}{2}\right]\left|\frac{\partial v_m}{\partial r}\right|^2 e^{-2\rho}\,\mathrm{d}x \tag{34}$$

$$+ r^{s-1}\int_{S_r}\left[\bar{a}_1 - t + \bar{\delta}\frac{m}{r} + o(1)\right]\frac{\partial v_m}{\partial r}v_m e^{-2\rho}\,\mathrm{d}x \tag{35}$$

$$+ r^{s-1}\int_{S_r}\left[\left(\lambda - \frac{a_4^2}{4}\right)\left(\frac{s}{2} + \frac{\bar{\delta}}{2}\right) - \frac{\bar{a}_2}{2} - \frac{1}{4}a_4\bar{\delta}_1 + o(1)\right]v_m^2e^{-2\rho}\,\mathrm{d}x \tag{36}$$

$$+ r^{s-1}\int_{S_r}\frac{m(m+1)}{r^2}\left[\frac{s-2}{2} + \frac{\bar{\delta}}{2}\right]v_m^2e^{-2\rho}\,\mathrm{d}x. \tag{37}$$

Theorem 6.1. Under the conditions of Theorem 1.8, there exist $s_0, R_0, m_0 > 0$ such that for $m \geq m_0$ and $r \geq R_0$,

$$\frac{\partial F(m, r, t, s_0)}{\partial r} > 0.$$

Proof. Let s_0 be such that $s_0 < 2a_3 - \delta$ and sufficiently close to $2a_3 - \delta$. By the assumption $a_3 > 1 + \delta$ (see (6)), one has

$$r(\nabla \mathrm{d}r) - \left(\frac{s_0}{2} + \frac{1}{2}\bar{\delta}\right)\hat{g} \geq 0,$$

for large r, which implies

$$(33) > 0. \tag{38}$$

By assumption $a_3 > 1 + \delta$ (see (6)) and $\lambda > \frac{a_4^2}{4} + \frac{2a_2 + a_4\delta_1}{4(a_3 - \delta)}$ (see (7)), one has

$$\left(\lambda - \frac{a_4^2}{4}\right)\left(\frac{s_0}{2} + \frac{\bar{\delta}}{2}\right) - \frac{\bar{a}_2}{2} - \frac{a_4}{4}\bar{\delta}_1 > 0,$$

and

$$\frac{s_0 - 2}{2} + \frac{\bar{\delta}}{2} > 0.$$

By Cauchy–Schwartz inequality again, one has

$$\left[\frac{m(m+1)}{r^2}\left(\frac{s_0 - 2}{2} + \frac{\bar{\delta}}{2}\right) + \left(\lambda - \frac{a_4^2}{4}\right)\left(\frac{s_0}{2} + \frac{\bar{\delta}}{2}\right) - \frac{\bar{a}_2}{2} - \frac{a_4}{4}\bar{\delta}_1\right] v_m^2 + \left[2m - \frac{\bar{\delta}}{2} + \frac{s_0}{2}\right]\left|\frac{\partial v_m}{\partial r}\right|^2 > \left|\left[\frac{\bar{a}_1}{2} - t + \bar{\delta}\frac{m}{r}\right]\frac{\partial v_m}{\partial r} v_m\right|$$

for large m and r. Thus, one has

$$|(35)| < (34) + (36) + (37) \tag{39}$$

for large m and r.

Now Theorem 6.1 follows from (38) and (39). ∎

Theorem 6.2. Let $s < \mu$ and s be sufficiently close to μ. Then under the conditions of Theorem 1.8, we have

$$\frac{\partial F(0, r, 0, s)}{\partial r} > 0,$$

for large r.

Proof. Let $m = 0$, $t = 0$ in (33), one has

$$\begin{aligned}\frac{\partial F(0,r,0,s)}{\partial r} &= \int_{S_r} r^{s-1}\left[r(\nabla \mathrm{d}r) - \left(\frac{s}{2} + \frac{1}{2}\bar{\delta}\right)\hat{g}\right)(\nabla v, \nabla v)\Bigg] e^{-2\rho}\,\mathrm{d}x \\ &+ r^{s-1}\int_{S_r}\left[-\frac{\bar{\delta}}{2} + \frac{s}{2}\right]\left|\frac{\partial v}{\partial r}\right|^2 e^{-2\rho}\,\mathrm{d}x \\ &+ r^{s-1}\int_{S_r}[\bar{a}_1 + o(1)]\frac{\partial v}{\partial r} v e^{-2\rho}\,\mathrm{d}x \\ &+ r^{s-1}\int_{S_r}\left[\left(\lambda - \frac{a_4^2}{4}\right)\left(\frac{s}{2} + \frac{\bar{\delta}}{2}\right) - \frac{\bar{a}_2}{2} - \frac{a_4}{4}\bar{\delta}_1 + o(1)\right] v^2.\end{aligned}$$

We will show that for large r,

$$\frac{\partial F(0,r,0,s)}{\partial r} > 0.$$

By Cauchy–Schwartz inequality, it suffices to prove

$$4\left[\left(\lambda - \frac{a_4^2}{4}\right)\left(\frac{s}{2} + \frac{\bar{\delta}}{2}\right) - \frac{\bar{a}_2}{2} - \frac{a_4}{4}\bar{\delta}_1\right]\left[-\frac{\bar{\delta}}{2} + \frac{s}{2}\right] > |\bar{a}_1|^2, \tag{40}$$

which holds by assumption (3) and s is close to μ. ■

Proof of Theorem 1.6.

Proof. The proof follows from the proof of Theorems 1.1 and 1.6. We only need to replace Theorems 4.1 and 4.2, with Theorems 6.1 and 6.2. ■

Proof of Corollary 1.9.

Proof. The proof follows from Theorem 1.6 and Lemma 4.5. ■

Proof of Theorem 1.11. In the proof Theorems 1.1 and 1.8, we let $q_2 = -\frac{a_4^2}{4} - \frac{a_4 a_5}{2r} - V_2$ and $q_2 = -\frac{a_4^2}{4} - \frac{a_4 a_5}{2r} - V_2 - \frac{a_4\bar{\delta}}{2r}$, respectively. Now we only need to let $q_2 = -\frac{a_4^2}{4} - \frac{a_4 a_5}{2r} - V_2 - (1-\sigma)\frac{a_4\bar{\delta}}{2r}$, and following the proof of Theorems 1.1 and 1.8, we can prove Theorem 1.11. We omit the details. ■

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