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Inhomogeneous Diophantine approximation in the coprime setting

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ABSTRACT

Given $n \in \mathbb{N}$ and $x, \gamma \in \mathbb{R}$, let

$$||\gamma - nx||' = \min\{|\gamma - nx + m| : m \in \mathbb{Z}, \gcd(n, m) = 1\},$$

Two conjectures in the coprime inhomogeneous Diophantine approximation state that for any irrational number α and almost every $\gamma \in \mathbb{R}$,

$$\liminf_{n\to\infty} n||\gamma - n\alpha||' = 0$$

and that there exists $C > 0$, such that for all $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and $\gamma \in [0, 1)$,

$$\liminf_{n\to\infty} n||\gamma - n\alpha||' < C.$$

We prove the first conjecture and disprove the second one.

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1. Introduction

Let $\psi : \mathbb{N} \to \mathbb{R}^+$ and $\alpha \in \mathbb{R}\backslash\mathbb{Q}, \gamma \neq 0$. A classical Diophantine approximation problem studies the existence of infinitely many pairs (p, q) of integers such that

$$|\gamma - q\alpha + p| \leq \psi(q). \tag{1}$$

It is referred to as homogeneous if $\gamma = 0$ and inhomogeneous if $\gamma \neq 0$. See [3] for the discussion of known results and references.

Questions of this type have applications, among other things, to several areas of dynamical systems and to the spectral theory of quasiperiodic Schrödinger operators (e.g. [1,2,12,13]). The inhomogeneous problem above can be understood in the metric sense: a.e. γ, and in the uniform sense: all γ.

Coprime inhomogeneous approximation asks the same questions about infinitely many coprime pairs (p, q). This question has been linked to the density exponents of lattice orbits in $\mathbb{R}^2$, in [15].

For the classical uniform setting, Minkowski Theorem guarantees that for any irrational $\alpha \in \mathbb{R}$ and $\gamma \notin \alpha\mathbb{Z} + \mathbb{Z}$, there are infinitely many pairs (p, q) of integers such that

$$|\gamma - q\alpha + p| \leq \frac{1}{4|q|}. \tag{2}$$

Grace [10] showed that 1/4 in (2) is sharp, and Khintchine [6] showed that

$$\liminf_{|q|\to\infty} |q| \, ||q\alpha - \gamma|| \leq \frac{1}{4}(1 - 4\lambda(\alpha)^2)^{\frac{1}{2}},$$

where

$$\lambda(\alpha) = \liminf_{|q|\to\infty} |q| \, ||q\alpha||,$$

and $||x|| = \text{dist}(x, \mathbb{Z})$.

Uniform inhomogeneous coprime approximation was studied by Chalk and Erdós who proved [5] that for any irrational $\alpha \in \mathbb{R}$ and for any γ there are infinitely many pairs of coprime integers (p, q) such that (1) holds with $\psi(q) = (\frac{\log q}{\log\log q})^2 \frac{1}{q}$.

Laurent and Nogueira [15] conjectured that a result similar to Minkowski's theorem holds also for the inhomogeneous coprime approximation, namely that there exists $C > 0$ such that for any irrational $\alpha \in \mathbb{R}$ and for any γ there are infinitely many pairs of coprime integers (p, q) with

$$|\gamma - q\alpha + p| \leq \frac{C}{|q|}.$$

In other words the conjecture is that $(\frac{\log q}{\log\log q})^2$ in Chalk-Erdős can be replaced by C. Such a result would clearly be optimal up to determining the optimal C.

The work of Chalk and Erdős was forgotten by the community until recently and the problem was studied in several papers (e.g. [11]), where results somewhat weaker than in [5] were obtained by different methods. The best positive result towards this conjecture remains the one in [5].

Our first result in the present paper is to show that such C does not exist. This shows that the coprime requirement leads to fundamental differences in the quality of approximation for the inhomogeneous setting.

Theorem 1.1. *For any constant C, there exists $(\alpha,\gamma)\in[0,1)^2$ with $\alpha\in\mathbb{R}\backslash\mathbb{Q}$ and $\gamma\notin\alpha\mathbb{Z}+\mathbb{Z}$ such that the inequality*

$$|\gamma-q\alpha+p|\leq\frac{C}{q},$$

only has finitely many coprime solutions $(p,q)\in\mathbb{N}^2$.

Remark 1.2. Actually, for both α and γ corresponding bad sets can be shown to be dense and uncountable, see Theorem 4.8.

Given $n\in\mathbb{N}$ and $x\in\mathbb{R}$, define

$$||x-n\alpha||'=\min\{|x-n\alpha-m|:m\in\mathbb{Z},(n,m)=1\},$$

where (n,m) is the largest positive common divisor of n and m.

Addressing the inhomogeneous coprime approximation problem from the metric point, Laurent and Nogueira proved that (1) has infinitely many coprime solutions (p,q) for almost every $(\alpha,\gamma)\in\mathbb{R}^2$ provided $\sum\psi(n)=\infty$. In particular, there are infinitely many coprime solutions for almost every $(\alpha,\gamma)\in\mathbb{R}^2$ for $\psi(n)=c/n$. Laurent and Nougeira [15] conjectured that the same is true on each fiber for a fixed α, and they proved that

$$\liminf_{n\to\infty}|n|||\gamma-n\alpha||'\leq 2 \tag{3}$$

for α such that $\sum_{k\geq0}\frac{1}{\max(1,\log q_k)}=\infty$, where q_k are denominators of continued fraction approximants of α, and a.e. γ. Condition (3) is essential for the proof of [15] because it requires an application of Gallaher's theorem.

Our second result in this paper is a proof of (a stronger version of) the above conjecture for all irrational α.

We prove

Theorem 1.3. *For any irrational number α,*

$$\liminf_{n\to\infty}n||\gamma-n\alpha||'=0$$

holds for almost every $\gamma\in\mathbb{R}$.

Remark 1.4.

- Since $||\gamma - n\alpha||'$ is 1-periodic with respect to α, we always assume $\alpha \in (0, 1)$ in this paper.
- It is known [14] that for almost every γ,

$$\liminf_{n\to\infty} n||\gamma - n\alpha|| = 0. \tag{4}$$

  However, the exceptional set of γ of (4) has full Hausdorff measure [4]. A necessary and sufficient condition on ψ and α so that $\liminf_{n\to\infty} \psi(n)||\gamma - n\alpha|| = 0$ holds for a.e. γ was given in [9].

Except the generalized Borel-Cantelli lemma, basic facts on the distribution of prime numbers and some basic ergodic arguments, the present paper is self-contained.

The rest of the paper is organized as follows: In §2, we obtain the asymptotics of coprime pairs. In §3, we will give the proof of Theorem 1.3. In §4, we will give the proof of Theorem 1.1.

The following standard notations will be used. Let (n, m) be the largest common divisor of n and m, and $\{x\} = x - \lfloor x \rfloor$, the fractional part of x. Denote by $|A|$ the Lebesgue measure of A and by $\#S$ the number of elements in S. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Without loss of generality, we always assume $\alpha \in (0, 1)$ is irrational.

2. The asymptotics of coprime pairs

For $n \in \mathbb{N}$, let $\pi(n)$ be the number of prime number less than n. It is well known that the prime numbers satisfy the following asymptotics [17]

$$\pi(n) = \frac{n}{\ln n}(1 + O(\frac{1}{\ln n})). \tag{5}$$

By the distribution of prime numbers, we also have the following well known results: a weaker version of Mertens' second theorem,

$$\sum_{\substack{2\le p<n \\ p \text{ is prime}}} \frac{1}{p} = \ln\ln n + O(1), \tag{6}$$

and a weaker version of Rosser's theorem (see [16]),

$$\sum_{p \text{ is prime}} \frac{1}{p\ln p} < \infty. \tag{7}$$

For any $\alpha \in [0,1)\backslash\mathbb{Q}$, we denote its continued fraction expansion by

$$\alpha = [a_1, a_2, \cdots, a_n, \cdots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\cdots}}}}.$$

Let

$$\frac{p_n}{q_n} = [a_1, a_2, \cdots, a_n] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\cdots + \frac{1}{a_n}}}}},$$

where $(p_n, q_n) = 1$.

By the properties of continued fraction expansion (see [8] for example), one has

$$\min_{p\in\mathbb{Z}} |k\alpha - p| \geq |q_n\alpha - p_n| \tag{8}$$

for any $1 \leq k < q_{n+1}$, and

$$\frac{1}{q_n + q_{n+1}} \leq |q_n\alpha - p_n| \leq \frac{1}{q_{n+1}}. \tag{9}$$

Moreover,

$$q_n\alpha - p_n = (-1)^n |q_n\alpha - p_n|. \tag{10}$$

In the following Sections 2 and 3, C is a large absolute constant. Let

$$\kappa = \prod_{p \text{ is prime}} (1 - \frac{1}{p^2}).$$

It is well known that

$$\kappa = \frac{6}{\pi^2}.$$

Set

$$J_n = \{k \in [1, q_{n+1} - 1] \cap \mathbb{Z} : (\lfloor k\alpha \rfloor, k) = 1\}, \tag{11}$$

where $\lfloor x \rfloor$ is the largest integer less or equal than x.

For an interval $I \subset [0,1)$, let

$$\hat{J}_n = \{k : k \in J_n, \{k\alpha\} \in I\}.$$

The following Theorem is crucial in our proof.

Theorem 2.1. *There exists a sequence $\{n_k\}$ (independent of I) such that the following asymptotics hold as $k \to \infty$,*

$$\#\hat{J}_{n_k} = (\kappa|I| + o(1))q_{n_k+1}. \tag{12}$$

Proof. Set $J_n(p)$,

$$J_n(p) = \{k \in [1, q_{n+1} - 1] \cap \mathbb{Z} : p|(\lfloor k\alpha \rfloor, k)\},$$

and

$$\hat{J}_n(p) = \{k \in [1, q_{n+1} - 1] \cap \mathbb{Z} : p|(\lfloor k\alpha \rfloor, k), \{k\alpha\} \in I\}.$$

Claim 1: $k \in \hat{J}_n(p)$ if and only if there exists some $k_1 \in \mathbb{N}$ such that

$$k = pk_1 \tag{13}$$

and

$$\{k_1\alpha\} \in \frac{I}{p}, \tag{14}$$

where

$$\frac{I}{p} = \{\frac{x}{p} : x \in I\}.$$

See a proof of the Claim 1 at the end of this Section.

Fix $\xi > 0$ (small enough). Let $\varepsilon > 0$ be sufficiently small. Now we distinguish the cases $p \leq \frac{1}{\xi}$ and $p > \frac{1}{\xi}$.

By ergodic theorem, for large n (depending on ξ, ε) one has

$$|I|\frac{q_{n+1}}{p^2} - \varepsilon q_{n+1} \leq \#\{k_1 : 1 \leq k_1 < \frac{q_{n+1}}{p}, \{k_1\alpha\} \in \frac{I}{p}\} \leq |I|\frac{q_{n+1}}{p^2} + \varepsilon q_{n+1} \tag{15}$$

for all $p \leq \frac{1}{\xi}$. By Claim 1, one has

$$|I|\frac{q_{n+1}}{p^2} - \varepsilon q_{n+1} \leq \#\hat{J}_n(p) \leq |I|\frac{q_{n+1}}{p^2} + \varepsilon q_{n+1}.$$

By the definition of $J_n(p)$, we have for any prime numbers $p_1, p_2, \cdots, p_s$,

$$\hat{J}_n(p_1) \cap \hat{J}_n(p_2) \cap \cdots \cap \hat{J}_n(p_s) = \hat{J}_n(p_1 p_2 \cdots p_s).$$

Thus by the inclusion-exclusion principle, we have

$$|I|(1-\varepsilon-\prod_{\substack{p\le\frac{1}{\xi}\\ p\text{ is prime}}}(1-\frac{1}{p^2}))q_{n+1}\le \#\bigcup_{\substack{p\le\frac{1}{\xi}\\ p\text{ is prime}}}\hat{J}_n(p)\le |I|(1+\varepsilon-\prod_{\substack{p\le\frac{1}{\xi}\\ p\text{ is prime}}}(1-\frac{1}{p^2}))q_{n+1}.$$

This implies (letting ε go to zero) that as $n\to\infty$,

$$\begin{aligned}&\#\{k:1\le k<q_{n+1},\text{ there exists some prime number }2\le p\le\frac{1}{\xi}\text{ such that }p|(\lfloor k\alpha\rfloor,k),\\ &\quad\text{and }k\alpha\in I\}\\ &\qquad=|I|q_{n+1}[1+o(1)-\prod_{\substack{p\le\frac{1}{\xi}\\ p\text{ is prime}}}(1-\frac{1}{p^2})].\end{aligned}\tag{16}$$

Now we are in a position to study the case $p>\frac{1}{\xi}$. We will prove that there exists a sequence $\{n_k\}$ such that

$$\#\bigcup_{\substack{p>\frac{1}{\xi}\\ p\text{ is prime}}}J_{n_k}(p)=\varphi(\xi)q_{n_k+1}\tag{17}$$

as $k\to\infty$, where $\varphi(\xi)$ goes to 0 as $\xi\to 0$.

We will split all primes p into the cases $\frac{q_{n+1}}{C}<p<q_{n+1}$, $Cq_n\le p\le\frac{q_{n+1}}{C}$, $\frac{q_n}{C}\le p\le Cq_n$ and $\frac{1}{\xi}<p\le\frac{q_n}{C}$, where C is a large constant.

Case 1: $\frac{q_{n+1}}{C}<p<q_{n+1}$.

By (13), one has

$$k_1\le\frac{q_{n+1}}{p}.\tag{18}$$

This leads to $k_1\le C$ in the current case. By (5), we have

$$\begin{aligned}\#\bigcup_{\substack{\frac{q_{n+1}}{C}<p<q_{n+1}\\ p\text{ is prime}}}J_n(p)&\le C\#\{p:\frac{q_{n+1}}{C}<p<q_{n+1}\text{ and }p\text{ is prime}\}\\ &\le C\frac{q_{n+1}}{\ln q_{n+1}}=o(1)q_{n+1}.\end{aligned}$$

Case 2: $\frac{q_n}{C}\le p\le Cq_n$.

By (5) and (18) again, one has

$$\begin{aligned}\#\bigcup_{\substack{\frac{q_n}{C}\le p\le Cq_n\\ p\text{ is prime}}}J_n(p)&\le C\frac{q_{n+1}}{q_n}\#\{p:\frac{q_n}{C}\le p\le Cq_n\text{ and }p\text{ is prime}\}\\ &\le C\frac{q_{n+1}}{q_n}\frac{q_n}{\ln q_n}=o(1)q_{n+1}.\end{aligned}$$

Case 3: $Cq_n\le p\le\frac{q_{n+1}}{C}$.

If $q_{n+1} \leq Cq_n$, there is no such p. We are done. Thus, we assume

$$q_{n+1} \geq Cq_n. \tag{19}$$

In this case, one has

$$k_1 = \ell q_n \tag{20}$$

for some $\ell \in \mathbb{N}$. Indeed, suppose $k_1 = \ell q_n + j_k$ with $1 \leq j_k < q_n$. By (8), (9) and $\ell \leq \frac{q_{n+1}}{Cq_n^2}$, one has

$$\begin{aligned}\{k_1\alpha\} &= \{\ell q_n \alpha + j_k \alpha\} \\ &\geq ||j_k\alpha|| - \ell ||q_n\alpha|| \\ &\geq \frac{1}{2q_n} - \frac{q_{n+1}}{Cq_n^2}\frac{1}{q_{n+1}} \\ &\geq \frac{1}{4q_n}.\end{aligned}$$

This contradicts (14).

If n is odd, by (9) and (10), we have

$$\begin{aligned}\{k_1\alpha\} &= 1 - ||k_1\alpha|| \\ &= 1 - ||\ell q_n \alpha|| \\ &\geq 1 - \frac{q_{n+1}}{pq_n}\frac{1}{q_{n+1}} \\ &\geq \frac{1}{2},\end{aligned}$$

which is impossible since $\{k_1\alpha\} < \frac{1}{p}$. This means there is no such k in the current case.

Assume n is even. Suppose $k \in J_n(p)$ and $k \notin J_n(p')$ for prime $p' < p$.

Claim 2:

$$k = \ell p q_n, \text{ and } p' \nmid \ell.$$

See a proof at the end of this Section.

Thus for any prime p in this case, we have

$$\#(J_n(p) \backslash \bigcup_{\substack{Cq_n < p' < p \\ p' \text{ is prime}}} J_n(p')) \leq \#\{\ell : \ell \leq \frac{q_{n+1}}{pq_n}, \ell \text{ does not have any divisor } p'$$
$$\text{with } Cq_n < p' < p\}$$

$$
\begin{aligned}
&\leq \frac{q_{n+1}}{pq_n}\prod_{\substack{Cq_n<p'<p\\ p' \text{ is prime}}}(1-\frac{1}{p'})\\
&\leq C\frac{q_{n+1}}{pq_n}\frac{\ln q_n}{\ln p}\\
&\leq C\frac{q_{n+1}\ln q_n}{q_n}\frac{1}{p\ln p},
\end{aligned}
$$

where the third inequality holds by (6).

Thus by (7), we have

$$
\begin{aligned}
\#\bigcup_{\substack{Cq_n<p<\frac{q_{n+1}}{C}\\ p \text{ is prime}}} J_n(p) &\leq C\frac{q_{n+1}\ln q_n}{q_n}\sum_{\substack{Cq_n<p<\frac{q_{n+1}}{C}\\ p \text{ is prime}}}\frac{1}{p\ln p}\\
&= o(1)q_{n+1}.
\end{aligned}
$$

Case 4: $\frac{1}{\xi}\leq p\leq \frac{q_n}{C}$.

For each $k_1<\frac{q_{n+1}}{p}$, rewrite $k_1=\ell q_n+\ell_{k_1}$ with $1\leq \ell_{k_1}<q_n$. Then by (9), one has

$$||\ell q_n\alpha||=\ell||q_n\alpha||<\frac{1}{p}.$$

In this case, we must have

$$||\ell_{k_1}\alpha||\leq \frac{C}{p}. \tag{21}$$

Indeed, if $||\ell_{k_1}\alpha||\geq \frac{C}{p}$, by (9), one has

$$
\begin{aligned}
\{k_1\alpha\} &= \{\ell q_n\alpha+\ell_{k_1}\alpha\}\\
&\geq ||\ell_{k_1}\alpha||-||\ell q_n\alpha||\\
&\geq ||\ell_{k_1}\alpha||-\frac{1}{p}\\
&\geq \frac{C}{p}.
\end{aligned}
$$

This is impossible since $\{k_1\alpha\}<\frac{1}{p}$.

Also, by (8) and (9), one has

$$\#\{j:1\leq j<q_n,||j\alpha||<\frac{C}{p}\}\leq C\frac{q_n}{p}+1\leq C\frac{q_n}{p}. \tag{22}$$

By (21) and (22), we have

$$\begin{aligned}\#\bigcup_{\substack{\frac{1}{\xi}<p<\frac{q_n}{C}\\ p \text{ is prime}}} J_n(p) &\leq \sum_{\substack{\frac{1}{\xi}<p<\frac{q_n}{C}\\ p \text{ is prime}}} C(\frac{q_{n+1}}{pq_n}+1)\frac{q_n}{p} \\ &\leq \sum_{\substack{\frac{1}{\xi}<p<\frac{q_n}{C}\\ p \text{ is prime}}} C(\frac{q_{n+1}}{pq_n})\frac{q_n}{p} + C\sum_{\substack{\frac{1}{\xi}<p<\frac{q_n}{C}\\ p \text{ is prime}}} \frac{q_n}{p} \\ &= \varphi(\xi)q_{n+1} + Cq_n \ln\ln q_n, \end{aligned} \tag{23}$$

where $\varphi(\xi) \to 0$ as $\xi \to 0$.

Suppose there exists an infinite sequence $\{n_k\}$ such that

$$q_{n_k+1} \geq q_{n_k}(\ln\ln q_{n_k})^2. \tag{24}$$

Then by (23), we have

$$\#\bigcup_{\substack{\frac{1}{\xi}<p<\frac{q_{n_k}}{C}\\ p \text{ is prime}}} J_{n_k}(p) = \varphi(\xi)q_{n_k+1}.$$

Putting the other cases together, this completes the proof of (17).

Otherwise for all large s, we have

$$q_{s+1} \leq q_s(\ln\ln q_s)^2. \tag{25}$$

For any p, let s be the unique positive integer such that

$$q_s \leq p < q_{s+1}. \tag{26}$$

Suppose $k_1, k_1' \in J_n(p)$. We must have

$$|k_1 - k_1'| \geq q_{s-2}. \tag{27}$$

Otherwise, by (8) and (9), one has

$$||k_1\alpha - k_1'\alpha|| > \frac{1}{2q_{s-2}}.$$

This is impossible since $\{k_1\alpha\} < \frac{1}{p}$, $\{k_1'\alpha\} < \frac{1}{p}$ and $p \geq q_s$.

By (25), (26) and (27), for any $k_1, k_1' \in J_n(p)$, one has

$$|k_1 - k_1'| \geq p^{\frac{1}{2}}.$$

Thus

$$\# \bigcup_{\substack{\frac{1}{\xi}<p<\frac{q_n}{C} \\ p \text{ is prime}}} J_n(p) \leq \sum_{\substack{p>\frac{1}{\xi} \\ p \text{ is prime}}} C\frac{q_{n+1}}{pp^{\frac{1}{2}}}$$
$$= \varphi(\xi)q_{n+1}.$$

Putting the other cases together, we finish the proof of (17).

Now the Theorem follows from (16) and (17) by letting $\xi \to 0$. □

Proof of Claim 1. Suppose $k \in \hat{J}_n(p)$. Then there exist $k_1, q_1 \in \mathbb{N}$ such that

$$k = pk_1$$

and

$$\lfloor k\alpha \rfloor = pq_1.$$

Using $k_1\alpha = \lfloor k_1\alpha \rfloor + \{k_1\alpha\}$, one has

$$\lfloor k\alpha \rfloor = \lfloor pk_1\alpha \rfloor = p\lfloor k_1\alpha \rfloor + \lfloor p\{k_1\alpha\} \rfloor.$$

This implies

$$p | \lfloor p\{k_1\alpha\} \rfloor.$$

Noting that $0 \leq \lfloor p\{k_1\alpha\} \rfloor \leq p - 1$, one has

$$\lfloor p\{k_1\alpha\} \rfloor = 0. \tag{28}$$

Thus

$$\{k_1\alpha\} < \frac{1}{p}.$$

Combining with the assumption that $\{k\alpha\} \in I$, one has

$$p\{k_1\alpha\} = \{k\alpha\} \in I.$$

This yields that

$$\{k_1\alpha\} \in \frac{I}{p}.$$

The proof of the other side is similar. We omit the details. □

Proof of Claim 2. Otherwise, $k = p'pmq_n$ for some $p' < p$. Since n is even, by (10), one has

$$||q_n\alpha|| = \{q_n\alpha\} < \frac{1}{q_{n+1}}.$$

Thus

$$\begin{aligned}\{pmq_n\alpha\} &= pm||q_n\alpha|| \\ &< \frac{q_{n+1}}{p'q_n}\frac{1}{q_{n+1}} = \frac{1}{p'q_n}.\end{aligned}$$

This implies

$$\begin{aligned}\lfloor k\alpha\rfloor &= \lfloor p'pmq_n\alpha\rfloor \\ &= p'\lfloor pmq_n\alpha\rfloor + \lfloor p'\{pmq_n\alpha\}\rfloor \\ &= p'\lfloor pmq_n\alpha\rfloor.\end{aligned}$$

Thus

$$p'|(\lfloor k\alpha\rfloor, k).$$

We get a contradiction since $(\lfloor k\alpha\rfloor, k) \notin J(p')$. □

Remark 2.2. We should mention that the sequence $\{n_k\}$ in Theorem 2.1 is either defined by (24) or is the entire sequence $n \in \mathbb{N}$ in case (25). So it does not depend on the interval I.

3. Proof of Theorem 1.3

We present the general form of the Borel-Cantelli Lemma first, which is the key technique in this part of the argument. See [7,18] for details.

Lemma 3.1. *Let E_k, $k = 1, 2, \cdots$, be a sequence of Lebesgue measurable sets in $[0,1]$ and suppose that*

$$\sum_{k=1}^{\infty} |E_k| = \infty. \tag{29}$$

Then the Lebesgue measure of $E := \limsup_{N\to\infty} E_N := \bigcap_{N=1}^{\infty}\bigcup_{k=N}^{\infty} E_k$ satisfies

$$|E| \geq \limsup_{N\to\infty} \frac{(\sum_{k=1}^{N}|E_k|)^2}{\sum_{k=1}^{N}\sum_{l=1}^{N}|E_k \cap E_l|}.$$

Lemma 3.1 immediately implies

Corollary 3.2. *Suppose the sets* $\{E_k\}$ *are pairwise quasi-independent with respect to constant* $A > 0$*, that is*

$$|E_k \cap E_l| \leq A|E_k||E_l| + C2^{-(k+l)},$$

for all $k \neq l$*, and* $\sum_{k=1}^{\infty} |E_k| = \infty$*. Let* $E := \limsup_{N\to\infty} E_N := \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} E_k$*. Then*

$$|E| \geq \frac{1}{A}.$$

Define

$$I_n = \bigcup_{k\in J_n} (\{k\alpha\} - \frac{\tau}{q_{n+1}}, \{k\alpha\} + \frac{\tau}{q_{n+1}}), \tag{30}$$

where $0 < \tau < \frac{1}{3}$, and J_n is given by (11).

By (8) and (9), we have $I_n \subset [0, 1]$ and I_n contains exactly $\#J_n$ intervals.

Letting $I = [0, 1)$ in Theorem 2.1, we have

Lemma 3.3. *Let sequence* $\{n_k\}$ *be given by Theorem 2.1. Then we have*

$$\#J_{n_k} = (\kappa + o(1))q_{n_k+1}$$

and

$$|I_{n_k}| = 2\tau\kappa + o(1).$$

Now we want to show the quasi-independence of a subsequence of $\{I_{n_k}\}$.

Theorem 3.4. *Fixing* $n_{l_1} \in \{n_k\}$*, we have*

$$|I_{n_{l_1}} \cap I_{n_{l_2}}| = |I_{n_{l_1}}|\,|I_{n_{l_2}}| + o(1), \tag{31}$$

as $n_{l_2} \in \{n_k\}$ *goes to infinity.*

Proof. Recall that I_n contains $\#J_n$ intervals. Let $I_n = \bigcup I_n^i$, $i = 1, 2, \cdots, \#J_n$. Fix one interval $I_{n_{l_1}}^i$ with $1 \leq i \leq \#J_{n_{l_1}}$. Now let l_2 go to infinity. By Theorem 2.1, one has

$$\#\{k : 1 \leq k < q_{n_{l_2}+1} : \{k\alpha\} \in I_{n_{l_1}}^i, (\lfloor k\alpha \rfloor, k) = 1\} = q_{n_{l_2}+1}\kappa|I_{n_{l_1}}^i| + o(1)q_{n_{l_2}+1}.$$

By (30), one has

$$|I_{n_{l_2}} \cap I_{n_{l_1}}^i| = 2\tau\kappa|I_{n_{l_1}}^i| + o(1).$$

By Lemma 3.3, we have

$$|I_{n_{l_2}} \cap I^i_{n_{l_1}}| = |I_{n_{l_2}}|\,|I^i_{n_{l_1}}| + o(1).$$

Summing up all the $i \in \#J_{n_{l_1}}$, we obtain the Theorem. □

Proof of Theorem 1.3. We give the proof of $\gamma \in [0,1)$ first. Applying Theorem 3.4, there exists a sequence $\{n_{k_l}\}$ such that

$$|I_{n_{k_i}} \cap I_{n_{k_j}}| = |I_{n_{k_i}}|\,|I_{n_{k_j}}| + \frac{O(1)}{2^{i+j}}.$$

Letting $E_l = I_{n_{k_l}}$, by Lemma 3.3, one has

$$|E_l| = 2\tau\kappa + o(1).$$

Applying Corollary 3.2 with $A = 1$, we get $|\limsup E_l| = 1$.

By the definition of I_n, we have for any $\gamma \in I_n$, there exists some $1 \le k < q_{n+1}$ such that $(\lfloor k\alpha \rfloor, k) = 1$ and

$$|\gamma - \{k\alpha\}| = |\gamma - k\alpha + \lfloor k\alpha \rfloor| \le \frac{\tau}{q_{n+1}}.$$

This implies for any n_{k_l}, there exists some $1 \le j < q_{n_{k_l}+1}$ such that

$$||\gamma - j\alpha||' \le \frac{\tau}{q_{n_k+1}} \le \frac{\tau}{j}.$$

Since τ is arbitrary, we have for almost every $\gamma \in [0,1)$,

$$\liminf_{k\to\infty} k||\gamma - k\alpha||' = 0.$$

Let us now consider $\gamma \in [m, m+1)$ for some $m \in \mathbb{Z}$. In this case, we only need to modify the definition of I_n in (30) as

$$I_n^m = \bigcup_{k\in J_n} (m + \{k\alpha\} - \frac{\tau}{q_{n+1}}, m + \{k\alpha\} + \frac{\tau}{q_{n+1}}).$$

By the same proof as for $\gamma \in [0,1)$, we have for almost every $\gamma \in [m, m+1)$,

$$\liminf_{k\to\infty} k||\gamma - k\alpha||' = 0.$$

This completes the proof. □

4. Proof of Theorem 1.1

In this section, we will prove the following theorem, which is a finer version of Theorem 1.1.

Theorem 4.1. *For any positive constant M, there exist $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and an uncountable set $\Omega \subset \mathbb{R}$ (depending on α) such that for all $\gamma \in \Omega$, the inequality*

$$|\gamma - q\alpha + p| \leq \frac{M}{q}, \tag{32}$$

only has finitely many coprime solutions $(p, q) \in \mathbb{N}^2$.

We need some preparations first. In this section, all the large constants C, C_1 and C_2 only depend on M. In the following arguments, we assume $C_2, C_1, C \in \mathbb{N}$ and

$$C_2 >> C_1 >> C.$$

Let $p^1 = 2, p^2, p^3, \cdots, p^n$ be the successive prime numbers with some $n = (2C_1 + 1)^2$ and let

$$P = p^1 p^2 \cdots p^n.$$

Define $a_k = \hat{l}_k P$ for $\hat{l}_k \in \mathbb{N}$, $k = 1, 2, \cdots$. In the following construction, we need that $\hat{l}_k > C_2$. Let $\alpha = [a_1, a_2, \cdots, a_k, \cdots]$ and $\frac{p_k}{q_k} = [a_1, a_2, \cdots, a_k]$. Then $p_0 = 0, p_1 = 1, q_0 = 1, q_1 = a_1$ and

$$p_k = a_k p_{k-1} + p_{k-2}, \tag{33}$$

and

$$q_k = a_k q_{k-1} + q_{k-2}. \tag{34}$$

Thus, we have

$$q_k \equiv 1 \mod P, \tag{35}$$

and

$$p_{2k} \equiv 0 \mod P, \tag{36}$$

and

$$p_{2k+1} \equiv 1 \mod P. \tag{37}$$

Since $\hat{l}_k > C_2$, we have

$$q_{k+1} \geq C_2 q_k. \tag{38}$$

We assign each pair (t, j), $|t| \leq C_1, |j| \leq C_1$ a different prime number $p^{t,j}$. We can randomly choose $p^{t,j}$ so that $p^{t,j}$ is a permutation of a subset of prime numbers $p^1, p^2, \cdots, p^n$ with $n = (2C_1 + 1)^2$. We also assume α is given by (33)-(38).

The plan is to construct a sequence $\{b_k\}$ such that for all k, $b_k \equiv t \mod p^{t,j}$, $\lfloor b_k\alpha\rfloor \equiv j \mod p^{t,j}$, for all $|t| \leq C_1, |j| \leq C_1$. This will be done by induction. We then construct nested intervals $\{I_k\} \subset [0, 1)$ centered at $b_k\alpha \mod \mathbb{Z}$ and $\lim |I_k| = 0$. We will show that for $\gamma = \cap I_k$ there are only finitely many coprime solutions to (32). Here is the sketch of the argument.

Suppose (32) has infinitely many coprime solutions. We will show (Theorem 4.7) that solutions (p, q) must have the structure $p = \lfloor q\alpha\rfloor$ and $q = b_k + d_k q_k + r_k q_{k-1}$ with $|d_k| \leq C$ and $|r_k| \leq C$ for some k. By (35)-(37), the remainders of $d_k q_k + r_k q_{k-1}$, $\lfloor(d_k q_k + r_k q_{k-1})\alpha\rfloor \mod p^{t,j}$ for all $|t|, |j| \leq C_1$ are bounded by $(2C + 1)^2$. It will imply that for some $(t_0, j_0) \in [-C_1, C_1] \times [-C_1, C_1]$, both $\lfloor q\alpha\rfloor \mod p^{t_0,j_0}$ and $q \mod p^{t_0,j_0}$ are zero. This is a contradiction.

To start with the construction of b_k, clearly, we can find $b_1 \equiv t \mod p^{t,j}$ by the Chinese Remainder Theorem. Simultaneously achieving $\lfloor b_1\alpha\rfloor \equiv j \mod p^{t,j}$ requires $b_1\alpha/p^{t,j} \mod \mathbb{Z}$ belonging to a certain interval of length $1/p^{t,j}$. In fact, in order to proceed with inductive construction of b_k, we will need a little more: that we can guarantee $b_1\alpha/p^{t,j} \mod \mathbb{Z}$ in slightly shrunk intervals. The following lemma is a preparation for that.

Lemma 4.2. *Suppose $\hat{p}^1, \hat{p}^2, \cdots \hat{p}^k$ are distinct prime numbers. Let $\hat{P} = \hat{p}^1\hat{p}^2 \cdots \hat{p}^k$. Then there exists a small $\delta > 0$ and a large constant $\bar{L} > 0$ (both depending on $\hat{p}^1, \hat{p}^2, \cdots \hat{p}^k$) such that for any α with $a_1 \geq \bar{L}$, any given box $I = I_1 \times I_2 \times \cdots \times I_k \subset \mathbb{T}^k$ with $|I_i| \geq \frac{1}{\hat{p}^i} - \delta$ for $i = 1, 2, \cdots, k$, any $L \geq a_1 + 1$ and any L_0, there exists some $j \in \{L_0, L_0 + 1, \cdots, L_0 + L\}$ such that*

$$(j\frac{\hat{P}}{\hat{p}^1}\alpha, j\frac{\hat{P}}{\hat{p}^2}\alpha, \cdots, j\frac{\hat{P}}{\hat{p}^k}\alpha) \in I \mod \mathbb{Z}^k.$$

Proof. Let us consider the map $\psi_k : \mathbb{R} \to \mathbb{T}^k$,

$$\psi_k(t) = (t, \frac{\hat{p}^1}{\hat{p}^2}t, \cdots, \frac{\hat{p}^1}{\hat{p}^k}t) \mod \mathbb{Z}^k. \tag{39}$$

See Fig. 1. Since $\psi_k(\hat{p}^2\hat{p}^3 \cdots \hat{p}^k) = 0 \mod \mathbb{Z}^k$, identifying $\mathbb{T}^k$ with $[0, 1)^k$, we have that there exists some $N = N(\hat{p}^1, \hat{p}^2, \cdots, \hat{p}^k)$ such that the image of $\psi_k \subset [0, 1)^k$ consists of at most N segments.

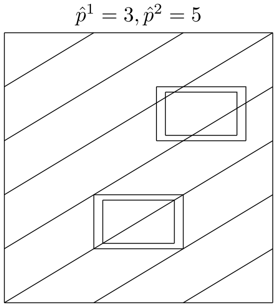

Fig. 1. Two prime numbers.

Claim 3: For any closed box $\hat I^k = \hat I_1 \times \hat I_2 \times \cdots \times \hat I_k$ with $|\hat I_i| = \frac{1}{\hat p^i}$ for $i = 1, 2, \cdots, k$, and for any $t_1 \in \hat I_1$, there exists some $t \equiv t_1 \mod \mathbb{Z}$ such that $\psi_k(t) \in \hat I^k \mod \mathbb{Z}$.

We will prove Claim 3 by induction. For $k = 1$, it is trivial. Suppose it holds for k. Thus for any $t_1 \in \hat I_1$, there exists some t_2 such that $t_2 \equiv t_1 \mod \mathbb{Z}$ and $\psi_k(t) \in \hat I^k$. Since all the $\hat p^i$ are prime numbers, there exists q such that $q\hat p^1 \hat p^2 \cdots \hat p^k \equiv 1 \mod \hat p^{k+1}$. Then there exists some $0 \le j \le \hat p^{k+1} - 1$ such that

$$j q \hat p^1 \hat p^2 \cdots \hat p^k \frac{1}{\hat p^{k+1}} + \frac{\hat p^1}{\hat p^{k+1}} t_2 \in \hat I_{k+1} \mod \mathbb{Z},$$

since $|\hat I_{k+1}| = \frac{1}{\hat p^{k+1}}$ and $\hat I_{k+1}$ is closed. Let $t_3 = jq\hat p^2 \cdots \hat p^k + t_2$. Then $\frac{\hat p^1}{\hat p^{k+1}} t_3 \in \hat I_{k+1}$ and $\psi_k(t_3) \in \hat I_k$ since $t_3 \equiv t_2 \mod \mathbb{Z}$. Thus $\psi_{k+1}(t_3) \in \hat I^{k+1}$. $\square$

Thus by Claim 3 and the fact that the image of $\psi_k \subset [0, 1)^k$ consists of at most N segments, for any closed interval $\hat I^k = \hat I_1 \times \hat I_2 \times \cdots \times \hat I_k$ with $|\hat I_i| = \frac{1}{\hat p^i}$ for $i = 1, 2, \cdots, k$, there exists some $\hat{\hat I}_1 \subset [0, 1)$ with $|\hat{\hat I}_1| \ge \frac{1}{2N\hat p^1}$ and $0 \le \hat j_0 < \hat p^2 \hat p^3 \cdots \hat p^k$ such that $\psi_k(t) \mod \mathbb{Z}^k \in \hat I^k$ for all $t \in \hat{\hat I}_1 + \hat j_0$. We mention that we use the fact that ψ_k is a map with period $\hat p^2 \hat p^3 \cdots \hat p^k$.

Let $\bar L = 3N\hat P$ and take α with $a_1 \ge L$. By the continued fraction expansion the set $\{\alpha, 2\alpha, \cdots, L\alpha\}$ is $\frac{1}{3N\hat P}$ dense on the torus $\mathbb{T}$ if $L \ge a_1 + 1$. Let $0 < \delta << \frac{1}{12N\hat P}$. Now we will show that L and δ satisfy the requirements of Lemma 4.2.

Indeed, suppose box $I = I_1 \times I_2 \times \cdots \times I_k$ has $|I_i| \ge \frac{1}{\hat p^i} - \delta$ for $i = 1, 2, \cdots, k$. Then, since the slopes of components of ψ_k are bounded from below by $\min\{\frac{\hat p^1}{\hat p^k}\} \ge \frac{\hat p^1}{\hat P}$, there exists some $\tilde I_1 \subset \hat{\hat I}_1$ with $|\tilde I_1| \ge \frac{1}{3N\hat p^1}$ such that for any $m \in \mathbb{Z}$, $\psi_k(t) \in I \mod \mathbb{Z}^k$ for all $t \in m\hat p^2 \hat p^3 \cdots \hat p^k + j_0 + \tilde I_1$. We mention that we use again the fact that ψ_k is a map with period $\hat p^2 \hat p^3 \cdots \hat p^k$. By the fact that the set $\{\alpha, 2\alpha, \cdots, L\alpha\}$ is $\frac{1}{3N\hat P}$ dense on torus $\mathbb{T}$, we have that there exists some $j \in \{L_0, L_0 + 1, \cdots, L_0 + L\}$ and $m_0 \in \mathbb{Z}$ such that

$$j\alpha \in m_0 + \frac{j_0}{\hat p^2 \hat p^3 \cdots \hat p^k} + \frac{\tilde I_1}{\hat p^2 \hat p^3 \cdots \hat p^k}.$$

This implies

$$t_j = j\frac{\hat{P}}{\hat{p}^1}\alpha \in m_0\hat{p}^2\hat{p}^3\cdots\hat{p}^k + j_0 + \tilde{I}_1,$$

and then

$$\psi_k(t_j) = (j\frac{\hat{P}}{\hat{p}^1}\alpha, j\frac{\hat{P}}{\hat{p}^2}\alpha, \cdots, j\frac{\hat{P}}{\hat{p}^k}\alpha) \in I \mod \mathbb{Z}^k. \quad \square$$

Lemma 4.3. *Let p be a prime number. Suppose*

$$b \equiv t \mod p.$$

Then $\lfloor b\alpha + \gamma_j \rfloor \equiv j \mod p$ iff

$$\frac{(b-t)}{p}\alpha \in [\frac{j - t\alpha - \gamma_j}{p}, \frac{j+1-t\alpha-\gamma_j}{p}) \mod \mathbb{Z}.$$

Proof. Let $b = kp + t$. Suppose $\lfloor b\alpha + \gamma_j \rfloor \equiv j \mod p$. Using $k\alpha = \lfloor k\alpha \rfloor + \{k\alpha\}$, one has

$$\lfloor b\alpha + \gamma_j \rfloor = p\lfloor k\alpha \rfloor + \lfloor p\{k\alpha\} + t\alpha + \gamma_j \rfloor \equiv j \mod p$$

This implies

$$k\alpha \in [\frac{j - t\alpha - \gamma_j}{p}, \frac{j+1-t\alpha-\gamma_j}{p}) \mod \mathbb{Z}.$$

The proof of the other side is similar. We omit the details. $\square$

In the following, we always assume $a_1 \geq \bar{L}$.

Lemma 4.4. *There exist a small $\delta > 0$ (independent of α) and $b_1 \in \mathbb{N}$ such that for all $|t|, |j| \leq C_1$,*

$$b_1 \equiv t \mod p^{t,j} \tag{40}$$

and

$$\frac{(b_1 - t)}{p^{t,j}}\alpha \in (\frac{j+\delta-t\alpha}{p^{t,j}}, \frac{j+1-\delta-t\alpha}{p^{t,j}}) \mod \mathbb{Z}. \tag{41}$$

Proof. By the Chinese remainder theorem, there exists b such that

$$b_0 \equiv t \mod p^{t,j}$$

for all $|t|, |j| \leq C_1$.

If $b_1 = b_0 + lP$, we also have

$$b_1 \equiv t \mod p^{t,j}$$

for all $|t|, |j| \leq C_1$.

Suppose $\delta > 0$ is small enough (only depends on $p^{t,j}$). We only need to choose proper $l \in \mathbb{N}$ such that for all t and j,

$$\frac{(b_1 - t)}{p^{t,j}}\alpha \in (\frac{j + \delta - t\alpha}{p^{t,j}}, \frac{j + 1 - \delta - t\alpha}{p^{t,j}}) \mod \mathbb{Z}. \tag{42}$$

Let $P^{t,j} = \frac{P}{p^{t,j}}$. Thus (42) is equivalent to

$$P^{t,j} l\alpha \in (\frac{j + \delta - t\alpha}{p^{t,j}}, \frac{j + 1 - \delta - t\alpha}{p^{t,j}}) \mod \mathbb{Z}.$$

The existence of such l is guaranteed by Lemma 4.2. □

Now we will construct nested intervals $\{I_k\} \subset [0, 1)$ such that $I_{k+1} \subset I_k$ and $\lim |I_k| = 0$. Here is the detail. Below, l_k is always in $\mathbb{N}$.

Let b_1 be given by Lemma 4.4. Using the fact that $\hat{l}_k > C_2$, one has $\frac{q_2}{2} > b_1$. Thus we can define

$$b_2 = b_1 + l_1 P q_1$$

such that

$$|b_2 - \frac{q_2}{2}| \leq CPq_1.$$

Inductively, for $k \geq 2$, define

$$b_{k+1} = b_k + l_k P q_k \tag{43}$$

such that

$$|b_{k+1} - \frac{q_{k+1}}{2}| \leq CPq_k. \tag{44}$$

Let us define

$$I_k = \{\gamma : \gamma \in [0, 1), ||\gamma - b_k\alpha|| \leq \frac{1}{q_k}\}, \tag{45}$$

where b_k is given by (43).

Let

$$\gamma = \cap I_k. \tag{46}$$

Remark 4.5. By modifying b_1 and l_k in construction of b_{k+1}, we can get a dense and uncountable set of γ.

Lemma 4.6. *Under the construction of (43), we have that for all t, j,*

$$b_{k+1} \equiv t \mod p^{t,j} \tag{47}$$

and

$$\frac{(b_{k+1}-t)}{p^{t,j}}\alpha \in (\frac{j+\frac{\delta}{2}-t\alpha}{p^{t,j}}, \frac{j+1-\frac{\delta}{2}-t\alpha}{p^{t,j}}) \mod \mathbb{Z}. \tag{48}$$

Proof. The proof of (47) follows from the definition of (40) and (43).

We will give the proof of (48) by induction. The base case holds by Lemma 4.4. Suppose

$$\frac{(b_k-t)}{p^{t,j}}\alpha \in (\frac{j+\frac{\delta}{2}+2^{-(k+2)}\delta-t\alpha}{p^{t,j}}, \frac{j+1-\frac{\delta}{2}-2^{-(k+2)}\delta-t\alpha}{p^{t,j}}) \mod \mathbb{Z}. \tag{49}$$

By (8) and (9), one has

$$\begin{aligned} ||\frac{b_{k+1}-b_k}{p^{t,j}}\alpha|| &= ||l_k P^{t,j} q_k\alpha|| \\ &\leq \frac{q_{k+1}}{2q_k}\frac{1}{q_{k+1}} \\ &\leq \frac{1}{2q_k} \leq \frac{1}{C_2^k}, \end{aligned}$$

where the last inequality holds by (38). By (49), for appropriately large C_1, we have

$$\frac{(b_{k+1}-t)}{p^{t,j}}\alpha \in (\frac{j+\frac{\delta}{2}+2^{-k-3}\delta-t\alpha}{p^{t,j}}, \frac{j+1-\frac{\delta}{2}-2^{-k-3}\delta-t\alpha}{p^{t,j}}) \mod \mathbb{Z}.$$

Then by induction, we finish the proof. □

Thus in order to prove Theorem 4.1, we only need to show that for the γ given by (46), the inequality

$$|\gamma - q\alpha + p| \leq \frac{M}{q},$$

only has finitely coprime solutions $(p, q) \in \mathbb{N}^2$.

Before, we give the proof, we need another theorem.

Theorem 4.7. *Suppose γ is given by (46). Suppose $b_k \leq q < b_{k+1}$ and*

$$||\gamma - q\alpha|| \leq \frac{M}{q}.$$

Then q must have the following form

- *Case I: $q = b_{k+1} - aq_k$ with $0 \leq a \leq C$.*
- *Case II: $q = b_k + aq_k + bq_{k-1}$ with $0 \leq a \leq C$ and $|b| \leq C$.*

Proof. Suppose $b_k \leq q < b_{k+1}$ and

$$||\gamma - q\alpha|| \leq \frac{M}{q}. \tag{50}$$

Case I: $q > Cq_k$.

In this case, we claim that $q = b_{k+1} - aq_k$ for some $a \geq 0$. Otherwise $q = b_{k+1} - aq_k + l$ for some $1 \leq l < q_k$. Thus

$$\begin{aligned} ||\gamma - q\alpha|| &= ||\gamma - b_{k+1}\alpha + aq_k\alpha - l\alpha|| \\ &\geq ||l\alpha|| - ||aq_k\alpha|| - ||\gamma - b_{k+1}\alpha|| \\ &\geq \frac{1}{q_k + q_{k-1}} - \frac{b_{k+1}}{q_k}\frac{1}{q_{k+1}} - \frac{1}{q_{k+1}} \\ &\geq \frac{1}{4q_k}, \end{aligned}$$

where the second inequality holds by (8), (9) and (45), and the third inequality holds by the fact $q_{k+1} > C_2 q_k$. This contradicts (50) since $q \geq Cq_k$.

Now we are in a position to show $0 \leq a \leq C$. Suppose $a > C$. By (44) and $q = b_{k+1} - aq_k$, one has

$$a \leq \frac{q_{k+1}}{2q_k} + CP. \tag{51}$$

By (50), we have

$$||\gamma - q\alpha|| \leq \frac{M}{b_{k+1} - aq_k}$$

and also, using (51),

$$\begin{aligned} ||\gamma - q\alpha|| &\geq ||aq_k\alpha|| - ||\gamma - b_{k+1}\alpha|| \\ &\geq \frac{a}{q_{k+1} + q_k} - \frac{1}{q_{k+1}} \\ &\geq \frac{a}{2q_{k+1}}. \end{aligned}$$

Thus we have

$$\frac{a}{2q_{k+1}} \leq \frac{M}{b_{k+1} - aq_k}. \tag{52}$$

Solving quadratic inequality (52), we have by (44),

$$a \geq \frac{b_{k+1} + \sqrt{b_{k+1}^2 - 8Mq_kq_{k+1}}}{2q_k} = \frac{b_{k+1}}{q_k} + O(1), \tag{53}$$

or

$$a \leq \frac{b_{k+1} - \sqrt{b_{k+1}^2 - 8Mq_kq_{k+1}}}{2q_k} = O(1). \tag{54}$$

Inequality (53) can not happen since $q = b_{k+1} - aq_k \geq Cq_k$. Inequality (54) does not hold since we assume $a > C$. This implies Case I.

Case II: $b_k \leq q \leq Cq_k$. Rewrite q as $q = b_k + aq_k + bq_{k-1} + l$, where $|bq_{k-1} + l| \leq \frac{1}{2}q_k$ and $|l| < q_{k-1}$. Notice that $|b| \leq \frac{q_k}{2q_{k-1}}$.

We claim that $l = 0$. Indeed, assume $|l| > 0$. Then

$$\begin{aligned}
||\gamma - q\alpha|| &= ||\gamma - b_k\alpha - aq_k\alpha - bq_{k-1}\alpha - l\alpha|| \\
&\geq ||l\alpha|| - ||\gamma - b_k\alpha|| - ||aq_k\alpha|| - ||bq_{k-1}\alpha|| \\
&\geq \frac{1}{q_{k-1} + q_{k-2}} - \frac{1}{q_k} - \frac{a}{q_{k+1}} - \frac{q_k}{2q_{k-1}}\frac{1}{q_k} \\
&\geq \frac{1}{3q_{k-1}}
\end{aligned}$$

where the second inequality holds by (8), (9) and (45), and the third inequality holds by the fact $q_{k+1} > C_2q_k$. This contradicts (50) since $q \geq b_k$.

In this case ($q \leq Cq_k$), it is immediate that $0 \leq a \leq C$. Thus we only need to prove $|b| \leq C$. Since $q = b_k + aq_k + bq_{k-1}$ and $q \geq b_k$, we have using also (44) that

$$||\gamma - q\alpha|| \leq \frac{M}{b_k} \leq \frac{3M}{q_k}$$

and also have

$$\begin{aligned}
||\gamma - q\alpha|| &\geq ||bq_{k-1}\alpha|| - ||\gamma - b_k\alpha|| - ||aq_k\alpha|| \\
&\geq \frac{|b|}{q_{k-1} + q_k} - \frac{1}{q_{k+1}} - \frac{C}{q_k}
\end{aligned}$$

Thus we have

$$|b| \leq C. \quad \square$$

Proof of Theorem 4.1. Let γ be given by (46). Suppose $b_k \leq q < b_{k+1}$ is such that (p, q) are coprime and

$$|\gamma - q\alpha + p| \leq \frac{M}{q}.$$

Since $\gamma \neq 0$, this implies $p = \lfloor q\alpha \rfloor$ and $||\gamma - q\alpha|| \leq \frac{M}{q}$. By Theorem 4.7, we have $q = b_{k+1} - aq_k$ with $0 \leq a \leq C$ or $q = b_k + aq_k + bq_{k-1}$ with $0 \leq a \leq C$ and $|b| \leq C$.

We will show that $(q, \lfloor q\alpha \rfloor)$ is not coprime for all such q.

Without loss of generality, assume $q = b_{k+1} - aq_k$ and $0 \leq a \leq C$, the other part of the argument being similar.

Suppose $l > 0$ and $l < C$. By (36), (37) and (10), we have for odd k

$$\lfloor lq_k\alpha \rfloor = lp_k - 1 \equiv l - 1 \mod P \tag{55}$$

and for even k

$$\lfloor lq_k\alpha \rfloor = lp_k \equiv 0 \mod P. \tag{56}$$

Suppose $l < 0$ and $|l| < C$. Similarly, we have for odd k

$$\lfloor lq_k\alpha \rfloor = lp_k \equiv l \mod P \tag{57}$$

and for even k

$$\lfloor lq_k\alpha \rfloor = lp_k - 1 \equiv -1 \mod P. \tag{58}$$

Let $\langle x \rangle$ be the unique number in $[-1/2, 1/2)$ such that $x - \langle x \rangle$ is an integer. Let $t = a$. Let $0 \leq -j < P$ be such that $-j \equiv -aq_k\alpha - \langle -aq_k\alpha \rangle \mod P$ and $\gamma_j = \langle -aq_k\alpha \rangle$.

By (55)-(58), we have $0 \leq |t|, |j| \leq C$.

By (8) and (9), one has

$$|\gamma_j| = || - aq_k\alpha|| \leq \frac{C}{q_{k+1}}. \tag{59}$$

By (47), we have for all $|t| \leq C_1$,

$$b_{k+1} \equiv t \mod p^{t,j},$$

which implies for some $t \in [-C, C]$ (using $t = a$ and (35))

$$b_{k+1} - aq_k \equiv 0 \mod p^{t,j}. \tag{60}$$

Applying Lemma 4.6 and (59), one has for large k,

$$\frac{(b_{k+1}-t)}{p^{t,j}}\alpha \in (\frac{j+\gamma_j - t\alpha}{p^{t,j}}, \frac{j+1-\gamma_j - t\alpha}{p^{t,j}}) \mod \mathbb{Z}.$$

By Lemma 4.3, one has for all $|j| \le C_1$

$$\lfloor b_{k+1}\alpha + \gamma_j \rfloor \equiv j \mod p^{t,j}.$$

This implies for some $j \in [-C, C]$,

$$\lfloor b_{k+1}\alpha - aq_k\alpha \rfloor \equiv \lfloor b_{k+1}\alpha + \gamma_j \rfloor - j \equiv 0 \mod p^{t,j}. \tag{61}$$

Thus by (60) and (61), we have that $(b_{k+1} - aq_k, \lfloor b_{k+1} - aq_k \rfloor)$ is not coprime. This implies for such γ given by (46), the inequality

$$|\gamma - q\alpha + p| \le \frac{M}{q},$$

only has finitely many coprime solutions $(p, q) \in \mathbb{N}^2$. By Remark 4.5, this completes the proof. □

Actually, we have proved the following more general result.

Theorem 4.8. *For any positive constant M, there exist large constants $\bar{C}_1$ and $\bar{C}_2$ (depending on M) such that the following statement holds: Let $P = p^1 p^2 \cdots p^n$, where $n = \bar{C}_1^3 + 1$. Let $\frac{p_k}{q_k}$ be the continued fraction expansion to α. Let*

$$\Lambda = \{\alpha : \text{ there exists some } S = \{a_1, a_2, \cdots, a_m\} \subset \mathbb{N} \cap [0, P] \text{ with } m \le \bar{C}_1 \text{ such that,}$$
$$\text{eventually for all } k,\ q_k, p_k \in S \mod P \text{ and } q_{k+1} \ge \bar{C}_2 P q_k\}.$$

Then for any $\alpha \in \Lambda$, there exists a dense uncountable set $\Omega(\alpha) \subset [0, 1)$ such that for all $\gamma \in \Omega(\alpha)$, inequality

$$|\gamma - q\alpha + p| \le \frac{M}{q},$$

only has finitely many coprime solutions $(p, q) \in \mathbb{N}^2$.

Remark 4.9. Λ is a dense uncountable set.

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