UPPER BOUNDS ON THE SPECTRAL GAPS OF QUASI-PERIODIC SCHRÖDINGER OPERATORS WITH LIOUVILLE FREQUENCIES

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Abstract. We prove that the size of the spectral gaps of weakly coupled quasi-periodic Schrödinger operators with Liouville frequencies decays exponentially. As an application, we obtain the homogeneity of the spectrum.

1. Introduction and main results

Let us consider a quasi-periodic Schrödinger operator given by

$$(H_{\lambda f,\alpha,\theta}x)_n = x_{n+1} + x_{n-1} + \lambda f(\theta + n\alpha)x_n, \tag{1.1}$$

where $x = \{x_n\}_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$ and f is a real analytic function on $\mathbb{R}/\mathbb{Z}$. We will refer to such operators as QPS. The QPS depend on three parameters $(\alpha, \lambda, \theta) \in \mathbb{R}^3$. Usually, we call α the frequency, θ the phase and λ the coupling. In particular, if $f(x) = 2\cos(2\pi x)$, we call (1.1) an almost Mathieu operator (AMO). We will denote AMO by $H_{\lambda,\alpha,\theta}$.

For rational frequency α, the spectrum consists of finite number of intervals by Floquet theory. For irrational frequency α and nonzero coupling constant, it is well-known that the spectrum does not depend on θ and we denote it by $\Sigma_{\lambda f,\alpha}$ ($\Sigma_{\lambda,\alpha}$ for AMO). From now on, we always assume α is irrational. Each connected component of $[E_{\min}, E_{\max}]\backslash\Sigma_{\lambda f,\alpha}$ is called a spectral gap, where $E_{\min} = \min\{E : E \in \Sigma_{\lambda f,\alpha}\}$ and $E_{\max} = \max\{E : E \in \Sigma_{\lambda f,\alpha}\}$. If $\lambda = 0$, the spectrum $\Sigma_{\lambda f,\alpha} = [-2, 2]$ so that there is no spectral gap. Then it is interesting to study upper bounds of the size of the spectral gaps under small perturbation (i.e. λ is small). In order to state the results, we introduce the fibered rotation number $\rho_{\lambda f,\alpha}(E)$ (see section 2.3) of QPS, which has the following properties [27]:

- **(i):** $\rho_{\lambda f,\alpha}(\cdot)$ is a continuous non-increasing surjective function with $\rho_{\lambda f,\alpha}\colon \mathbb{R} \to [0, \frac{1}{2}]$, $\rho_{\lambda f,\alpha}(E) = \frac{1}{2}$ for $E \le E_{\min}$ and $\rho_{\lambda f,\alpha}(E) = 0$ for $E \ge E_{\max}$.
- **(ii):** For each spectral gap G, there exists a unique integer $m \in \mathbb{Z}\backslash\{0\}$, such that the fibered rotation number restricted to the spectral gap satisfies $2\rho_{\lambda f,\alpha}|_G \equiv m\alpha \mod \mathbb{Z}$.

For any $m \in \mathbb{Z}\backslash\{0\}$, let us define

$$[E_m^-, E_m^+] = \{E \in \mathbb{R} : 2\rho_{\lambda f,\alpha}(E) = m\alpha \mod \mathbb{Z}\}. \tag{1.2}$$

Now we distinguish two cases.

1. $E_m^- < E_m^+$. In this case, letting $G_m = (E_m^-, E_m^+)$, then G_m is a spectral gap.
2. $E_m^- = E_m^+$. In this case, let $G_m = \{E_m^-\}$ and we call $G_m = \{E_m^-\}$ a collapsed spectral gap.

Thus for any $m \in \mathbb{Z}\backslash\{0\}$, there exists a corresponding (possibly collapsed) spectral gap G_m.

We give some history of the results on the lower bounds of the spectral gaps first, which originate from the study of the dry Ten Martini Problem. The spectrum $\Sigma_{\lambda,\alpha}$ of AMO has been conjectured to be a Cantor set (dubbed the Ten Martini Problem), which was finally proved by Avila-Jitomirskaya [5], after an a.e. result by Puig [33]. The dry Ten Martini Problem asserts that AMO has no collapsed spectral gap for all $\lambda \neq 0$ and $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, which is stronger than the Ten Martini Problem by properties of the fibered rotation number $\rho_{\lambda,\alpha}$. In [12], Choi-Elliott-Yui showed that AMO has no collapsed spectral gap by setting up the lower bounds of the spectral gaps if α is Liouville and λ satisfies some assumption. Here, α is Liouville means $\beta(\alpha) > 0$, where

$$\beta(\alpha) = \limsup_{k\to\infty} \frac{-\ln ||k\alpha||_{\mathbb{R}/\mathbb{Z}}}{|k|}, \tag{1.3}$$

and $||x||_{\mathbb{R}/\mathbb{Z}} = \min_{k\in\mathbb{Z}} |x-k|$. To the contrary, if $\beta(\alpha) = 0$, we say α is weak Diophantine [1].

Later, Puig proved the dry Ten Martini Problem for a set of (α, λ) of positive Lebesgue measure [33, 34]. Puig approached it by conjugating the Schrödinger cocycle $S_{\lambda f,E}$ to a parabolic matrix $\begin{bmatrix} \pm 1 & \mu \\ 0 & \pm 1 \end{bmatrix}$ and perturbing $S_{\lambda f,E}$ to $S_{\lambda f,E+\varepsilon}$, where $E = E_m^-$ or $E = E_m^+$. This idea is significantly developed by Avila and Jitomirskaya [1, 6, 7], in which they were able to deal with all Diophantine α and $\lambda \neq \pm 1$. Avila-Jitomirskaya's reducibility result also holds for general analytical potentials with small coupling constant. For the quantitative lower bounds of the spectral gaps, see [28, 29] and the references therein.

Now, let us move to the upper bounds on spectral gaps. Moser and Pöschel have shown that for a small analytic potential and a Diophantine vector of frequencies, the spectral gap with some certain label k decays exponentially. For the continuous quasi-periodic operators, the breakthrough is from Damanik and Goldstein where they obtained very precise exponential decay in terms of the smallness of potentials, and the decay rate is bounded by the size of the analytic strip [13]. As an application, the homogeneity of the spectrum can also be obtained [14]. For the discrete case, Leguil-You-Zhao-Zhou [29] showed that the rotation number $\rho_{\lambda f,\alpha}(E+\varepsilon)$ of $S_{\lambda f,E+\varepsilon}$ will change under large perturbation ε based on the reducibility result of [6], where $E = E_m^-$ or $E = E_m^+$. This leads to an upper bound of the spectral gap. We will say more after the statements of our main results. Before that, Amor [20] got an upper bound on the spectral gap for small coupling constant and Diophantine frequency by KAM theory stemming from [16]. Finally, we mention that there are some direct results about the homogeneity of the spectrum, see [17–19].

However, all of the results for general analytic potentials on the upper bounds of spectral gaps are focused on (weak) Diophantine frequencies. Recently, there has been a significant interest in extending various Diophantine results to the case of Liouville frequencies, as phase transitions in the behaviors of various objects happen in this regime [9, 11, 21, 23–25]. The contribution of the present paper is to investigate the upper bounds on gaps for the Liouville frequencies.

Theorem 1.1. *Let $H_{\lambda f,\alpha,\theta}$ be given by (1.1) and E_m^-, E_m^+ be given by (1.2). Suppose $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ satisfies $0 \le \beta(\alpha) < \infty$. Then there exists an absolute constant $C > 0$ such that if f is analytic on the strip $\{x \in \mathbb{C}/\mathbb{Z} : |\Im x| < h\}$ and $\beta(\alpha) \le \frac{h}{C^2}$, then there exist $\lambda_0 = \lambda_0(f, h, \beta(\alpha)) > 0$ and $m_\star = m_\star(\lambda, f, h, \alpha) > 0$ such that for any $|\lambda| \le \lambda_0$, the following estimate holds*

$$E_m^+ - E_m^- \le e^{-\frac{h}{C}|m|}$$

for $|m| \ge m_\star$. In the particular case of AMO, $\lambda_0 = e^{-C^2 h}$.

For trigonometric polynomial potential, one has

Theorem 1.2. *Let $H_{\lambda f,\alpha,\theta}$ be given by (1.1) and E_m^-, E_m^+ be given by (1.2). Suppose $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ satisfies $0 \le \beta(\alpha) < \infty$ and f is a trigonometric polynomial. Then for any $\eta > 0$, there exist*

[1] We say α is Diophantine if there exists $\kappa, \tau > 0$ such that $||k\alpha||_{\mathbb{R}/\mathbb{Z}} \ge \frac{\tau}{|k|^\kappa}$ for any $k \neq 0$.

$\lambda_0 = \lambda_0(f, \eta, \beta(\alpha)) > 0$ *and* $m_\star = m_\star(\lambda, f, \eta, \alpha) > 0$*, such that for* $|\lambda| \le \lambda_0$*, the following estimate holds*

$$E_m^+ - E_m^- \le e^{-\eta|m|}$$

for $|m| \ge m_\star > 0$.

For AMO, we have the following refinement.

Theorem 1.3. *Let* $H_{\lambda,\alpha,\theta}$ *be an almost Mathieu operator and* E_m^-, E_m^+ *be given by* (1.2)*. Suppose* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ *satisfies* $0 \le \beta(\alpha) < \infty$*. Then there exists an absolute constant* $C > 0$ *such that for any* $|\lambda| \le e^{-C\beta(\alpha)}$*, the following estimate holds*

$$E_m^+ - E_m^- \le |\lambda|^{\frac{1}{C}|m|} \tag{1.4}$$

for $|m| \ge m_\star(\lambda, \alpha)$.

Remark 1.4. (1) The case of $\beta(\alpha) = 0$ in Theorem 1.3 with explicit C was proved in [29].

(2) Under the assumption $0 \le |\lambda| \le e^{-C\beta(\alpha)}$, Liu and Yuan [31] proved that there is no collapsed spectral gap, i.e., $E_m^+ > E_m^-$ for any nonzero integer m.

(3). Under the condition of Theorem 1.3, we have $|\lambda|^{\frac{1}{C}|m|} \le e^{\frac{\ln|\lambda|}{C}|m|}$. So the size of the spectral gap $G_m = (E_m^-, E_m^+)$ decays exponentially with respect to the label m.

(4). We expect the optimal decay in (1.4) to be $C|\lambda|^{|m|}$.

As an application, we obtain

Theorem 1.5. *Under the condition of Theorem 1.1, for any* $\epsilon > 0$*, there exists* $\sigma_\star = \sigma_\star(\lambda, f, \alpha, \epsilon) > 0$ *such that for all* $E \in \Sigma_{\lambda f,\alpha}$ *and* $\sigma \in (0, \sigma_\star)$*, we have*

$$\mathrm{Leb}\left((E - \sigma, E + \sigma) \cap \Sigma_{\lambda f,\alpha}\right) \ge (1 - \epsilon)\sigma,$$

where $\mathrm{Leb}(\cdot)$ *is the Lebesgue measure.*

Remark 1.6. By letting E be a point on the boundary of a spectral gap, we see that the lower bound $1 - \epsilon$ is optimal.

We want to explain the motivations for results, and also explain the new challenge for the Liouville case. Recently, the global theory of one-frequency cocycles has been proposed. The spectrum of the quasi-periodic operator (or the corresponding Schrödinger cocycle) can be classified into three regimes:

- Supercritical regime if the Lyapunov exponent is positive.
- Subcritical if the corresponding transfer matrices $A_n(z)$ are uniformly subexponentially bounded through some strip $|\Im z| \le h$.
- Critical regime otherwise.

See [2, 3] the formal definition and generalization. The three regimes have very important spectral features. Roughly speaking, the (almost) reducibility in subcritical regime is the competition between h and $\beta(\alpha)$ and it relates to absolutely continuous spectrum [1, 2, 4, 15, 16, 20, 22]. The (almost) localization in supercritical regime is the competition between the positive Lyapunov exponent and resonance (it is governed by the frequency resonance $\beta(\alpha)$ and the phase resonance) and it relates to the singular continuous spectrum and the pure point spectrum [5, 24–26]. The critical regime relates to the singular continuous spectrum [8, 10]. The supercritical regime and subcritical regime can be connected by Aubry duality, and then the (almost) localization and (almost) reducibility are connected [11, 23, 31, 33, 34]. However, most of the previous references focus on Diophantine frequencies. The motivation of the results in this paper is to set up the *quantitative* almost reducibility by the almost localization in the dual model so that we can deal with upper bounds of spectral gaps. Roughly speaking, in order to balance the small divisor from the frequency α, we need the subcritical regime at least in a strip of width $h > C^2\beta(\alpha)$ and the upper bounds of spectral gaps are controlled by the decaying rate $\gamma = \frac{h}{C}$, where C is a large absolute constant. In this paper, we do not focus on the explicit value of C but it is doable. In particular, we only need $h > 0$ in the case of weak Diophantine frequencies ($\beta(\alpha)$=0). It is very difficult to address the spectral gap by the approach $\beta(\alpha) \to h \to \gamma$. As aforementioned, the recent results for general analytic potentials are to deal with Diophantine frequencies [29]. For one dimensional case with general analytic potentials and weak Diophantine frequencies ($\beta(\alpha) = 0$), the authors obtained that $\gamma > 0$ depends on the strip width h in [29]. We are able to give the explicit formula for γ for any frequency with $\beta(\alpha) < \infty$, that is $\gamma = \frac{h}{C}$. We should mention that the results in [29] hold in higher dimensions and the explicit λ_0 is given. The most challenged part in this paper is to deal with Liouville frequencies ($\beta(\alpha) > 0$). The problems of Liouville frequencies are very hard to deal with. The traditional KAM theory is not able to set up the reducibility for the corresponding cocycle. Recently, there are several big progresses to deal with Liouville frequencies [4, 5, 11, 22, 24, 31]. Our method is based on several combinations of previous methods plus the delicate quantitative estimate. Our approach from $h \to \gamma$ is inspired by [29]. Some challenges related to Liouville frequencies from $\beta \to h \to \gamma$ have been solved in [30, 31], where the reducibility results in [6] were extended to Liouville frequencies. Here, we obtain a more delicate and *quantitative* version of the results of [30, 31] in order to establish the upper bounds of the spectral gaps.

The present paper is organized as follows. In section 2, we give some basic concepts and notations. In section 3, we construct a conjugacy by Aubry duality in order to reduce the cocycle. In section 4, we perturb the cocycle near the boundary of a spectral gap. In section 5, we complete the proofs of Theorems 1.1, 1.2, 1.3 and 1.5.

2. Some basic concepts and notations

2.1. **Cocycle and transfer matrix.** $C^\omega_\delta(\mathbb{R},\mathcal{B})$ be the set of all analytic mappings from $\mathbb{R}$ to some Banach space $(\mathcal{B}, ||\cdot||)$, which admit an analytic extension to the strip $|\Im z| \leq \delta$. Denote by $C^\omega_\delta(\mathbb{R}/\mathbb{Z},\mathcal{B}) \subset C^\omega_\delta(\mathbb{R},\mathcal{B})$ the subspace of 1-periodic mappings. Sometimes, we omit δ for simplicity. By a cocycle, we mean a pair $(\alpha, A) \in (\mathbb{R}\setminus\mathbb{Q}) \times C^\omega_\delta(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ and we can regard it as a dynamical system on $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2$ with

$$(\alpha, A) : (x, v) \longmapsto (x + \alpha, A(x)v),\ (x, v) \in (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2.$$

For $k > 0$, we define the k-step transfer matrix as

$$A_k(x) = \prod_{l=k}^{1} A(x + (l-1)\alpha).$$

2.2. **Conjugacy and reducibility.** Given two cocycles (α, A) and (α, B) with $A, B \in C^\omega_\delta(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, a conjugacy between them is a cocycle (α, R) with $R \in C^\omega_\delta(\mathbb{R}/\mathbb{Z}, \mathrm{PSL}(2,\mathbb{R}))$ such that

$$R^{-1}(x + \alpha)A(x)R(x) = B(x).$$

We say (α, A) is reducible if it conjugates to a constant cocycle (α, B).

Given $R \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{PSL}(2,\mathbb{R}))$, we say the degree of R is k and denote dy $\deg(R) = k$, if R is homotopic to $R_{\frac{k}{2}x}$ for some $k \in \mathbb{Z}$, where

$$R_x = \begin{bmatrix} \cos 2\pi x & -\sin 2\pi x \\ \sin 2\pi x & \cos 2\pi x \end{bmatrix}.$$

2.3. **The fibered rotation number.** Suppose $A \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ is homotopic to the identity. Then the fibered rotation number $\rho_{\alpha,A}$ of the cocycle (α, A) is well defined. We refer to papers [6, 29] for the definition of the fibered rotation number. If $A, B : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ and $R : \mathbb{R}/\mathbb{Z} \to \mathrm{PSL}(2,\mathbb{R})$ are such that A is homotopic to the identity and $R^{-1}(x+\alpha)A(x)R(x) = B$, then B is homotopic to the identity and

$$2\rho(\alpha, A) - 2\rho(\alpha, B) = \deg(R)\alpha. \tag{2.1}$$

Moreover, there is some absolute constant $C > 0$ such that

$$|\rho(\alpha, A) - \theta| \le C \sup_{x\in\mathbb{R}/\mathbb{Z}} ||A(x) - R_\theta||. \tag{2.2}$$

In this paper, we consider the Schrödinger cocycle $(\alpha, S_{\lambda f,E})$, where

$$S_{\lambda f,E}(x) = \begin{bmatrix} E - \lambda f(x) & -1 \\ 1 & 0 \end{bmatrix}.$$

If $f = 2\cos(2\pi x)$, we call $(\alpha, S_{\lambda f,E})$ an almost Mathieu cocycle which is denoted by $(\alpha, S_{\lambda,E})$ for simplicity. It is easy to see that $S_{\lambda f,E}$ is homotopic to the identity. Thus the fibered rotation number of $(\alpha, S_{\lambda f,E})$ is well defined and denoted by $\rho_{\lambda f,\alpha}(E)$ ($\rho_{\lambda,\alpha}(E)$ for AMO).

2.4. **Aubry Duality.** For Schrödinger operator $H_{\lambda f,\alpha,\theta}$, we define the dual Schrödinger operator by $\widehat{H}_{\lambda f,\alpha,\theta}$,

$$(\widehat{H}_{\lambda f,\alpha,\theta}x)_n = \sum_{k\in\mathbb{Z}} \lambda \widehat{f}_k x_{n-k} + 2\cos 2\pi(\theta + n\alpha)x_n,$$

where $\widehat{f}_k$ is the Fourier coefficient of the potential f. Note that the spectrum of $\widehat{H}_{\lambda f,\alpha,\theta}$ is equal to $\Sigma_{\lambda f,\alpha}$.

Aubry duality expresses an algebraic relation between the families of operators $\{\widehat{H}_{\lambda f,\alpha,\theta}\}_{\theta\in\mathbb{R}}$ and $\{H_{\lambda f,\alpha,\theta}\}_{\theta\in\mathbb{R}}$ by Bloch waves, i.e., if $u : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ is an L^2 function whose Fourier coefficients $\widehat{u}$ satisfy $\widehat{H}_{\lambda f,\alpha,\theta}\hat{u} = E\hat{u}$, then $\mathcal{U}(x) = \begin{pmatrix} e^{2\pi i\theta}u(x) \\ u(x-\alpha) \end{pmatrix}$ satisfies

$$S_{\lambda f,E}(x) \cdot \mathcal{U}(x) = e^{2\pi i\theta}\mathcal{U}(x+\alpha). \tag{2.3}$$

2.5. **Some notations and assupmtions.** We briefly comment on the constants and norms in the following proofs. We assume α is irrational and $\lambda \neq 0$. We let C (resp. c) be large (resp. small) positive absolute constant and C (resp. c) may be different even in the same formula. C_2 (resp. C_1) denotes a fixed (resp. any) constant, which is larger than all the constants C, c^{-1} appearing in this paper.

Let $C(\alpha)$ be a large constant depending on α (and f) and $C_\star$ (resp. $c_\star$) be a large (resp. small) constant depending on λ, f and α. Define for $\delta \ge 0$ the strip $\Delta_\delta = \{z \in \mathbb{C}/\mathbb{Z} : |\Im z| \le \delta\}$ and let $||v||_\delta = \sup_{\delta\in\Delta_s} ||v(z)||$, where v is a mapping from Δ_δ to some Banach space $(\mathcal{B}, ||\cdot||)$. For any mapping v defined on $\mathbb{R}/\mathbb{Z}$, we let $[v] = \int_{\mathbb{R}/\mathbb{Z}} v(x)\mathrm{d}x$. In this paper, $\mathcal{B}$ may be $\mathbb{C}$, $\mathbb{C}^2$ or $\mathrm{SL}(2,\mathbb{C})$.

3. The construction of reducibility by Aubry duality

In order to state our reducibility result, we introduce some Lemmas first.

Lemma 3.1 (Theorem 3.3, [6])**.** *Let $E \in \Sigma_{\lambda f,\alpha}$. Then there exist some $\theta = \theta(E) \in \mathbb{R}/\mathbb{Z}$ and $\widehat{u} = \{\widehat{u}_k\}_{k\in\mathbb{Z}}$ with $\widehat{u}_0 = 1, |\widehat{u}_k| \le 1$ such that $\widehat{H}_{\lambda f,\alpha,\theta}\widehat{u} = E\widehat{u}$.*

Suppose η satisfies

$$\eta > C_1\beta(\alpha),$$

where C_1 is a large absolute constant.

Lemma 3.2 (Theorems 4.7 and 5.2, [31])**.** *Suppose $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ satisfies $0 \le \beta(\alpha) < \infty$. Then there exists an absolute constant $C_2 > 0$ such that if f is analytic on the strip $\Delta_{C_2\eta}$, then there exists $\lambda_0(f,\eta,\alpha) > 0$ (depending only on f,η,α) such that if $0 < |\lambda| \le \lambda_0(f,\eta,\alpha)$ and $E \in \Sigma_{\lambda f,\alpha}$ with $2\rho_{\lambda f,\alpha}(E) = m\alpha \mod \mathbb{Z}$, then there is some $\widetilde{n} \in \mathbb{Z}$ such that*

$$2\theta(E) = \widetilde{n}\alpha \mod \mathbb{Z}, \tag{3.1}$$

and for $|m| \ge m_\star$ [2]

$$|m| \le C|\widetilde{n}|. \tag{3.2}$$

Moreover,

$$|\widehat{u}_k| \le C_\star e^{-2\pi\eta|k|}, \quad \text{for } |k| \ge 3|\widetilde{n}|, \tag{3.3}$$

where $\theta(E)$ and $\{\widehat{u}_k\}$ are given by Lemma 3.1. In particular, $\lambda_0 = e^{-C_2\eta}$ for AMO.

Remark 3.3. The proof of this lemma for AMO can be found in [31]. It is easy to extend this result to general QPS following the arguments in [30].

In the following, we fix λ_0 as in Lemma 3.2. In order to avoid the repetition, we only give the proof of $\beta(\alpha) > 0$. Actually, the proof of $\beta(\alpha) = 0$ is much easier.

From now on, we focus on a specific gap G_m. Let $E = E_m^+ \in \Sigma_{\lambda f,\alpha}$ and $A^E(x) = S_{\lambda f,E}(x)$ (sometimes we omit dependence on λ and f for simplicity). We will reduce (α, A^E) to a parabolic matrix $\begin{bmatrix} \pm 1 & \mu \\ 0 & \pm 1 \end{bmatrix}$. In order to study the size of spectral gap by reducibility, we will set up subtle estimates on the coefficient μ and the conjugacy. We attach E with $\theta(E)$ and find the localized solution for the Aubry dual operator. Then we use the localized solution given by (3.3) to construct conjugacies which reduce the cocycle. We always assume the conditions in Lemma 3.2 are satisfied so that

$$n = |\widetilde{n}| < \infty.$$

Our main theorem in this section is

Theorem 3.4. *Suppose $0 < |\lambda| \le \lambda_0$. Then for $E = E_m^+$, there exists $R(x) \in C^\omega_{20\beta}(\mathbb{R}/\mathbb{Z}, \mathrm{PSL}(2,\mathbb{R}))$ such that*

$$R^{-1}(x+\alpha)A^E(x)R(x) = \begin{bmatrix} \pm 1 & \mu_m \\ 0 & \pm 1 \end{bmatrix}, \tag{3.4}$$

where

$$|\mu_m| \le C_\star e^{-\frac{\eta}{2}n}, \tag{3.5}$$

and

$$||R||_{20\beta(\alpha)} \le C_\star e^{C\beta(\alpha)n}. \tag{3.6}$$

Remark 3.5. Actually R in Theorem 3.4 depends on the label m. We ignore the dependence for simplicity. By some results in [29, 31], we can say more about μ_m,

(i): For general analytic potential f, μ_m may be equal to zero. By Proposition 18 in [34], the gap G_m is collapsed for $\mu_m = 0$ and there is nothing to prove in this case. Thus we assume $\mu_m \ne 0$ in the following.

[2] Recall that $m_\star$ is a large constant depending on λ, f and α.

(ii): If $E = E_m^+$ and $\mu_m \neq 0$, then the reduced matrix can only be $\begin{bmatrix} 1 & \mu_m \\ 0 & 1 \end{bmatrix}$ with $\mu_m > 0$ or $\begin{bmatrix} -1 & \mu_m \\ 0 & -1 \end{bmatrix}$ with $\mu_m < 0$ [Theorem 6.1, [31]].

In [31], Liu and Yuan got the reducibility (3.4) for AMO without the estimates of (3.5) and (3.6). Thus, the strategy of the proof of Theorem 3.4 is to follow the arguments of Liu and Yuan with quantitative analysis. For simplicity, we omit the dependence on m in the proof of (3.4) and (3.5) in this section.

Here, we give another lemma, which controls the growth of the cocycle.

Lemma 3.6 (Theorem 5.1, [30])**.** *Suppose* $|\lambda| \leq \lambda_0$. *Then*

$$\sup_{0 \leq k \leq e^{\eta n}} ||A_k^E||_\eta \leq C_\star e^{C\beta(\alpha) n}. \tag{3.7}$$

We define

$$\mathcal{U}(x) = \begin{pmatrix} e^{2\pi i \theta} u(x) \\ u(x-\alpha) \end{pmatrix}, \tag{3.8}$$

where $u(x) = \sum_{k \in \mathbb{Z}} \widehat{u}_k e^{2\pi i k x}$ and $\theta = \theta(E), \widehat{u} = \{\widehat{u}_k\}$ are given by Lemmas 3.1 and 3.2.

Let

$$\widehat{\mathcal{U}}(x) = e^{i\pi \widetilde{n} x} \mathcal{U}(x). \tag{3.9}$$

Lemma 3.7. *Let $\widehat{\mathcal{U}}(x)$ be given by (3.9). Then $\widehat{\mathcal{U}}(x)$ is well defined on $\mathbb{R}/2\mathbb{Z}$ and analytic on $\Delta_{40\beta(\alpha)}$, and*

$$||\widehat{\mathcal{U}}||_{40\beta(\alpha)} \leq C_\star e^{C\beta(\alpha) n}. \tag{3.10}$$

Proof. This follows from (3.3) and the fact that $|\widehat{u}_k| \leq 1$ directly. □

Remark 3.8. Actually, $\widehat{\mathcal{U}}(x)$ is analytic on Δ_η. However $40\beta(\alpha)$ is enough for our goal.

By (2.3), we have

$$A^E(x)\widehat{\mathcal{U}}(x) = \pm \widehat{\mathcal{U}}(x+\alpha). \tag{3.11}$$

For the $z \in 40\beta(\alpha)$, define

$$\Re\widehat{\mathcal{U}}(z) = \frac{\widehat{\mathcal{U}}(z) + \overline{\widehat{\mathcal{U}}(\overline{z})}}{2}; \Im\widehat{\mathcal{U}}(z) = \frac{\widehat{\mathcal{U}}(z) - \overline{\widehat{\mathcal{U}}(\overline{z})}}{2i}.$$

Then for $x \in \mathbb{R}/\mathbb{Z}$, we have

$$\widehat{\mathcal{U}}(x) = \Re\widehat{\mathcal{U}}(x) + i\Im\widehat{\mathcal{U}}(x) \in \mathbb{R}^2 + i\mathbb{R}^2,$$

and it follows from (3.11) that for $x \in \mathbb{R}/\mathbb{Z}$

$$A^E(x)\Re\widehat{\mathcal{U}}(x) = \pm \Re\widehat{\mathcal{U}}(x+\alpha); \tag{3.12}$$

$$A^E(x)\Im\widehat{\mathcal{U}}(x) = \pm \Im\widehat{\mathcal{U}}(x+\alpha). \tag{3.13}$$

Note that $\Re\widehat{\mathcal{U}}(x)$ and $\Im\widehat{\mathcal{U}}(x)$ are well defined on $\mathbb{R}/2\mathbb{Z}$ and analytic in the strip $\Delta_{40\beta(\alpha)}$.

Lemma 3.9. *We can select $\mathcal{V} = \Re\widehat{\mathcal{U}}$ or $\mathcal{V} = \Im\widehat{\mathcal{U}}$ such that $\mathcal{V}$ is real analytic on $\Delta_{40\beta(\alpha)}$ and*

$$\inf_{|\Im x| \leq 40\beta(\alpha)} ||\mathcal{V}(x)|| \geq c_\star e^{-C\beta(\alpha) n}. \tag{3.14}$$

Proof. Since $\widehat{u}_0 = 1$, we have

$$||\int_{\mathbb{R}/2\mathbb{Z}} \left(e^{-\widetilde{n}\pi i x}\Re\widehat{\mathcal{U}}(x) + ie^{-\widetilde{n}\pi i x}\Im\widehat{\mathcal{U}}(x)\right) \mathrm{d}x|| = 2\sqrt{2}.$$

Thus we can choose $\mathcal{V} = \Re\widehat{\mathcal{U}}$ or $\mathcal{V} = \Im\widehat{\mathcal{U}}$ such that

$$||\int_{\mathbb{R}/2\mathbb{Z}} e^{-\widetilde{n}\pi i x}\mathcal{V}(x)\mathrm{d}x|| \geq \sqrt{2}. \tag{3.15}$$

Suppose (3.14) is not true. Then there must be some $x_0 \in \Delta_{40\beta(\alpha)}$ with $\Im x_0 = t$ such that

$$||\mathcal{V}(x_0)|| \leq c_\star e^{-C\beta(\alpha)n}. \tag{3.16}$$

By Lemma 3.6, (3.12) and (3.13), and following the arguments of the proof of Theorem 4.5 in [31], one has

$$\sup_{x\in\mathbb{R}} ||\mathcal{V}(x+it)|| \leq C_\star e^{-C\beta(\alpha)n}.$$

Thus, we obtain

$$||\int_{\mathbb{R}/2\mathbb{Z}} e^{-\widetilde{n}\pi i(x+it)}\mathcal{V}(x+it)\mathrm{d}x|| \leq C_\star e^{-C\beta(\alpha)n},$$

which contradicts (3.15).

□

Proof of Theorem 3.4

Proof. Let

$$R^{(1)}(x) = \left[\begin{array}{cc} \mathcal{V}(x) & T\frac{\mathcal{V}(x)}{||\mathcal{V}(x)||^2} \end{array}\right], \tag{3.17}$$

where $T\left(\begin{array}{c} x \\ y \end{array}\right) = \left(\begin{array}{c} -y \\ x \end{array}\right)$ and $\mathcal{V}$ is given by Lemma 3.9. By Lemma 3.7, it is easy to see that $R^{(1)} \in C^\omega_{40\beta}(\mathbb{R}/\mathbb{Z}, \mathrm{PSL}(2,\mathbb{R}))$. From (3.10), (3.14) and (3.17), we have

$$||(R^{(1)})^{-1}||_{40\beta(\alpha)}, ||R^{(1)}||_{40\beta(\alpha)} \leq C_\star e^{C\beta(\alpha)n}. \tag{3.18}$$

By (3.12), (3.13), (3.17) and (3.18), one has

$$(R^{(1)})^{-1}(x+\alpha)A^E(x)R^{(1)}(x) = \left[\begin{array}{cc} \pm 1 & \nu(x) \\ 0 & \pm 1 \end{array}\right], \tag{3.19}$$

where

$$||\nu||_{40\beta(\alpha)} \leq C_\star e^{C\beta(\alpha)n}. \tag{3.20}$$

Now we will reduce the right side hand of (3.19) to a constant cocycle by solving a homological equation. More concretely, let $\phi(x)$ be a function defined on $\mathbb{R}/\mathbb{Z}$ with $[\phi] = 0$ and

$$\left[\begin{array}{cc} 1 & \phi(x+\alpha) \\ 0 & 1 \end{array}\right]^{-1} \left[\begin{array}{cc} \pm 1 & \nu(x) \\ 0 & \pm 1 \end{array}\right] \left[\begin{array}{cc} 1 & \phi(x) \\ 0 & 1 \end{array}\right] = \left[\begin{array}{cc} \pm 1 & [\nu] \\ 0 & \pm 1 \end{array}\right].$$

This can be done if we let

$$\pm\phi(x+\alpha) \mp \phi(x) = \nu(x) - [\nu]. \tag{3.21}$$

By considering the Fourier series of (3.21), one has

$$\widehat{\phi}_k = \pm\frac{\widehat{\nu}_k}{e^{2\pi i k\alpha} - 1} \ (k \neq 0), \tag{3.22}$$

where $\widehat{\phi}_k$ and $\widehat{\nu}_k$ are Fourier coefficients of $\phi(x), \nu(x)$ respectively.

By the definition of $\beta(\alpha)$, we have the following small divisor condition

$$||k\alpha||_{\mathbb{R}/\mathbb{Z}} \geq C(\alpha)e^{-2\beta(\alpha)|k|}, k \neq 0. \tag{3.23}$$

Combining with (3.22) and (3.20), one has

$$||\phi||_{20\beta(\alpha)} \leq C_\star e^{C\beta(\alpha)n}. \tag{3.24}$$

Let

$$R(x) = R^{(1)}(x) \begin{bmatrix} 1 & \phi(x) \\ 0 & 1 \end{bmatrix}. \tag{3.25}$$

By (3.18) and (3.24), one has

$$||R||_{20\beta(\alpha)}, ||R^{-1}||_{20\beta(\alpha)} \leq C_\star e^{C\beta(\alpha)n}. \tag{3.26}$$

This implies (3.6). Now we are in the position to give a estimate on μ. From (3.19) and (3.25), we obtain

$$R^{-1}(x+\alpha)A^E(x)R(x) = \begin{bmatrix} \pm 1 & \mu \\ 0 & \pm 1 \end{bmatrix},$$

and thus for any $l \in \mathbb{N}$

$$R^{-1}(x+l\alpha)A_l^E(x)R(x) = \begin{bmatrix} \pm 1 & l\mu \\ 0 & \pm 1 \end{bmatrix}. \tag{3.27}$$

Let $l = l_0 = \lfloor e^{\frac{3}{4}\eta n} \rfloor$ in (3.27), one has

$$\begin{aligned} l_0|\mu| &\leq ||R^{-1}||_{20\beta(\alpha)}||A_{l_0}^E||_{20\beta(\alpha)}||R||_{20\beta(\alpha)} \\ &\leq C_\star e^{C\beta(\alpha)n}, \end{aligned} \tag{3.28}$$

where the second inequality holds by (3.7) and (3.26).

(3.5) follows from (3.28) directly. □

We will give more details about

$$R(x) = \begin{bmatrix} R_{11}(x) & R_{12}(x) \\ R_{21}(x) & R_{22}(x) \end{bmatrix}, \tag{3.29}$$

which is defined in Theorem 3.4.

Theorem 3.10. *Let $[R_{ij}(x)]_{i,j\in\{1,2\}}$ be in Theorem 3.4. Then we have*

(i)

$$\begin{aligned} &R_{21}(x+\alpha) = R_{11}(x), \\ &R_{22}(x+\alpha) = R_{12}(x) - \mu R_{11}(x), \\ &R_{11}(x+\alpha)R_{12}(x) - R_{12}(x+\alpha)R_{11}(x) = 1 + \mu R_{11}(x+\alpha)R_{11}(x); \end{aligned} \tag{3.30}$$

(ii)

$$[R_{11}^2] = [R_{21}^2] \geq \frac{1}{2||R||_0}; \tag{3.31}$$

(iii)

$$\frac{[R_{11}^2]}{[R_{11}^2][R_{12}^2] - [R_{11}R_{12}]^2} \leq C_\star e^{C\beta(\alpha)n}, \tag{3.32}$$

$$[R_{11}^2][R_{12}^2] - [R_{11}R_{12}]^2 \geq c_\star e^{-C\beta(\alpha)n}. \tag{3.33}$$

Proof. (i). This is done by direct computations and the details can be found in the proof of Lemma 6.3 in [29].

(ii). See the proof of Lemma 6.2 in [29].

(iii). The proof is similar to that in [29]. Note

$$\frac{[R_{11}^2][R_{12}^2]-[R_{11}R_{12}]^2}{[R_{11}^2]}=\left[\left(R_{12}-\frac{[R_{11}R_{12}]}{[R_{11}^2]}R_{11}\right)^2\right],$$

and define

$$\hat{R}(x)=R_{12}(x)-\frac{[R_{11}R_{12}]}{[R_{11}^2]}R_{11}(x). \tag{3.34}$$

By (3.30) and (3.34), we have

$$R_{11}(x+\alpha)\hat{R}(x)-R_{11}(x)\hat{R}(x+\alpha)=1+\mu R_{11}(x+\alpha)R_{11}(x). \tag{3.35}$$

By Cauchy-Schwarz inequality, one has

$$\Big[|R_{11}(\cdot+\alpha)\hat{R}(\cdot)-R_{11}(\cdot)\hat{R}(\cdot+\alpha)|\Big]\le 2||R||_0\sqrt{[\hat{R}^2]}. \tag{3.36}$$

By (3.5) and (3.6) in Theorem 3.4, we get for $n\ge n_\star$[3]

$$[|1+\mu R_{11}(\cdot+\alpha)R_{11}(\cdot)|]\ge\frac{1}{2}. \tag{3.37}$$

By (3.35), (3.36) and (3.37), one has

$$[\hat{R}^2]\ge\frac{1}{16||R||_0^2}\ge e^{-C\beta(\alpha)n},$$

which implies (3.32). Now (3.33) follows from (3.6), (3.31) and (3.32). □

4. Perturbation near the boundary of a spectral gap

In this section, we will perturb the cocycle (α, A^E) near the boundary $E=E_m^+$ of a spectral gap G_m with $m\in\mathbb{Z}\setminus\{0\}$. Without loss of generality, we assume the reduced cocycle given by Theorem 3.4 is

$$P=\begin{bmatrix}1 & \mu_m\\ 0 & 1\end{bmatrix}. \tag{4.1}$$

Lemma 4.1 ([29, 33, 34])**.** *Let $R(x)$ be as in Theorem 3.4 and P in (4.1). Then for $\epsilon\in\mathbb{R}, x\in\mathbb{R}/\mathbb{Z}$, we have*

$$R^{-1}(x+\alpha)A^{E+\epsilon}(x)R(x)=P+\epsilon\widetilde{P}(x), \tag{4.2}$$

where

$$\widetilde{P}(x)=\begin{bmatrix}R_{11}(x)R_{12}(x)-\mu_m R_{11}^2(x) & R_{12}^2(x)-\mu_m R_{11}(x)R_{12}(x)\\ -R_{11}^2(x) & -R_{11}(x)R_{12}(x)\end{bmatrix}. \tag{4.3}$$

Next, we will tackle the perturbed cocycle $(\alpha, P+\epsilon\widetilde{P})$ given in (4.2) by averaging method, which was originally from [32], and well developed in [20, 33, 34] for Diophantine frequencies and [31] for Liouville frequencies. We develop the averaging method to reduce the cocycle $(\alpha, P+\epsilon\widetilde{P})$ with the Liouville frequency α to a new constant cocycle plus a smaller perturbation, that is

Theorem 4.2. *Let $\delta=5\beta(\alpha)$. Then the following statements hold*

[3] $n_\star$ is a large constant depending on λ, f and α.

(i) *for any* $|\epsilon| \leq \frac{1}{C(\alpha)||R||_{2\delta}^2}$, *there exist* $R_{1,\epsilon}, \widetilde{P}_{1,\epsilon} \in C_\delta^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ *and* $P_{1,\epsilon} \in \mathrm{SL}(2,\mathbb{R})$ *such that*

$$R_{1,\epsilon}^{-1}(x+\alpha)(P+\epsilon\widetilde{P}(x))R_{1,\epsilon}(x) = P_{1,\epsilon} + \epsilon^2\widetilde{P}_{1,\epsilon}(x)$$

and

$$||R_{1,\epsilon} - I||_\delta \leq C(\alpha)||R||_{2\delta}^2|\epsilon|, \tag{4.4}$$

$$||P_{1,\epsilon} - P|| \leq C(\alpha)||R||_{2\delta}^2|\epsilon|, \tag{4.5}$$

$$||\widetilde{P}_{1,\epsilon}||_\delta \leq C(\alpha)||R||_{2\delta}^4, \tag{4.6}$$

$$P_{1,\epsilon} = P + \epsilon[\widetilde{P}]; \tag{4.7}$$

(ii) *for any* $|\epsilon| \leq \frac{1}{C(\alpha)||R||_{2\delta}^4}$, *there exist* $R_{2,\epsilon}, \widetilde{P}_{2,\epsilon} \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ *and* $P_{2,\epsilon} \in \mathrm{SL}(2,\mathbb{R})$ *such that*

$$R_{2,\epsilon}^{-1}(x+\alpha)(P_{1,\epsilon} + \epsilon^2\widetilde{P}_{1,\epsilon}(x))R_{2,\epsilon}(x) = P_{2,\epsilon} + \epsilon^3\widetilde{P}_{2,\epsilon}(x), \tag{4.8}$$

and

$$\begin{aligned} &||R_{2,\epsilon} - I||_0 \leq C(\alpha)||R||_{2\delta}^4\epsilon^2, \\ &||P_{2,\epsilon} - P_{1,\epsilon}|| \leq C(\alpha)||R||_{2\delta}^4\epsilon^2, \\ &||\widetilde{P}_{2,\epsilon}||_0 \leq C(\alpha)||R||_{2\delta}^8, \\ &P_{2,\epsilon} = P_{1,\epsilon} + \epsilon^2[\widetilde{P}_{1,\epsilon}]. \end{aligned} \tag{4.9}$$

The proof of Theorem 4.2 is similar to that in [20, 29, 31, 33, 34] with some modifications. We present the proof in the Appendix. We should mention that it is necessary to shrink the strip to overcome the small divisor condition (3.23) when we solve the homological equation. In the proof of cases (i) and (ii) of Theorem 4.2, we shrink the strip from 2δ to δ and δ to 0 respectively.

Now we can state our main result of perturbation near the spectral gap.

Theorem 4.3. *Let* $\delta = 5\beta(\alpha)$ *and suppose* $|\epsilon| \leq \frac{1}{C(\alpha)||R||_{2\delta}^4}$. *Let* $\widehat{R}_\epsilon(x) = R_\epsilon(x)R_{1,\epsilon}(x)R_{2,\epsilon}(x) \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{PSL}(2,\mathbb{R}))$, *where* $R_{1,\epsilon}(x)$ *and* $R_{2,\epsilon}(x)$ *are given by Theorem 4.2. Then we have*

$$\widehat{R}_\epsilon^{-1}(x+\alpha)A^{E+\epsilon}(x)\widehat{R}_\epsilon(x) = e^{\mathfrak{P}+\epsilon\mathfrak{P}_1+\epsilon^2\mathfrak{P}_2+\epsilon^3\mathfrak{R}_\epsilon(x)}, \tag{4.10}$$

where

$$\begin{aligned} &\mathfrak{P} = \begin{bmatrix} 0 & \mu_m \\ 0 & 0 \end{bmatrix}, \\ &\mathfrak{P}_1 = \begin{bmatrix} -\frac{\mu_m}{2}[R_{11}^2] + [R_{11}R_{12}] & -\mu_m[R_{11}R_{12}] + [R_{12}^2] \\ -[R_{11}^2] & \frac{\mu_m}{2}[R_{11}^2] - [R_{11}R_{12}] \end{bmatrix}, \\ &\mathfrak{P}_2 \in \mathrm{sl}(2,\mathbb{R}), \\ &||\mathfrak{P}_2|| \leq C(\alpha)||R||_{2\delta}^4, \\ &||\mathfrak{R}_\epsilon||_0 \leq C(\alpha)||R||_{2\delta}^8. \end{aligned}$$

Moreover,

$$\deg(\widehat{R}_\epsilon) = \deg(R). \tag{4.11}$$

Proof. (4.10) follows from (4.8) and some simple computations.

It suffices to prove (4.11). From (4.4) and (4.9), we obtain for $|\epsilon| \leq \frac{1}{C(\alpha)||R||_{2\delta}^4}$

$$||R_{1,\epsilon} - I||_0 \leq \frac{1}{4}, ||R_{2,\epsilon} - I||_0 \leq \frac{1}{4},$$

so that both $R_{1,\epsilon}$ and $R_{2,\epsilon}$ are homotopic to the identity. This implies (4.11). $\square$

5. Proof of the main theorems

In this section, we will complete the proofs of Theorems 1.1, 1.2, 1.3 and 1.5. We assume $|m| \geq m_\star$. Then by (3.2), n is large enough.

Theorem 5.1. *Suppose* $0 < |\lambda| \leq \lambda_0$. *We have*

$$E_m^+ - E_m^- \leq e^{-\frac{\eta}{3}n}. \tag{5.1}$$

Proof. We let $\delta = 5\beta(\alpha)$. From (3.6), one has

$$||R||_{2\delta} \leq C_\star e^{C\beta(\alpha)n}$$

and so that

$$\frac{1}{C(\alpha)||R||_{2\delta}^4} \geq e^{-C\beta(\alpha)n}. \tag{5.2}$$

We define

$$\epsilon_m = \frac{-2\mu_m[R_{11}^2]}{[R_{11}^2][R_{12}^2] - [R_{11}R_{12}]^2} < 0,$$

since $\mu_m > 0$ by Remark 3.5.

Following from (3.5) and (3.32), we have

$$\begin{aligned} |\epsilon_m| &\leq C_\star e^{-\frac{\eta}{2}n + C\beta(\alpha)n} \\ &\leq \frac{1}{C(\alpha)||R||_{2\delta}^4}, \end{aligned}$$

where the second inequality holds by (5.2).

Thus we can apply Theorem 4.3 with $\epsilon = \epsilon_m < 0$. Let

$$\begin{aligned} \mathfrak{D} &= \mathfrak{P} + \epsilon_m \mathfrak{P}_1 + \epsilon_m^2 \mathfrak{P}_2 \\ &:= \begin{bmatrix} D_1 & D_2 \\ D_3 & -D_1 \end{bmatrix} \in \mathrm{sl}(2, \mathbb{R}), \end{aligned}$$

where

$$\begin{aligned} D_1 &= \epsilon_m \left([R_{11}R_{12}] - \frac{\mu_m}{2}[R_{11}^2] \right) + O(\epsilon_m^2 ||\mathfrak{P}_2||), \\ D_2 &= \mu_m + \epsilon_m \left([R_{12}^2] - \mu_m[R_{11}R_{12}] \right) + O(\epsilon_m^2 ||\mathfrak{P}_2||), \\ D_3 &= -\epsilon_m[R_{11}^2] + O(\epsilon_m^2 ||\mathfrak{P}_2||), \end{aligned}$$

and

$$\Delta = \det(\mathfrak{D}) = \frac{{\epsilon_m}^2}{2}([R_{11}^2][R_{12}^2] - [R_{11}R_{12}]^2) + O(|\epsilon_m|^3 ||R||_0^2 ||\mathfrak{P}_2||^2 + \mu_m \epsilon_m^2 ||R||_0^4 ||\mathfrak{P}_2||).$$

Recalling (3.33) and by direct computations, one has

$$\begin{aligned} |D_1| &\leq e^{C\beta(\alpha)n}\mu_m, \\ |D_2| &\geq e^{-C\beta(\alpha)n}\mu_m, D_2 < 0, \\ \Delta &\geq e^{-C\beta(\alpha)n}\mu_m^2 > 0. \end{aligned}$$

Let

$$Q = \begin{bmatrix} 0 & \frac{\sqrt{-D_2}}{\Delta^{\frac{1}{4}}} \\ \frac{-\Delta^{\frac{1}{4}}}{\sqrt{-D_2}} & \frac{D_1}{\Delta^{\frac{1}{4}}\sqrt{-D_2}} \end{bmatrix}, \; Q^{-1} = \begin{bmatrix} \frac{D_1}{\Delta^{\frac{1}{4}}\sqrt{-D_2}} & -\frac{\sqrt{-D_2}}{\Delta^{\frac{1}{4}}} \\ \frac{\Delta^{\frac{1}{4}}}{\sqrt{-D_2}} & 0 \end{bmatrix}.$$

Then

$$Q^{-1}\mathfrak{D}Q = \begin{bmatrix} 0 & -\sqrt{\Delta} \\ \sqrt{\Delta} & 0 \end{bmatrix}.$$

Using the estimates that

$$||Q||, ||Q^{-1}|| \leq \frac{1}{\Delta^{\frac{1}{4}}\sqrt{-D_2}},$$

we obtain

$$(\widehat{R}_{\epsilon_m}(x+\alpha)Q)^{-1}A^{E_m^+ + \epsilon_m}(x)\widehat{R}_{\epsilon_m}(x)Q = e^{\sqrt{\Delta}\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \epsilon_m^3 \mathfrak{S}(x)\right)}, \tag{5.3}$$

where

$$\mathfrak{S}(x) = \frac{Q^{-1}(\mathfrak{R}_{\epsilon_m}(x))Q}{\sqrt{\Delta}}$$

and

$$\begin{aligned} ||\epsilon_m^3 \mathfrak{S}||_0 &\leq C_\star e^{C\beta(\alpha)n} \frac{|\epsilon_m|^3 ||R||_{2\delta}^8}{\mu_m^2} \\ &\leq e^{-\frac{\eta}{4}n} \ll 1. \end{aligned} \tag{5.4}$$

Let ρ' be the fibered rotation number of the right hand side of (5.3). Then ρ' is small and $\rho' \neq 0$ by (2.2) and (5.4). Recalling (2.1), (4.11) and (5.3), we have

$$2\rho_{\lambda,\alpha}(E_m^+ + \epsilon_m) = 2\rho' + \deg{(R)}\alpha \text{ mod } \mathbb{Z}$$

and

$$2\rho_{\lambda f,\alpha}(E_m^+) = \deg{(R)}\alpha \text{ mod } \mathbb{Z}.$$

This means $\rho_{\lambda f,\alpha}(E_m^+ + \epsilon_m) \neq \rho_{\lambda f,\alpha}(E_m^+)$. Then $E_m^+ + \epsilon_m \notin G_m$, that is

$$E_m^+ - E_m^- \leq |\epsilon_m| \leq e^{-\frac{\eta}{3}n}.$$

□

Proof of Theorem 1.1. In Lemma 3.2, let $h = C_2\eta$. Theorem 1.1 follows from Theorem 5.1 and the fact that $|m| \leq Cn$ by (3.2). □

Proof of Theorem 1.2. Theorem 1.2 follows from Theorem 1.1 and the fact that any trigonometric polynomial is analytic on $\mathbb{C}$. □

Proof of Theorem 1.3. For AMO, by Lemma 3.2, $\lambda_0 = e^{-C_2\eta}$ with $\eta > C_1\beta(\alpha)$. Let $\eta = \frac{-\ln|\lambda|}{C_2}$ so that $|\lambda| \leq \lambda_0$. By Theorem 5.1, we have for $|m| \geq m_\star$,

$$\begin{aligned} E_m^+ - E_m^- &\leq e^{-\frac{1}{3}\eta n} \\ &\leq |\lambda|^{\frac{1}{3C_2}n} \\ &\leq |\lambda|^{\frac{1}{3CC_2}|m|}, \end{aligned} \tag{5.5}$$

where the third inequality holds by (3.2). This implies Theorem 1.3. □

In order to prove Theorem 1.5, we need two lemmas.

Lemma 5.2 (Corollary 6.1, [30])**.** *Let* $|\lambda| \leq \lambda_0$. *Then*

$$|\rho_{\lambda f,\alpha}(E_1) - \rho_{\lambda f,\alpha}(E_2)| \leq C_\star |E_1 - E_2|^{\frac{1}{2}}, \textit{ for all } \quad E_1, E_2 \in \mathbb{R}. \tag{5.6}$$

Lemma 5.3. *Let $G_m = (E_m^-, E_m^+)$ for $m \in \mathbb{Z}\setminus\{0\}$ and $G_0 = (-\infty, E_{\min}) \cup (E_{\max}, +\infty)$. Then for $m' \neq m \in \mathbb{Z}\setminus\{0\}$ with $|m'| \geq |m|$, we have*

$$\operatorname{dist}(G_m, G_{m'}) = \inf_{x \in G_m, x' \in G_{m'}} |x - x'| \geq c_\star e^{-8\beta(\alpha)|m'|}, \tag{5.7}$$

and for $m \in \mathbb{Z}\setminus\{0\}$

$$\operatorname{dist}(G_m, G_0) \geq c_\star e^{-8\beta(\alpha)|m|}. \tag{5.8}$$

Proof. We start with the proof of (5.7). From the small divisor condition (3.23), one has

$$\begin{aligned} ||(m-m')\alpha||_{\mathbb{R}/\mathbb{Z}} &\geq \frac{1}{C(\alpha)} e^{-2\beta(\alpha)|m-m'|} \\ &\geq \frac{1}{C(\alpha)} e^{-4\beta(\alpha)|m'|}, \end{aligned} \tag{5.9}$$

for $|m'| \geq |m|$.

Without loss of generality, we assume $E_m^+ \leq E_{m'}^-$. By Lemma 5.2, we have

$$\begin{aligned} \operatorname{dist}(G_m, G_{m'}) &= |E_{m'}^- - E_m^+| \\ &\geq \left(\frac{1}{C_\star} |\rho_{\lambda f,\alpha}(E_{m'}^-) - \rho_{\lambda f,\alpha}(E_m^+)| \right)^2 \\ &\geq c_\star ||(m-m')\alpha||_{\mathbb{R}/\mathbb{Z}}^2, \\ &\geq c_\star e^{-8\beta(\alpha)|m'|}, \end{aligned} \tag{5.10}$$

where the second inequality holds by (1.2) and the third inequality holds by (5.9). We finish the proof of (5.7). The proof of (5.8) is similar. □

Proof of Theorem 1.5. Let $\eta = C_1\beta(\alpha)$. We assume $0 < \sigma \leq \sigma_\star(\lambda, f, \alpha, \epsilon)$. For $E \in \Sigma_{\lambda f,\alpha}$ and σ, let

$$\mathcal{R}(E,\sigma) = \{m \in \mathbb{Z}\setminus\{0\} : (E-\sigma, E+\sigma) \cap G_m \neq \emptyset\}.$$

Define $m_0 \in \mathbb{Z}\setminus\{0\}$ with $|m_0| = \min_{m \in \mathcal{R}(E,\sigma)} |m|$. For any $m \in \mathcal{R}(E,\sigma)$, one has

$$\operatorname{dist}(G_m, G_{m_0}) \leq 2\sigma. \tag{5.11}$$

We first assume $(E-\sigma, E+\sigma) \cap G_0 = \emptyset$. Recalling (5.7), we have for any $m \in \mathcal{R}(E,\sigma)$ with $m \neq m_0$,

$$2\sigma \geq c_\star e^{-8\beta(\alpha)|m|},$$

that is

$$|m| \geq \frac{-\ln(C_\star\sigma)}{8\beta(\alpha)}. \tag{5.12}$$

Then by (5.1), we obtain

$$\begin{aligned} &\sum_{m \in \mathcal{R}(E,\sigma), m \neq m_0} \operatorname{Leb}((E-\sigma, E+\sigma) \cap G_m) \\ &\qquad \leq \sum_{m \in \mathcal{R}(E,\sigma), m \neq m_0} (E_m^+ - E_m^-) \\ &\qquad \leq \sum_{|m| \geq \frac{-\ln(C_\star\sigma)}{8\beta(\alpha)}} C_\star e^{-c\eta|m|} \\ &\qquad \leq \epsilon\sigma. \end{aligned} \tag{5.13}$$

On the other hand, $E\in\Sigma_{\lambda f,\alpha}$ implies $E\notin G_{m_0}$. Thus we have

$$\mathrm{Leb}((E-\sigma,E+\sigma)\cap G_{m_0})\leq\sigma. \tag{5.14}$$

In this case, (5.13) and (5.14) imply

$$\begin{aligned}
&\mathrm{Leb}((E-\sigma,E+\sigma)\cap\Sigma_{\lambda,\alpha})\\
\geq\;& 2\sigma-\mathrm{Leb}((E-\sigma,E+\sigma)\cap G_{m_0})\\
&-\sum_{m\in\mathcal{R}(E,\sigma),m\neq m_0}\mathrm{Leb}((E-\sigma,E+\sigma)\cap G_m)\\
\geq\;& 2\sigma-\sigma-\epsilon\sigma\geq(1-\epsilon)\sigma.
\end{aligned} \tag{5.15}$$

In the case $(E-\sigma,E+\sigma)\cap G_0\neq\emptyset$, without loss of generality, we assume $(E-\sigma,E+\sigma)\cap(-\infty,E_{\min})\neq\emptyset$. Then we have

$$0<E_m^- - E_{\min}\leq 2\sigma$$

for any $m\in\mathcal{R}(E,\sigma)$. Thus, (5.12) also holds for any $m\in\mathcal{R}(E,\sigma)$ by (5.8). From the proof of (5.13), we have

$$\sum_{m\in\mathcal{R}(E,\sigma)}\mathrm{Leb}((E-\sigma,E+\sigma)\cap G_m)\leq\epsilon\sigma. \tag{5.16}$$

Noticing that $E\in\Sigma_{\lambda f,\alpha}$ and $E\notin G_0$, one has

$$\mathrm{Leb}((E-\sigma,E+\sigma)\cap G_0)\leq\sigma. \tag{5.17}$$

By (5.16) and (5.17), we obtain

$$\begin{aligned}
&\mathrm{Leb}((E-\sigma,E+\sigma)\cap\Sigma_{\lambda,\alpha})\\
\geq\;& 2\sigma-\mathrm{Leb}((E-\sigma,E+\sigma)\cap G_0)\\
&-\sum_{m\in\mathcal{R}(E,\sigma)}\mathrm{Leb}((E-\sigma,E+\sigma)\cap G_m)\\
\geq\;& 2\sigma-\sigma-\epsilon\sigma\geq(1-\epsilon)\sigma.
\end{aligned}$$

Putting all the cases together, we complete the proof of Theorem 1.5. □

APPENDIX

Proof of Theorem 4.2. By (4.3), we have the upper bound

$$||\widetilde{P}||_{2\delta}\leq C||R||_{2\delta}^2. \tag{5.18}$$

(i). Notice that (4.5) follows from (4.7) and (5.18) directly. It suffices to prove (4.4) and (4.6). The strategy employs Newton's iteration. Let us consider the cocycles of the form

$$R_{1,\epsilon}(x)=e^{\epsilon\mathfrak{Y}(x)}, \tag{5.19}$$

where $\mathfrak{Y}(x)\in\mathrm{sl}(2,\mathbb{R})$ will be specified later. Under the conjugacy of $R_{1,\epsilon}(x)$ in (5.19), we have

$$\begin{aligned}
&e^{-\epsilon\mathfrak{Y}(x+\alpha)}(P+\epsilon\widetilde{P}(x))e^{\epsilon\mathfrak{Y}(x)}\\
&=\left(I-\epsilon\mathfrak{Y}(x+\alpha)+O(\epsilon^2)\right)\left(P+\epsilon\widetilde{P}(x)\right)\left(I+\epsilon\mathfrak{Y}(x)+O(\epsilon^2)\right).
\end{aligned} \tag{5.20}$$

In order to make the nonconstant terms of order ϵ in (5.20) vanish, we need to solve

$$\mathfrak{Y}(x+\alpha)P-P\mathfrak{Y}(x)=\widetilde{P}-[\widetilde{P}]. \tag{5.21}$$

We can solve equation (5.21) by using Fourier coefficients. For this reason, let

$$\widehat{\mathfrak{Y}}_{21}(k)=\frac{\widehat{\widetilde{P}_{21}}(k)}{e^{2k\pi i\alpha}-1}\ (k\neq 0), \tag{5.22}$$

$$\widehat{\mathfrak{Y}}_{11}(k)=\frac{\mu_m\widehat{\widetilde{P}_{21}}(k)+(e^{2k\pi i\alpha}-1)\widehat{\widetilde{P}_{11}}(k)}{(e^{2k\pi i\alpha}-1)^2}\ (k\neq 0), \tag{5.23}$$

$$\widehat{\mathfrak{Y}}_{22}(k)=\frac{(e^{2k\pi i\alpha}-1)\widehat{\widetilde{P}_{22}}(k)-\mu_m e^{2k\pi i\alpha}\widehat{\widetilde{P}_{21}}(k)}{(e^{2k\pi i\alpha}-1)^2}\ (k\neq 0), \tag{5.24}$$

$$\widehat{\mathfrak{Y}}_{12}(k)=\frac{\widehat{\widetilde{P}_{12}}(k)+\mu_m\left(\widehat{\mathfrak{Y}}_{22}(k)-e^{2k\pi i\alpha}\widehat{\mathfrak{Y}}_{11}(k)\right)}{e^{2k\pi i\alpha}-1}\ (k\neq 0), \tag{5.25}$$

and

$$\widehat{\mathfrak{Y}}_{ij}(0)=0\ (\text{for any } 1\leq i,j\leq 2), \tag{5.26}$$

where $\mathfrak{Y}(x)=(\mathfrak{Y}_{ij}(x))_{1\leq i,j\leq 2}$, $\widetilde{P}(x)=(\widetilde{P}_{ij}(x))_{1\leq i,j\leq 2}$ and

$$\mathfrak{Y}_{ij}(x)=\sum_{k\in\mathbb{Z}}\widehat{\mathfrak{Y}}_{ij}(k)e^{2\pi kix},\ \widetilde{P}_{ij}(x)=\sum_{k\in\mathbb{Z}}\widehat{\widetilde{P}_{ij}}(k)e^{2\pi kix}.$$

It is easy to check that $\mathfrak{Y}(x)$ given by (5.22)-(5.25) solves (5.21) and belongs to $\mathrm{sl}(2,\mathbb{R})$.

From the small divisor condition (3.23) and (5.22)-(5.25), we have

$$||\mathfrak{Y}||_\delta\leq C(\alpha)||\widetilde{P}||_{2\delta}\leq C(\alpha)||R||_{2\delta}^2, \tag{5.27}$$

where the second equality holds by (5.18).

By the definition of $\mathfrak{Y}(x)$ and (5.19), one has

$$R_{1,\epsilon}^{-1}(x+\alpha)(P+\epsilon\widetilde{P}(x))R_{1,\epsilon}(x)=P_{1,\epsilon}+\epsilon^2\widetilde{P}_{1,\epsilon}(x).$$

Now we are in the position to get the estimates.

Assume

$$|\epsilon|\leq\frac{1}{C(\alpha)||R||_{2\delta}^2}. \tag{5.28}$$

From (5.27) and (5.28), one has

$$|\epsilon|\cdot||\mathfrak{Y}||_\delta\leq c(\alpha),$$

and then

$$\begin{aligned}||R_{1,\epsilon}-I||_\delta &\leq \sum_{k=1}^{+\infty}\frac{\epsilon^k||\mathfrak{Y}||_\delta^k}{k!}\\ &\leq C|\epsilon|\cdot||\mathfrak{Y}||_\delta\\ &\leq C(\alpha)||R||_{2\delta}^2|\epsilon|,\end{aligned} \tag{5.29}$$

which implies (4.4). By direct computations, we obtain

$$\begin{aligned}
&\epsilon^2\widetilde{P}_{1,\epsilon}(x)\\
(5.30)\qquad &= \epsilon^2(\widetilde{P}(x)\mathfrak{Y}(x) - \mathfrak{Y}(x+\alpha)P\mathfrak{Y}(x) - \mathfrak{Y}(x+\alpha)\widetilde{P}(x) - \epsilon\mathfrak{Y}(x+\alpha)\widetilde{P}(x)\mathfrak{Y}(x))\\
(5.31)\qquad &+\sum_{k=2}^{+\infty}\frac{(-\epsilon)^k\mathfrak{Y}^k(x+\alpha)}{k!}\left(P+\epsilon\widetilde{P}(x)\right)e^{\epsilon\mathfrak{Y}(x)}\\
(5.32)\qquad &+(1-\epsilon\mathfrak{Y}(x+\alpha))\left(P+\epsilon\widetilde{P}(x)\right)\sum_{k=2}^{+\infty}\frac{\epsilon^k\mathfrak{Y}^k(x)}{k!}.
\end{aligned}$$

By (5.18) and (5.27), we obtain the following estimates,

$$\begin{aligned}
||(5.30)||_\delta &\leq \epsilon^2(2||\widetilde{P}||_\delta\cdot||\mathfrak{Y}||_\delta + ||P||\cdot||\mathfrak{Y}||_\delta^2 + |\epsilon|\cdot||\widetilde{P}||_\delta\cdot||\mathfrak{Y}||_\delta^2)\\
&\leq C(\alpha)||R||_{2\delta}^4\epsilon^2,
\end{aligned}$$

$$\begin{aligned}
||(5.31)||_\delta &\leq (e^{|\epsilon|\cdot||\mathfrak{Y}||_\delta} - 1 - |\epsilon|\cdot||\mathfrak{Y}||_\delta)(2+|\epsilon|\cdot||\widetilde{P}||_\delta)e^{|\epsilon|\cdot||\mathfrak{Y}||_\delta}\\
&\leq C\epsilon^2\cdot||\mathfrak{Y}||_\delta^2(2+|\epsilon|\cdot||R||_\delta^2)(1+|\epsilon|\cdot||\mathfrak{Y}||_\delta)\\
&\leq C(\alpha)||R||_{2\delta}^4\epsilon^2,
\end{aligned}$$

and

$$||(5.32)||_\delta \leq C(\alpha)||R||_{2\delta}^4\epsilon^2.$$

This implies (4.6).

(ii). The proof of (ii) is similar to the proof (i). Let

$$R_{2,\epsilon}(x) = e^{\epsilon^2\mathfrak{X}(x)}, \mathfrak{X}(x)\in \mathrm{sl}(2,\mathbb{R}),$$

and the homological equation becomes

$$\mathfrak{X}(x+\alpha)P - P\mathfrak{X}(x) = P_{1,\epsilon}^*(x),$$

where

$$P_{1,\epsilon}^*(x) = \widetilde{P}_{1,\epsilon}(x) - [\widetilde{P}_{1,\epsilon}].$$

Thus under the conjugacy $R_{2,\epsilon}$, we have

$$R_{2,\epsilon}^{-1}(x+\alpha)(P_{1,\epsilon}+\epsilon^2\widetilde{P}_{1,\epsilon}(x))R_{2,\epsilon}(x) = P_{2,\epsilon}+\epsilon^3\widetilde{P}_{2,\epsilon}(x),$$

and the estimates are similar to those in (i).

□

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