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ORIGINAL PAPER

Sharp bounds for finitely many embedded eigenvalues of perturbed Stark type operators

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Abstract
For perturbed Stark operators $Hu = -u'' - xu + qu$, the author has proved that $\limsup_{x\to\infty} x^{\frac{1}{2}}|q(x)|$ must be larger than $\frac{1}{\sqrt{2}}N^{\frac{1}{2}}$ in order to create N linearly independent eigensolutions in $L^2(\mathbb{R}^+)$ [29]. In this paper, we apply generalized Wigner–von Neumann type functions to construct embedded eigenvalues for a class of Schrödinger operators, including a proof that the bound $\frac{1}{\sqrt{2}}N^{\frac{1}{2}}$ is sharp.

KEYWORDS
embedded eigenvalues, Rudin–Shapiro sequence, stark operators, Wigner–von Neumann functions

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47A10, 47A75

1 | INTRODUCTION

The Stark operator $Hu = -u'' - xu + qu$ describes a charged quantum particle in a constant electric field with an additional electric potential q. It has attracted a lot of attentions from both mathematics and physics [1, 2, 4, 6, 8, 9, 11, 13, 20, 21, 23, 24, 40, 42].

In this paper, we consider a class of more general operators, Stark type operators on $L^2(\mathbb{R}^+)$:

$$Hu = -u'' - x^{\alpha}u + qu, \tag{1.1}$$

where $0 < \alpha < 2$. Denote by $H_0u = -u'' - x^{\alpha}u$ and regard q as a perturbation.

It is well known that for any $0 < \alpha < 2$, $\sigma_{\mathrm{ess}}(H_0) = \sigma_{\mathrm{ac}}(H_0) = \mathbb{R}$ and H_0 does not have any eigenvalue. The criteria for the perturbation such that the associated perturbed Stark type operator has single eigenvalue, finitely many eigenvalues or countably many eigenvalues have been obtained in [29].

Define $P \subset \mathbb{R}$ as

$$P = \{E \in \mathbb{R} : -u'' - x^{\alpha}u + qu = Eu \text{ has an } L^2(\mathbb{R}^+) \text{ solution}\}.$$

In [29], the author proved that

Theorem 1.1. *[29, Theorem 1.5] Let a be given by*

$$a = \limsup_{x\to\infty} x^{1-\frac{\alpha}{2}}|q(x)|. \tag{1.2}$$

Then we have

$$a \geq \frac{2-\alpha}{\sqrt{2}}(\#P)^{\frac{1}{2}}. \tag{1.3}$$

Theorem 1.2. *[29, Theorem 1.6] For any $\{E_j\}_{j=1}^N \subset \mathbb{R}$ and any $\{\theta_j\}_{j=1}^N \subset [0,\pi]$, there exist potentials $q \in C^\infty[0,+\infty)$ such that*

$$\limsup_{x\to\infty} x^{1-\frac{\alpha}{2}}|q(x)| \leq (2-\alpha)e^{2\sqrt{\ln N}}N,$$

and for any $j = 1,2,\dots,N$, $-u'' - x^\alpha u + qu = E_j u$ has an $L^2(\mathbb{R}^+)$ solution u with the boundary condition

$$\frac{u'(0)}{u(0)} = \tan\theta_j.$$

Theorem 1.1 implies that in order to create N linearly independent eigensolutions in $L^2(\mathbb{R}^+)$, the quantity a given by (1.2) must be equal or larger than $\frac{2-\alpha}{\sqrt{2}}N^{\frac{1}{2}}$. However, Theorem 1.2 shows that if we allow $a \geq (2-\alpha)e^{2\sqrt{\ln N}}N$, one can create N eigensolutions in $L^2(\mathbb{R}^+)$ for arbitrary N. There is a gap between $N^{\frac{1}{2}}$ and $e^{2\sqrt{\ln N}}N$. It is natural to ask what is the sharp bound of a to create N linearly independent eigensolutions in $L^2(\mathbb{R}^+)$.

Question 1.3. What is the minimum of γ such that for any N, there is a potential q on $\mathbb{R}^+$ such that $\#P \geq N$ and

$$\limsup_{x\to\infty} x^{1-\frac{\alpha}{2}}|q(x)| \leq C(\gamma)N^\gamma.$$

Theorems 1.1 and 1.2 imply $\gamma \in \left[\frac{1}{2}, 1\right]$.

Our first result in this paper is to show that for any α satisfying $\frac{2}{3} < \alpha < 2$, $\gamma = \frac{1}{2}$ is the solution to Question 1.3.

Theorem 1.4. *Suppose $\frac{2}{3} < \alpha < 2$. Then for any $N > 0$, there exists a potential q on $\mathbb{R}^+$ such that*

$$\limsup_{x\to\infty} x^{1-\frac{\alpha}{2}}|q(x)| \leq 20(2-\alpha)\sqrt{N} \tag{1.4}$$

and $\#P = N$.

For some technical reasons, currently we can only give the proof for $\frac{2}{3} < \alpha < 2$. We believe it that $\gamma = \frac{1}{2}$ is the solution to Question 1.3 for all $0 < \alpha < 2$.

Question 1.3 and Theorem 1.4 do not care about the locations of the corresponding energies. If we take the distribution of energies into consideration, what is the sharp upper bound? We formulate it as the following question.

Question 1.5. What is the minimum of γ such that for any $\{E_j\}_{j=1}^N$, there exists a potential q on $\mathbb{R}^+$ such that $-u'' - x^\alpha u + qu = E_j u$ has an $L^2(\mathbb{R}^+)$ solution for each $j = 1,2,\dots,N$ and

$$\limsup_{x\to\infty} x^{1-\frac{\alpha}{2}}|q(x)| \leq C(\gamma)N^\gamma.$$

Theorems 1.1 and 1.2 imply $\gamma \in \left[\frac{1}{2}, 1\right]$. We conjecture that $\gamma = 1$ is the solution to Question 1.5.

During the proof Theorem 1.4, we are able to improve the bound in Theorem 1.2.

Theorem 1.6. *For any $\varepsilon > 0$, $\{E_j\}_{j=1}^N \subset \mathbb{R}$ and $\{\theta_j\}_{j=1}^N \subset [0,\pi]$, there exist local $L^1(\mathbb{R}^+)$ potentials q such that*

$$\limsup_{x\to\infty} x^{1-\frac{\alpha}{2}}|q(x)| \leq (2-\alpha+\varepsilon)N, \tag{1.5}$$

and for each $j = 1,2,\ldots,N$, $-u'' - x^\alpha u + qu = E_j u$ has an $L^2(\mathbb{R}^+)$ solution u with the boundary condition

$$\frac{u'(0)}{u(0)} = \tan\theta_j.$$

Remark 1.7.

- Theorem 1.6 gives better bounds than those in Theorem 1.1, but less regularity in potentials.
- Applying additional piecewise constructions in the proof of Theorem 1.6, it is possible to show that the upper bound in (1.5) can be improved to $(2-\alpha)N$. We refer the readers to the critical case of [29, Theorem 1.2] for details.
- In the main part of this paper, we only consider the relations between L^2 solutions and limit bound of $q(x)$. In the last section, we will mention similar results presented in the bounds of integrals.

The proof of both Theorems 1.4 and 1.6 are inspired by the methods tackling perturbed free Schrödinger operators. Let us turn to perturbed free Schrödinger operators $-D^2 + V$ first. Naboko [35] and Simon [39] constructed power-decaying potentials V such that $-D^2 + V$ has dense eigenvalues. Before that, Wigner–von Neumann type functions can only create one L^2 solution [41]. Recently, there have been several important developments on the problem of embedded eigenvalues for Schrödinger operators, Laplacians on manifolds or other models [12, 14–17, 25, 27, 29–32, 34]. For perturbed Stark type operators, under the rational independence assumption of set $\{E_j\}$, Naboko and Pushnitskii [36] constructed operators with given a set $\{E_j\}$ as embedded eigenvalues. The author [29] constructed perturbed Stark type operators with any given $\{E_j\}$ as a set of eigenvalues with the quantitative bound (see Theorem 1.2). However, the potential cannot be given explicitly. One of the motivations of this paper is to approach the problem in an explicit way.

In [39], Simon used Wigner–von Neumann type functions $V(x) = \frac{a}{1+x}\sum_j \sin(2\lambda_j x + 2\phi_j)\chi_{[a_j,\infty)}$, to complete his constructions. It turns out that Wigner–von Neumann type function is a good way to create embedded eigenvalues [15, 17, 26, 31–33]. Moreover, Wigner–von Neumann type functions can also be used to achieve the optimal bounds. Denote by

$$S = \{E > 0 : -u''(x) + V(x)u(x) = Eu(x) \text{ has an } L^2(\mathbb{R}^+) \text{ solution}\}.$$

Kiselev–Last–Simon [22] proved if $\limsup_{x\to\infty} x|V(x)| < \infty$, then the set S is countable and

$$\sum_{E_i\in S} E_i < \infty. \tag{1.6}$$

This result has been extended to perturbed periodic operators by the author [28]. By Wigner–von Neumann type functions and additional probability arguments from [18], Remling [37] proved that there are potentials $V(x) = O(x^{-1})$ with $\sum_{E_i\in S} E_i^p = \infty$ for every $p < 1$. Remling's result implies that (1.6) cannot be improved in some sense, which answers a question in [22].

Another motivation of the present paper is to find the substitution of Wigner–von Neumann type functions to deal with perturbed Stark type operators, so that we can use the ideas of Simon, Remling, and among others to address our problems. We found a good type of substituted functions $\frac{\sin\left(\int_0^\xi \sqrt{1-\beta_{E_j}(x)}\,dx + t_j\right)}{\xi}$, which is called the *generalized Wigner–von Neumann type function*. See the definition of $\beta_{E_j}(x)$ below.

We want to highlight another ingredient in our proof. Our goal is to create cancellations of the sum of generalized Wigner–von Neumann type functions as many as we can. This problem is very similar to the constructions of uniformly bounded bases in the spaces of complex homogenous polynomials [3, 38][1]. It turns that the Rudin–Shapiro sequence, which is a key ingredient of [3], is also useful to our problem.

Although the arguments in this paper are inspired by those in dealing with perturbed free Schrödinger operators, the details are much more delicate and difficult, in particular, oscillated integrals and resonant phenomena. The spectra of perturbed free Schrödinger operators and Stark operators are quite different. Under the assumption $V(x) = O(x^{-1})$, perturbed free Schrödinger operators can have infinitely many eigenvalues. However, under the corresponding assumption, perturbed Stark operators can only have finitely many eigenvalues by Theorem 1.1. Moreover, for free cases, there is only one leading entry dominating each Prüfer angle and the leading entries are distinguished by energies. For Stark type cases, there are finitely many entries (the number of entries depends on α) dominating each Prüfer angle, and the first/leading entry is 1 for any Prüfer angle, which leads to resonance. Similar resonance has been studied in [29]. In the proof of Theorem 1.6, we are able to deal with the resonance coming from the first dominating entry and all the other dominating entries at the same time. However, for the technical reason, we can only deal with the first two dominating entry for the topic in Theorem 1.4, and the assumption $\alpha > \frac{2}{3}$ will guarantee that there are exact two entries to dominate Prüfer angles.

2 | PREPARATIONS

Let $v_\alpha(x) = x^\alpha$ for $x \in \mathbb{R}^+$ and consider the Schrödinger equation on $\mathbb{R}^+$,

$$-u''(x) - x^\alpha u(x) + q(x)u(x) = Eu(x). \tag{2.1}$$

The Liouville transformation (see [4, 36]) is given by

$$\xi(x) = \int_0^x \sqrt{v_\alpha(t)}\,dt, \quad \phi(\xi) = v_\alpha(x(\xi))^{\frac{1}{4}} u(x(\xi)). \tag{2.2}$$

We define a weight function $p_\alpha(\xi)$ by

$$p_\alpha(\xi) = \frac{1}{v_\alpha(x(\xi))}. \tag{2.3}$$

We also define a potential by

$$Q_\alpha(\xi, E) = -\frac{5}{16}\frac{|v_\alpha'(x(\xi))|^2}{v_\alpha(x(\xi))^3} + \frac{1}{4}\frac{v_\alpha''(x(\xi))}{v_\alpha(x(\xi))^2} + \frac{q(x(\xi)) - E}{v_\alpha(x(\xi))}. \tag{2.4}$$

Let $c = \left(1 + \frac{\alpha}{2}\right)^{\frac{2}{2+\alpha}}$. Direct computations imply that

$$x = c\xi^{\frac{2}{2+\alpha}}, \ \phi(\xi, E) = c^{\frac{\alpha}{4}} \xi^{\frac{\alpha}{2(2+\alpha)}} u\left(c\xi^{\frac{2}{2+\alpha}}\right), \tag{2.5}$$

$$p(\xi) = \frac{1}{c^\alpha \xi^{\frac{2\alpha}{2+\alpha}}}, \tag{2.6}$$

and

$$Q_\alpha(\xi, E) = -\frac{5}{4}\frac{\alpha^2}{(2+\alpha)^2}\frac{1}{\xi^2} + \frac{\alpha(\alpha-1)}{(2+\alpha)^2}\frac{1}{\xi^2} + \frac{q\left(c\xi^{\frac{2}{2+\alpha}}\right) - E}{c^\alpha \xi^{\frac{2\alpha}{2+\alpha}}}. \tag{2.7}$$

Notice that the potential $Q_\alpha(\xi, E)$ depends on q, α and E. In the following, we always fix $\alpha \in (0, 2)$. For simplicity, we drop off its dependence. Let

$$V(\xi) = \frac{q\left(c\xi^{\frac{2}{2+\alpha}}\right)}{c^\alpha \xi^{\frac{2\alpha}{2+\alpha}}}. \tag{2.8}$$

Then

$$Q(\xi,E) = -\frac{5}{4}\frac{\alpha^2}{(2+\alpha)^2}\frac{1}{\xi^2} + \frac{\alpha(\alpha-1)}{(2+\alpha)^2}\frac{1}{\xi^2} - \frac{E}{c^\alpha \xi^{\frac{2\alpha}{2+\alpha}}} + V(\xi) \tag{2.9}$$
$$= -\frac{E}{c^\alpha \xi^{\frac{2\alpha}{2+\alpha}}} + V(\xi) + \frac{O(1)}{\xi^2}.$$

Suppose $u \in L^2(\mathbb{R}^+)$ is a solution of (2.1). It follows that ϕ satisfies

$$-\frac{d^2\phi}{d\xi^2} + Q(\xi,E)\phi = \phi, \tag{2.10}$$

and $\phi \in L^2(\mathbb{R}^+, p(\xi)d\xi)$.

Below, $\epsilon > 0$ always depends on α in an explicit way. Denote by

$$\beta_E(\xi) = -\frac{E}{c^\alpha \xi^{\frac{2\alpha}{2+\alpha}}}.$$

When ξ is large, one has $|\beta_E(\xi)| < 1$. By shifting the equation, we always assume $\beta_E(\xi)$ is sufficiently small. So $\sqrt{1-\beta_E(\xi)}$ is well defined.

Proposition 2.1. *Suppose $a \neq 0$. Then following estimates hold for $\xi > \xi_0 > 1$ and $\gamma \in \mathbb{R}$:*

1.

$$\int_{\xi_0}^{\xi} \frac{\sin\left(a\int_0^s \sqrt{1-\beta_E(x)}\,dx + \gamma\right)}{s}\,ds = \frac{O(1)}{\xi_0^\epsilon}. \tag{2.11}$$

2. for any $E_1 \neq E_2 \in \mathbb{R}$,

$$\int_{\xi_0}^{\xi} \frac{\sin\left(a\int_0^s \sqrt{1-\beta_{E_1}(x)}\,dx \pm a\int_0^s \sqrt{1-\beta_{E_2}(x)}\,dx + \gamma\right)}{s}\,ds = \frac{O(1)}{\xi_0^\epsilon}. \tag{2.12}$$

Proof. We only give the proof of case "−" in (2.12). The rest can be proceeded in a similar way.

Denote by

$$\beta(\xi) = a\int_0^\xi \left[\sqrt{1-\beta_{E_1}(x)}\,dx - a\int_0^\xi \sqrt{1-\beta_{E_2}(x)}\right] dx + \gamma.$$

Then

$$\beta'(\xi) = a\sqrt{1-\beta_{E_1}(\xi)} - a\sqrt{1-\beta_{E_2}(\xi)} = a\frac{E_1-E_2}{2c^\alpha \xi^{\frac{2\alpha}{2+\alpha}}} + \frac{o(1)}{\xi^{\frac{2\alpha}{2+\alpha}}},$$

and

$$\beta''(\xi) = \frac{O(1)}{\xi^{1+\frac{2\alpha}{2+\alpha}}}.$$

Integration by part, we have

$$\int_{\xi_0}^{\xi} \frac{\sin\left(a\int_0^s \sqrt{1-\beta_{E_1}(x)}\,dx - a\int_0^s \sqrt{1-\beta_{E_2}(x)}\,dx + \gamma\right)}{s} = \int_{\xi_0}^{\xi} \frac{\sin\beta(s)}{s}\,ds = \int_{\xi_0}^{\xi} \frac{\beta'(s)\sin\beta(s)}{\beta'(s)s}\,ds$$
$$= O(\xi_0^{-\epsilon}) + O(1)\int_{\xi_0}^{\xi} \frac{\cos\beta(s)\beta''}{s\beta'^2}\,ds$$
$$= O(\xi_0^{-\epsilon}). \qquad \square$$

3 | ASYMPTOTICAL BEHAVIOR OF SOLUTIONS FOR A CLASS OF LINEAR SYSTEMS

The proof of this section is inspired by the WKB method. We refer the readers to papers [5, 7 10] for arguments.

Theorem 3.1. *Suppose $a > 0$ is a constant. Suppose $\{E_j\} \in \mathbb{R}$ are distinct. Define $V(\xi) = 0$ for $\xi \in [0,1]$ and*

$$V(\xi) = \frac{4a}{\xi}\sum_{j=1}^{N} \sin\left(\int_0^{\xi} 2\sqrt{1-\beta_{E_j}(x)}\,dx + 2t_j\right), \tag{3.1}$$

for $\xi > 1$.

Define $q(x)$ on $[0,\infty)$ such that (2.8) holds for (3.1). Let $Q(\xi, E)$ be given by (2.9). Then the following asymptotics hold as ξ goes to infinity,

1. *if $E \neq E_j$ for any $j = 1, 2, \ldots N$, then there exists a fundamental system of solutions $\{y_1(\xi), y_2(\xi)\}$ of (2.10) such that*

$$\begin{bmatrix} y_1(\xi) \\ y_1'(\xi) \end{bmatrix} = \begin{bmatrix} \cos\left(\int_0^{\xi} \sqrt{1-\beta_E(x)}\,dx + t_j\right) \\ -\sin\left(\int_0^{\xi} \sqrt{1-\beta_E(x)}\,dx + t_j\right) \end{bmatrix} + O(\xi^{-\epsilon})$$

and

$$\begin{bmatrix} y_2(\xi) \\ y_2'(\xi) \end{bmatrix} = \begin{bmatrix} \sin\left(\int_0^{\xi} \sqrt{1-\beta_E(x)}\,dx + t_j\right) \\ \cos\left(\int_0^{\xi} \sqrt{1-\beta_E(x)}\,dx + t_j\right) \end{bmatrix} + O(\xi^{-\epsilon}).$$

2. *if $E = E_j$ for some j, then there exists a fundamental system of solutions $\{y_1(\xi), y_2(\xi)\}$ of (2.10) such that*

$$\begin{bmatrix} y_1(\xi) \\ y_1'(\xi) \end{bmatrix} = \xi^{a}\begin{bmatrix} \cos\left(\int_0^{\xi} \sqrt{1-\beta_{E_j}(x)}\,dx + t_j\right) \\ -\sin\left(\int_0^{\xi} \sqrt{1-\beta_{E_j}(x)}\,dx + t_j\right) \end{bmatrix} + O(\xi^{a-\epsilon})$$

and

$$\begin{bmatrix} y_2(\xi) \\ y_2'(\xi) \end{bmatrix} = \xi^{-a}\begin{bmatrix} \sin\left(\int_0^{\xi} \sqrt{1-\beta_{E_j}(x)}\,dx + t_j\right) \\ \cos\left(\int_0^{\xi} \sqrt{1-\beta_{E_j}(x)}\,dx + t_j\right) \end{bmatrix} + O(\xi^{-a-\epsilon}).$$

Proof. In order to avoid repetition, we only give the proof of the case the case $E = E_j$ for some $j = 1, 2, \dots, N$. Denote by

$$\tilde{V}(\xi) = \frac{4a}{\xi} \sum_{i=1, i\neq j}^{N} \sin\left(2\int_0^{\xi} \sqrt{1-\beta_{E_i}(x)}\, dx + 2t_i\right). \tag{3.2}$$

Rewrite the second order differential equation of (2.10) as the linearly differential equations,

$$\begin{bmatrix} y \\ y' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ \beta_{E_j}(\xi) + V(\xi) + \frac{O(1)}{\xi^2} - 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}.$$

Let

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \sqrt{1-\beta_{E_j}(\xi)} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}.$$

We obtain a new equation

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \left(\begin{bmatrix} 0 & \sqrt{1-\beta_{E_j}(\xi)} \\ -\sqrt{1-\beta_{E_j}(\xi)} + \frac{V}{\sqrt{1-\beta_{E_j}(\xi)}} & 1 \end{bmatrix} + \frac{O(1)}{\xi^{1+\epsilon}} \right) \begin{bmatrix} y \\ y' \end{bmatrix}.$$

Let

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\left(\int_0^{\xi} \sqrt{1-\beta_{E_j}(x)}\, dx + t_j\right) & -\sin\left(\int_0^{\xi} \sqrt{1-\beta_{E_j}(x)}\, dx + t_j\right) \\ \sin\left(\int_0^{\xi} \sqrt{1-\beta_{E_j}(x)}\, dx + t_j\right) & \cos\left(\int_0^{\xi} \sqrt{1-\beta_{E_j}(x)}\, dx + t_j\right) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Obviously, one has

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\left(\int_0^{\xi} \sqrt{1-\beta_{E_j}(x)}\, dx + t_j\right) & \sin\left(\int_0^{\xi} \sqrt{1-\beta_{E_j}(x)}\, dx + t_j\right) \\ -\sin\left(\int_0^{\xi} \sqrt{1-\beta_{E_j}(x)}\, dx + t_j\right) & \cos\left(\int_0^{\xi} \sqrt{1-\beta_{E_j}(x)}\, dx + t_j\right) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

After some calculations, we have

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \left(\Lambda(\xi) + H(\xi) + O\left(\xi^{-1-\epsilon}\right)\right) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

where

$$\Lambda(\xi) = \begin{bmatrix} -2a \frac{\sin^2\left(2\int_0^{\xi} \sqrt{1-\beta_{E_j}(x)} dx + 2t_j\right)}{\xi} & 0 \\ 0 & 2a \frac{\sin^2\left(2\int_0^{\xi} \sqrt{1-\beta_{E_j}(x)}\, dx + 2t_j\right)}{\xi} \end{bmatrix} \tag{3.3}$$

and

$$H(\xi)=\begin{bmatrix}H_{11}(\xi) & H_{12}(\xi)\\ H_{21}(\xi) & H_{22}(\xi)\end{bmatrix}.$$

The explicit formulas for H_{ij}, $i,j=1,2$, are

$$H_{11}=-\frac{1}{2}\tilde{V}(\xi)\sin\left(2\int_0^{\xi}\sqrt{1-\beta_{E_j}(x)}\,dx+2t_j\right),$$

$$H_{22}(\xi)=\frac{1}{2}\tilde{V}(\xi)\sin\left(2\int_0^{\xi}\sqrt{1-\beta_{E_j}(x)}\,dx+2t_j\right),$$

$$H_{12}(\xi)=-\frac{1}{2}V(\xi)\left(1-\cos\left(2\int_0^{\xi}\sqrt{1-\beta_{E_j}(x)}\,dx+2t_j\right)\right),$$

and

$$H_{21}(\xi)=\frac{1}{2}V(\xi)\left(1+\cos\left(2\int_0^{\xi}\sqrt{1-\beta_{E_j}(x)}\,dx+2t_j\right)\right).$$

By Proposition 2.1, one has

$$Q(\xi)\equiv-\int_{\xi}^{\infty}H(s)\,ds=O(\xi^{-\epsilon}).$$

Assume ξ is large, then $||Q||\leq\frac{1}{2}$. Let

$$\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\end{bmatrix}=(I+Q)^{-1}\begin{bmatrix}y_1\\ y_2\end{bmatrix}.$$

We obtain

$$\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\end{bmatrix}'=(\Lambda(\xi)+R(\xi))\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\end{bmatrix}, \tag{3.4}$$

where $R(\xi)=\left(R_{ij}\right)=O\left(\xi^{-1-\epsilon}\right)$. Let

$$\varphi(\xi)=\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\end{bmatrix}\quad\text{and}\quad\lambda(\xi)=2a\frac{\sin^2\left(2\int_0^{\xi}\sqrt{1-\beta_{E_j}(x)}\,dx+2t_j\right)}{\xi}.$$

Let us consider the integral equation,

$$\varphi(\xi)=\begin{bmatrix}e^{-\int_1^{\xi}\lambda(s)\,ds}\\ 0\end{bmatrix}-\int_{\xi}^{\infty}\begin{bmatrix}e^{-\int_{\xi}^{y}\lambda(s)\,ds} & 0\\ 0 & e^{\int_{\xi}^{y}\lambda(s)\,ds}\end{bmatrix}R(y)\varphi(y)\,dy. \tag{3.5}$$

If (3.5) has a solution $||\varphi(\xi)|| \le 2e^{-\int_1^\xi \lambda(s)\,ds}$, then (by direct computation) $\varphi(\xi)$ is a solution of Equation (3.4). Moreover,

$$\left\| \int_\xi^\infty \begin{bmatrix} e^{-\int_\xi^y \lambda(s)\,ds} & 0 \\ 0 & e^{\int_\xi^y \lambda(s)\,ds} \end{bmatrix} R(y)\varphi(y)\,dy \right\| = O(1)\int_\xi^\infty e^{\int_\xi^y \lambda(s)} ||R(y)|| e^{-\int_1^y \lambda(s)\,ds}$$
$$= O(1)e^{-\int_1^\xi \lambda(s)\,ds}\int_\xi^\infty ||R(y)||$$
$$= e^{-\int_1^\xi \lambda(s)\,ds}O(\xi^{-\epsilon}). \tag{3.6}$$

Define iteration equations:

$$\varphi_k(\xi) = \begin{bmatrix} e^{-\int_1^\xi \lambda(s)\,ds} \\ 0 \end{bmatrix} - \int_\xi^\infty \begin{bmatrix} e^{-\int_\xi^y \lambda(s)\,ds} & 0 \\ 0 & e^{\int_\xi^y \lambda(s)\,ds} \end{bmatrix} R(y)\varphi_{k-1}(y)\,dy, \tag{3.7}$$

with $\varphi = 0$. By induction that $||\varphi_k(\xi) - \varphi_{k-1}(\xi)|| \le \frac{1}{2^{k+1}}e^{-\int_1^\xi \lambda(s)\,ds}$, one can show (3.5) has a solution $||\varphi(\xi)|| \le 2e^{-\int_1^\xi \lambda(s)\,ds}$ (see p. 94 in [5] for all the details).

By (3.4), (3.5) and (3.6), we get a solution

$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = e^{-\int_1^\xi \lambda(s)\,ds}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + O(\xi^{-\epsilon})\right).$$

By Proposition 2.1 again, we have

$$\int_1^\xi \lambda(s)\,ds = 2a\int_1^\xi \frac{\sin^2\left(2\int_0^s \sqrt{1-\beta_{E_j}(x)}\,dx + 2t_j\right)}{s}\,ds$$
$$= a\int_1^\xi \frac{1}{s}\,ds - a\int_1^\xi \frac{\cos\left(4\int_0^s \sqrt{1-\beta_{E_j}(x)}\,dx + 4t_j\right)}{s}\,ds$$
$$= a\int_1^\xi \frac{1}{s}\,ds - a\int_1^\infty \frac{\cos\left(4\int_0^s \sqrt{1-\beta_{E_j}(x)}\,dx + 4t_j\right)}{s}\,ds + a\int_\xi^\infty \frac{\cos\left(4\int_0^s \sqrt{1-\beta_{E_j}(x)}\,dx + 4t_j\right)}{s}\,ds$$
$$= \ln \xi^a - c + O(\xi^{-\epsilon}),$$

where the constant c equals

$$a\int_1^\infty \frac{\cos\left(4\int_0^s \sqrt{1-\beta_{E_j}(x)}\,dx + 4t_j\right)}{s}\,ds.$$

Thus (3.4) has a solution

$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = \xi^{-a}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + O(\xi^{-a-\epsilon}). \tag{3.8}$$

By the similar argument (see [5] again), we obtain that (3.4) has a solution

$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = \xi^a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + O(\xi^{a-\epsilon}). \tag{3.9}$$

Now the theorem follows from (3.8) and (3.9). □

4 | PROOF OF THEOREMS 1.4 AND 1.6

Proof of Theorem 1.6. Let $a > \frac{2-\alpha}{2(2+\alpha)}$. Define potentials

$$V(\xi) = \frac{4a}{\xi} \sum_{j=1}^{N} \sin\left(2 \int_0^{\xi} \sqrt{1 - \beta_{E_j}(x)}\, dx + 2t_j\right) \chi_{[a_j,\infty)},$$

with large a_j.

Obviously,

$$|\xi V(\xi)| \le 4aN,$$

so that

$$\left|\xi^{1-\frac{\alpha}{2}} q(\xi)\right| \le 2(2+\alpha)aN. \tag{4.1}$$

By (2.6),

$$p(\xi)\left|\phi\left(\xi, E_j\right)\right|^2 \le O(1)\xi^{-2a-\frac{2\alpha}{2+\alpha}} \le O(1)\xi^{-1-\epsilon}. \tag{4.2}$$

By (4.2), (2.10) has a solution $\phi\left(\xi, E_j\right) \in L^2(\mathbb{R}^+, p(\xi)d\xi)$ for each $j = 1, 2, \ldots, N$. However, $\phi\left(\xi, E_j\right)$ may not satisfy the given boundary condition. This can be done by adjusting a_j and additional functions W with support in (1,2). We refer readers to [39] for rigorous arguments. Now the Theorem follows from (4.1). □

Let

$$\tau = \frac{2+\alpha}{2(2-\alpha)} \frac{1}{\left(1+\frac{\alpha}{2}\right)}^{\frac{2\alpha}{2+\alpha}},$$

so that

$$\tau \frac{d\xi^{\frac{2-\alpha}{2+\alpha}}}{d\xi} = \frac{1}{2} \frac{1}{c^{\alpha} \xi^{\frac{2\alpha}{2+\alpha}}}.$$

For any given N, let

$$E_j = \frac{j}{N\tau}, \text{ for } j = 1, 2, \ldots, N.$$

Before giving the proof of Theorem 1.4, several lemmas about Rudin–Shapiro sequence are needed. Suppose $\left\{\sigma_j\right\}_{j=0}^{n}$ is $n+1$ consecutive numbers in the Rudin–Shapiro sequence. We should mention that $\sigma_j \in \{\pm 1\}$, $j = 0, 1, 2, \ldots, n$.

Lemma 4.1. *[[19, p. 46], [3, Lemma 1]] Assume $a, b \in \mathbb{Z}$ satisfy $b - a = 2^m - 1$ with $m \in \mathbb{Z}^+$. Then*

$$\sup_{\theta\in\mathbb{R}} \left| \sum_{j=a}^{b} \sigma_j e^{2\pi i j\theta} \right| \leq \sqrt{2} 2^{m/2}.$$

By dividing the general interval $[a, b]$ into dyadic intervals as in Lemma 4.1, we have

Lemma 4.2. *For any $a, b \in \mathbb{Z}$ and $a < b$, we have*

$$\sup_{\theta\in\mathbb{R}} \left| \sum_{j=a}^{b} \sigma_j e^{2\pi i j\theta} \right| \leq 5\sqrt{b - a + 1}. \tag{4.3}$$

Proof. Let n be such that $2^n \leq b - a + 1 < 2^{n+1}$. Rewrite $b - a$ as

$$b - a + 1 = \sum_{k=0}^{n} \eta_k 2^k,$$

where $\eta_k \in \{0, 1\}$. For k with $\eta_k = 1$, applying Lemma 4.1 with $m = k$, we have

$$\sup_{\theta\in\mathbb{R}} \left| \sum_{j=a}^{b} \sigma_j e^{2\pi i j\theta} \right| \leq \sqrt{2} \sum_{k=0}^{n} \eta_k 2^{k/2} \leq \frac{\sqrt{2}}{\sqrt{2} - 1} \left(2^{\frac{n+1}{2}} - 1 \right) \leq 5\sqrt{b - a + 1}.$$

□

Lemma 4.2 appears in [3] without the quantitative factor 5 in (4.3). Lemma 4.2 immediately implies,

Lemma 4.3. *There exist $\theta_j \in \left\{0, \frac{1}{2}\right\}$, $j = 1, 2, \dots, N$, such that for any $\xi > 0$*

$$\left| \sum_{j=1}^{N} \sin\left(2\tau E_j \xi + 2\pi\theta_j\right) \right| + \left| \sum_{j=1}^{N} \cos\left(2\tau E_j \xi + 2\pi\theta_j\right) \right| \leq 10\sqrt{N}. \tag{4.4}$$

Proof of Theorem 1.4. It suffices to prove the case that $N \geq 2$. For any given $N \geq 2$, let

$$E_j = \frac{j}{N\tau}, \text{ for } j = 1, 2, \dots, N.$$

Let $a > \frac{2-\alpha}{2(2+\alpha)}$. By Lemma 4.3, there exist $t_j \in [0, \pi)$ such that for any $\xi > 0$,

$$\left| \sum_{j=1}^{N} \sin\left(2\tau E_j \xi + 2t_j\right) \right| + \left| \sum_{j=1}^{N} \cos\left(2\tau E_j \xi + 2t_j\right) \right| \leq 10\sqrt{N}. \tag{4.5}$$

Let

$$V(\xi) = \frac{4a}{\xi} \sum_{j=1}^{N} \sin\left(2 \int_0^{\xi} \sqrt{1 - \beta_{E_j}(x)}\, dx + 2t_j\right). \tag{4.6}$$

By Tayor series, one has

$$\begin{aligned}\int_0^{\xi} \sqrt{1-\beta_{E_j}(x)}\,dx &= \int_0^{\xi} \left(1-\frac{1}{2}\beta_{E_j}(x)+O(1)\beta_{E_j}(x)^2\right)dx \\ &= \xi + \tau E_j \xi^{\frac{2-\alpha}{2+\alpha}} + \frac{O(1)}{\xi^{\frac{4\alpha}{2+\alpha}-1}} + \tilde{t}_j \\ &= \xi + \tau E_j \xi^{\frac{2-\alpha}{2+\alpha}} + O(\xi^{-\epsilon}) + \tilde{t}_j, \end{aligned} \tag{4.7}$$

since $\alpha > \frac{2}{3}$.

By (4.5), (4.6) and (4.7), one has

$$\begin{aligned}|\xi V(\xi)| &= 4a\left|\sum_{j=1}^{N} \sin\left(2\int_0^{\xi}\sqrt{1-\beta_{E_j}(x)}\,dx + 2t_j\right)\right| \\ &= 4a\left|\sum_{j=1}^{N} \sin\left(2\xi + 2\tau E_j \xi^{\frac{\alpha}{2+\alpha}} + 2t_j + 2\tilde{t}_j\right)\right| + O(\xi^{-\epsilon}) \\ &\le 4a\left|\sum_{j=1}^{N} \sin\left(2\tau E_j \xi^{\frac{\alpha}{2+\alpha}} + 2t_j + 2\tilde{t}_j\right)\right| + 4a\left|\sum_{j=1}^{N} \cos\left(2\tau E_j \xi^{\frac{\alpha}{2+\alpha}} + 2t_j + 2\tilde{t}_j\right)\right| + O(\xi^{-\epsilon}) \\ &\le 40a\sqrt{N} + O(\xi^{-\epsilon}). \end{aligned} \tag{4.8}$$

Define $q(x)$ on $[0,\infty)$ such that (2.8) holds for (4.6). Then by (4.8), we have

$$\xi^{1-\frac{\alpha}{2}}|q(\xi)| \le (2+\alpha)20a\sqrt{N} + O(\xi^{-\epsilon}). \tag{4.9}$$

By Theorem 3.1, for any E_j, $j=1,2,\dots,N$, (2.10) has a solution $\phi(\xi, E_j)$ satisfying

$$\left|\phi(\xi, E_j)\right| \le 2\xi^{-a}$$

for large ξ. Let $a = \frac{2-\alpha}{2+\alpha}$. By (2.6),

$$p(\xi)\left|\phi(\xi, E_j)\right|^2 \le O(1)\xi^{-2a-\frac{2\alpha}{2+\alpha}} \le O(1)\xi^{-1-\epsilon}. \tag{4.10}$$

It implies $\phi(\xi, E_j) \in L^2(\mathbb{R}^+, p(\xi)d\xi)$ and then $u \in L^2(\mathbb{R}^+)$. Now Theorem 1.4 follows from (4.9). □

5 | PROBLEMS ABOUT THE SHARP BOUNDS IN THE INTEGRAL FORM

Motivated by [36], we will discuss the sharp bounds in the integral form in this section. Define

$$\begin{aligned} I_q^{\alpha}(\rho) &= \int_1^{\infty} \frac{dx}{\sqrt{v_{\alpha}(x)}} \exp\left(-\rho\int_1^{x} \frac{|q(t)|}{\sqrt{v_{\alpha}(t)}}\,dt\right) \\ &= \int_1^{\infty} \frac{dx}{x^{\alpha/2}} \exp\left(-\rho\int_1^{x} \frac{|q(t)|}{t^{\alpha/2}}\,dt\right). \end{aligned}$$

We drop the dependence of α for simplicity. Let

$$I_q = \sup_{\rho} \{\rho \geq 0 : I_q(\rho) = \infty\} = \inf_{\rho} \{\rho \geq 0 : I_q(\rho) < \infty\}.$$

Obviously, $|q_1(x)| \leq |q_2(x)|$ implies

$$I_{q_1} \geq I_{q_2}.$$

Regarding one L^2 solution, Naboko and Pushnitskii [36] proved the following statement

Theorem 5.1. *[36, Theorem 1] Suppose $I_q > 1$. Then $-u'' - x^\alpha u + qu = Eu$ has no $L^2(\mathbb{R}^+)$ solution for any $E \in \mathbb{R}$.*

Theorem 5.2. *[36, Theorem 2] For any $E \in \mathbb{R}$ and $0 < \epsilon < 1$, there exists q such that $I_q = 1 - \epsilon$ and $-u'' - x^\alpha u + qu = Eu$ has an $L^2(\mathbb{R}^+)$ solution.*

Remark 5.3. Theorems 5.1 and 5.2 hold for more general cases [36].

It is natural to ask what is the analog version of Theorems 5.1 and 5.2 for the N linearly independent $L^2(\mathbb{R}^+)$ solutions. By choosing test functions $q(x) = \frac{C}{x^{1-\frac{\alpha}{2}}}$ and also motivating from Questions 1.3 and 1.5, we post another two questions.

Question 5.4. What is the minimum of γ_1 such that for any N, there is a potential q on $\mathbb{R}^+$ such that $\#P \geq N$ and

$$I_q \geq \frac{c(\gamma_1)}{N^{\gamma_1}}.$$

Question 5.5. What is the minimum of γ_2 such that for any $\{E_j\}_{j=1}^N$, there exists a potential q on $\mathbb{R}^+$ such that $-u'' - x^\alpha u + qu = E_j u$ has an $L^2(\mathbb{R}^+)$ solution for each $j = 1, 2, \ldots, N$ and

$$I_q \geq \frac{c(\gamma_2)}{N^{\gamma_2}}.$$

By Theorems 1.4 and 1.6, one has following corollaries.

Corollary 5.6. *Suppose $\frac{2}{3} < \alpha < 2$. Then for any $N > 0$, there exists a potential q on $\mathbb{R}^+$ such that*

$$I_q \geq \frac{1}{20(2-\alpha)\sqrt{N}}$$

and $\#P = N$.

Corollary 5.7. *For any $\epsilon > 0$, $\{E_j\}_{j=1}^N \subset \mathbb{R}$ and $\{\theta_j\}_{j=1}^N \subset [0, \pi]$, there exist potentials q such that*

$$I_q \geq \frac{2-\alpha}{2(2-\alpha+\epsilon)N}$$

and for each $j = 1, 2, \ldots, N$, $-u'' - x^\alpha u + qu = E_j u$ has an $L^2(\mathbb{R}^+)$ solution u with the boundary condition

$$\frac{u'(0)}{u(0)} = \tan\theta_j.$$

By Corollaries 5.6 and 5.7, we have for $0 < \alpha < 2$,

$$\gamma_2 \leq 1$$

and for $\frac{2}{3} < \alpha < 2$,

$$\gamma_1 \leq \frac{1}{2}.$$

We conjecture that $\gamma_1 = \frac{1}{2}$ and $\gamma_2 = 1$ are the answers to Questions 1.3 and 1.5 respectively.

Finally, we remark that the method in [36] is based on the integral of "piecewise constant" functions. More precisely, in each piece, $f(x) \approx \frac{C}{1+x}$, which has the same flavor of Naboko [35] Our proof in the present paper (also [29]) is very different. We used the delicate estimate of the oscillated integrals $\left(f(x) \approx \frac{\sin\theta(x)}{1+x}\right)$ stemming from Prüfer transformation. Since the two methods are very different, we are not sure wether the corresponding integral form of Theorem 1.1 holds or not.

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