Coarse models of homogeneous spaces and translations-like actions

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Abstract

In 1999, K. Whyte introduced translation-like actions of a group H on a group G as a dynamical/geometric generalization of H being a subgroup of G. In this paper, our interest lies in when lattices in closed Lie subgroups acts translation-like on lattices in the ambient Lie group. Extending work of D. Cohen, we show that cocompact lattices in non-compact simple Lie groups $\mathbf{G}$ not isogenous to $\mathrm{SL}(2,\mathbb{R})$ admit translation-like actions by $\mathbb{Z}^2$. This result follows from a more general result. Namely, we prove that any cocompact lattice in the unipotent radical $\mathbf{N}$ of the Borel subgroup $\mathbf{AN}$ of $\mathbf{G}$ acts translation-like on any cocompact lattice in $\mathbf{G}$. We also prove that for non-compact simple Lie groups G,H with $H < G$ and lattices $\Gamma < G$ and $\Delta < H$, that Δ admits a translation-like action on Γ such that Γ/Δ is quasi-isometric to G/H where Γ/Δ is the quotient (metric space) via a translation-like action of Δ on Γ and word metric on Γ.

1 Introduction

Given a Lie group $\mathbf{G}$ equipped with a bi-invariant metric, every cocompact lattice $\Gamma < \mathbf{G}$ with a finite word metric is a coarse geometric model of $\mathbf{G}$ (e.g. the inclusion map is a quasi-isometry). One theme in the study of lattices is how much of the structure of $\mathbf{G}$ is captured in the structures on the lattices Γ. When $\mathbf{G}$ is a non-compact real simple Lie group of real rank at least two, Margulis established that these lattices are arithmetic which is one of the strongest ways that Γ can capture the structure of $\mathbf{G}$. He also directly related the finite dimensional representation theory of Γ with that of $\mathbf{G}$ via super-rigidity. These lattices are also conjectured by Serre to have the congruence subgroup property, which shows that the finite representation theory of Γ functions through the structure of $\mathbf{G}$.

Given a Lie group $\mathbf{G}$ and closed subgroup $\mathbf{H} \leq \mathbf{G}$, two associated geometric objects are the homogenous space $\mathbf{G}/\mathbf{H}$ and the foliation of $\mathbf{G}$ via the $\mathbf{H}$–cosets. Given a cocompact lattice $\Gamma \leq \mathbf{G}$, we define $\Delta = \mathbf{H} \cap \Gamma$ and ask if Γ/Δ is a coarse model for $\mathbf{G}/\mathbf{H}$? When $\Delta \leq \mathbf{H}$ is a cocompact lattice, Γ/Δ is a coarse model for $\mathbf{G}/\mathbf{H}$. Likewise, the coset foliation on Γ via $\gamma\Delta$ is a coarse model for the $\mathbf{H}$–coset foliation. Unfortunately, the intersection $\Delta = \mathbf{H} \cap \Gamma$ can vary (depending on Γ and $\mathbf{H}$) from trivial to a cocompact lattice in $\mathbf{H}$. For instance, there are infinitely many commensurability classes of arithmetic lattices $\Gamma < \mathrm{SL}(2,\mathbb{C})$ such that $\mathbf{H} \cap \Gamma$ is a cocompact lattice for countably many $\mathbf{H}$ that are conjugate to $\mathrm{SL}(2,\mathbb{R})$. However, there also exist infinitely many commensurability classes of arithmetic lattices $\Gamma < \mathrm{SL}(2,\mathbb{C})$ such that $\mathbf{H} \cap \Gamma$ is never a lattice for any $\mathbf{H}$ that is conjugate to $\mathrm{SL}(2,\mathbb{R})$. By Kahn–Markovic [13], all of these lattices have quasi-isometric surface subgroups Δ. For these subgroups Δ, the space Γ/Δ gives a coarse model for $\mathbf{G}/\mathbf{H}$ despite Δ not being a subgroup of some $\mathbf{H}$. We take an alternative approach to finding models for $\mathbf{G}/\mathbf{H}$.

Given a group H and a metric space (X,d) with a free (left) H–action, we say that H acts **translation-like** on X if $\sup\{d(x,h\cdot x) : x \in X\} < \infty$ for each $h \in H$; an action satisfying this condition is called **wobbling**. Our present interest is when $X = G$ is a finitely generated group equipped with a word metric associated to a finite generating subset. Whyte [22] introduced translation-like actions as a geometric coarsification of subgroups. Indeed, when $H \leq G$, the right action of H on G is free and translation-like for any finite generating subset of G. In an effort to justify this view, Whyte established a coarse geometric result in relation to the von Neumann–Day conjecture. The conjecture asserts that a group G is non-amenable if and only if G contains a non-abelian free subgroup, which by Ol'shanskii [17] is known to be false. On the other hand, Whyte [22] proved a coarsification of this conjecture, establishing that G is non-amenable if and only if G admits a translation-like action by a non-abelian free group.

In 1902, Burnside asked if every infinite, finitely generated group G contains an element of infinite order, and Golod–Shafarevich [8] answered Burnside's question in the negative by providing examples of finitely generated infinite torsion groups. Seward [20] took a similar approach as Whyte to Burnside's problem, proving that a finitely generated group G is infinite if and only if G admits a translation-like action by $\mathbb{Z}$.

With translation-like actions that are sufficiently well behaved, we provide a method to construct a model for the homogeneous space $\mathbf{G}/\mathbf{H}$ that is compatible with models for the Lie groups $\mathbf{G}$ and $\mathbf{H}$ given by cocompact lattices $\Delta < \mathbf{H}$ and $\Gamma < \mathbf{G}$. Suppose that Δ admits a translation-like action on Γ where the orbits of the action of Δ on Γ are coarsely embedded and are contained in cosets of $\mathbf{H}$ in $\mathbf{G}$. Moreover, suppose that the quotient of Γ by the translation-like action of Δ admits a natural metric with a natural inclusion into $\mathbf{G}/\mathbf{H}$ that is coarsely dense. We then say that the translation-like action of Δ on Γ gives rise to a coarse model of $\mathbf{G}/\mathbf{H}$ and denote it as Γ/Δ.

Following Seward and Whyte, Cohen [6] investigated the geometric coarsification of a question due to Gersten–Gromov (see [1, Ques 1.1]). Specifically, if G admits a finite $K(G,1)$ and contains no Baumslag–Solitar subgroups $BS(m,n)$, then is G hyperbolic? Like the von Neumann–Day conjecture and Burnside's question, this question is known to have a negative answer, and in fact, there are many counterexamples to the Gersten–Gromov question. For example, Rips [19] proved that there exists a $C'(1/6)$ small cancellation group with a finitely generated but not finitely presentable subgroup H. Since $C'(1/6)$ small cancellation groups are hyperbolic, the subgroup H cannot contain any Baumslag–Solitar subgroups which gives a counterexample to the Gersten conjecture. Even if we restrict ourselves to the class of finitely presentable groups, we have counterexamples. Brady [4] using branched coverings of cubical complexes to produce a hyperbolic group with a finitely presented subgroup that is not hyperbolic which provides finitely presentable counterexample to the Gersten conjecture.

The geometric coarsification of the Gersten–Gromov question is that a group G with a finite $K(G,1)$ is hyperbolic if and only if G does not admit a translation-like action by any Baumslag–Solitar group. The main result of [6] proved that cocompact lattices in $\mathrm{SO}(3,1)$ admit translation-like actions by $\mathbb{Z}^2$, proving that the geometric coarsification of the Gersten–Gromov question is false. Moreover, by inspecting the construction in [6], we see that the translation-like action of $\mathbb{Z}^2$ on cocompact lattices in $\mathrm{SO}(3,1)$ gives rise to a coarse model for $\mathrm{SO}(3,1)/\mathbb{R}^2$ which can be seen as the space of horospheres of 3-dimensional hyperbolic space.

Our first result extends [6] to all cocompact lattices in all semisimple Lie groups. Fixing an Iwasawa decomposition of $\mathbf{G} = \mathbf{KAN}$, when $\Gamma < \mathbf{G}$ is a non-cocompact lattice, then $\Delta = \Gamma \cap \mathbf{N}$ is a cocompact lattice in $\mathbf{N}$. The Lie group $\mathbf{N}$ is a connected, simply connected nilpotent Lie group and so $\Delta < \mathbf{N}$ is a torsion-free, finitely generated nilpotent group. When $\Gamma < \mathbf{G}$ is a cocompact lattice, then $\Gamma \cap \mathbf{N}$ is trivial. Despite it being impossible for Γ to have torsion-free nilpotent subgroups besides $\mathbb{Z}$, the lattices Γ do admit translation-like actions by the lattices in $\mathbf{N}$ that give rise to coarse models for $\mathbf{G}/\mathbf{N}$.

Theorem 1.1. *Let $\mathbf{G}$ be a semisimple Lie group with an Iwasawa decomposition $\mathbf{G} = \mathbf{KAN}$. If $\Gamma < \mathbf{G}$ and $\Delta < \mathbf{N}$ are cocompact lattices, then Γ admits a translation-like action by Δ. Moreover, we can choose this* *translation-like action to give rise to a coarse mode* Γ/Λ *of the homogeneous space* $\boldsymbol{G}/\boldsymbol{N}$*. Finally, given distinct lattices* $\Gamma_1,\Gamma_2 < \mathbf{G}$ *and* $\Delta_1,\Delta_2 < \mathbf{N}$*, we have the coarse models* Γ_1/Δ_1 *and* Γ_2/Δ_2 *for* $\mathbf{G}/\mathbf{N}$ *are bi-Lipschitz.*

One immediate corollary of this theorem is the following.

Corollary 1.2. *Let* $\mathbf{G}$ *be a noncompact simple Lie group which is not isogenous to* $\mathrm{SL}(2,\mathbb{R})$*. If* $\Gamma < \mathbf{G}$ *is cocompact lattice, then* Γ *admits a translation-like action by* $\mathbb{Z}^2$*.*

This corollary generalizes the main result of [6]. More recently, Jiang [12] proved that the lamplighter group admits no translation-like actions by Baumslag–Solitar groups. As the lamplighter group is not finitely presentable, it cannot be hyperbolic. Hence, this provides a counterexample for the other direction of the geometric coarsification of the Gersten–Gromov question. In particular, there are hyperbolic groups that admit actions by Baumslag–Solitar groups and there exist non-hyperbolic groups which do not admit any translation-like actions by a Baumslag–Solitar group.

Question 1. *Does there exist a non-hyperbolic, finitely presentable group that does not admit a translation-like action by any Baumslag–Solitar group?*

We give an outline of the proof of our first theorem which follows the proof of the main theorem of [6]. Using unipotent flows, we construct a net in $\mathbf{G}/\mathbf{K}$ which is bi-Lipschitz to our group Γ on which Δ admits a translation-like action. The unipotent subgroups of the Iwasawa decomposition with the induced metric are bi-Lipschitz to $\mathbf{N}$ with a left invariant metric in which Δ is a cocompact lattice. The nilpotent Lie groups $\mathbf{N}$ admit natural scaling automorphisms which we use to shrink or expand the copy of Δ in each coset $(a,\mathbf{N})$ where $a \in \mathbb{Z}^{\mathrm{rank}(\mathbf{G})}$ as a varies to account for the changes in the induced geometry of each translate of the unipotent subgroup. Since each layer of this net is a copy of Δ, we act on these layers by right translation. The actions on the layers combine together to give an action on the entire net that is translation-like. Through the bi-Lipschitz equivalence of Γ with this net, we obtain a translation-like action of the group Δ on Γ.

The last theorem of our note constructs coarse models for homogeneous spaces of the form $\mathbf{G}/\mathbf{H}$ where both $\mathbf{G}$ and $\mathbf{H}$ are noncompact real simple Lie groups using cocompact lattices in $\mathbf{G}$ and $\mathbf{H}$. We refer the reader to Definition 2.11 for the definition of a coarse model.

Theorem 1.3. *Let* $\mathbf{G}$ *and* $\mathbf{H}$ *be* $\mathbb{Q}$*–defined noncompact real simple Lie groups such that* $\mathbf{H} \leq \mathbf{G}$*. If* $\Delta < \mathbf{H}$ *and* $\Gamma < \mathbf{G}$ *are cocompact lattices, then* Δ *admits a translation-like action on* Γ*. Moreover, we can choose this translation-like such that* Γ/Δ *is a coarse model for* $\mathbf{G}/\mathbf{H}$*. Finally, given distinct lattices* $\Gamma_1,\Gamma_2 < \mathbf{G}$ *and* $\Delta_1,\Delta_2 < \mathbf{H}$*, the spaces* Γ_1/Δ_1 *and* Γ_2/Δ_2 *for* $\mathbf{G}/\mathbf{H}$ *are bi-Lipschitz.*

The proof of this theorem follows from basic structural results of simple Lie groups.

2 Background

For a group G and $g,h \in G$, the commutator is denoted by g and h as $[g,h] = g^{-1}h^{-1}gh$. For subgroups $A,B \leq G$, the subgroup generated by $\{[a,b] \,:\, a \in A,\, b \in B\}$ is denoted by $[A,B]$. The i-th step of the lower central series of G is denoted as G_i. When N is a nilpotent group, we denote its step length as $c(N)$.

2.1 Lie groups and Lie algebras

Lie groups will be typically denoted by $\mathbf{G}$ with Lie algebras given by $\mathfrak{g}$. The Lie bracket of X and Y will be denoted by $[X,Y]$. Inner products will be denoted $\langle \cdot,\cdot \rangle$. Left translation by a group element $g \in \mathbf{G}$ will be denoted by L_g. The i-th step of the lower central series of a Lie algebra $\mathfrak{g}$ will be denoted by $\mathfrak{g}_i$. The tangent space of $\mathbf{G}$ at any element $g \in \mathbf{G}$ will be denoted by $T_g(\mathbf{G})$.

Given a connected Lie group $\mathbf{G}$ with Lie algebra $\mathfrak{g}$ and $g \in \mathbf{G}$, the map $L_g \colon \mathbf{G} \to \mathbf{G}$ given by $L_g(x) = g \cdot x$ is a diffeomorphism of $\mathbf{G}$ for all $g \in \mathbf{G}$. Thus, the tangent space $T_g(\mathbf{G})$ can be identified as $(dL_g)_1(T_1(\mathbf{G}))$ where $(dL_g)_1$ is the linear isomorphism from $T_1(\mathbf{G})$ to $T_g(\mathbf{G})$. Fixing a positive definite bilinear form $\langle \cdot,\cdot \rangle$ on $\mathfrak{g} = T_1(\mathbf{G})$, we have a left invariant Riemannian metric on $\mathbf{G}$ defined via

$$\langle X,Y \rangle_g = \left\langle dL_{g^{-1}}(X), dL_{g^{-1}}(Y) \right\rangle$$

for all $X,Y \in T_g(\mathbf{G})$ and for all $g \in \mathbf{G}$. For $X \in \mathfrak{g}$, we have the linear endomorphism $\mathrm{ad}_X \colon \mathfrak{g} \to \mathfrak{g}$ given by $\mathrm{ad}_X(Y) = [X,Y]$.

Given a group G, we define the **lower central series of** G recursively by $G_1 = G$ and $G_i = [G, G_{i-1}]$ for $i > 1$. We say that G is **nilpotent of step size** c if c is the minimal natural number such that $G_{c+1} = \{1\}$. If the step size is unspecified, we just say that G is a nilpotent group. The **lower central series for a Lie algebra** $\mathfrak{g}$ is defined recursively by $\mathfrak{g}_1 = \mathfrak{g}$ and $\mathfrak{g}_i = [\mathfrak{g}, \mathfrak{g}_{i-1}]$ for $i > 1$. We say that $\mathfrak{n}$ is **nilpotent of step length** c if c is the minimal natural number such that $\mathfrak{n}_{c+1} = \{0\}$. If the step size is unspecified, we just say that $\mathfrak{n}$ is a nilpotent Lie algebra.

Given a Lie group $\mathbf{G}$ and a left Haar measure μ, we say that a discrete subgroup $\Gamma < G$ is a **lattice** if $\mu(\Gamma \backslash \mathbf{G}) < \infty$. When $\Gamma \backslash \mathbf{G}$ is compact, we say Γ is **cocompact**. If $\mathbf{G} < \mathrm{GL}(n,\mathbb{C})$ is a $\mathbb{Q}$-defined linear group, the group of integral points is defined by $\mathbf{G}(\mathbb{Z}) = \mathbf{G} \cap \mathrm{GL}(n,\mathbb{Z})$.

2.2 Coarse Geometry and UDBG spaces

Given metric spaces (X_1,d_1) and (X_2,d_2), we say X_1 and X_2 are **quasi-isometric** if there exists a function $f \colon (X_1,d_1) \to (X_2,d_2)$ and constants $A \geq 1$, $B \geq 0$, and $C \geq 0$ such that

$$\frac{1}{A} d_1(x,y) - B \leq d_2(f(x),f(y)) \leq A d_1(x,y) + B,$$

for all $x,y \in X_1$, and for each $z \in X_2$, there exists an element $x \in X_1$ such that $d_2(z,f(x)) \leq C$. We call the map f a **quasi-isometry** between (X_1,d_1) and (X_2,d_2). If the above map is bijective and if $B = 0$, we call the map f a **bi-Lipschitz map** and say that the metric spaces (X_1,d_1) and (X_2,d_2) are **bi-Lipschitz**.

We introduce some conditions on discrete metric spaces that induce some regularity. We say a metric space (X,d) is **uniformly discrete** if

$$\inf\{d(x_1,x_2) \ : \ x_1,x_2 \in X \text{ and } x_1 \neq x_2\} > 0.$$

A discrete metric space (X,d) has **bounded geometry** if for all $r > 0$, there exists a constant $C_r > 0$ such that $|B_r(x)| \leq C_r$ for all $x \in X$. We call a uniformly discrete metric space of bounded geometry a **UDBG space**.

We are interested in a particular class of UDBG spaces seen in the following definition

Definition 2.1. Let X be a UDBG space. If $F \subset X$ and $r \in \mathbb{N}$, then the r**–boundary of** F **in** X is given by

$$\partial_r^X(F) \stackrel{\text{def}}{=} \{x \in X - F \ : \ \text{there exists } y \in Y \text{ such that } d(x,y) \leq r\}.$$

A **Følner sequence for** X is a sequence $\{F_i\}_{i\in\mathbb{N}}$ of non-empty finite subsets of X such that for all $r \in N$, we have

$$\lim_{n\to\infty} \frac{\left|\partial_r^X(F_n)\right|}{|F_n|} = 0.$$

We say that a UDBG space is **non-amenable** if it admits no Følner sequences.

The following property of non-amenable UDBG spaces is of particular importance to us.

Proposition 2.2. *Let (X_1,d_1) and (X_2,d_2) be non-amenable UDBG spaces, and suppose that $f\colon X_1 \to X_2$ is a quasi-isometry. Then f is bounded distance from a bi-Lipschitz map $F\colon X_1 \to X_2$.*

Proof. Since X_1 and X_2 are non-amenable, we have that $H_0^{uf}(X_1) = 0$ and $H_0^{uf}(X_2) = 0$ by [2, Thm 3.1] where $H_0^{uf}(X_1)$ and $H_0^{uf}(X_2)$ denote the 0-th uniformly finite homology groups of X_1 and X_2. Denoting $[X_1]$ and $[X_2]$ as the characteristic classes of X_1 and X_2, we have that $[X_1] = 0$ and $[X_2] = 0$. Thus, if $f_*\colon H_0^{uf}(X_1) \to H_0^{uf}(X_2)$ is the map of 0-th uniformly finite homology induced by the quasi-isometry f, we have $f_*([X_1]) = [X_2]$. Hence, [22, Thm 1.1] implies that f is bounded distance from a bi-Lipschitz map. $\square$

We finish this section by noting some straightforward properties of translation-like actions. In particular, translation-like actions respect bi-Lipschitz equivalences of metric spaces and satisfy transitivity properties as seen in the following lemmas. As these lemmas are straightforward, we omit the proofs for brevity.

Lemma 2.3. *Let G be a finitely generated group that acts translation-like on (X_1,d_1), and suppose that (X_1,d_1) is bi-Lipschitz to (X_2,d_2) via the map F. Then G admits a translation-like action on (X_2,d_2) via the action $g\cdot x = F(g\cdot F^{-1}(x))$.*

Lemma 2.4. *Let H,G be finitely generated groups equipped with word metrics, and let (X,d) be a metric space. Suppose that H that is bi-Lipschitz to G via the map F and that G acts translation-like on (X,d). If Λ is a set of orbit representatives of the action of G on X, then H acts translation-like on (X,d) via $h\cdot(x\cdot g) = x\cdot F(F^{-1}(g)\cdot h)$ for $x \in \Lambda$ where we write the action on the right.*

Lemma 2.5. *Let H,G be finitely generated groups equipped with word metrics, and let (X,d) be a metric space. Suppose that H acts translation-like on G and that G acts translation-like on (X,d). Then H acts translation-like on (X,d).*

2.3 Coarse models for homogeneous spaces

We start this subsection with the following definition.

Definition 2.6. Let X be a metric space and suppose that G is a finitely generated group that admits at translation-like action on X. A **chain** between x and y in X is a sequence of points $\{x_i,y_i\}_{i=1}^k$ such that $x = x_1$, $y = y_k$, and for each $1 \le i \le k-1$, there exists a $g_i \in G$ such that $g_i \cdot y_i = x_{i+1}$.

With the notion of chains between points in a metric space being acted on translation-like, we can define a natural quotient of metric space by the translation-like action by some finitely generated group.

Definition 2.7. Let (X,d) be a metric space, and suppose that G is a finitely generated group that admits a translation-like action on X. We define a distance function $d\colon X\times X \to \mathbb{R}_{\ge 0}$ on the quotient $X/\sim$ by

$$d_{X/G}([x],[y]) = \inf\left\{ \sum_{i=1}^k d(x_i,y_i) \;:\; \{x_i,y_i\}_{i=1}^k \text{ is a chain from } x \text{ to } y \right\}.$$

The space $X/\sim$ endowed with the function $d_{X/G}(\cdot,\cdot)$ is call the **translation-like geometric quotient of** X **by** G.

For a general metric space (X,d) which admits a translation-like action by a group G, we have that X/G is not necessarily a metric space. However, when X is a UDBG space, the X/G is a metric space as seen in the following proposition.

Proposition 2.8. *Let X be a UDBG space, and suppose that G admits a translation-like action on X. Then X/G is a metric space.*

Proof. To begin, $d_{X/G}([x],[y]) = d_{X/G}([y],[x])$ is clear. As X is a UDBG space, we have that

$$\inf\{d(x,y) \ : \ x,y \in X, x \neq y\} > 0.$$

In particular, if $[x]$, $[y]$ are distinct equivalence classes in X/G, then $d_{X/G}([x],[y]) > 0$. For the triangle inequality, let $\{p_i,q_i\}_{i=1}^{k}$ be a chain from x to y, and let $\{p'_t,q'_t\}_{t=1}^{s}$ be a chain from y to z. We then have that $\{p_i,q_i\}_{i=1}^{k} \cup \{p'_t,q'_t\}_{t=1}^{s}$ is a chain from x to z. We may write

$$d_{X/G}([x],[y]) \leq \sum_{i=1}^{k} d(p_i,p_i) + \sum_{t=1}^{s} d(p'_t,q'_t).$$

By definition, we note that

$$\begin{aligned} d_{X/G}([x],[y]) + d_{X/G}([y],[z]) \quad &= \quad \inf\left\{\sum_{i=1}^{k} d(p_i,q_i) \ : \ \{p_i,q_i\}_{i=1}^{k} \text{ is a chain from } x \text{ to } y\right\} \\ &+ \quad \inf\left\{\sum_{i=1}^{s} d(p'_t,q'_t) \ : \ \{p'_t,q'_t\}_{t=1}^{k} \text{ is a chain from } y \text{ to } z\right\}. \end{aligned}$$

Therefore, by definition that

$$d_{X/G}([x],[y]) \leq d_{X/G}([x],[y]) + d_{X/G}([y],[z]).$$

Thus, X/G is a metric space. ☐

When given a finitely generated group G with a finite generated subgroup $H \leq G$, we note that H acts translation-like on G in a natural way by left multiplication; moreover, we have that the translation-like geometric of G by H is bi-Lipschitz to the coset space of H in G. In general, a translation-like geometric quotient of a finitely generated group G by a finitely generated group H will not necessarily be bi-Lipschitz to the coset space of a subgroup $K \leq G$. Therefore, we may view the translation-like geometric quotient of G by a finitely generated group H is a generalization of coset spaces of subgroups.

The next propositions show that if given a UDBG space X with a translation-like action by a group G, then the translation-like action geometric quotient is well-defined up to the bi-Lipschitz classes of G and X.

Proposition 2.9. *Let X and Y are UDBG spaces with a bi-Lipschitz equivalence $F: X \to Y$, and suppose that G is a finitely generated group that acts translation-like on X. If we equip Y with the translation-like action of G induced by the bi-Lipschitz equivalence, then X/G is bi-Lipschitz to Y/G.*

Proof. Let d_X and d_Y be the metrics of X and Y, respectively. We claim that F descends to a bijection between X/G and Y/G. By Lemma 2.3, we have that the action of G on Y is given by $g \cdot y = F(g \cdot F^{-1}(y))$. If $g \cdot x_1 = x_2$ for $x_1, x_2 \in X$, we have that

$$g \cdot F(x_1) = F(g \cdot F^{-1}(F(x_1))) = F(g \cdot x_1) = F(x_2).$$

Thus, the map F preserves equivalence classes, and since the induced map $\bar{F} : X/G \to Y/G$ is clearly a bijection, we have our claim.

There exists a constant $C \geq 1$ such that for all elements $x, y \in X$, we have that

$$\frac{1}{C} d_X(x,y) \leq d_Y(F(x), F(y)) \leq C d_X(x,y).$$

If $(p_1, q_1), \cdots, (p_n, q_n)$ is a chain from x to y in X, then $(F(p_1), F(q_1)), \cdots, (F(p_n), F(q_n))$ is a chain from $F(x)$ to $F(y)$. In particular, we have that

$$d_{Y/G}([\bar{F}(x)], [\bar{F}(y)]) \leq \sum_{i=1}^{n} d_Y(F(x), F(y)) \leq C \sum_{i=1}^{n} d_X(x,y).$$

By taking the infimum over all n-chains from x to y, we have that

$$d_{Y/G}([\bar{F}(x)], [\bar{F}(y)]) \leq C d_{X/G}([x], [y]).$$

Using similar arguments, we have that

$$\frac{1}{C} d_{X/G}([x], [y]) \leq d_{Y/G}([\bar{F}(x)], [\bar{F}(y)]). \qquad \square$$

Proposition 2.10. *Let X be a UDBG space, and suppose that G is a finitely generated group that admits a translation-like action on X. If H is bi-Lipschitz to G via the map F, then with the induced translation-like action of H on X, we have that X/G is bi-Lipschitz to X/H.*

Proof. For simplicity in this proof, we go with the right action. Letting Λ be a set of orbit representatives of the action of G on H, we have that $X = \bigsqcup_{x \in \Lambda} x \cdot G$. We have that H acts on itself via right multiplication, and thus, the action of H on X is given by

$$h \cdot (x \cdot g) = x \cdot (F(F^{-1}(g) \cdot h^{-1})).$$

We claim that $y_1 \sim y_2$ via the G-action if and only if $y_1 \sim y_2$ via the H–action. Suppose that x represents the equivalence class of y_1 and y_2. There exist elements $g_1, g_2 \in G$ such that $x \cdot g_1 = y_1$ and $x \cdot g_2 = y_2$. Since H acts transitively on G, there exists an element $h \in H$ such that $g_1 \cdot h = g_2$. Therefore, $y_1 \cdot h = y_2$. The other direction is similar. As a consequence, we have that $(p_1, q_1), \cdots, (p_n, q_n)$ is a chain from x to y with respect to the G-action if and only if it is a chain from x to y with respect to the H–action. In particular

$$d_{X/G}([x]_G, [y]_G) = d_{X/H}([x]_H, [y]_H).$$

By the above arguments, we have that the identity map from X to itself descends to a map of the orbit spaces $F : X/G \to X/H$ which is a bi-Lipschitz equivalence. $\square$

Definition 2.11. Let $\mathbf{G}$ be a Lie group with a Lie subgroup $\mathbf{H} \leq \mathbf{G}$. Let $\Gamma < \mathbf{G}$ and $\Delta < \mathbf{H}$ be cocompact lattices. We say that a translation-like action of Δ on Γ gives rise to a **coarse model of the homogeneous space $\mathbf{G}/\mathbf{H}$** if there exists a UDBG space $X \subset \mathbf{G}$ that bi-Lipschitz to Γ such that the orbits of the induced translation-like action of Δ on X are coarsely embedded and contained in cosets of $\mathbf{H}$ in $\mathbf{G}$ and where there exists a natural bi-Lipschitz embedding from X/Δ to $\mathbf{G}/\mathbf{H}$ that is a quasi-isometry.

2.4 Carnot Lie groups

We are interested in a special class of nilpotent Lie algebras that admit natural dilations which act as a generalized notion of scaling.

Definition 2.12. Let $\mathfrak{g}$ be a nilpotent Lie algebra of step length c. We say that $\mathfrak{n}$ is a **stratified** nilpotent Lie algebra if it admits a grading $\mathfrak{n} = \bigoplus_{i=1}^{c} \mathfrak{v}_i$ where $\mathfrak{v}_1$ generates $\mathfrak{n}$. We say that a nilpotent Lie group $\mathbf{N}$ is **stratified** if its Lie algebra is stratified.

Let $\mathfrak{n}$ be a stratified nilpotent Lie algebra of step size c with grading $\bigoplus_{i=1}^{c} \mathfrak{v}_i$. Observe that the linear maps $d\delta_t \colon \mathfrak{n} \to \mathfrak{n}$ given by

$$d\delta_t X_1, \cdots, X_c) = \left(t \cdot X_1, t^2 \cdot X_2, \cdots, t^c \cdot X_c\right)$$

satisfy $d\delta_t([X,Y]) = [d\delta_t(X), d\delta_t(Y)]$ and $d\delta_{ts} = d\delta_t \circ d\delta_s$ for $X,Y \in \mathfrak{g}$ and $t,s > 0$. Thus, $\{d\delta_t \ : \ t > 0\}$ gives a one parameter family of Lie automorphisms of $\mathfrak{n}$. If $\mathbf{N}$ is a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$, then by exponentiating $d\delta_t$ we have an one parameter family of automorphisms denoted δ_t. The **dilation on N of factor** t is the Lie automorphism δ_t.

We have the following lemma whose proof is an exercise in basic differential topology.

Lemma 2.13. *Let $\mathbf{N}$ be a connected, simply connected stratified nilpotent Lie group with Lie algebra $\mathfrak{n}$. Let $X \in \mathfrak{n}$, $t > 0$, and $x \in \mathbf{N}$. If $V = L_x(X)$, then $(d\delta_t)_x(V) = (dL_x \circ d\delta_t)_1(X)$.*

Proof. Since $\mathbf{N}$ is a connected, simply connected nilpotent Lie group, the exponential map exp is a diffeomorphism whose inverse we formally denote as Log. Letting U be a small neighborhood about the identity, we have that (U, Log) is a local chart around the identity. Thus, we have that $(L_x(U), \varphi_x)$ is a local chart about x where $\varphi_x = \mathrm{Log} \circ L_{x^{-1}}$. We then have that the map given by $\varphi_x^{-1} \circ (d\delta_t)_1 \circ \varphi_x \colon L_x(U) \to \delta_t(L_x(U))$ is a local coordinate representation of δ_t at x. Thus,

$$(d\delta_t)_x = (d\varphi_x)^{-1} \circ (d\delta_t)_1 \circ (d\varphi_x) = (d(L_x \circ \exp)) \circ (d\delta_t)_1 \circ d(\mathrm{Log} \circ L_{x^{-1}}).$$

Observing that $\mathbf{N} \subset \mathrm{GL}(n,\mathbb{R})$ and $\mathfrak{n} \subset \mathfrak{gl}(n,\mathbb{R})$ for some n, we may write

$$(d\delta_t)_x(V) = x\,(d\exp)_1 \circ (d\delta_t)_1 \circ (d\,\mathrm{Log})_1(x^{-1}\,V).$$

There exist vectors $X_i \in \mathfrak{v}_i$ such that $V = \sum_{i=1}^{c} xX_i$. Since $\delta_t \circ \exp = \exp \circ \delta_t$, we have that $\mathrm{Log} \circ \delta_t = \delta_t \circ \mathrm{Log}$. In particular, we may write $(d\delta_t)_1 \circ (d\,\mathrm{Log})_1 = (d\,\mathrm{Log})_1 \circ (d\delta_t)_1$. Thus,

$$(d\delta_t)_x(V) = (d\delta_t)_x\,(xX) = x(d\exp)_1 \circ (d\delta_t)_1 \circ (d\,\mathrm{Log})_1 \circ L_{x^{-1}}(xX) = \left(\sum_{i=1}^{c} x\,(d\delta_t)_1\,X_i\right).$$

Hence,

$$(d\delta_t)_x(V) = x\left(\sum_{i=1}^{c} t^i\,X_i\right) = \sum_{i=1}^{c}(dL_x)_1(t^i\,X_i) = (dL_x)_1\left(\sum_{i=1}^{c} t^i\,X_i\right) = (dL_x)_1 \circ (d\delta_t)_1(X).$$

Therefore, $(d\delta_t)_x(V) = (dL_x \circ \delta_t)_1(X)$. $\square$

2.5 Semisimple Lie groups

We recall standard facts in the theory of semisimple Lie groups which can be found in [7, 11, 14, 21].

Definition 2.14. Given a real Lie algebra $\mathfrak{g}$, the **Killing form** is the symmetric bilinear form $B\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ given by

$$B_{\mathfrak{g}}(X,Y)=\mathrm{Tr}(\mathrm{ad}_X\circ\mathrm{ad}_Y).$$

We write $B=B_{\mathfrak{g}}$ when $\mathfrak{g}$ is clear from context. If B is non-degenerate, we say that $\mathfrak{g}$ is a **semisimple Lie algebra**. If the Lie algebra of the Lie group $\mathbf{G}$ is semisimple, we say that $\mathbf{G}$ is a **semisimple Lie group**.

2.5.1 Iwasawa decomposition of a semisimple Lie group

The Iwasawa decomposition of a semisimple Lie group $\mathbf{G}$ arises from considerations of an involutive automorphism of the Lie algebra $\mathfrak{g}$.

Definition 2.15. An involution $\theta\colon\mathfrak{g}\to\mathfrak{g}$ is called a **Cartan involution** if the bilinear form given by $B_\theta(X,Y)=-B(X,\theta(Y))$ is positive definite. We call the bilinear form B_θ the **Cartan-Killing metric** on $\mathbf{G}$. Every real semisimple Lie algebra admits a Cartan involution, and any two Cartan involutions of a real semisimple Lie algebra differ by an inner automorphism.

If θ is a Cartan involution of the semisimple Lie algebra $\mathfrak{g}$, then the Cartan decomposition is given by the vector space direct sum $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ where $\mathfrak{k}$ and $\mathfrak{p}$ are the eigenspaces relative to the eigenvalues 1 and -1 of θ. We fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, with $\dim\mathfrak{a}=\mathrm{rank}(\mathbf{G})$. The Cartan decomposition is orthogonal with respect to the bilinear form $B_\theta(X,Y)$. We fix an order on the system $R\subseteq\mathfrak{a}'$ of non-zero restricted roots of $(\mathfrak{g},\mathfrak{a})$. Let

$$\mathfrak{m}=\{X\in\mathfrak{k}\,:\,[X,Y]=0\text{ for all }Y\in\mathfrak{a}\}.$$

The Lie algebra $\mathfrak{g}$ decomposes as

$$\mathfrak{g}=\mathfrak{m}+\mathfrak{a}+\bigoplus_{\alpha\in R}\mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha$ is the root space relative to the root α. We denote Π^+ as the subset of positive roots. If $\mathbf{K}$, $\mathbf{A}$, and $\mathbf{N}$ are the Lie subgroups with Lie algebras $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}=\oplus_{\alpha\in\Pi_+}\mathfrak{g}_\alpha$, then the map from $\mathbf{K}\times\mathbf{A}\times\mathbf{N}$ to $\mathbf{G}$ given by $(k,a,n)\to kan$ is a diffeomorphism. In particular, we write $\mathbf{G}=\mathbf{KAN}$ and call this the **Iwasawa decomposition of G.** We have that $\mathbf{K}$ is a compact Lie group, $\mathbf{A}$ is a connected, simply connected abelian Lie group, and $\mathbf{N}$ is a connected, simply connected nilpotent Lie group. Moreover, we have that $\mathbf{N}$ has additional structure in that $\mathbf{N}$ is a stratified nilpotent group as shown below.

Denote by Φ the subset of positive simple roots. Given that root spaces satisfy $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]\subseteq\mathfrak{g}_{\alpha+\beta}$, the subspace $V\subseteq\mathfrak{n}$ given by $V=\bigoplus_{\delta\in\Phi}\mathfrak{g}_\delta$ provides a stratification of $\mathfrak{n}$. In particular, $\mathfrak{n}$ is a stratified nilpotent Lie algebra and thus, $\mathbf{N}$ is a stratified nilpotent Lie group. We write this down as a proposition.

Proposition 2.16. *Let $\mathbf{G}$ be a connected, semisimple Lie group, and let $\mathbf{G}=\boldsymbol{KAN}$ be an Iwasawa decomposition. Then $\mathbf{N}$ is a stratified nilpotent Lie group.*

We introduce some notation. Assuming that $\mathbf{N}$ has step length c, we denote Φ_i as the set of roots such that $\mathfrak{n}_i/\mathfrak{n}_{i+1}=\bigoplus_{\beta\in\Pi_i}\mathfrak{g}_\beta$ as vector spaces with some ordering on the roots. Since $\mathbf{N}$ is a connected, simply connected nilpotent Lie group, the exponential map is a diffeomorphism. In particular, the Baker-Campbell-Hausdorff formula implies that $\mathbf{N}$ is diffeomorphic to $\prod_{i=1}^{c}\prod_{\beta\in\Pi_i}\exp(\mathfrak{g}_\beta)$.

3 Metrics on semisimple Lie groups

For semisimple Lie groups $\mathbf{G}$ with maximal compact subgroup $\mathbf{K}$, we have that $\mathbf{G}/\mathbf{K}=\mathbb{R}^{\mathrm{rank}(\mathbf{G})}\times\mathbf{N}$ as smooth spaces. If g is the Cartan-Killing metric on $\mathbf{G}$, then at the identity coset of $\mathbf{G}/\mathbf{K}$, we have by [3, Section 4] for $(a,n) \in \mathbf{G}/\mathbf{K}$ that

$$g_{a,n} = \sum_{i=1}^{\operatorname{rank}(\mathbf{G})} da_i^2 + \sum_{i=1}^{c(\mathbf{N})} \sum_{\beta \in \Pi_i} \beta(a)(g_\beta)_n$$

where $\sum_{\beta \in \Phi} g_\beta$ is a left-invariant metric on $\mathfrak{n}$, the Lie algebra of $\mathbf{N}$. If $c\colon [0,1] \to \mathbf{G}/\mathbf{K}$ is a smooth curve, we may write

$$c(t) = \left(c_a(t), \left(c_{\beta,1}(t)\right)_{\beta \in \Phi_1}, \cdots, \left(c_{\beta,c(\mathbf{N})}(t)\right)_{\beta \in \Phi_{c(\mathbf{N})}} \right)$$

where $c_a\colon [0,1] \to \mathbb{R}^{\operatorname{rank}(\mathbf{G})}$ is a smooth math and $c_{\beta,i}\colon [0,1] \to \exp(\mathfrak{g}_\beta)$ is a smooth map for all $\beta \in \Pi_i$ and $1 \leq i \leq c(\mathbf{N})$. Thus, it is evident that $\mathbb{R}^{\operatorname{rank}(\mathbf{G})}$ with the standard flat metric, which we denote as $|\cdot|$, is isometrically embedded. Since any vector $X \in \mathfrak{n}$ may be written as

$$X = \sum_{i=1}^{c} \sum_{\beta \in \Pi_i} X_\beta$$

where $X_\beta \in \mathfrak{g}_\beta$, we may write the length of c with respect to the metric $g_{a,n}$ as

$$\ell_{\mathbf{G}/\mathbf{K}}(c) = \int_0^1 \sqrt{\sum_{j=1}^{\operatorname{rank}(\mathbf{G})} (dc_{a_j}(t))^2 + \sum_{t=1}^{c(\mathbf{N})} \sum_{\beta \in \Pi_i} \beta(a) g_\beta(dc_\beta(t), dc_\beta(t))}dt.$$

The associated distance function on $\mathbf{G}/\mathbf{K}$ is given by

$$d_{\mathbf{G}/\mathbf{K}} = \inf\left\{\ell_{\mathbf{G}/\mathbf{K}}(c) \ : \ c \text{ is a smooth path in } \mathbf{G}/\mathbf{K} \text{ from } x \text{ to } y\right\}.$$

For $a \in \mathbb{R}^{\operatorname{rank}(\mathbf{G})}$, we denote $\mathbf{N}_a$ as the nilpotent Lie group $\mathbf{N}$ equipped with the left invariant metric

$$\sum_{i=1}^{c(\mathbf{N})} \sum_{\beta \in \Pi_i} \beta(a)(g_\beta)_n$$

which we will identify with $\{a\} \times \mathbf{N}$ in $\mathbf{G}/\mathbf{K}$. Any smooth curve $c\colon [0,1] \to \mathbf{N}_a$ has the form

$$c(t) = \left(\left(c_{\beta,1}(t)\right)_{\beta \in \Phi_1}, \cdots, \left(c_{\beta,c(\mathbf{N})}(t)\right)_{\beta \in \Phi_{c(\mathbf{N})}} \right)$$

where $c_{\beta,i}(t) \in \exp(\mathfrak{g}_\beta)$ for all $t \in [0,1]$. Therefore, the length of c in $\mathbf{N}_a$ is given by

$$\ell_a(c) = \int_0^1 \sqrt{\sum_{i=1}^{c(\mathbf{N}_a)} \sum_{\beta \in \Pi_i} \beta(a) g_\beta(dc_{\beta,i}(t), dc_{\beta,i}(t))}dt.$$

As before, the associated distance function is given by

$$d_a(x,y) = \inf\{\ell_a(c) \ : \ c \text{ is a smooth path in } \mathbf{N}_a \text{ from } x \text{ to } y\}.$$

We have the following smooth diffeomorphism which dilates $\mathbf{N}$ based on the point a in $\mathbb{R}^{\operatorname{rank}(\mathbf{G})}$.

Definition 3.1. For each $1 \leq i \leq c(\mathbf{N})$ and $\beta \in \Pi_i$, we denote $f_{\beta,i}(a) = 1/\sqrt[2i]{\beta(a)}$. With this value, we denote the following map $F_a\colon \mathbf{N} \to \mathbf{N}$ as

$$F_a(x) = \left(\delta_{f_{\beta,i}(a)}(x_{\beta,i})\right)_{1 \leq i \leq c(\mathbf{N}), \beta \in \Pi_i}.$$

Since $\delta_{\beta,i}$ is a smooth map for all $\beta \in \Pi_i$ and each $1 \leq i \leq c(\mathbf{N})$, we have that F is a diffeomorphism.

We note for all elements $a \in \mathbb{R}^{\mathrm{rank}(\mathbf{G})}$ and roots $\beta \in \Phi$ that $\beta(a) > 0$. In particular, we have that $\beta(\vec{0}) = 1$ for all $\beta \in \Phi$. With this observation in mind, we have the following proposition which relates the length of the path c in $\mathbf{N}_{\vec{0}}$ to length of the path in $F_a(c)$ in $\mathbf{N}_a$.

Proposition 3.2. *If $c\colon [0,1] \to \boldsymbol{N}$ is a smooth curve, then for all $a \in \mathbb{R}^{rank(\boldsymbol{G})}$ we have that $\ell_a(F_a(c)) = \ell_0(c)$.*

Proof. We have that

$$
\begin{aligned}
c(t) &= \left(\left(c_{\beta,1}(t)\right)_{\beta\in\Phi_1}, \cdots, \left(c_{\beta,c(\mathbf{N})}(t)\right)_{\beta\in\Phi_{c(\mathbf{N})}}\right)\\
dc(t) &= \left(\left(dc_{\beta,1}(t)\right)_{\beta\in\Phi_1}, \cdots, \left(dc_{\beta,c(\mathbf{N})}(t)\right)_{\beta\in\Phi_{c(\mathbf{N})}}\right).
\end{aligned}
$$

We may write

$$dc_{\beta,i}(t) = dL_{c_{\beta,i}(t)}(X_{\beta,i}(t))$$

where $X_{\beta,i}\colon [0,1] \to \mathfrak{g}_\beta$ is a smooth function. For notational simplicity, we let $\rho_{\beta,i,a}(t) = \delta_{f_{\beta,i}(a)} \circ c_{\beta,i}(t)$. Thus, Lemma 2.13 implies that

$$d(\delta_{f_{\beta,i}(a)} \circ c_{\beta,i})(t) = (\delta_{f_{\beta,i}(a)})_1(dL_{\rho_{\beta,i,a}(t)}(X_{\beta,i}(t))) = (1/\sqrt[2]{\beta(a)})dL_{c_{\beta,i}(t)}(X_{\beta,i}(t)).$$

Therefore, we may write

$$
\begin{aligned}
\beta(a)g_\beta(d(\rho_{\beta,i,a}(t)), d(\rho_{\beta,i,a}(t))_{\rho_{\beta,i,a}(t)} &= g_\beta(dL_{c_{\beta,i}(t)}(X_{\beta,i}(t)), dL_{c_{\beta,i}(t)}(X_{\beta,i}(t)))_{\rho_{\beta,i,a}(t)}\\
&= g_\beta(X_\beta(t), X_\beta(t))_1\\
&= g_\beta(dL_{c_{\beta,i}}(X(t)), dL_{c_{\beta,i}}(t))_1\\
&= g_\beta(dc_{\beta,i}(t), dc_{\beta,i}(t)).
\end{aligned}
$$

Combining everything together, we may write

$$
\begin{aligned}
\ell_a(F_a(c)) &= \int_0^1 \sqrt{\sum_{i=1}^{c(\mathbf{N}_a)} \sum_{\beta\in\Pi_i} \beta(a)g_\beta(d\rho_{\beta,i,a}(t), d\rho_{\beta,i,a}(t))_{\rho_{\beta,i,a}(t)}}dt\\
&= \int_0^1 \sqrt{\sum_{i=1}^{c(\mathbf{N}_a)} \sum_{\beta\in\Pi_i} \beta(a)g_\beta(dc_{\beta,i}(t), dc_{\beta,i}(t))_{c_{\beta,i}(t)}}dt\\
&= \ell_0(c). \quad \square
\end{aligned}
$$

As a natural consequence, we have the following corollary.

Corollary 3.3. *Let $x, y \in \boldsymbol{N}$, and let $a \in \mathbb{R}^{rank(\boldsymbol{G})}$. Then $d_a(F_a(x), F_a(y)) = d_0(x,y)$.*

Proof. Let c be a smooth path from x to y. We have by the above proposition that $\ell_0(c) = \ell_a(F_a \circ c)$. Since $F_a \circ c$ is a path from $F_a(x)$ to $F_a(y)$, we have that

$$d_a(F_a(x), F_a(y)) \le \ell_a(F_a \circ c) = \ell_0(c).$$

Therefore, by definition, we have that $d_{\vec{0}}(x,y) \le d_a(F_a(x), F_a(y))$. Using a similar argument, we also have that $d_a(F_a(x), F_a(y)) \le d_0(x,y)$. Therefore, $d_0(x,y) = d_a(F_a(x), F_a(y))$. $\square$

We now provide a lower bound for the distance between points in distinct cosets of $\mathbf{N}$ in terms of the distance between of the the coordinates of the coset representatives.

Lemma 3.4. *Let x,y be distinct points in $\mathbb{R}^{rank(\mathbf{G})}$, and let $g,h \in \mathbf{N}$. We then have that $d_{\mathbf{G}/\mathbf{K}}((x,g),(y,h)) \geq |x-y|$. Moreover, if $g = h$, then $d_{\mathbf{G}/\mathbf{K}}((x,g),(y,g)) = |x-y|$.*

Proof. Let c be a path between (x,g) and (y,h). We may write

$$\begin{aligned} \ell_{\mathbf{G}/\mathbf{K}}(c) &= \int_0^1 \sqrt{\sum_{j=1}^{\text{rank}(\mathbf{G})} (dc_{a_j}(t))^2 + \sum_{i=1}^{c(\mathbf{N})} \sum_{\beta \in \Pi_i} \beta(a) g_\beta(dc_{\beta,i}(t), dc_{\beta,i}(t))dt} \\ &\geq \int_0^1 \sqrt{\sum_{j=1}^{c(\mathbf{N})} (dc_{a_{\beta,i}}(t))^2 dt} = \int_0^1 |dc_a(t)| \geq |x-y|. \end{aligned}$$

Therefore, we have by definition that $d_{\mathbf{G}/\mathbf{K}}((x,g),(y,h)) \geq |x-y|$.

Let $\gamma\colon [0,1] \to \mathbb{R}^{\text{rank}(\mathbf{G})}$ be a straight line path from x to y, and let $c\colon [0,1] \to \mathbf{G}/\mathbf{K}$ be the path given by $c(t) = (\gamma(t), g)$. We may express the length of c as

$$\ell_{\mathbf{G}/\mathbf{K}}(c) = \int_0^1 \sqrt{\sum_{j=1}^{\text{rank}(\mathbf{G})} (dc_{a_i}(t))^2 dt} = \int_0^1 |d\gamma(t)|\, dt = |x-y|.$$

In particular, we have that $d_{\mathbf{G}/\mathbf{K}}((x,g),(y,g)) \leq |x-y|$. Using the above inequality, we have that

$$d_{\mathbf{G}/\mathbf{K}}((x,g),(y,g)) = |x-y|. \qquad \square$$

The last proposition of this section relates the distance between $(\vec{0},x)$ and $(\vec{0},y)$ in $\mathbf{N}_{\vec{0}}$ with the distance between $(a, F_a(x))$ and $(a, F_a(y))$ for any $a \in \mathbb{R}^{\text{rank}(\mathbf{G})}$ as points in $\mathbf{G}/\mathbf{K}$.

Proposition 3.5. *Let $g,h \in \mathbf{N}$, and let $a \in \mathbb{R}^{\text{rank}(\mathbf{G})}$. Then*

$$C_1 \ln(d_0(x,y)) \leq d_{\mathbf{G}/\mathbf{K}}((a, F_a(x)), (a, F_a(y))) \leq C_2 \ln(d_0(x,y))$$

for some constants $C_1, C_2 > 0$.

Proof. By [9, 3.C_1'], we have that there exist constants $C_1, C_2 > 0$ such that

$$C_1 \ln(d_a(F_a(x), F_a(y))) \leq d_{\mathbf{G}/\mathbf{K}}((a, F_a(x)), (a, F_a(y))) \leq C_2 \ln(d_a(F_a(x), F_a(y))).$$

By Corollary 3.3, we have that $d_a(F_a(x), F_a(y)) = d_0(x,y)$. Thus, we have that

$$C_1 \ln(d_0(x,y)) \leq d_{\mathbf{G}/\mathbf{K}}((a, F_a(x)), (a, F_a(y))) \leq C_2 \ln(d_0(x,y)). \qquad \square$$

4 Lipchitz models for cocompact lattices in semisimple Lie groups

We now introduce a model for the Lipschitz geometry of cocompact lattices in an arbitrary semisimple Lie group $\mathbf{G}$ with an Iwasawa decomposition $\mathbf{G} = \mathbf{KAN}$. For a cocompact lattice $\Delta \subset \mathbf{N}$, we let $\mathbf{X}(\Delta) \subset \mathbf{G}/\mathbf{K}$ be the subset given by

$$\mathbf{X}(\Delta) = \left\{ (a, F_a(g)) \,:\, a \in \mathbb{Z}^{\text{rank}(\mathbf{G})}, g \in \Delta \right\}$$

with the induced metric.

Proposition 4.1. *If $\Delta < N$ be a cocompact lattice such that*

$$\inf\{d_0(x,y) \ : \ x,y \in \Delta, x \neq y\} > 1,$$

then $\boldsymbol{X}(\Delta)$ is a UDBG space.

Proof. We first show that $\mathbf{X}(\Delta)$ is uniformly discrete. If $x,y \in \mathbb{R}^{\operatorname{rank}(\mathbf{G})}$ such that $x \neq y$, then Lemma 3.4 implies for any $g,h \in \Delta$ that

$$d_{\mathbf{G}/\mathbf{K}}((x,F_x(g)),(y,F_y(h))) \geq |x-y| \geq 1.$$

For $z = x = y$, Proposition 3.5 implies that there exists a constant $C_1 > 0$ such that

$$d_{\mathbf{G}/\mathbf{K}}((z,F_z(g)),(z,F_z(h))) \geq C_1 \ln(d_0(g,h)) \geq C_1 \ln(\inf\{d_0(a,b) \ : \ a,b \in \Delta, a \neq b\}).$$

Therefore, for all $(x,F_x(g)),(y,F_y(h)) \in \mathbf{X}(\Delta)$, we have that

$$d_{\mathbf{G}/\mathbf{K}}((x,F_x(g)),(y,F_y(h))) \geq \min\{1, C_1 \ln(\inf\{d_0(a,b) \ : \ a,b \in \Delta, a \neq b\})\}.$$

In particular, we have that

$$\inf\left\{d_{\mathbf{G}/\mathbf{K}}((x,F_x(g)),(y,F_y(h))) \ : \ (x,F_x(g)) \neq (y,F_y(h)) \text{ in } \mathbf{X}(\Delta)\right\} > 0$$

showing that $\mathbf{X}(\Delta)$ is uniformly discrete.

We now demonstrate that $\mathbf{X}(\Delta)$ has bounded geometry. To do that, we show for all $r > 0$ that there exists a constant C_r such that $\left|B_{\mathbf{X}(\Delta)}((x,F_x(g)))\right| \leq C_r$ for all $g \in \Delta$ and $x \in \mathbb{Z}^{\operatorname{rank}(\mathbf{G})}$. We start by showing that there exists a universal constant M_r such that any r–ball in $\mathbf{X}(\Delta)$ intersects at most M_r sets of the form $(x,\mathbf{N})$ where $x \in \mathbb{Z}^{\operatorname{rank}(\mathbf{G})}$. We also need to show that there exists a constant $C_r > 0$ such that

$$\left|B_{\mathbf{X}(\Delta),r}(x,F_x(g)) \cap (y,\mathbf{N})\right| \leq C_r$$

for $y \in \mathbb{Z}^{\operatorname{rank}(\mathbf{G})}$.

If $(y,F_y(h)) \in B_{\mathbf{X}(\Delta),r}((x,F_x(g)))$ such that $|x-y| > r$, then Proposition 3.4 implies that

$$d_{\mathbf{G}/\mathbf{K}}((y,F_y(h)),(x,F_x(g))) \geq |x-y| > r$$

which is a contradiction. Therefore, we have that $|x-y| \leq r$. Since x is fixed, it is easy to see that exists a constant $M > 0$ such that there are at most M sets of the form $(y,\mathbf{N})$ such that

$$B_{\mathbf{X}(\Delta),r}((x,F_x(g))) \cap (y,\mathbf{N}) \neq \emptyset.$$

First consider $(x,F_x(h)) \in B_{\mathbf{X}(\Delta),r}((x,F_x(g)))$. We have by the above reasoning that $|x-y| \leq r$, and thus, the triangle inequality and Corollary 3.3 imply that

$$d_0(g,h) \leq C_1 e^{d_{\mathbf{G}/\mathbf{K}}((x,F_x(h)),(x,F_x(g)))} \leq C_1 e^r.$$

Thus, $h \in B_{\Delta,C_1 e^r}(g)$, and by Gromov's polynomial growth theorem, we have that there exists a constant $C_2 > 0$ and a natural number d such that $\left|B_{\Delta,C_1 e^r}(g)\right| \leq C_2 C_1^d e^{dr}$. Now consider $(y,F_y(h)) \in B_{\mathbf{X}(\Delta),r}((x,F_x(g)))$ where $x \neq y$. That implies

$$(y,\mathbf{N}) \cap B_{\mathbf{X}(\Delta),r}((x,F_x(g)) \neq \emptyset,$$

and by the above statement, we have that there exist at most M_r such points y. Taking these statements together, we have that

$$\left|B_{\mathbf{X}(\Delta),r}((x,F_x(g))\right| \leq C_3 e^{dr}$$

for some constant $C_3 > 0$. Therefore, $\mathbf{X}(\Delta)$ is a UDBG space. □

The following proposition demonstrates that under appropriate assumptions on $\Delta < \mathbf{N}$ that $\mathbf{X}(\Delta)$ can be thought of as a model for the Lipschitz geometry of Γ where $\Gamma < \mathbf{G}$ is a cocompact lattice.

Proposition 4.2. *Let $\Gamma < \boldsymbol{G}$ be a cocompact lattice, and let $\Delta < \boldsymbol{N}$ be a cocompact lattice satisfying*

$$\inf\{d_0(g,h) \ : \ g,h \in \Delta, g \neq h\} > 1.$$

Then $\boldsymbol{X}(\Delta)$ is bi-Lipschitz to Γ.

Proof. Let $C_1, C_2 > 0$ be the constants from Proposition 3.5, and let $C = \max\{C_1, C_2\}$. Since Δ is a cocompact lattice in $\mathbf{N}$, we have that Δ is quasi-isometric to $\mathbf{N}$. In particular, there exists a constant $\varepsilon_1 > 0$ such that if $g \in \mathbf{N}$, then there exists an element $h \in \Delta$ such that $d_0(g,h) \leq \varepsilon_1$. Letting $\varepsilon = C\,\varepsilon_1 + \mathrm{rank}(\mathbf{G})$, we claim that $\mathbf{X}(\Delta)$ is ε–dense in $\mathbf{G}/\mathbf{K}$. Let $(x,g) \in \mathbf{X}$ where $x \in \mathbb{R}^{\mathrm{rank}(\mathbf{G})}$ and $g \in \mathbf{N}$.

Suppose that $x \in \mathbb{Z}^{\mathrm{rank}(\mathbf{G})}$. There exists an element $h \in \Delta$ such that $d_1(F_{-x}(g),h) \leq \varepsilon_1$. Proposition 3.5 implies that

$$d_{\mathbf{G}/\mathbf{K}}((x,F_x(h)),(x,g)) \leq d_0(h,(F_{-x}(g)) \leq C\,\varepsilon_1.$$

Suppose that $x \in \mathbb{R}^{\mathrm{rank}(\mathbf{G})} \backslash \mathbb{Z}^{\mathrm{rank}(\mathbf{G})}$. There exists an element $y \in \mathbb{Z}^{\mathrm{rank}(\mathbf{G})}$ such that $|x-y| \leq 2\,\mathrm{rank}(\mathbf{G})$, and thus, there exists an element $h \in \Delta$ such that $d_0(F_{-y}(g),h) \leq \varepsilon_1$. By the triangle inequality, Lemma 3.4, and Proposition 3.5, we have that

$$\begin{aligned}
d_{\mathbf{G}/\mathbf{K}}((x,g),(y,F_y(h))) &\leq d_{\mathbf{X}}((x,g),(y,g)) + d_{\mathbf{X}}((y,g),(y,F_y(h))) \\
&\leq 2\,\mathrm{rank}(\mathbf{G}) + d_{\bar{0}}(F_{-y}(g),h) \\
&\leq 2\,\mathrm{rank}(\mathbf{G}) + \varepsilon_1 = \varepsilon.
\end{aligned}$$

Therefore, $\mathbf{X}(\Delta)$ is ε-dense in $\mathbf{G}/\mathbf{K}$, and subsequently, $\mathbf{X}(\Delta)$ and Γ are quasi-isometric. Since $\mathbf{X}(\Delta)$ and Γ are quasi-isometric non-amenable spaces, Proposition 2.2 implies that they are bi-lipschitz. $\square$

5 Proof of Theorem 1.1

For the readers convenience, we restate our Theorem 1.1.

Theorem 1.1. *Let $\mathbf{G}$ be a semisimple Lie group with an Iwasawa decomposition $\mathbf{G} = \mathbf{KAN}$. If $\Gamma < \mathbf{G}$ and $\Delta < \mathbf{N}$ are cocompact lattices, then Γ admits a translation-like action by Δ. Moreover, we can choose this translation-like action to give rise to a coarse model Γ/Δ of the homogeneous space $\mathbf{G}/\mathbf{N}$. Finally, given distinct lattices $\Gamma_1, \Gamma_2 < \mathbf{G}$ and $\Delta_1, \Delta_2 < \mathbf{N}$, we have the coarse models Γ_1/Δ_1 and Γ_2/Δ_2 for $\mathbf{G}/\mathbf{N}$ are bi-Lipschitz.*

Proof. It is evident that there exists a cocompact lattice Δ' in $\mathbf{N}$ satisfying

$$\inf\left\{d_0(g,h) \,|\, g,h \in \Delta', g \neq h\right\} > 1.$$

We first demonstrate that Δ' admits a translation-like action on $\mathbf{X}(\Delta')$. For $g \in \Delta'$ and $(x,F_x(h)) \in \mathbf{X}(\Delta')$, we let $g \cdot (x,F_x(h)) = (x,F_x(hg^{-1}))$. It is easy to see that this is a free action. Therefore, we need to demonstrate that we have a wobbling action. We have by Proposition 3.5 that there exists a constant $C > 0$ such that

$$d_{\mathbf{G}/\mathbf{K}}((x,F_x(h))(x,F_x(hg^{-1}))) \leq C\ln(d_0(h,hg^{-1})) \leq C\ln(d_0(1,g)).$$

Therefore, Δ' admits a translation-like action on $\mathbf{X}(\Delta')$.

To finish, we note that Δ is a cocompact lattice in $\mathbf{N}$, and by the discussion after [5, Ques 2], we have that Δ and Δ' are bi-Lipschitz. We have that Δ acts on itself by right multiplication, and thus, Lemma 2.3 implies that Δ admits a translation-like action on Δ'. Lemma 2.4 implies that Δ admits a translation-like action on $\mathbf{X}(\Delta')$. Since $\mathbf{X}(\Delta')$ is bi-Lipschitz to Γ, we have by Lemma 2.3 that Δ admits a translation-like action on Γ as desired. It is evident that the given translation-like action gives rise to a coarse model for $\mathbf{G}/\mathbf{N}$.

If $\Gamma, \Gamma' < \mathbf{G}$ are cocompact lattices, we have that Γ and Γ' are bi-Lipschitz by Proposition 2.2. Moreover, if $\Delta, \Delta' < \mathbf{N}$ are cocompact lattices, then since $\mathbf{N}$ is a Carnot group, we have by the remark after [5, Ques 2] that Δ and Δ' are blipschitz. hus, by applying Proposition 2.9 and Proposition 2.10, we see that Γ/Δ and Γ'/Δ' are bi-Lipschitz. $\square$

For the proof of Corollary 1.2, we note that if $\mathbf{G}$ is not isogenous to $\mathrm{SL}(2,\mathbb{R})$, then $\mathbb{Z}^2 \leq \Delta$. Since $\mathbb{Z}^2$ acts translation-like on Δ by virtue of being a subgroup, we have by Lemma 2.5 that $\mathbb{Z}^2$ acts translation-like Γ.

6 Proof of Theorem 1.3

We restate Theorem 1.3 for the reader's convenience.

Theorem 1.3. *Let $\mathbf{G}$ and $\mathbf{H}$ be $\mathbb{Q}$–defined noncompact real simple Lie groups such that $\mathbf{H} \leq \mathbf{G}$. If $\Delta < \mathbf{H}$ and $\Gamma < \mathbf{G}$ are cocompact lattices, then Δ admits a translation-like action on Γ. Moreover, we can choose this translation-like such that Γ/Δ is a coarse model for $\mathbf{G}/\mathbf{H}$. Finally, given distinct lattices $\Gamma_1, \Gamma_2 < \mathbf{G}$ and $\Delta_1, \Delta_2 < \mathbf{H}$, the spaces Γ_1/Δ_1 and Γ_2/Δ_2 for $\mathbf{G}/\mathbf{H}$ are bi-Lipschitz.*

Proof. Since the inclusion of $\mathbf{H}$ into $\mathbf{G}$ is $\mathbb{Q}$–defined, we have by [18, 10.14. Corollary (iii)] that $\mathbf{H}(\mathbb{Z})$ is a subgroup of a cocompact lattice Λ that is commensurable with $\mathbf{G}(\mathbb{Z})$. We have that $\mathbf{H}(\mathbb{Z}) \leq \mathbf{H} \cap \Lambda \leq \Lambda$, and thus, $\mathbf{H} \cap \Lambda$ is a cocompact lattice in $\mathbf{H}$. Hence, we have that $\Lambda/\mathbb{Z} \cap \Lambda$ naturally embeds into $\mathbf{G}/\mathbf{H}$ as a coarse dense subset. Thus, subgroup containment of $\mathbf{H} \cap \Lambda$ into Λ is a translation-like action that gives rise to a coarse model for the homogeneous space $\mathbf{G}/\mathbf{H}$. Since Γ and Λ are quasi-isometric non-amenable spaces, Proposition 2.2 implies that Γ and Λ are bi-Lipschitz, and thus, $\mathbf{H} \cap \Lambda$ admits a translation-like action by Lemma 2.3. Hence, Proposition 2.9 implies that $\Gamma/\mathbf{H} \cap \Lambda$ is bi-Lipschitz to $\Lambda/\mathbf{H} \cap \Lambda$, and thus, the translation-like action of $\mathbf{H} \cap \Lambda$ on Γ gives rise to a coarse model of $\mathbf{G}/\mathbf{H}$. Additionally, Δ and $\mathbf{H} \cap \Lambda$ are quasi-isometric nonamenable spaces, and thus, by Proposition 2.2, we have that they are bi-Lipschitz. Thus, Lemma 2.4 implies that Δ admits a natural translation-like action on Γ. Moreover, we have by Proposition 2.10 that Γ/Δ is bi-Lipschitz to $\Gamma/\mathbf{H} \cap \Lambda$. Subsequently, Δ admits a translation-like action on Γ that gives rise to a coarse model for $\mathbf{G}/\mathbf{H}$. Finally, we note that if $\Gamma' < \mathbf{G}$ and $\Delta' < \mathbf{H}$ are different cocompact lattices, then by Proposition 2.2, we have that Δ and Δ' are bi-Lipschitz and that Γ and Γ' are bi-Lipschitz. Thus, by applying Proposition 2.9 and Proposition 2.10, we see that Γ/Δ and Γ'/Δ' are bi-Lipschitz. $\square$

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