Multirate Sampled-Data Observer Design Based on a Continuous-Time Design

Chen Ling and Costas Kravaris

***Abstract*—A multirate sampled-data observer for nonlinear systems under asynchronous sampling is developed in this paper. The proposed multirate observer is based on a continuous-time design coupled with intersample output predictors for the sampled measurements. The sampled-data system with the multirate observer forms a hybrid system and it is shown that the error dynamics of the overall system is input-to-output stable with respect to measurement errors, by applying the Karafyllis–Jiang vector small-gain theorem. This sampled-data design also offers robustness with respect to perturbations in the sampling schedule. The proposed method is evaluated through linear systems and a nonlinear batch reactor example.**

***Index Terms*—Asynchronous sampling, input-to-output stability (IOS), multirate observers, nonlinear sampled-data systems, vector small-gain theorem.**

I. Introduction

The objective of this paper is to develop a state observer in a multirate sampled-data system under asynchronous sampling. The problem of nonlinear observer design has been intensively studied for systems under fast sampling [1]–[5], which can be potentially applied to estimation, process control, and fault detection and identification. Motivated by practical implementation needs, however, one of the biggest challenges is to design a state observer for general multirate systems (e.g., chemical processes, biological systems, and networked control systems), where different sampling rates of sensors need to be accommodated in the observer design framework.

The observer design for linear multirate systems was studied in [6] and [7], where an available continuous-time observer design was adopted but different methods (i.e., sample-and-hold strategy [6] and model-based prediction [7]) were employed to approximate the intersample behavior between consecutive measurements. Both approaches provided robustness with respect to perturbations in the sampling schedule.

In nonlinear systems, the focus has been primarily on single-rate sampled-data observer design based on a mixed continuous and discrete strategy, which was inspired by the continuous-discrete Kalman filter in [8]. In [9], an observer for state affine systems with regularly persistent inputs was designed. The results in [9] were extended to observer design for state affine systems up to output injection in [10] and adaptive observer design in [11]. The continuous-discrete approach has been applied to high-gain observers in [12]–[17]. In general, the continuous-discrete observer consists of two steps: open-loop prediction when no measurements are available and impulsive correction once a new measurement is obtained. A similar idea was also used in [18] and [19].

The problem of nonlinear multirate observer design has received little attention, given the challenge in stability analysis that arises from the asynchronous nature of different sensors and uncertainty in the sampling schedule. A nonlinear multirate observer design, based on an approximate discrete-time model, was developed in [20], with guaranteed semiglobal practical input-to-state stability (ISS) from exogenous disturbances to estimation errors. An artificial fast-rate sampler and a hold device were introduced to reconstruct the missing outputs as well as inputs between sampling times, which were then fed to a single-rate observer working at a base sampling period of the plant. The results were extended to one-sided Lipschitz systems in [21]. Recently, a hybrid observer was reported for a class of nonlinear systems with multirate sampled and delayed measurements [22], with global exponential stability of the error dynamics. However, it assumes a certain special structure of the nonlinear system for the method to be applicable.

In this paper, the proposed multirate sampled-data observer design adopts the idea in [23] and [24] of using a state predictor to approximate the intersample behavior, but in a more general context, where multiple intersample predictors are used for the multirate system. These predictors will be running asynchronously at the same time. Each predictor generates an estimate of the evolution of a sampled output between consecutive measurements, in the same spirit as [7] for linear systems. The existence of a continuous-time observer is a prerequisite for a multirate observer design. This is a common assumption in the continuous-discrete observer design for sampled-data systems [15]. Taking the measurement errors as inputs and the estimation errors as outputs, the notion of input-to-output stability (IOS), originated from [25] for systems described by ordinary differential equations, is adopted for stability analysis of the sampled-data system and the multirate observer. Since the overall system is a hybrid system in the sense that the classical semigroup property does not hold, the notion of weak semigroup property introduced in [26] and [27] will be utilized, as it is more relaxed than the semigroup property and allows to study a very general class of systems (e.g., sampled-data systems, networked control systems, and hybrid systems). A direct product from this system theoretic framework is a small-gain theorem for two interconnected feedback systems [28], which played an important role in the stability analysis of the single-output sampled-data observer in [23]. This result was further generalized to a vector small-gain theorem in [29], which allows to study IOS and ISS properties for large-scale systems consisting of multiple interacting subsystems, such as the proposed multirate sampled-data observer in Section II.

The rest of this paper is organized as follows. The representation of a multirate sampled-data observer is formulated in Section II. The main results are stated in Section III and applied to linear and nonlinear systems in Section IV.

Manuscript received December 19, 2018; accepted July 7, 2019. Date of publication July 17, 2019; date of current version December 3, 2019. This work was supported by the National Science Foundation under Grant CBET-1706201. Recommended by Associate Editor F. Mazenc. *(Corresponding author: Costas Kravaris.)*

The authors are with the Artie McFerrin Department of Chemical Engineering, Texas A&M University, College Station, TX 77843-3122 USA (e-mail: ling0408@tamu.edu; kravaris@mail.che.tamu.edu).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2019.2929295

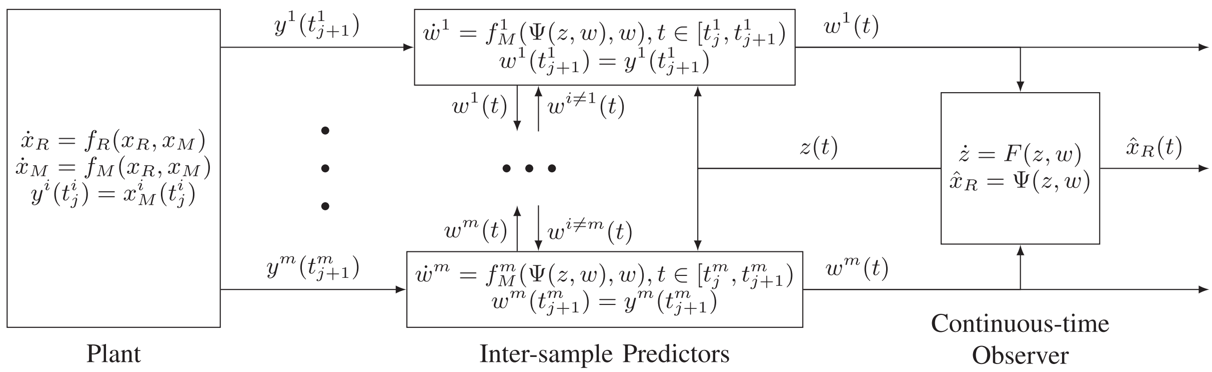

Fig. 1. Schematic of the multirate sampled-data observer with the plant (no measurement error).

II. Formulation of the Sampled-Data Observer

A. Notations

1) $\mathcal{K}^+$ denotes the class of positive, continuous functions defined on $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geqslant 0\}$. We adopt the notations of class $\mathcal{K}$, class $\mathcal{K}_\infty$, and class $\mathcal{KL}$ functions in [30]. The set of nonnegative integers is denoted by $\mathbb{Z}_+$.
2) $\mathbb{R}^n_+ := \{[x_1, \ldots, x_n]' \in \mathbb{R}^n : x_1 \geqslant 0, \ldots, x_n \geqslant 0\}$. Let $x, y \in \mathbb{R}^n$. We say that $x \leqslant y$ if and only if $(y - x) \in \mathbb{R}^n_+$. We say that a function $\rho : \mathbb{R}^n_+ \to \mathbb{R}_+$ is of class $\mathcal{N}_n$, if ρ is continuous with $\rho(0) = 0$ and such that $\rho(x) \leqslant \rho(y)$ for all $x, y \in \mathbb{R}^n_+$ with $x \leqslant y$.
3) For every positive integer l and an open, nonempty set $A \subseteq \mathbb{R}^n$, $C^l(A; \Omega)$ denotes the class of continuous functions on A with continuous derivatives of order l, which take values in $\Omega \subseteq \mathbb{R}^m$. $C^0(A; \Omega)$ denotes the class of continuous functions on A, which take values in Ω.
4) We denote by $\|\cdot\|_{\mathcal{X}}$ the norm of the normed linear space $\mathcal{X}$. By $|\cdot|$, we denote the ℓ_1-norm of $\mathbb{R}^n$. Let $I \subseteq \mathbb{R}_+$ be an interval and $D \subseteq \mathbb{R}^l$ be a nonempty set. By $\mathcal{L}^\infty_{loc}(I; D)$, we denote the class of all Lebesgue measurable and locally bounded functions $u : I \to D$. For $u \in \mathcal{L}^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^n)$, we define the norm $\|u(t)\|_{\mathcal{U}} := \sum_{i=1}^n \sup_{\tau\in[0,t]} |u_i(\tau)|$. Notice that $\sup_{\tau\in[0,t]} |u_i(\tau)|$ denotes the actual supremum of $|u_i(t)|$ on $[0, t]$.

B. Problem Formulation

Consider a multioutput continuous-time autonomous system, where without loss of generality, the output is assumed to be a part of the state vector

$$\begin{aligned}\dot{x}_R(t) &= f_R(x_R(t), x_M(t))\\ \dot{x}_M(t) &= f_M(x_R(t), x_M(t))\\ y(t) &= x_M(t)\end{aligned} \tag{1}$$

with $x_R \in \mathbb{R}^{n-m}$ being the unmeasured state vector, $x_M \in \mathbb{R}^m$ being the remaining state vector that is directly measured, y denoting the output vector, and $f_R \in C^1(\mathbb{R}^{n-m} \times \mathbb{R}^m; \mathbb{R}^{n-m})$ and $f_M \in C^1(\mathbb{R}^{n-m} \times \mathbb{R}^m; \mathbb{R}^m)$ with $f_R(0,0) = 0$ and $f_M(0,0) = 0$.

In the presence of multiple measurements, it makes more sense to use a reduced-order observer so that a significantly lower dimensionality can ease implementation of the observer. Therefore, a reduced-order observer formulation will be the focus of this paper. Suppose that a continuous-time reduced-order observer design is available for system (1)

$$\begin{aligned}\dot{z}(t) &= F(z(t), y(t))\\ \hat{x}_R(t) &= \Psi(z(t), y(t))\end{aligned} \tag{2}$$

with $z \in \mathbb{R}^k$ being the observer state, $\hat{x}_R \in \mathbb{R}^{n-m}$ being the state estimates, and $F \in C^1(\mathbb{R}^k \times \mathbb{R}^m; \mathbb{R}^k)$ and $\Psi \in C^1(\mathbb{R}^k \times \mathbb{R}^m; \mathbb{R}^{n-m})$ with $F(0,0) = 0$ and $\Psi(0,0) = 0$.

The output equation of system (1) should be modified under slow-sampled measurements, which yields the following multirate sampled-data system:

$$\begin{aligned}\dot{x}_R(t) &= f_R(x_R(t), x_M(t))\\ \dot{x}_M(t) &= f_M(x_R(t), x_M(t))\\ y^i(t^i_j) &= x^i_M(t^i_j), \qquad j \in \mathbb{Z}_+, i = 1, 2, \ldots, m\end{aligned} \tag{3}$$

where t^i_j denotes the jth sampling time for the ith component in x_M, at some sequence of time instants $S = \{t_k\}_{k=0}^\infty$. The sampling times of the ith sensor form an infinite subsequence that tends to infinity. These sampling times are not necessarily uniformly spaced, but satisfying $0 < t^i_{j+1} - t^i_j \leqslant r$ for all $j \in \mathbb{Z}_+$, where r is the maximum sampling period among all the sensors. The sampling times from all the subsequences will be considered as the sampling times of the multirate sampled-data system (3). Notice that S is the sequence of all sampling times in ascending order. There is a one-to-one mapping from $\{t^i_j : j \in \mathbb{Z}_+, i = 1, 2, \ldots, m\}$ to $\{t_k\}_{k=0}^\infty$. Finally, we assume that there is no measurement available at the initial time t_0.

As mentioned in Section I, a continuous-time design (2) will be the basis of a multirate observer design in the presence of asynchronous sampled measurements, as long as the intersample behavior is taken care of. In this way, a continuous-time design from the literature can be reused in the context of a multirate observer so that we do not need to design it from scratch. System (3) can be used to predict the evolution of the output between consecutive measurements. As depicted in Fig. 1, we propose the multirate sampled-data observer design

$$\begin{aligned}\dot{z}(t) &= F(z(t), w(t)), & t \in [t_k, t_{k+1})\\ \dot{w}(t) &= f_M(\Psi(z(t), w(t)), w(t)), & t \in [t_k, t_{k+1})\\ w^i(t_{k+1}) &= y^i(t_{k+1}) &\\ \hat{x}_R(t) &= \Psi(z(t), w(t)), & \hat{x}_R \in \mathbb{R}^{n-m}.\end{aligned} \tag{4}$$

This observer has the same dynamics as the continuous-time observer (2). The predictors operate continuously at different time horizons, which generate additional signals $w(t)$ to approximate and replace the output $y(t)$ in the implementation of continuous-time observer (2).

$w^i(t)$ will be reinitialized once a new measurement $y^i(t_{k+1})$ becomes available, whereas the rest of the predictor states do not change until their measurements are obtained. By integrating the predictor equations, a model-based correction is applied on the most-recent measurement. Notice that t_k and t_{k+1} are not necessarily the sampling times from the same sensor. It was seen in [7] that the model-based prediction offers a more meaningful approach to approximate the intersample behavior instead of a simple sample-and-hold strategy, especially under large sampling period.

It is important to point out that the sampled-data system (3) together with the multirate observer (4) is a hybrid system, which does not satisfy the classical semigroup property. However, the weak semigroup property still holds (see [26] and [27]). The recent results in [26]–[29] for a wide class of systems will be used in the proof of the main theorem to be stated in the following section. From the main theorem, other important features of this multirate sampled-data observer include: First, the missing intersample behavior can be reconstructed by using the intersample predictors and second, as long as the maximum sampling period is sufficiently small, if the continuous-time observer implementation guarantees stability of the error dynamics and robustness with respect to measurement errors, then the multirate observer will inherit these properties as well. These properties are unaffected by perturbations in the sampling schedule, which is a major advantage of the proposed hybrid implementation as opposed to an approximate discrete-time observer approach. Moreover, this continuous-discrete observer approach is able to use all possible measurements with different sampling rates, without making common assumptions when a discrete-time observer is used, such as that the sampling periods of the sensors are uniform and/or their ratios are rational numbers.

C. Basic Notions

We require that the following assumption holds.

Assumption 1: System (1) is forward complete.

Note that Assumption 1, according to the main results in [31], implies the existence of functions $\mu \in \mathcal{K}^+$ and $a \in \mathcal{K}_\infty$ such that for every $(x_{R,0}, x_{M,0}) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$, the solution $(x_R(t), x_M(t))$ of (1) with initial condition $(x_R(0), x_M(0)) = (x_{R,0}, x_{M,0})$ exists for all $t \geqslant 0$ and satisfies

$$|(x_R(t), x_M(t))| \leqslant \mu(t) a(|(x_{R,0}, x_{M,0})|) \quad \forall t \geqslant 0. \tag{5}$$

In other words, a finite dimensional system described by ordinary differential equations is forward complete if and only if the corresponding solution exists for all $t \geqslant 0$ and for every initial condition [24].

Similar to [23, Definition 2.1] but in the context of a reduced-order observer, it is necessary to define the following notion of robust observer for system (1) with respect to measurement errors, which is important to develop the main results of this paper.

Definition 1: The system

$$\begin{aligned} \dot{z}(t) &= F(z(t), y(t)), \quad z \in \mathbb{R}^k \\ \hat{x}_R(t) &= \Psi(z(t), y(t)), \quad \hat{x}_R \in \mathbb{R}^{n-m} \end{aligned} \tag{6}$$

where $F \in C^1(\mathbb{R}^k \times \mathbb{R}^m; \mathbb{R}^k)$ and $\Psi \in C^1(\mathbb{R}^k \times \mathbb{R}^m; \mathbb{R}^{n-m})$ with $F(0,0) = 0$ and $\Psi(0,0) = 0$ is called a *robust observer* for system (1) with respect to measurement errors, if the following conditions are met.

1) There exist functions $\sigma \in \mathcal{KL}$, $\gamma, p \in \mathcal{N}_1$, $\mu \in \mathcal{K}^+$, and $a \in \mathcal{K}_\infty$ such that for every $(x_{R,0}, x_{M,0}, z_0, v) \in \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^k \times \mathcal{L}^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^m)$, the solution $(x_R(t), x_M(t), z(t))$ of

$$\begin{aligned} \dot{x}_R(t) &= f_R(x_R(t), x_M(t)) \\ \dot{x}_M(t) &= f_M(x_R(t), x_M(t)) \\ \dot{z}(t) &= F(z(t), x_M(t) + v(t)) \\ \hat{x}_R(t) &= \Psi(z(t), x_M(t) + v(t)) \end{aligned} \tag{7}$$

with initial condition $(x_R(0), x_M(0), z(0)) = (x_{R,0}, x_{M,0}, z_0)$ corresponding to $v \in \mathcal{L}^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^m)$ exists for all $t \geqslant 0$ and satisfies the following estimates:

$$\begin{aligned} |\hat{x}_R(t) - x_R(t)| \leqslant \sigma(|(x_{R,0}, x_{M,0}, z_0)|, t) \\ + \gamma(\|v(t)\|_{\mathcal{U}}) \quad \forall t \geqslant 0 \end{aligned} \tag{8a}$$

$$\begin{aligned} |z(t)| \leqslant \mu(t)[a(|(x_{R,0}, x_{M,0}, z_0)|) \\ + p(\|v(t)\|_{\mathcal{U}})] \quad \forall t \geqslant 0. \end{aligned} \tag{8b}$$

2) For every $(x_{R,0}, x_{M,0}) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$, there exists $z_0 \in \mathbb{R}^k$ such that the solution $(x_R(t), x_M(t), z(t))$ of system (7) with initial condition $(x_R(0), x_M(0), z(0)) = (x_{R,0}, x_{M,0}, z_0)$ corresponding to $v \equiv 0$ satisfies $x_R(t) = \Psi(z(t), x_M(t))$ for all $t \geqslant 0$.

Remark 1: If system (6) is a robust observer for system (1) with respect to measurement errors, then system (7) with the output $Y = \Psi(z, x_M + v) - x_R$ satisfies the uniform IOS (UIOS) property from the input $v \in \mathcal{L}^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^m)$ with gain $\gamma \in \mathcal{N}_1$ (see [28]).

Instead of using online, continuous-time outputs, a multirate sampled-data observer only uses outputs at discrete sampling times in $S = \{t_k\}_{k=0}^\infty$. The sampling partition is not necessarily uniform, but there exists a maximum sampling period r acting as the upper bound on each sampling interval. Next, we define the notion of a robust multirate sampled-data observer.

Definition 2: The system

$$\begin{aligned} \dot{\zeta}(t) &= g(\zeta(t), \zeta(t_k)), \quad t \in [t_k, t_{k+1}) \\ \zeta(t_{k+1}) &= G\left(\lim_{t \to t_{k+1}^-} \zeta(t), y^i(t_{k+1})\right) \\ \hat{x}_R(t) &= \Psi(\zeta(t)) \end{aligned} \tag{9}$$

where $g \in C^1(\mathbb{R}^k \times \mathbb{R}^k; \mathbb{R}^k)$, $G \in C^0(\mathbb{R}^k \times \mathbb{R}; \mathbb{R}^k)$, and $\Psi \in C^1(\mathbb{R}^k; \mathbb{R}^{n-m})$ with $g(0,0) = 0$, $G(0,0) = 0$, and $\Psi(0) = 0$ is called a *robust multirate sampled-data observer* for system (3) with respect to measurement errors, if the following conditions are met:

1) There exist functions $\sigma \in \mathcal{KL}$, $\gamma, p \in \mathcal{N}_1$, $\mu \in \mathcal{K}^+$, and $a \in \mathcal{K}_\infty$ such that for every $(x_{R,0}, x_{M,0}, \zeta_0, d, v) \in \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^k \times \mathcal{L}^\infty_{loc}(\mathbb{R}_+; [0,1]) \times \mathcal{L}^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^m)$, the solution $(x_R(t), x_M(t), \zeta(t))$ of

$$\begin{aligned} \dot{x}_R(t) &= f_R(x_R(t), x_M(t)) \\ \dot{x}_M(t) &= f_M(x_R(t), x_M(t)) \\ \dot{\zeta}(t) &= g(\zeta(t), \zeta(t_k)), \quad t \in [t_k, t_{k+1}) \\ \zeta(t_{k+1}) &= G\left(\lim_{t \to t_{k+1}^-} \zeta(t), x_M^i(t_{k+1}) + v^i(t_{k+1})\right) \\ t_{k+1} &= t_k + r d(t_k) \\ \hat{x}_R(t) &= \Psi(\zeta(t)) \end{aligned} \tag{10}$$

with initial condition $(x_R(0), x_M(0), \zeta(0)) = (x_{R,0}, x_{M,0}, \zeta_0)$ corresponding to $d \in \mathcal{L}^\infty_{loc}(\mathbb{R}_+; [0,1])$ and $v \in \mathcal{L}^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^m)$ exists for all $t \geqslant 0$ and satisfies the following estimates:

$$|\hat{x}_R(t) - x_R(t)| \leqslant \sigma(|(x_{R,0}, x_{M,0}, \zeta_0)|, t) + \gamma(\|v(t)\|_{\mathcal{U}}) \quad \forall t \geqslant 0 \tag{11a}$$

$$|\zeta(t)| \leqslant \mu(t)[a(|(x_{R,0}, x_{M,0}, \zeta_0)|) + p(\|v(t)\|_{\mathcal{U}})] \quad \forall t \geqslant 0. \tag{11b}$$

2) For every $(x_{R,0}, x_{M,0}) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$, there exists $\zeta_0 \in \mathbb{R}^k$ such that for all $d \in \mathcal{L}^\infty_{loc}(\mathbb{R}_+; [0,1])$, the solution $(x_R(t), x_M(t), \zeta(t))$ of (10) with initial condition $(x_R(0), x_M(0), \zeta(0)) = (x_{R,0}, x_{M,0}, \zeta_0)$ corresponding to $d \in \mathcal{L}^\infty_{loc}(\mathbb{R}_+; [0,1])$ and $v \equiv 0$, satisfies $x_R(t) = \Psi(\zeta(t))$ for all $t \geqslant 0$.

Remark 2: If system (9) is a robust multirate sampled-data observer for system (3) with respect to measurement errors, then system (10) with the output $Y = \Psi(\zeta) - x_R$ satisfies the UIOS property from the input $v \in \mathcal{L}^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^m)$, with gain $\gamma \in \mathcal{N}_1$ (see [28], where the notion of UIOS is defined for hybrid systems, e.g., system (10), which does not satisfy the classical semigroup property).

Remark 3: The sampling period in each subsequence is allowed to be time varying. The equation $t_{k+1} = t_k + rd(t_k)$ generates the sampling instants in S sequentially with $0 \leqslant t_{k+1} - t_k \leqslant r$ for all $k \in \mathbb{Z}_+$. The value that $d(t_k)$ takes introduces uncertainty to the end point of each sampling interval. Proving stability for any disturbance $d \in \mathcal{L}^\infty_{loc}(\mathbb{R}_+; [0,1])$ will guarantee stability for all sampling schedules of system (10).

Remark 4: At a specific time t_k, there could be measurement of more than one output or the sampling of one sensor may coincide with another (i.e., $d(t_k) = 0$). Hence, some sampling instants may appear more than once in the sequence S, where the reinitialization step will occur repeatedly but on different elements in $w(t)$.

III. Main Results

Recently, the nonlinear small-gain theorem was generalized from two interconnected systems to large-scale complex systems consisting of multiple, interacting input-to-output stable (or input-to-state stable) subsystems in [32]–[34]. In [29], a generalization of several previously developed nonlinear small-gain theorems was obtained. Uniform and nonuniform IOS and ISS properties were studied for a wide class of nonlinear feedback systems that do not satisfy the semigroup property, such as hybrid and switched systems.

In this section, we assume that there exists a robust observer for system (1) in the sense of Definition 1, and would like to give conditions so that stability of the error dynamics and robustness with respect to measurement errors still hold for the multirate design.

Theorem 1: Consider system (1) under Assumption 1 and suppose that system (6) is a robust observer for system (1) with respect to measurement errors. Moreover, suppose that there exist constants $C^i \geqslant 0$ and functions $\bar{\sigma}^i \in \mathcal{KL}$ for all $i = 1, 2, \ldots, m$, such that for every $(x_{R,0}, x_{M,0}, z_0, v) \in \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^k \times \mathcal{L}^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^m)$, the solution $(x_R(t), x_M(t), z(t))$ of (7) with initial condition $(x_R(0), x_M(0), z(0)) = (x_{R,0}, x_{M,0}, z_0)$ corresponding to $v \in \mathcal{L}^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^m)$ exists for all $t \geqslant 0$ and satisfies the following estimate:

$$\begin{aligned} &|f^i_M(\Psi(z(t), x_M(t) + v(t)), x_M(t) + v(t)) \\ &- f^i_M(x_R(t), x_M(t))| \\ &\leqslant \bar{\sigma}^i(|(x_{R,0}, x_{M,0}, z_0)|, t) + C^i \|v(t)\|_{\mathcal{U}} \quad \forall t \geqslant 0. \end{aligned} \tag{12}$$

Additionally, suppose that $3rC^i m < 1$ for $i = 1, 2, \ldots, m$ and $3\gamma(ms) < s$ for all $s > 0$, where $\gamma \in \mathcal{N}_1$ is the gain function in the estimate (8a) of the robust observer. Then, (4) is a robust multirate sampled-data observer for system (3) with respect to measurement errors.

Proof: We focus on the following hybrid system, consisting of a sampled-data system and a multirate observer

$$\begin{aligned} \dot{x}_R(t) &= f_R(x_R(t), x_M(t)) \\ \dot{x}_M(t) &= f_M(x_R(t), x_M(t)) \\ \dot{z}(t) &= F(z(t), w(t)), & t \in [t_k, t_{k+1}) \\ \dot{w}(t) &= f_M(\Psi(z(t), w(t)), w(t)), & t \in [t_k, t_{k+1}) \\ w^i(t_{k+1}) &= x^i_M(t_{k+1}) + v^i(t_{k+1}) \\ t_{k+1} &= t_k + rd(t_k) \\ Y(t) &= \Psi(z(t), w(t)) - x_R(t). \end{aligned} \tag{13}$$

By virtue of Definition 2, it is necessary to show that system (13) with the output $Y = \Psi(z, w) - x_R$ satisfies the UIOS property from the input $v \in \mathcal{L}^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^m)$.

Note that the hybrid system (13) has a distributed structure, where each intersample predictor is a subsystem operating asynchronously. Each subsystem receives the associated system output and reinitializes its own intersample predictor. These subsystems also communicate with each other as well as the continuous-time observer by transmitting the predicted outputs. We focus on the ith subsystem and treat $w^j(t)$ ($j \neq i$) and $v^i(t)$ as inputs to this subsystem. First, the boundedness of $\|w(t) - x_M(t)\|_{\mathcal{U}}$ will be established. Next, we will focus on the overall hybrid system (13) and study the UIOS property from the actual input $v \in \mathcal{L}^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^m)$. A vector small-gain theorem (see [29]) will be used to complete the proof.

Consider the ith subsystem from the distributed structure, where $w^j(t)$ ($j \neq i$) and $v^i(t)$ are considered as input

$$\begin{aligned} \dot{x}_R(t) &= f_R(x_R(t), x_M(t)) \\ \dot{x}_M(t) &= f_M(x_R(t), x_M(t)) \\ \dot{z}(t) &= F(z(t), w(t)), & t \in [t^i_j, t^i_{j+1}) \\ \dot{w}^i(t) &= f^i_M(\Psi(z(t), w(t)), w(t)), & t \in [t^i_j, t^i_{j+1}) \\ w^i(t^i_{j+1}) &= x^i_M(t^i_{j+1}) + v^i(t^i_{j+1}) \\ t^i_{j+1} &= t^i_j + rd(t^i_j) \\ Y(t) &= \Psi(z(t), w(t)) - x_R(t) \end{aligned} \tag{14}$$

which satisfies the weak semigroup property. Since two different samplings can never occur at the same time in a subsystem (i.e., $t^i_{j_1} \neq t^i_{j_2}$ if $j_1 \neq j_2$), disturbance $d \in \mathcal{L}^\infty_{loc}(\mathbb{R}_+; (0,1])$ is used to introduce perturbations in the sampling schedule of the ith subsystem, which rules out the Zeno phenomenon. Because system (6) is a robust observer for (1) with respect to measurement errors, it follows from (8a), (8b), and (12) that for every $(x_{R,0}, x_{M,0}, z_0, w^i_0, d) \in \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R} \times \mathcal{L}^\infty_{loc}(\mathbb{R}_+; (0,1])$, the solution $(x_R(t), x_M(t), z(t), w^i(t))$ of (14) with initial condition $(x_R(0), x_M(0), z(0), w^i(0)) = (x_{R,0}, x_{M,0}, z_0, w^i_0)$ corresponding to the inputs $w^j(t)$ where $j \neq i$ (i.e., predicted outputs from other intersample predictors) satisfies the following estimates for all $t \in [0, t_{\max})$:

$$|Y(t)| \leqslant \hat{\sigma}(|(x_{R,0}, x_{M,0}, z_0)|, t) + \gamma(\|w(t) - x_M(t)\|_{\mathcal{U}}) \tag{15}$$

$$|z(t)| \leqslant \mu(t)[a(|(x_{R,0}, x_{M,0}, z_0)|) + p(\|w(t) - x_M(t)\|_{\mathcal{U}})] \tag{16}$$

$$\begin{aligned} &|f_M^i(\Psi(z(t), w(t)), w(t)) - f_M^i(x_R(t), x_M(t))| \\ &\leqslant \bar{\sigma}^i(|(x_{R,0}, x_{M,0}, z_0)|, t) + C^i \|w(t) - x_M(t)\|_{\mathcal{U}}. \end{aligned} \tag{17}$$

Since Assumption 1 holds, we obtain from (5) and (16)

$$\begin{aligned} |(x_R(t), x_M(t), z(t))| &\leqslant \bar{\mu}(t)[\bar{a}(|(x_{R,0}, x_{M,0}, z_0)|) \\ &\quad + p(\|w(t) - x_M(t)\|_{\mathcal{U}})] \quad \forall t \in [0, t_{\max}) \end{aligned} \tag{18}$$

for appropriate functions $\hat{\sigma}, \bar{\sigma}^i \in \mathcal{KL}$, $\gamma, p \in \mathcal{N}_1$, $\mu, \bar{\mu} \in \mathcal{K}^+$, and $a, \bar{a} \in \mathcal{K}_\infty$, where $t_{\max} \in (0, +\infty]$ is the maximal existence time of the solution.

Consider those time intervals where the reinitialization step occurs at the beginning. For all $t \in [t_j^i, t_{j+1}^i) \cap [0, t_{\max})$ with $j \geqslant 1$, we have

$$\begin{aligned} |w^i(t) - x_M^i(t)| &\leqslant \int_{t_j^i}^t |f_M^i(\Psi(z(s), w(s)), w(s)) \\ &\quad - f_M^i(x_R(s), x_M(s))| ds + |v^i(t_j^i)| \\ &\leqslant r\bar{\sigma}^i(|(x_{R,0}, x_{M,0}, z_0)|, t_j^i) + rC^i \|w(t) - x_M(t)\|_{\mathcal{U}} \\ &\quad + |v^i(t_j^i)| \\ &\leqslant \sigma_1^i(|(x_{R,0}, x_{M,0}, z_0)|, t) + rC^i \|w(t) - x_M(t)\|_{\mathcal{U}} \\ &\quad + \sup_{0 \leqslant \tau \leqslant t} |v^i(\tau)| \end{aligned} \tag{19}$$

where $\sigma_1^i(s, t) = r\bar{\sigma}^i(s, t - r)$ for $t \geqslant r$ and $\sigma_1^i(s, t) = \exp(r - t)r\bar{\sigma}^i(s, 0)$ for $t < r$. Note that $\sigma_1^i(s, t) \in \mathcal{KL}$. Since there is no measurement available at the initial time $t_0^i = 0$, we make an initial guess $w^i(t_0^i) = w^i(0) = w_0^i$ for the ith state of the predictor. For all $t \in [0, t_1^i) \cap [0, t_{\max})$, we get

$$\begin{aligned} |w^i(t) - x_M^i(t)| &\leqslant |w_0^i - x_{M,0}^i| \\ &\quad + r \sup_{0 \leqslant s \leqslant t} |f_M^i(\Psi(z(s), w(s)), w(s)) - f_M^i(x_R(s), x_M(s))| \\ &\leqslant |w_0^i| + |x_{M,0}^i| + r\bar{\sigma}^i(|(x_{R,0}, x_{M,0}, z_0)|, 0) \\ &\quad + rC^i \|w(t) - x_M(t)\|_{\mathcal{U}} \\ &\leqslant \sigma_2^i(|(x_{R,0}, x_{M,0}, z_0, w_0^i)|, t) + rC^i \|w(t) - x_M(t)\|_{\mathcal{U}} \end{aligned} \tag{20}$$

where $\sigma_2^i(s, t) = [r\bar{\sigma}^i(s, 0) + s]\exp(r - t)$ and $\sigma_2^i(s, t) \in \mathcal{KL}$. Combining (19) and (20), we conclude that the following estimate holds for all $t \in [0, t_{\max})$ and for $i = 1, 2, \ldots, m$

$$\begin{aligned} |w^i(t) - x_M^i(t)| &\leqslant \sigma^i(|(x_{R,0}, x_{M,0}, z_0, w_0^i)|, t) \\ &\quad + rC^i \|w(t) - x_M(t)\|_{\mathcal{U}} + \sup_{0 \leqslant \tau \leqslant t} |v^i(\tau)|. \end{aligned} \tag{21}$$

From (21), the fact that $\sum_{i=1}^m rC^i < 1/3$, and the assumption that $t_{\max}$ is finite, it suffices to show the boundedness of $\|w(t) - x_M(t)\|_{\mathcal{U}}$ for all $t \in [0, t_{\max})$. In fact

$$\begin{aligned} &\|w(t) - x_M(t)\|_{\mathcal{U}} \\ &\leqslant \frac{\sum_{i=1}^m \sigma^i(|(x_{R,0}, x_{M,0}, z_0, w_0^i)|, 0) + \|v(t)\|_{\mathcal{U}}}{1 - \sum_{i=1}^m rC^i}. \end{aligned} \tag{22}$$

From (18), (21), (22), and the boundedness-implies-continuation (BIC) property (see [26]) for system (14), we conclude that $t_{\max} = +\infty$. Hence, all the aforementioned inequalities hold for all $t \geqslant 0$. Therefore, $t \in [0, t_{\max})$ can be replaced by $t \geqslant 0$ in the inequalities. In addition, the BIC property of each subsystem implies that the BIC property holds true for the overall hybrid system (13) with $t_{\max} = +\infty$.

Now, we are in a position to study the UIOS property of the overall hybrid, multirate system (13) from the actual input $v \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$. A vector small-gain theorem will be used to check stability of the large-scale hybrid systems composed of multiple interconnected subsystems.

Without loss of generality, we may assume that $\bar{\mu}(t) \geqslant 1$ in (18). From (18), (22), and the triangle inequality $|w(t)| \leqslant |w(t) - x_M(t)| + |x_M(t)|$, we obtain for all $t \geqslant 0$

$$\begin{aligned} &|(x_R(t), x_M(t), z(t), w(t))| \\ &\leqslant 2\bar{\mu}(t)[\hat{a}(|(x_{R,0}, x_{M,0}, z_0, w_0)|) + \hat{p}(\|v(t)\|_{\mathcal{U}})] \end{aligned} \tag{23}$$

for appropriate functions $\hat{a} \in \mathcal{K}_\infty$ and $\hat{p} \in \mathcal{N}_1$. Furthermore, (23) and the BIC property of the hybrid system (13) imply the following important properties.

1) System (13) is robustly forward complete from the input $v \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$.
2) $0 \in \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^m$ is a robust equilibrium point from the input $v \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$, with any output function $H(t, x_R, x_M, z, w, v)$ with $H(t, 0, 0, 0, 0, 0) = 0$ for all $t \geqslant 0$.

Now, Hypotheses (H1)–(H4) of the vector small-gain theorem are satisfied by virtue of the inequalities (15), (21), (22), and (23). We conclude that the hybrid system (13) satisfies the UIOS property from the input $v \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$. In other words, there exist functions $\tilde{\sigma} \in \mathcal{KL}$, $\tilde{\gamma}, \tilde{p} \in \mathcal{N}_1$, $\tilde{\mu} \in \mathcal{K}^+$, and $\tilde{a} \in \mathcal{K}_\infty$ such that for every $(x_{R,0}, x_{M,0}, z_0, w_0, d, v) \in \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^m \times \mathcal{L}_{loc}^\infty(\mathbb{R}_+; [0,1]) \times \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$, the solution $(x_R(t), x_M(t), z(t), w(t))$ of (13) with initial condition $(x_R(0), x_M(0), z(0), w(0)) = (x_{R,0}, x_{M,0}, z_0, w_0)$ corresponding to $d \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; [0,1])$ and $v \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$ satisfies the following estimates for all $t \geqslant 0$:

$$|Y(t)| \leqslant \tilde{\sigma}(|(x_{R,0}, x_{M,0}, z_0, w_0)|, t) + \tilde{\gamma}(\|v(t)\|_{\mathcal{U}}) \tag{24a}$$

$$|(z(t), w(t))| \leqslant \tilde{\mu}(t)[\tilde{a}(|(x_{R,0}, x_{M,0}, z_0, w_0)|) + \tilde{p}(\|v(t)\|_{\mathcal{U}})]. \tag{24b}$$

Moreover, for every $(x_{R,0}, x_{M,0}) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$, there exists $(z_0, w_0) \in \mathbb{R}^k \times \mathbb{R}^m$ with $w_0 = x_{M,0}$ such that for all $d \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; [0,1])$, the solution $(x_R(t), x_M(t), z(t), w(t))$ of (13) with initial condition $(x_R(0), x_M(0), z(0), w(0)) = (x_{R,0}, x_{M,0}, z_0, w_0)$ corresponding to $d \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; [0,1])$ and $v \equiv 0$, satisfies $x_R(t) = \Psi(z(t), w(t))$ for all $t \geqslant 0$.

Remark 5: If we consider the hybrid system (13) with the prediction error as the output map, i.e., $Y'(t) = w(t) - x_M(t)$, by applying the vector small-gain theorem again, we conclude that with this output, system (13) also satisfies the UIOS property from the input $v \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$. In other words, there exist functions $\tilde{\sigma} \in \mathcal{KL}$ and $\tilde{\gamma} \in \mathcal{N}_1$ such that for every $(x_{R,0}, x_{M,0}, z_0, w_0, d, v) \in \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^m \times \mathcal{L}_{loc}^\infty(\mathbb{R}_+; [0,1]) \times \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$, the solution $(x_R(t), x_M(t), z(t), w(t))$ of (13) with initial condition $(x_R(0), x_M(0), z(0), w(0)) = (x_{R,0}, x_{M,0}, z_0, w_0)$ corresponding to $d \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; [0,1])$ and $v \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$ satisfies

$$\begin{aligned} |w(t) - x_M(t)| &\leqslant \tilde{\sigma}(|(x_{R,0}, x_{M,0}, z_0, w_0)|, t) \\ &\quad + \tilde{\gamma}(\|v(t)\|_{\mathcal{U}}) \quad \forall t \geqslant 0. \end{aligned} \tag{25}$$

Without measurement errors, the error dynamics of the multirate sampled-data observer, including the intersample predictors, will converge to zero asymptotically.

Remark 6: The continuous-time observer design coupled with intersample output predictors can be applied to multirate full-order observer design, under appropriate modifications. The vector small-gain theorem is applicable to study the UIOS property of the overall system from measurement errors.

Remark 7: It would be possible to include exogenous inputs in the system definition (3) and (4), and subsequently extend the stability analysis mutatis mutandis. This is not included for brevity.

IV. Applications

In this section, the performance of the proposed multirate sampled-data observer is illustrated through linear systems and a nonlinear reactor example. An explicit formula for estimating the maximum sampling period is derived for linear detectable systems with application to an oscillator example.

A. Linear Detectable Systems

Consider a linear detectable system, where without loss of generality, the output is assumed to be a part of the state vector

$$\begin{aligned} \dot{x}_R(t) &= A_{11}x_R(t) + A_{12}x_M(t), \quad x_R \in \mathbb{R}^{n-m} \\ \dot{x}_M(t) &= A_{21}x_R(t) + A_{22}x_M(t), \quad x_M \in \mathbb{R}^{m} \\ y(t) &= x_M(t). \end{aligned} \tag{26}$$

A reduced-order Luenberger observer design is available

$$\begin{aligned} \dot{z}(t) &= Fz(t) + Hy(t) \\ \hat{x}_R(t) &= T_R^{-1}(z(t) - T_M y(t)) \end{aligned} \tag{27}$$

where F is a Hurwitz matrix with desired eigenvalues and forms a controllable pair with H. The transformation matrices T_R and T_M satisfy the following Sylvester equation:

$$\begin{bmatrix} T_R & T_M \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = F\begin{bmatrix} T_R & T_M \end{bmatrix} + H\begin{bmatrix} 0 & I \end{bmatrix}. \tag{28}$$

Consequently, there exists a positive definite matrix P such that $F'P + PF$ is negative definite and there exist constants $\alpha, \delta > 0$ such that

$$x'(F'P + PF)x + 2x'PHv \leqslant -2\alpha x'Px + \delta|v|^2 \tag{29}$$

for all $(x, v) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$. Inequality (29) implies for every $(x_{R,0}, x_{M,0}, z_0, v) \in \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathcal{L}^{\infty}_{loc}(\mathbb{R}_+; \mathbb{R}^m)$ the solution of (26) with

$$\dot{z}(t) = Fz(t) + H(y(t) + v(t)) \tag{30}$$

with initial condition $(x_R(0), x_M(0), z(0)) = (x_{R,0}, x_{M,0}, z_0)$ corresponding to $v \in \mathcal{L}^{\infty}_{loc}(\mathbb{R}_+; \mathbb{R}^m)$ satisfies the estimates for $i = 1, 2, \ldots, m$

$$\begin{aligned} |\hat{x}_R - x_R| &\leqslant |T_R^{-1}| \exp(-\alpha t)\sqrt{\frac{K_2}{K_1}}|z_0 - T_R x_{R,0} - T_M x_{M,0}| \\ &+ \left(|T_R^{-1}|\sqrt{\frac{\delta}{2\alpha K_1}} + |T_R^{-1}T_M|\right) \sup_{0\leqslant\tau\leqslant t}|v(\tau)| \quad \forall t \geqslant 0 \end{aligned}$$

$$\begin{aligned} &|A_{21}^i(\hat{x}_R - x_R) + A_{22}^i v| \\ &\leqslant |A_{21}^i||T_R^{-1}| \exp(-\alpha t)\sqrt{\frac{K_2}{K_1}}|z_0 - T_R x_{R,0} - T_M x_{M,0}| \\ &+ \left(|A_{21}^i||T_R^{-1}|\sqrt{\frac{\delta}{2\alpha K_1}} + |A_{21}^i||T_R^{-1}T_M| + |A_{22}^i|\right) \\ &\times \sup_{0\leqslant\tau\leqslant t}|v(\tau)| \quad \forall t \geqslant 0 \end{aligned}$$

where $K_1, K_2 > 0$ are constants such that $K_1|x|^2 \leqslant x'Px \leqslant K_2|x|^2$ for all $x \in \mathbb{R}^{n-m}$, and A_{21}^i and A_{22}^i are the ith row of the matrices A_{21} and A_{22}, respectively. We conclude from Theorem 1 that the following system

$$\begin{aligned} \dot{z}(t) &= Fz(t) + Hw(t), & t \in [t_k, t_{k+1}) \\ \dot{w}(t) &= A_{21}\hat{x}_R(t) + A_{22}w(t), & t \in [t_k, t_{k+1}) \\ w^i(t_{k+1}) &= x_M^i(t_{k+1}) + v^i(t_{k+1}) \\ \hat{x}_R(t) &= T_R^{-1}(z(t) - T_M w(t)) \end{aligned} \tag{31}$$

is a robust multirate sampled-data observer for system (26) with respect to measurement errors given that the maximum sampling period r satisfies the inequalities

$$\begin{aligned} 3rm&\left(|A_{21}^i||T_R^{-1}|\sqrt{\frac{\delta}{2\alpha K_1}} + |A_{21}^i||T_R^{-1}T_M| + |A_{22}^i|\right) \\ &< 1, \quad i = 1, 2, \ldots, m \end{aligned} \tag{32}$$

and

$$3m\left(|T_R^{-1}|\sqrt{\frac{\delta}{2\alpha K_1}} + |T_R^{-1}T_M|\right) < 1. \tag{33}$$

Consider a third-order linear oscillator

$$\begin{aligned} &\dot{x}_1 = -0.1x_3, \quad \dot{x}_2 = 20x_1 - x_2, \quad \dot{x}_3 = 20x_1 \\ &y^1 = x_2, \qquad y^2 = x_3. \end{aligned} \tag{34}$$

A reduced-order Luenberger observer is designed with $F = -2$ and $H = \begin{bmatrix} 1 & 2 \end{bmatrix}$. Inequality (29) holds true with $P = K_1 = K_2 = 3$, $\alpha = 1$, and $\delta = 7.5$. We conclude from (32) and (33) that there exists a multirate sampled-data observer for system (34) provided that the maximum sampling period $r < 0.039$. Notice that conditions (32) and (33) are very conservative. Indeed, simulations show that the stability and robustness of the multirate observer will be preserved under much larger sampling period. The actual sampling subsequences of y^1 and y^2 are as follows:

$$t_j^1 = \{0.40, 2.15, 4.05, 5.88, 7.70\}$$

$$t_j^2 = \{0.20, 1.07, 2.17, 3.02, 4.07, 5.13, 6.20, 7.07\}$$

where perturbations in the sampling schedule are considered.

Fig. 2 shows the performance of the multirate sampled-data observer with initial conditions $x(0) = \begin{bmatrix} 0 & 0.5 & 0.3 \end{bmatrix}'$, $\hat{x}_1(0) = -0.5$, $w^1(0) = 3$, and $w^2(0) = -7.2$. It is clear that the observer provides reliable estimation of the state.

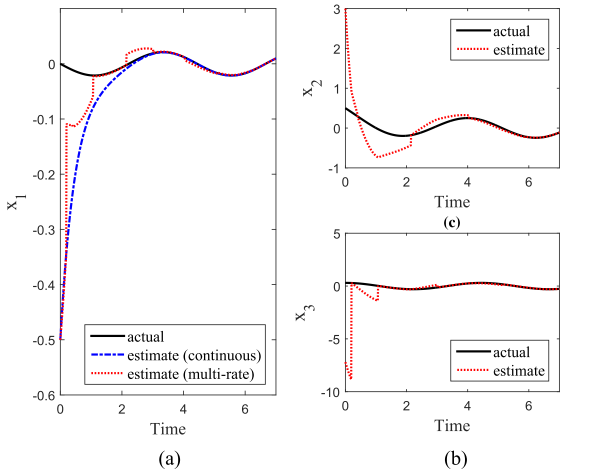

Fig. 2. The oscillator example: (a) Comparison of the multirate sampled-data observer with a continuous-time observer using continuous measurements. (b) and (c) Performance of the intersample output predictors for sampled measurements y^1 and y^2, respectively.

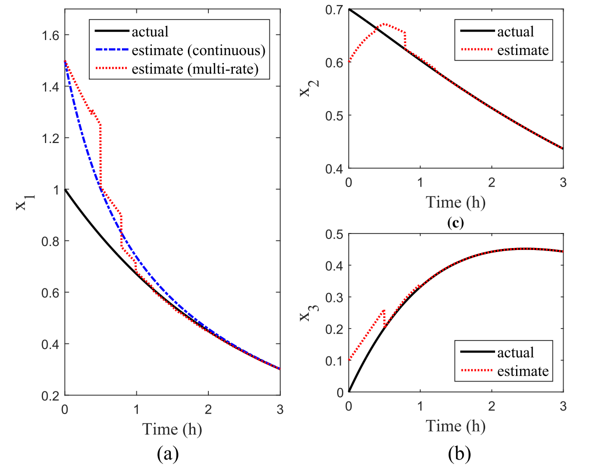

Fig. 3. The reactor example: (a) Comparison of the multirate sampled-data observer with a continuous-time observer using continuous measurements. (b) and (c) Performance of the intersample output predictors for sampled measurements y^1 and y^2, respectively.

B. Batch Chemical Reactor

Consider an isothermal batch reactor, where the following series reactions are taking place:

$$M \to P$$
$$2P \to Q \to R.$$

The reaction rates of M, P, and Q are assumed to be

$$r_M = -k_1 C_M, \qquad r_P = k_1 C_M - k_2 {C_P}^2$$
$$r_Q = k_2 {C_P}^2 - k_3 C_Q$$

where $k_1 = 0.4\,\mathrm{h}^{-1}$, $k_2 = 1$ L/(mol·h), and $k_3 = 0.5\,\mathrm{h}^{-1}$. The concentrations of P and Q can be measured by online analytical instruments, with different sampling rates. Let x_1, x_2, and x_3 represent the concentrations of M, P, and Q, respectively. The state-space model is given by

$$\dot{x}_1 = -0.4x_1, \qquad \dot{x}_2 = 0.4x_1 - {x_2}^2, \qquad \dot{x}_3 = {x_2}^2 - 0.5x_3$$
$$y^1(t_j^1) = x_2(t_j^1), \qquad y^2(t_j^2) = x_3(t_j^2), \qquad j \in \mathbb{Z}_+.$$

Sampling normally occurs every 0.4 h for y^1 and every 0.5 h for y^2. However, perturbations in the sampling schedule are considered and the actual sampling subsequences of y^1 and y^2 for $j = 1, 2, \ldots, 8$ are as follows:

$$t_j^1 = \{0.38, 0.79, 1.23, 1.60, 1.98, 2.41, 2.82, 3.20\}$$
$$t_j^2 = \{0.50, 0.99, 1.52, 2.01, 2.48, 3.01, 3.52, 4.01\}$$

A continuous-time observer, which serves as the basis of the multirate sampled-data observer, will be designed by using the exact error linearization method (see [5] for the full-order observer formulation, and [35] for the reduced-order observer formulation) as follows:

$$\dot{z} = Az + B\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{35}$$

where $A = -2$ and $B = \begin{bmatrix} 2 & 1.5 \end{bmatrix}$. The immersion map $z = T(x)$ satisfies

$$\frac{\partial T}{\partial x_1} f_1(x) + \frac{\partial T}{\partial x_2} f_2(x) + \frac{\partial T}{\partial x_3} f_3(x) = AT + B\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \tag{36}$$

which admits a global solution

$$T(x) = -0.25x_1 + x_2 + x_3 \tag{37}$$

which is solvable with respect to the unmeasured state x_1.

We design a multirate sampled-data observer based on (4) and the performance is shown in Fig. 3 with the initial conditions: $x(0) = \begin{bmatrix} 1 & 0.7 & 0 \end{bmatrix}'$, $\hat{x}_1(0) = 1.5$, $w^1(0) = 0.6$, and $w^2(0) = 0.1$. Fig. 3(a) shows that the speed of convergence of the multirate sampled-data observer and the continuous-time observer is comparable under the selected design parameters. From Fig. 3(b) and (c), the intersample predictors are able to predict the intersample behavior with high accuracy after a few samplings by using model-based prediction.

V. Conclusion

This paper developed a design method for nonlinear multirate sampled-data observer based on an available continuous-time design, coupled with intersample predictors. The main contributions are as follows.

i) The IOS property was established for the estimation and prediction errors with respect to measurement errors.
ii) The multirate design can handle nonuniform and asynchronous sampling without any assumption on the ratio of sampling periods to be an integer, as seen in the oscillator and reactor examples.
iii) As long as the maximum sampling period does not exceed a certain limit, the error dynamics of the proposed multirate observer is input-to-output stable, irrespective of perturbations in the sampling schedule.

The major contributions of the proposed hybrid observer over an approximate discrete-time observer are (ii) and (iii). Checkable sufficient conditions for stability and robustness were derived for linear detectable systems.

The theoretical framework of this study refers to a global observer design. For general nonlinear systems, a local observer formulation is of great interest. In addition, measurement delay and output feedback control based on a multirate observer will also be considered in our future work.

Acknowledgment

The authors would like to thank Prof. I. Karafyllis for his insightful discussions on the system-theoretic framework for hybrid uncertain systems.

References

[1] A. J. Krener and A. Isidori, "Linearization by output injection and nonlinear observers," *Syst. Control Lett.*, vol. 3, no. 1, pp. 47–52, 1983.

[2] M. Zeitz, "The extended Luenberger observer for nonlinear systems," *Syst. Control Lett.*, vol. 9, no. 2, pp. 149–156, 1987.

[3] J. P. Gauthier, H. Hammouri, and S. Othman, "A simple observer for nonlinear systems applications to bioreactors," *IEEE Trans. Autom. Control*, vol. 37, no. 6, pp. 875–880, Jun. 1992.

[4] H. Michalska and D. Q. Mayne, "Moving horizon observers and observer-based control," *IEEE Trans. Autom. Control*, vol. 40, no. 6, pp. 995–1006, Jun. 1995.

[5] N. Kazantzis and C. Kravaris, "Nonlinear observer design using Lyapunov's auxiliary theorem," *Syst. Control Lett.*, vol. 34, no. 5, pp. 241–247, 1998.

[6] M. Moarref and L. Rodrigues, "Observer design for linear multi-rate sampled-data systems," in *Proc. Amer. Control Conf.*, 2014, pp. 5319–5324.

[7] C. Ling and C. Kravaris, "Multi-rate observer design for process monitoring using asynchronous inter-sample output predictions," *AIChE J.*, vol. 63, no. 8, pp. 3384–3394, 2017.

[8] A. H. Jazwinski, *Stochastic Processes and Filtering Theory*. Chelmsford, MA, USA: Courier Corp., 2007.

[9] M. Nadri and H. Hammouri, "Design of a continuous-discrete observer for state affine systems," *Appl. Math. Lett.*, vol. 16, no. 6, pp. 967–974, 2003.

[10] M. Nadri, H. Hammouri, and C. Astorga, "Observer design for continuous-discrete time state affine systems up to output injection," *Eur. J. Control*, vol. 10, no. 3, pp. 252–263, 2004.

[11] T. Ahmed-Ali, R. Postoyan, and F. Lamnabhi-Lagarrigue, "Continuous–discrete adaptive observers for state affine systems," *Automatica*, vol. 45, no. 12, pp. 2986–2990, 2009.

[12] F. Deza, E. Busvelle, J. Gauthier, and D. Rakotopara, "High gain estimation for nonlinear systems," *Syst. Control Lett.*, vol. 18, no. 4, pp. 295–299, 1992.

[13] H. Hammouri, M. Nadri, and R. Mota, "Constant gain observer for continuous-discrete time uniformly observable systems," in *Proc. 45th IEEE Conf. Decis. Control*, 2006, pp. 5406–5411.

[14] M. Nadri, H. Hammouri, and R. M. Grajales, "Observer design for uniformly observable systems with sampled measurements," *IEEE Trans. Autom. Control*, vol. 58, no. 3, pp. 757–762, Mar. 2013.

[15] M. Farza, M. M'Saad, M. L. Fall, E. Pigeon, O. Gehan, and K. Busawon, "Continuous-discrete time observers for a class of MIMO nonlinear systems," *IEEE Trans. Autom. Control*, vol. 59, no. 4, pp. 1060–1065, Apr. 2014.

[16] M. Farza, I. Bouraoui, T. Menard, R. B. Abdennour, and M. M'Saad, "Adaptive observers for a class of uniformly observable systems with nonlinear parametrization and sampled outputs," *Automatica*, vol. 50, no. 11, pp. 2951–2960, 2014.

[17] V. Andrieu, M. Nadri, U. Serres, and J.-C. Vivalda, "Self-triggered continuous–discrete observer with updated sampling period," *Automatica*, vol. 62, pp. 106–113, 2015.

[18] T. Raff, M. Kögel, and F. Allgöwer, "Observer with sample-and-hold updating for Lipschitz nonlinear systems with nonuniformly sampled measurements," in *Proc. Amer. Control Conf.*, 2008, pp. 5254–5257.

[19] T. N. Dinh, V. Andrieu, M. Nadri, and U. Serres, "Continuous-discrete time observer design for Lipschitz systems with sampled measurements," *IEEE Trans. Autom. Control*, vol. 60, no. 3, pp. 787–792, Mar. 2015.

[20] H. Beikzadeh and H. J. Marquez, "Multirate observers for nonlinear sampled-data systems using input-to-state stability and discrete-time approximation," *IEEE Trans. Autom. Control*, vol. 59, no. 9, pp. 2469–2474, Sep. 2014.

[21] H. Beikzadeh and H. J. Marquez, "Sampled-data observer for one-sided Lipschitz systems: Single-rate and multirate cases," in *Proc. Amer. Control Conf.*, 2015, pp. 3386–3391.

[22] Y. Shen, D. Zhang, and X. Xia, "Continuous observer design for a class of multi-output nonlinear systems with multi-rate sampled and delayed output measurements," *Automatica*, vol. 75, pp. 127–132, 2017.

[23] I. Karafyllis and C. Kravaris, "From continuous-time design to sampled-data design of observers," *IEEE Trans. Autom. Control*, vol. 54, no. 9, pp. 2169–2174, Sep. 2009.

[24] T. Ahmed-Ali, I. Karafyllis, and F. Lamnabhi-Lagarrigue, "Global exponential sampled-data observers for nonlinear systems with delayed measurements," *Syst. Control Lett.*, vol. 62, no. 7, pp. 539–549, 2013.

[25] E. D. Sontag and Y. Wang, "Notions of input to output stability," *Syst. Control Lett.*, vol. 38, no. 4, pp. 235–248, 1999.

[26] I. Karafyllis, "A system-theoretic framework for a wide class of systems I: Applications to numerical analysis," *J. Math. Anal. Appl.*, vol. 328, no. 2, pp. 876–899, 2007.

[27] I. Karafyllis, "A system-theoretic framework for a wide class of systems II: Input-to-output stability," *J. Math. Anal. Appl.*, vol. 328, no. 1, pp. 466–486, 2007.

[28] I. Karafyllis and Z.-P. Jiang, "A small-gain theorem for a wide class of feedback systems with control applications," *SIAM J. Control Optim.*, vol. 46, no. 4, pp. 1483–1517, 2007.

[29] I. Karafyllis and Z.-P. Jiang, "A vector small-gain theorem for general non-linear control systems," *IMA J. Math. Control Inf.*, vol. 28, no. 3, pp. 309–344, 2011.

[30] H. K. Khalil, *Nonlinear Systems*. 2nd ed. Upper Saddle River, NJ, USA: Prentice-Hall, 1996.

[31] D. Angeli and E. D. Sontag, "Forward completeness, unboundedness observability, and their Lyapunov characterizations," *Syst. Control Lett.*, vol. 38, no. 4, pp. 209–217, 1999.

[32] S. N. Dashkovskiy, B. S. Rüffer, and F. R. Wirth, "An ISS small gain theorem for general networks," *Math. Control, Signals, Syst.*, vol. 19, no. 2, pp. 93–122, 2007.

[33] S. N. Dashkovskiy, B. S. Rüffer, and F. R. Wirth, "Small gain theorems for large scale systems and construction of ISS Lyapunov functions," *SIAM J. Control Optim.*, vol. 48, no. 6, pp. 4089–4118, 2010.

[34] Z.-P. Jiang and Y. Wang, "A generalization of the nonlinear small-gain theorem for large-scale complex systems," in *Proc. 7th World Congr. Intell. Control Autom.*, 2008, pp. 1188–1193.

[35] N. Kazantzis, C. Kravaris, and R. A. Wright, "Nonlinear observer design for process monitoring," *Ind. Eng. Chem. Res.*, vol. 39, no. 2, pp. 408–419, 2000.