Control oriented modeling of soft robots: the polynomial curvature case

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***Abstract*—The complex nature of soft robot dynamics calls for the development of models specifically tailored on the control application. In this paper we take a first step in this direction, by proposing a dynamic model for slender soft robots taking into account the fully infinite-dimensional dynamical structure of the system. We also contextually introduce a strategy to approximate this model at any level of detail through a finite dimensional system. First, we analyze the main mathematical properties of this model - in the case of lightweight and non lightweight soft robots. Then, we prove that using the constant term of curvature as control output produces a minimum phase system, in this way providing the theoretical support that existing curvature control techniques lack, and at the same time opening up to the use of advanced nonlinear control techniques. Finally, we propose a new controller, the PD-poly, which exploits information on high order deformations, to achieve zero steady state regulation error in presence of gravity and generic non constant curvature conditions.**

***Index Terms*—Modeling, Control, and Learning for Soft Robots; Motion Control; Dynamics.**

I. INTRODUCTION

In soft robots, the standard paradigm of stiff robotics is reverted by artificial bodies fully made of continuously deformable and compliant materials [1]. This design choice endows soft robots with several advantages, as being inherently safe in the interaction with humans and other animals, or the ability of squeezing within narrow spaces. This however comes at the price of making much harder the development of effective model based algorithms for managing these systems. This gap has been filled so far mostly and partially by learning based strategies [2]. Indeed, understanding the behavior of these systems from a more fundamental mathematical point of view has proven to be quite difficult, the main reason being the continuum nature of the problem.

In the last few years, a lot of effort has thus been put into developing finite dimensional approximations of soft robot's kinematics and dynamics. Among them, piecewise constant curvature models proved to work well in the practice - despite clearly being an over-simplification of the problem - with applications in kinematic control [3], feedforward dynamic control [4], feedback dynamic control [5]. However, the research in modeling soft robots goes now much further than constant curvature approximations, with fascinating theoretical and experimental results [6]–[10]. It is worth underlying that the present paper does not aim to compete with existing models for accurately or efficiently simulate the robot - task for which these methods are well suited. Indeed, in soft robotics the complex nature of the modeling problem calls for a distinction between models developed for simulating the robot, and models for model-based control. The goals of a good model for simulation are such as maximum accuracy, numerical efficiency, numerical stability. The works cited above very successfully go in this direction.

We aim instead at making a first step toward developing a formulation of the reduced dynamics that is better suited for designing model based controllers, and for assessing the theoretical properties of open loop and closed loop systems. An approach going in this direction is proposed in [11], [12], where the order of very high dimensional FEM discretization of the robot is reduction to achieve model based regulation. While promising, this approach lacks of interpretability of the results, and so far of nonlinear formulations.

This paper proposes a new modeling approach that, in continuity with constant curvature techniques, we call polynomial curvature model. Instead of operating a spatial discretization - as done by the above discussed techniques - we express the curvature function of the robot in the standard polynomial base of the Hilbert space. Each continuum shape is in this way expressed by an infinite dimensional vector having as first element the constant approximation of the curvature, as second the linear approximation, and so on. This allows for an exact infinite dimensional formulation of the problem, and at the same time it provides an easy way of approximating at any level of accuracy by order truncation. In doing so, we take inspiration from assumed mode technique, which has been developed for and successfully used in control-oriented modeling [13], and control [14] of flexible link robots. We discuss the structural properties of this model and we apply it for developing a high order curvature regulator. To conclude, this paper contributes with

- A new modeling technique suited for control purposes, allowing for a formulation of the dynamics with any level of precision - up to infinity.
- An in depth analysis of the main properties of this model.
- The proof that using the constant curvature of the robot as control output produces a minimum phase system.
- The PD-poly, a controller - extending the classic PD regulator - which can reach perfect steady state control in non constant curvature conditions by using the full knowledge of robot's shape.

II. GENERAL DEFINITIONS

Consider a planar soft robot with a rod-like shape. We call $L \in \mathbb{R}^+$ its length. We parametrize points along the robot through a normalized coordinate $s \in [0,1]$, such as the point at coordinate s is sL far from the base of the robot, with distance measured along the robot itself. One end of the

Manuscript received: September, 15, 2018; Accepted November, 12, 2019.
This paper was recommended for publication by Editor Kyu-Jin Cho upon evaluation of Reviewers' comments. This work was supported by the NSF under Grant Agreement NSF EFRI 1830901, and NSF 1226883. We are grateful for this support.

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Digital Object Identifier (DOI): see top of this page.

TABLE I
LIST OF MAIN SYMBOLS USED IN THE PAPER

Symbol	Meaning
$s \in [0,1]$	Coordinate along the robot
$L \in \mathbb{R}^+$	Robot's length
$\rho \in \mathbb{R}^+$	Robot's density
$\phi \in [0, 2\pi]$	Angle between the base and the gravity acceleration
$\tau : \mathbb{R}^+ \to \mathbb{R}$	Input torque applied in $s = 1$
$\kappa : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$	Local curvature at a given location and a given time
$\alpha : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$	Local orientation at a g.l. & g.t.
$(x,y) : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^2$	Cartesian coordinates at a g.l. & g.t.
$\boldsymbol{K}$	Sequence space of functions mapping $\mathbb{N} \times \mathbb{R}^+$ into $\mathbb{R}$
$\boldsymbol{M}$	Space of bounded linear operators mapping $\boldsymbol{K}$ into itself
$\Theta \in \boldsymbol{K}$	Modal configuration at a g.t.
$B : \boldsymbol{K} \to \boldsymbol{M}$	Inertia of the robot
$C : \boldsymbol{K} \times \boldsymbol{K} \to \boldsymbol{K}$	Coriolis and centrifugal forces
$K \in \boldsymbol{M}$	Stiffness in modal space
$D \in \boldsymbol{M}$	Damping in modal space
$A \in \boldsymbol{K}$	Input field
$m \in \mathbb{N}$	Order of the approximation
$[\,\cdot\,]_m : \boldsymbol{M} \to \mathbb{R}^{m+1 \times m+1}$	Matrix truncation
$[\,\cdot\,]_m : \boldsymbol{K} \to \mathbb{R}^{m+1}$	Vector truncation
$\theta_i \doteq \Theta(i,\cdot) : \mathbb{R}^+ \to \mathbb{R}$	$i-$th coordinate in modal space at a given time
$\theta \doteq [\Theta]_m : \mathbb{R}^+ \to \mathbb{R}^{m+1}$	Reduced dimensionality description of the robot at a given time

robot is fixed to an inertial system. This end has coordinate $s = 0$, the other end $s = 1$. We assume constant (normalized) density ρ through the robot length. We also considered the robot inextensible. Both the hypotheses are made for the sake of space and will be relaxed in future work.

The soft robot pose is fully specified by the curvature[1] function $\kappa(s,t)$, which we consider as configuration variable of the robot. We hypothesize this function to be analytic in s. Under this hypothesis, we can express it as an infinite expansion of monomials

$$\kappa(s,t) = \sum_{i=0}^{\infty} \theta_i(t) s^i \,. \tag{1}$$

We call $\theta_i \in \mathbb{R}^n$ modal component of order i. We also introduce $\Theta \in \boldsymbol{K}$ as the infinite sequence having θ_i as $i-$th element. This function provides an exact description of the robot shape, mathematically equivalent to κ itself. The core idea of this paper is to build a kinematic and dynamic description of the robot in Θ rather than in some space discretization, as typically done in the literature.

Table I reports a list of main symbols used through the paper.

A. Kinematics

The local orientation of the robot is obtained by direct integration of the curvature

$$\alpha(v,t) = \int_0^v \kappa(s,t) \mathrm{d}s = \sum_{i=0}^{\infty} \theta_i(t) \frac{v^{i+1}}{i+1}. \tag{2}$$

[1]Strictly speaking κ is the inverse of the curvature, and it is sometimes referred as to bending angle. We consider this variable instead of the actual curvature for reasons that can not be discussed here for the sake of space. This choice is done w.l.o.g. since the two are connected by a bijective map.

Knowing the angle function, the Cartesian coordinates of each point along the robot can be evaluated as

$$\frac{x(v,t)}{L} = \int_0^v \cos(\alpha(s,t))\,\mathrm{d}s, \quad \frac{y(v,t)}{L} = \int_0^v \sin(\alpha(s,t))\,\mathrm{d}s. \tag{3}$$

B. Finite dimension approximation and truncation operators

Working in Θ coordinates has the strong advantage of allowing for a finite dimensional approximation of the curvature in the form

$$\kappa(s,t) \simeq \sum_{0}^{m} \theta_i(t) s^i \,. \tag{4}$$

Note that θ_0 is the constant approximation of the robot's curvature. Thus the proposed model for $m = 0$ will be equivalent to the classic constant curvature model. Higher values of m will provide increasingly more precise finite dimensional descriptions of the robot's shape. So we call the models that can be obtained for any given value of m polynomial curvature models. The exact description can then be recovered as limit for m going to infinity.

We call $\theta \in \mathbb{R}^{m+1}$ the (approximated) modal description of the robot, having θ_i as $i-$th element. As $\Theta, \dot{\Theta}$ is the state of the exact model, $\theta, \dot{\theta}$ will serve as state of the model that can be derived by assuming (4).

To move from the infinite dimensional model to the finite dimensional one, we introduce the truncation operators $[\,\cdot\,]_m : \boldsymbol{M} \to \mathbb{R}^{m+1 \times m+1}$, $[\,\cdot\,]_m : \boldsymbol{K} \to \mathbb{R}^{m+1}$. They map infinite dimensional linear operators and vectors to their finite dimensional counterparts, by selecting the elements with indexes less or equal to m, and putting $\theta_{i>m} \equiv 0$. We use the same symbol for both operators to simplify the notation. In the following we will derive the model in the infinite dimensional case, extracting the finite dimensional representation when needed.

III. MODEL IN THE LIGHTWEIGHT CASE

In this section we derive all the non mass-related terms of the model, i.e. elastic forces, actuation, dissipative actions.

A. Elastic field

We consider a linear elastic field. The energy stored in the whole robot is $U_{\mathrm{E}}(t) = \int_0^1 \frac{k}{2} \kappa^2(s,t)\,\mathrm{d}s$, where k is the flexure rigidity of the rod. The elastic force field is evaluated as follows

$$\begin{aligned} G_{\mathrm{E},k}(\Theta) &= \frac{k}{2} \int_0^1 \frac{\partial}{\partial \theta_k} \kappa^2(s,\Theta)\,\mathrm{d}s = k \int_0^1 \kappa \frac{\partial}{\partial \theta_k} \kappa\,\mathrm{d}s \\ &= k \int_0^1 \left(\sum_{i=0}^{\infty} \theta_i s^i \right) s^k\,\mathrm{d}s = k \sum_{i=0}^{\infty} \theta_i \int_0^1 s^{i+k}\,\mathrm{d}s \\ &= k \sum_{i=0}^{\infty} \frac{1}{i+k+1} \theta_i \,. \end{aligned} \tag{5}$$

Thus the elastic field is linear in Θ and equal to $G_{\mathrm{E}}(\Theta) = K\Theta$ with $K \in \boldsymbol{M}$ infinite dimensional matrix with element (i,j) equal to

$$K_{i,j} = \frac{k}{i+j+1}. \tag{6}$$

This is an Hankel operator, and all its truncations $[K]_m$ are Hankel matrices [15].

Lemma 1. $\det([K]_m) \neq 0,\ \forall\ m \in \mathbb{N},\ k > 0.$

Proof. First, we recognize that $\det(\frac{1}{k}[K]_m)$ is a special case of the Cauchys double alternant (see for example [16, Sec. 2.1]), defined as

$$\det_{1\leq i,j\leq m}\left(\frac{1}{X_i+Y_j}\right) = \frac{\prod_{1\leq i<j\leq m}(X_i-X_j)(Y_i-Y_j)}{\prod_{1\leq i,j\leq m}(X_i+Y_j)}, \quad (7)$$

where X_i and Y_j are elements of two generic sequences of numbers. Considering $X_i = i$ and $Y_j = j+1$ yields

$$\begin{aligned}\det\left(\frac{1}{k}[K]_m\right) &= \det_{1\leq i,j\leq m+1}\left(\frac{1}{i+j+1}\right)\\ &= \frac{\prod_{1\leq i<j\leq m+1}(i-j)^2}{\prod_{1\leq i,j\leq m+1}(i+j+1)} > 0\ .\end{aligned} \quad (8)$$

The last step holds by considering that all the terms involved in the products are strictly positive. The lemma follows by considering that $\det(\frac{1}{k}[K]_m) = \frac{1}{k^m}\det([K]_m)$. □

Corollary 1. $[K]_m \succ 0,\ \forall\ m \in \mathbb{N},\ k > 0.$

Proof. The corollary follows by direct application of Sylvester's criterion [17], which states that a matrix is positive defined if and only if all its leading principal minors are positive, which is assured by (8) in Lemma 1. □

These results prove that $[K]_m$ is a well defined stiffness matrix, and they will be used later in the paper.

B. Input field

We contemplate a generic number of inputs applied along the robot structure. As typically hypothesized in the literature [18], [19], we consider here actuations in the form of a pure torque. To express the effect of a single action applied at coordinate s_{a}, we evaluate the Jacobian mapping $\dot{\Theta}$ to the derivative of the orientation in that location

$$\dot{\alpha}(s_{\mathrm{a}},t) = \sum_{i=0}^{\infty} \left.\frac{\partial \alpha(l,t)}{\partial \theta_i}\right|_{l=s_{\mathrm{a}}} \dot{\theta}_i = A^{\mathrm{T}}(s_{\mathrm{a}})\dot{\Theta}\ , \quad (9)$$

with $A \in \boldsymbol{K}$ being the transpose of the infinite dimensional Jacobian with $i-$th element equal to

$$A_i(s_{\mathrm{a}}) = \frac{s_{\mathrm{a}}^{i+1}}{i+1}\ . \quad (10)$$

We can now exploit kineto-static duality, to say that $A(s_{\mathrm{a},j})$ serves as an operator mapping the input τ_j in generalized forces in the modal space. The overall input action is thus $\sum A(s_{\mathrm{a},j})\tau_j$. For the sake of conciseness we consider in the following a single actuation acting at the tip of the robot, i.e. $A(1)\tau$. The argument will be omitted.

C. Damping forces

In analogy with the elastic field, we consider a damping force acting proportionally to the derivative of curvature. The force exerted on the infinitesimal element at coordinate s is $d\dot{\kappa}(s,\Theta) = d\sum_0^\infty \dot{\theta}_i s^i = dJ(s)\dot{\Theta}$, with $J(s)$ infinite dimensional Jacobian with $i-$th element equal to $J_i(s) = s^i$. This generalized force produces an equivalent action in Θ equal to $dJ^{\mathrm{T}}J\dot{\Theta}$ - evaluated using kinetostatic duality. The overall damping force is the sum of all the local effects, and can thus be evaluated by integrating the infinitesimal terms

$$\int_0^1 dJ^{\mathrm{T}}(s)J(s)\dot{\Theta}\mathrm{d}s = D\dot{\Theta}\ , \quad (11)$$

with D infinite dimensional matrix with element (i,j) equal to

$$D_{i,j} = \frac{d}{i+j+1}. \quad (12)$$

Note that D is equal to K, with the exception of a multiplicative constant. It is thus a Hankel operator too, and the following Corollaries hold.

Corollary 2. $\det([D]_m) \neq 0,\ \forall\ m \in \mathbb{N},\ d > 0.$

Proof. The proof adheres to the same arguments followed in proving Lemma 1. □

Corollary 3. $[D]_m \succ 0,\ \forall\ m \in \mathbb{N},\ d > 0.$

Proof. The proof adheres to the same arguments followed in proving Corollary 1. □

These results prove that $[D]_m$ is a well defined damping matrix, and they will be used later in the paper.

D. Overall model and steady state behavior

Combining results of the previous subsections yields the following infinite dimensional mechanical system

$$D\dot{\Theta} = -K\Theta + A\tau, \quad (13)$$

where $D, K \in \boldsymbol{M}$ are the damping and stiffness operators, and $A \in \boldsymbol{K}$ is the input field - as introduced in (6),(10),(12). This is an exact description of the system under the assumption that inertia-related forces are negligible, i.e. lightweight robot. It is worth underlying that this system is linear. Furthermore, the following lemma holds.

Lemma 2. *The system*

$$[D]_m\dot{\theta} = -[K]_m\theta + [A]_m\tau \quad (14)$$

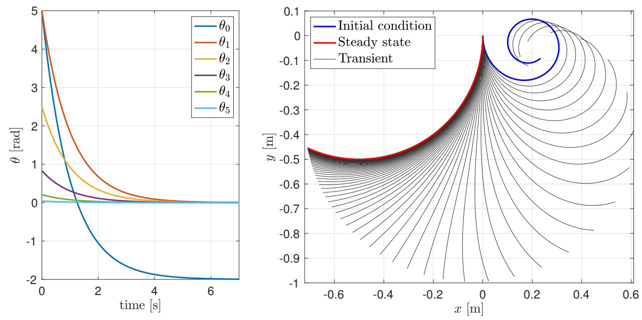
has the following unique globally asymptotically stable equilibrium; $\theta_0 = \frac{\tau}{k}$ *and* $\theta_i = 0,\ \forall i \neq 0,\ \forall m > 0.$

Proof. First note that since Lemma 1 holds, the solution of the equilibrium problem $[K]_m\theta = [A]_m\tau$ is unique for all τ. To prove that $\theta_0 = \frac{\tau}{k}$ and $\theta_i = 0$ is the equilibrium, it is sufficient to recognize that $[A]_m$ is the first column of $[K]_m$ scaled of a factor $\frac{1}{k}$. To prove the asymptotic stability we rewrite (14) as

$$\dot{\theta} = -[D]_m^{-1}[K]_m\theta + [D]_m^{-1}[A]_m\tau\ , \quad (15)$$

which is always possible thanks to Corollary 2. Finally the following holds $[D]_m^{-1}[K]_m \succ 0$, since $[D]_m, [K]_m \succ 0$ (Corollaries 1 and 3), and inverse and products of positive defined matrices are positive defined [20]. This assures the global asymptotic stability of the equilibrium, being the system linear. □

The thesis of Lemma 2 tells us that a pure torque produces steady state constant curvature behavior, as also predicted by classic theories in continuum mechanics.



(a) Modal coordinates (b) Cartesian

Fig. 1. Open loop lightweight case, with $\kappa(s,0) = 5e^s$, $\tau = -2$Nm, $\rho \simeq 0$, $k = 1$Nm, and $d = 1$Nms. The solution is exact, i.e. no truncation is applied. Panel (a) shows the first six components only because higher order ones would have been too small to be seen.

Corollary 4. *Eq.* (14) *is equivalent to the set of decoupled first order linear systems*

$$\dot{\theta}_0 = -\frac{k}{d}\theta_0 + \frac{1}{d}\tau\,, \quad \dot{\theta}_i = -\frac{k}{d}\theta_i\,, \quad \forall i > 0\,. \tag{16}$$

Proof. The proof follows directly from (15) by recognizing that $D = \frac{d}{k}K$, and by considering that the scalar multiplying τ should be such that the equilibrium is the one predicted by Lemma 2. □

Example 1. *Consider a soft limb with* $\rho \simeq 0$, $k = 1$Nm, *and* $d = 1$Nms*. The system starts from the initial condition* $\kappa(s,0) = 5e^s$*, which corresponds to the exact modal description* $\theta_i(0) = \frac{5}{i!}$, $\forall i$*. According to* (1) *and* (16) *the exact system evolution is*

$$\kappa(s,t) = \sum_0^\infty 5e^{-t}\frac{s^i}{i!} + \tau(1-e^{-t}) = 5e^{s-t} + \tau(1-e^{-t})\,.$$

Fig. 1 shows this evolution when $\tau = -2$Nm *and* $L = 1$m.

IV. Model in the Case of Non Negligible Mass

A. On the integrability of Cartesian coordinates

Eq. (3) contains integrals of trigonometric functions of polynomials, which are well know for not being integrable in closed form. This issue prevents an exact and analytical dynamical description of the general case, for reasons that will become later clear in the paper. In the following we will adhere to two parallel paths

i) Exact non analytical case; the integrals can be evaluated numerically at any level of precision. This enables using the exact expression of the model and of model-based controllers, but it makes harder to provide general theoretical results.

ii) Analytical approximated solution; as direct extension of the constant curvature case to the polynomial world, we introduce the following hypothesis of constant curvature dominance

$$\theta_0 >> \sum_{i=1}^{\infty}\frac{\theta_i}{i+1}. \tag{17}$$

We will explicitly state any time this hypothesis is used. Having closed form solutions, even if of local validity, helps in proving general theoretical results.

Under hypothesis (17), the Cartesian coordinates of each point of the robot can be approximated using a first order Taylor expansion

$$\begin{aligned}\frac{x(s,\Theta)}{L} &\simeq \left(\int_0^s \cos(\theta_0 l)\mathrm{d}l - \sum_{i=1}^{\infty}\frac{\theta_i}{i+1}\int_0^s l^i\sin(\theta_0 l)\mathrm{d}l\right),\\ \frac{y(s,\Theta)}{L} &\simeq \left(\int_0^s \sin(\theta_0 l)\mathrm{d}l + \sum_{i=1}^{\infty}\frac{\theta_i}{i+1}\int_0^s l^i\cos(\theta_0 l)\mathrm{d}l\right).\end{aligned} \tag{18}$$

Both terms of each sum have now solution in closed form, the first by direct integration and the second by iterative integration by parts. The result is

$$\begin{aligned}\frac{x(s,\Theta)}{L} \simeq\ & \frac{\sin(\theta_0 s)}{\theta_0}\\ &+ \sum_{i=1}^{\infty}\frac{\theta_i}{i+1}\left[\sum_{k=0}^{i}\frac{i!}{(i-k)!}\frac{s^{i-k}}{\theta_0^{k+1}}\cos(\theta_0 s + k\frac{\pi}{2})\right.\\ &\left. - \frac{i!}{\theta_0^{i+1}}\cos(i\frac{\pi}{2})\right],\\ \frac{y(s,\Theta)}{L} \simeq\ & \frac{1-\cos(\theta_0 s)}{\theta_0}\\ &+ \sum_{i=1}^{\infty}\frac{\theta_i}{i+1}\left[\sum_{k=0}^{i}\frac{i!}{(i-k)!}\frac{s^{i-k}}{\theta_0^{k+1}}\sin(\theta_0 s + k\frac{\pi}{2})\right.\\ &\left. - \frac{i!}{\theta_0^{i+1}}\sin(i\frac{\pi}{2})\right].\end{aligned} \tag{19}$$

The first term is the position of a constant curvature robot, and the second is a perturbation. Note that the perturbation itself is nonlinearly dependent on θ_0.

B. Gravitational field

Call $\phi \in [0, 2\pi]$ the angle that the base of the robot has w.r.t. the gravity field. The total gravitational potential energy is $U_{\mathrm{G}}(t,\phi) = \rho g\,(-\cos(\phi)C_{\mathrm{x}} + \sin(\phi)C_{\mathrm{y}})$, where $C_{\mathrm{x}}, C_{\mathrm{y}}$ are the Cartesian coordinates of the center of mass. Their dependency on κ is not reported for the sake of space. ρ is the normalized density of the robot - equal to the mass of the robot - and g is the gravity acceleration constant. The overall gravity force field is obtained through derivation

$$G_{\mathrm{G}} = \frac{\partial U_{\mathrm{G}}}{\partial \theta} = \rho g\left(\sin(\phi)\int_0^1 \nabla_{\mathrm{y}}^{\mathrm{T}}\mathrm{d}s - \cos(\phi)\int_0^1 \nabla_{\mathrm{x}}^{\mathrm{T}}\mathrm{d}s\right), \tag{20}$$

where we substituted the explicit definition of C_x and C_y, and where ∇_{x} and ∇_{y} are row vectors, with i-th element $\frac{\partial x}{\partial \theta_i}$ and $\frac{\partial y}{\partial \theta_i}$ respectively.

Under assumption (17), we can obtain the closed form solution in modal coordinates as shown by (21). For the sake of space, we do not report the steps of the derivation, and we take $\phi = 0$. The generic case assumes a similar form. It is easy to see either through manipulations of (20), or through direct inspection of (21), that under assumption (17) the following holds

$$\begin{aligned} G_{\mathrm{G}}(\Theta,\phi) &= G(\theta_0,\phi)\Theta + g_0(\theta_0,\phi),\\ G(\theta_0,\phi) &= \begin{bmatrix} 0 & g_{\mathrm{G},2}(\theta_0,\phi) & g_{\mathrm{G},3}(\theta_0,\phi) & \cdots\\ 0 & 0 & 0 & \cdots\\ \vdots & \vdots & \vdots & \end{bmatrix},\end{aligned} \tag{22}$$

where Θ is the modal description of the robot's shape, θ_0 is the constant curvature component of Θ. $g_0 : \mathbb{R}\times[0,2\pi] \to \boldsymbol{K}$

$$G_{\mathrm{G},0} = L\frac{\sin(\theta_0)\theta_0 - 2(1-\cos(\theta_0))}{\theta_0^3} + \frac{L}{\theta_0^2}\sum_{i=1}^{\infty}\frac{\theta_i}{i+1}\left\{\frac{(i+1)!}{\theta_0^i}\cos(i\frac{\pi}{2}) + \sum_{k=0}^{i}\frac{i!}{\theta_0^{k+1}}\left[\sum_{w=0}^{i-k}\frac{i-2k}{\theta_0^{i-k}}\sin(i\frac{\pi}{2})\right.\right.$$
$$\left.\left.+\ \frac{\sin(\theta_0 v + (k+w)\frac{\pi}{2})(k+1) + \theta_0\cos(\theta_0 + (k+w)\frac{\pi}{2}) - (w+1)\sin(\theta_0 v + (k+w)\frac{\pi}{2})}{(i-k-w)!\theta_0^w}\right]\right\}, \tag{21}$$
$$G_{\mathrm{G},i\neq 0} = \sum_{k=0}^{i}\frac{L}{i+1}\frac{i!}{\theta_0^{k+1}}\left[\sum_{w=0}^{i-k}\frac{1}{\theta_0^{w+1}}\frac{1}{(i-k-w)!}\sin(\theta_0 v + (k+w)\frac{\pi}{2}) - \frac{1}{\theta_0^{i-k+1}}\sin(i\frac{\pi}{2})\right] - \frac{L}{i+1}\frac{i!}{\theta_0^{i+1}}\cos(i\frac{\pi}{2}).$$

is a nonlinear force field describing how θ_0 affects itself and higher order modal terms through gravity. Vice versa, $G : \mathbb{R}\times[0,2\pi] \to \boldsymbol{M}$ describes how non constant curvature terms affect the dynamics of θ_0 through the gravity field. It is interesting to notice that non constant curvature terms do not directly affect themselves through gravity.

C. Inertia matrix

The general form of the inertia matrix can be derived from the kinetic energy

$$\begin{aligned} E_\mathrm{k} &= \frac{\rho}{2}\int_0^1 \dot{x}^\mathrm{T}\dot{x} + \dot{y}^\mathrm{T}\dot{y}\ \mathrm{d}s = \frac{1}{2}\dot{\Theta}^\mathrm{T}B(\Theta)\dot{\Theta}, \\ B(\Theta) &= \rho\int_0^1 \nabla_\mathrm{x}^\mathrm{T}\nabla_\mathrm{x} + \nabla_\mathrm{y}^\mathrm{T}\nabla_\mathrm{y}\ \mathrm{d}s, \end{aligned} \tag{23}$$

where ∇_x and ∇_y are defined as in (20). The following general result can be derived

Lemma 3. $[B]_m(\theta) \succeq 0$, $\forall\ m\in\mathbb{N}$, $\bar{\rho}>0$, $\theta\in\mathbb{R}^{m+1}$.

Proof. We test the thesis by applying the definition of positive definiteness, according to which the following must be true for all values of $\xi\in\mathbb{R}^{m+1}$

$$\begin{aligned} &\xi^\mathrm{T}[B]_m\xi \geq 0 \\ \iff& \bar{\rho}\int_0^1 \xi^\mathrm{T}\left(\nabla_\mathrm{x}^\mathrm{T}\nabla_\mathrm{x} + \nabla_\mathrm{y}^\mathrm{T}\nabla_\mathrm{y}\right)\xi\mathrm{d}s \geq 0 \\ \iff& \bar{\rho}\int_0^1 (\nabla_\mathrm{x}\xi)^\mathrm{T}(\nabla_\mathrm{x}\xi) + (\nabla_\mathrm{y}\xi)^\mathrm{T}(\nabla_\mathrm{y}\xi)\,\mathrm{d}s \geq 0 \\ \iff& \bar{\rho}\int_0^1 ||\nabla_\mathrm{x}\xi||^2 + ||\nabla_\mathrm{y}\xi||^2\mathrm{d}s \geq 0, \end{aligned} \tag{24}$$

which always holds true, being the last term an integral of positive elements. □

Again, (23) can be analytically evaluated under hypothesis (17). Note that this derivation is not an obvious one, since it involves integrations. However, it should result clear from a quick analysis of (23), that all the terms to be integrated are polynomials, trigonometric functions, or products of the two. We already provided above examples of integration of these kinds of function. We can not reported here the explicit form of (23) for the sake of space. However, the following general properties can be proven

Lemma 4. *Under hypothesis* (17)*, the elements of* $B(\Theta)$ *and* $[B]_m(\theta)$ *are such that*

i) $B_{i\neq 0,j\neq 0}$ *depends only and nonlinearly on* θ_0*,*
ii) $B_{i\neq 0,0}$ *and* $B_{0,j\neq 0}$ *are affine in* $\theta_{z\neq 0}$*, with coefficients nonlinear in* θ_0*,*
iii) $B_{0,0}$ *is quadratic in* $\theta_{z\neq 0}$*, with coefficients nonlinear in* θ_0.

Proof. The element (i,j) of B is

$$\frac{B_{i,j}}{\rho} = \int_0^1 \frac{\partial x}{\partial\theta_i}\frac{\partial x}{\partial\theta_j} + \frac{\partial y}{\partial\theta_i}\frac{\partial y}{\partial\theta_j}\ \mathrm{d}s. \tag{25}$$

Consider now that both x and y can be written as (see (19))

$$x = \Gamma_\mathrm{x}(\theta_0) + \Lambda_\mathrm{x}(\theta_0)\Theta, \quad y = \Gamma_\mathrm{y}(\theta_0) + \Lambda_\mathrm{y}(\theta_0)\Theta, \tag{26}$$

where $\Gamma_\mathrm{x},\Gamma_\mathrm{y},\Lambda_\mathrm{x},\Lambda_\mathrm{y}$ are functions of θ_0 only, with their opportune dimensions. Thus

$$\begin{aligned} \frac{\partial x}{\partial\theta_0} &= \frac{\partial\Gamma_\mathrm{x}}{\partial\theta_0}(\theta_0) + \frac{\partial\Lambda_\mathrm{x}}{\partial\theta_0}(\theta_0)\Theta, \quad \frac{\partial x}{\partial\theta_{i\neq 0}} = \Lambda_{\mathrm{x},i}(\theta_0) \\ \frac{\partial y}{\partial\theta_0} &= \frac{\partial\Gamma_\mathrm{y}}{\partial\theta_0}(\theta_0) + \frac{\partial\Lambda_\mathrm{y}}{\partial\theta_0}(\theta_0)\Theta, \quad \frac{\partial y}{\partial\theta_{i\neq 0}} = \Lambda_{\mathrm{y},i}(\theta_0). \end{aligned} \tag{27}$$

The thesis follows by direct substitution of these equations in (25), which yields (28).

□

D. Coriolis and centrifugal terms

Coriolis and centrifugal terms are collected into the following term

$$C(\Theta,\dot{\Theta}) = \dot{B}(\Theta)\dot{\Theta} - \frac{1}{2}\left(\frac{\partial}{\partial\Theta}E_\mathrm{k}(\Theta,\dot{\Theta})\right)^\mathrm{T}. \tag{29}$$

Note that, while potentially computationally heavy, the derivation of this form does not carry any fundamental difficulty. No integrations is indeed involved, and derivations can be evaluated both numerically and symbolically in an automatic fashion. For the latter, we used `Symbolic toolbox` of `MatLab2019b`. We can not report here the result for the sake of space. However, it is worth introducing the following general property.

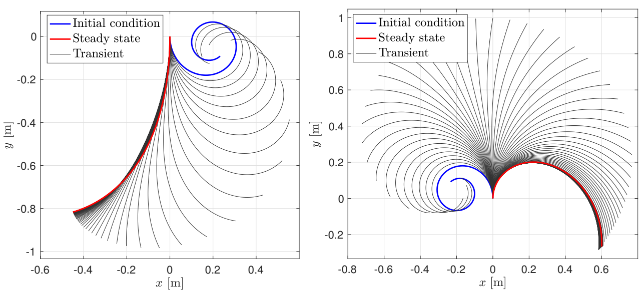
Lemma 5. *Under hypothesis* (17)*,* $[C]_m(\theta,\dot{\theta}) = 0$*,* $\forall\dot{\theta}$ *s.t.* $\dot{\theta}_0 = 0$*,* $\forall m > 0$.

Proof. The $i-$th element of $[C]_m$ can be written as

$$([C]_m)_i(\theta,\dot{\theta}) = \sum_{j=0}^{m}\sum_{k=0}^{m}\left(\frac{\partial B_{i,j}}{\partial\theta_k} - \frac{1}{2}\frac{\partial B_{j,k}}{\partial\theta_i}\right)\dot{\theta}_k\dot{\theta}_j\ . \tag{30}$$

When $\dot{\theta}_0 = 0$, the terms in $k\neq 0$ and $j\neq 0$ are the only ones to be not null. When $i\neq 0$, they are $\frac{\partial B_{i\neq 0,j\neq 0}}{\partial\theta_{k\neq 0}}$, which are null according to Lemma 4.

$$
\begin{aligned}
\frac{B_{0,0}}{\rho} &= \int_0^1 \frac{\partial \Gamma_{\mathrm{x}}}{\partial \theta_0}(\theta_0)^2 + \frac{\partial \Gamma_{\mathrm{y}}}{\partial \theta_0}(\theta_0)^2 \,\mathrm{d}s \\
&\quad + 2\left[\int_0^1 \frac{\partial \Gamma_{\mathrm{x}}}{\partial \theta_0}(\theta_0)\frac{\partial \Lambda_{\mathrm{x}}}{\partial \theta_0}(\theta_0) + \frac{\partial \Gamma_{\mathrm{y}}}{\partial \theta_0}(\theta_0)\frac{\partial \Lambda_{\mathrm{y}}}{\partial \theta_0}(\theta_0)\,\mathrm{d}s\right]\Theta + \Theta^{\mathrm{T}}\left[\int_0^1 \frac{\partial \Lambda_{\mathrm{x}}}{\partial \theta_0}^{\mathrm{T}}(\theta_0)\frac{\partial \Lambda_{\mathrm{x}}}{\partial \theta_0}(\theta_0) + \frac{\partial \Lambda_{\mathrm{y}}}{\partial \theta_0}^{\mathrm{T}}(\theta_0)\frac{\partial \Lambda_{\mathrm{y}}}{\partial \theta_0}(\theta_0)\,\mathrm{d}s\right]\Theta, \\
\frac{B_{0,j\neq 0}}{\rho} &= \int_0^1 \frac{\partial \Gamma_{\mathrm{x}}}{\partial \theta_0}(\theta_0)\Lambda_{\mathrm{x},j}(\theta_0) + \frac{\partial \Gamma_{\mathrm{y}}}{\partial \theta_0}(\theta_0)\Lambda_{\mathrm{y},j}(\theta_0)\,\mathrm{d}s + \left[\int_0^1 \frac{\partial \Lambda_{\mathrm{x}}}{\partial \theta_0}(\theta_0)\Lambda_{\mathrm{x},j}(\theta_0) + \frac{\partial \Lambda_{\mathrm{y}}}{\partial \theta_0}(\theta_0)\Lambda_{\mathrm{y},j}(\theta_0)\,\mathrm{d}s\right]\Theta, \\
\frac{B_{i\neq 0,j\neq 0}}{\rho} &= \int_0^1 \Lambda_{\mathrm{x},i}(\theta_0)\Lambda_{\mathrm{x},j}(\theta_0) + \Lambda_{\mathrm{y},i}(\theta_0)\Lambda_{\mathrm{y},j}(\theta_0)\,\mathrm{d}s.
\end{aligned}
\tag{28}
$$



(a) $\phi = 0$ (b) $\phi = \pi$

Fig. 2. Open loop evolution when mass is not negligible, with $\kappa(s,0) \simeq 5e^s$, $\dot\kappa(s,0) = 0$, $\tau = -2$Nm, $\rho = 1$Kg, $k = 1$Nm, $d = 1$Nms, and $m = 4$. Two orientations w.r.t. gravity are considered; $\phi = 0, \pi$.

To prove that $([C]_m)_0 = 0$, we start by considering the terms of the sum such that $j = \bar{j}$, $k = \bar{k}$ and $k = \bar{j}$, $j = \bar{k}$

$$
\begin{aligned}
\frac{\partial B_{0,\bar{j}\neq 0}}{\partial \theta_{\bar{k}\neq 0}} &= \int_0^1 \frac{\partial \Lambda_{\mathrm{x},\bar{k}}}{\partial \theta_0}\Lambda_{\mathrm{x},\bar{j}} + \frac{\partial \Lambda_{\mathrm{y},\bar{k}}}{\partial \theta_0}\Lambda_{\mathrm{y},\bar{j}}\,\mathrm{d}s, \\
\frac{\partial B_{0,\bar{k}\neq 0}}{\partial \theta_{\bar{j}\neq 0}} &= \int_0^1 \frac{\partial \Lambda_{\mathrm{x},\bar{j}}}{\partial \theta_0}\Lambda_{\mathrm{x},\bar{k}} + \frac{\partial \Lambda_{\mathrm{y},\bar{j}}}{\partial \theta_0}\Lambda_{\mathrm{y},\bar{k}}\,\mathrm{d}s, \\
\frac{\partial B_{\bar{k}\neq 0,\bar{j}\neq 0}}{\partial \theta_0} &= \frac{\partial B_{\bar{j}\neq 0,\bar{k}\neq 0}}{\partial \theta_0} = \int_0^1 \frac{\partial \Lambda_{\mathrm{x},\bar{i}}}{\partial \theta_0}\Lambda_{\mathrm{x},\bar{j}} \\
&\quad + \Lambda_{\mathrm{x},\bar{i}}\frac{\partial \Lambda_{\mathrm{x},\bar{j}}}{\partial \theta_0} + \frac{\partial \Lambda_{\mathrm{y},\bar{i}}}{\partial \theta_0}\Lambda_{\mathrm{y},\bar{j}} + \Lambda_{\mathrm{y},\bar{i}}\frac{\partial \Lambda_{\mathrm{y},\bar{j}}}{\partial \theta_0}\,\mathrm{d}s.
\end{aligned}
\tag{31}
$$

Thus

$$
\begin{aligned}
&\left(\frac{\partial B_{0,\bar{j}\neq 0}}{\partial \theta_{\bar{k}\neq 0}} - \frac{1}{2}\frac{\partial B_{\bar{j}\neq 0,\bar{k}\neq 0}}{\partial \theta_0}\right)\dot\theta_{\bar{k}\neq 0}\dot\theta_{\bar{j}\neq 0} \\
+&\left(\frac{\partial B_{0,\bar{j}\neq 0}}{\partial \theta_{\bar{k}\neq 0}} - \frac{1}{2}\frac{\partial B_{\bar{j}\neq 0,\bar{k}\neq 0}}{\partial \theta_0}\right)\dot\theta_{\bar{k}\neq 0}\dot\theta_{\bar{j}\neq 0} = 0.
\end{aligned}
\tag{32}
$$

Only the terms in $k = \bar{h}$ and $j = \bar{h}$ remain in the sum, for which we can prove following similar steps as the ones discussed above that $\frac{\partial B_{0,\bar{h}}}{\partial \theta_{\bar{h}}} = \frac{1}{2}\frac{\partial B_{\bar{h},\bar{h}}}{\partial \theta_0}$, concluding the proof. □

E. Overall Dynamical system

Finally, we write the infinite dimensional nonlinear dynamical system describing the behavior of the soft robot by combining (6) (10) (12) (20) (23) and (29)

$$B(\Theta)\ddot\Theta + C(\Theta,\dot\Theta) + G_{\mathrm{G}}(\Theta,\phi) + K\Theta + D\dot\Theta = A\tau, \tag{33}$$

where $B : \boldsymbol{K} \to \boldsymbol{M}$ is the inertia of the robot, $C : \boldsymbol{K} \times \boldsymbol{K} \to \boldsymbol{K}$ are Coriolis and centrifugal forces, $G_{\mathrm{G}} : \boldsymbol{K} \to \boldsymbol{K}$ are the gravitational forces, $K \in \boldsymbol{M}$ and $D \in \boldsymbol{M}$ are stiffness and damping matrices in modal space, and $A \in \boldsymbol{K}$ is the input field.

The finite dimensional approximation of (33) can be derived by applying the truncation operator $[\,\cdot\,]_m$ to all its terms, obtaining

$$[B]_m\ddot\theta + [C]_m + [K]_m\theta + [D]_m\dot\theta + [G_{\mathrm{G}}]_m = [A]_m\tau, \tag{34}$$

where dependencies on $\theta, \dot\theta, \phi$ are not reported for the sake of space.

Example 2. *We consider a soft robot as in example 1, with same input and initial conditions. We include here inertia related effects, with $\rho = 1$Kg, $\phi = 0$ and $\phi = \pi$. Fig. 2 shows the resulting evolution for $m = 4$.*

V. Model based control

As already discussed in the introduction, the main aim of this model is to provide a framework for advance model based control in soft robots, both in terms of controller design and theoretical assessment of structural properties. This section is a first example of the use of the proposed model in this direction.

A. Regulating the constant curvature term produces a minimum phase system

Given an output function representing some aspect of the system that we aim at directly control $y = h(\theta,\dot\theta)$, with $h : \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to \mathbb{R}^g$, the zero-dynamics is the collection of all trajectories such that $\dot y \equiv 0$, i.e. for any given perfect regulation of the output. A system is said to be non minimum phase if the trajectories of its zero dynamics are not divergent, vice versa it is minimum phase. An in depth introduction to the topic is provided by [21]. Note that in the context of nonlinear control, being minimum phase is a very important property for a system to have, since it enables formulating the regulation of the output y as a control goal. If this property is fulfilled, advanced techniques can be used to control the system, as high gain control, feedback passification, and feedback linearization. If the answer is negative, then there is no way of designing a controller - either with classic theories, or more recently developed data-driven strategies - that can regulate y. So, understanding if a system is minimum phase w.r.t. a meaningful output should be regarded as a major challenge in any emergent sub-fields of control theory, as control of soft robots.

As direct extension of curvature control in constant curvature robots, we analyze here the regulation of the constant approximation of κ. This function would be very complex to even formulate using other means of discretizing the dynamics, preventing any general solution. It is instead trivial to define in our framework, i.e.

$$y = \theta_0. \tag{35}$$

The dynamics of the output is obtained by deriving y two times, getting $\ddot{\theta}_0$. This is coincident with the first equation of (34). Consequently, we partition the complete reduced dynamics as follows

$$\begin{bmatrix} B_{0,0} & B_{0,\mathrm{zd}} \\ B_{\mathrm{zd},0} & B_{\mathrm{zd},\mathrm{zd}} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_0 \\ \ddot{\theta}_{\mathrm{zd}} \end{bmatrix} + \begin{bmatrix} C_0 \\ C_{\mathrm{zd}} \end{bmatrix} + \begin{bmatrix} K_{0,0} & K_{0,\mathrm{zd}} \\ K_{\mathrm{zd},0} & K_{\mathrm{zd},\mathrm{zd}} \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_{\mathrm{zd}} \end{bmatrix} + \begin{bmatrix} D_{0,0} & D_{0,\mathrm{zd}} \\ D_{\mathrm{zd},0} & D_{\mathrm{zd},\mathrm{zd}} \end{bmatrix} \begin{bmatrix} \dot{\theta}_0 \\ \dot{\theta}_{\mathrm{zd}} \end{bmatrix} + \begin{bmatrix} G_0 \\ G_{\mathrm{zd}} \end{bmatrix} = \begin{bmatrix} A_0 \\ A_{\mathrm{zd}} \end{bmatrix} \tau, \tag{36}$$

where dependencies on $\theta, \dot{\theta}, \phi$ are not reported for the sake of space. $\theta_{\mathrm{zd}} \in \mathbb{R}^m$ collects the modal coefficients going from θ_1 to θ_m, and $\dot{\theta}_{\mathrm{zd}} \in \mathbb{R}^m$ their derivatives. The block matrices result from partitioning the matrices appearing in (34).

Theorem 1. *In the hypothesis of dominant constant curvature* (17)*, all the trajectories of the zero-dynamics of* (34)*,* (35) *verifying the following conditions converge to a constant equilibrium, for all given and fixed* θ_0*. This is equivalent of saying that the system is (non strictly) minimum phase*[2]

i) $B_{\mathrm{zd},\mathrm{zd}}(\theta_0) \succ A_{\mathrm{zd}} B_{0,\mathrm{zd}}(\theta_0, \theta_{\mathrm{zd}})$,
ii) $D_{\mathrm{zd},\mathrm{zd}} \succ A_{\mathrm{zd}} D_{0,\mathrm{zd}} + \frac{1}{2} A_{\mathrm{zd}} \left[\frac{\partial B_{0,\mathrm{zd}}}{\partial \theta_{\mathrm{zd}}} \dot{\theta}_{\mathrm{zd}} \right]^{\mathrm{T}}$,
iii) $K_{\mathrm{zd},\mathrm{zd}} \succ A_{\mathrm{zd}} K_{0,\mathrm{zd}}$.

Proof. The zero dynamics of the robot is coincident with the m second order differential equations governing $\ddot{\theta}_{\mathrm{zd}}$, when $\theta_0 \equiv \bar{\theta}$, $\dot{\theta}_0 = 0$, $\ddot{\theta}_0 = 0$. This yields the following dynamical system, where dependencies are not reported for the sake of space $B_{\mathrm{zd},\mathrm{zd}} \ddot{\theta}_{\mathrm{zd}} + D_{\mathrm{zd},\mathrm{zd}} \dot{\theta}_{\mathrm{zd}} + K_{\mathrm{zd},\mathrm{zd}} \theta_{\mathrm{zd}} = -K_{\mathrm{zd},0} \bar{\theta}_0 - G_{\mathrm{zd}} + A_{\mathrm{zd}} \bar{\tau}$. Note that no Coriolis and centrifugal terms appear here, thanks to Lemma 5. $\bar{\tau}$ is the torque such that the conditions on θ_0 are enforced, which can be evaluated in closed form through direct substitution into the first equation of (36) as being $\bar{\tau} = B_{0,\mathrm{zd}} \ddot{\theta}_{\mathrm{zd}} + K_{0,0} \theta_0 + K_{0,\mathrm{zd}} \theta_{\mathrm{zd}} + D_{0,\mathrm{zd}} \dot{\theta}_{\mathrm{zd}} + G_0(\theta)$, where we exploited that $A_0 = 1$. The two can be combined, obtaining the following explicit expression of the zero dynamics

$$\begin{aligned} &(B_{\mathrm{zd},\mathrm{zd}} - A_{\mathrm{zd}} B_{0,\mathrm{zd}}) \ddot{\eta} + (D_{\mathrm{zd},\mathrm{zd}} - A_{\mathrm{zd}} D_{0,\mathrm{zd}}) \dot{\eta} \\ &\qquad + (K_{\mathrm{zd},\mathrm{zd}} - A_{\mathrm{zd}} K_{0,\mathrm{zd}}) \eta = 0, \end{aligned} \tag{37}$$

with $\eta = \theta_{\mathrm{zd}} - (K_{\mathrm{zd},\mathrm{zd}} - A_{\mathrm{zd}} K_{0,\mathrm{zd}})^{-1} c$, where c collects terms that are constants once θ_0 is fixed. Thus $\dot{\eta} = \dot{\theta}_{\mathrm{zd}}$, and $\ddot{\eta} = \ddot{\theta}_{\mathrm{zd}}$. Eq. (37) has the structure of a multi-dimensional mechanical system, with constant stiffness and damping, and configuration dependent inertia. This dependence is linear, as stated by Lemma 3. Furthermore both inertia and stiffness are positive defined by hypothesis (i) and (iii). The following Lyapunov candidate can thus be introduced; $V(\theta, \dot{\theta}) = \frac{1}{2} \dot{\eta}^{\mathrm{T}} (B_{\mathrm{zd},\mathrm{zd}} - A_{\mathrm{zd}} B_{0,\mathrm{zd}}) \dot{\eta} + \frac{1}{2} \eta^{\mathrm{T}} (K_{\mathrm{zd},\mathrm{zd}} - A_{\mathrm{zd}} K_{0,\mathrm{zd}}) \eta$. To evaluate the stability of (37), we derive V w.r.t. time

$$\begin{aligned} \dot{V} = &+ \dot{\theta}^{\mathrm{T}} (B_{\mathrm{zd},\mathrm{zd}} - A_{\mathrm{zd}} B_{0,\mathrm{zd}}) \ddot{\theta} + \frac{1}{2} \dot{\theta}^{\mathrm{T}} \frac{\mathrm{d}(B_{\mathrm{zd},\mathrm{zd}} - A_{\mathrm{zd}} B_{0,\mathrm{zd}})}{\mathrm{d}t} \dot{\theta} \\ &+ \dot{\theta}^{\mathrm{T}} (K_{\mathrm{zd},\mathrm{zd}} - A_{\mathrm{zd}} K_{0,\mathrm{zd}}) \theta \\ = &- \dot{\theta}^{\mathrm{T}} (K_{\mathrm{zd},\mathrm{zd}} - A_{\mathrm{zd}} K_{0,\mathrm{zd}}) \theta - \dot{\theta}^{\mathrm{T}} D_{\mathrm{zd},\mathrm{zd}} \dot{\theta} \\ &+ \frac{1}{2} \sum_{1}^{m} \left(\dot{\theta}^{\mathrm{T}} \frac{\partial (B_{\mathrm{zd},\mathrm{zd}} - A_{\mathrm{zd}} B_{0,\mathrm{zd}})}{\partial \theta_i} \dot{\theta} \right) \dot{\theta}_i \\ &+ \dot{\theta}^{\mathrm{T}} (K_{\mathrm{zd},\mathrm{zd}} - A_{\mathrm{zd}} K_{0,\mathrm{zd}}) \theta \\ = &- \dot{\theta}^{\mathrm{T}} \left(D_{\mathrm{zd},\mathrm{zd}} + \frac{1}{2} \sum_{1}^{m} \frac{\partial (B_{\mathrm{zd},\mathrm{zd}} - A_{\mathrm{zd}} B_{0,\mathrm{zd}})}{\partial \theta_i} \dot{\theta}_i \right) \dot{\theta} \\ = &- \dot{\theta}^{\mathrm{T}} \left(D_{\mathrm{zd},\mathrm{zd}} - A_{\mathrm{zd}} D_{0,\mathrm{zd}} + \frac{1}{2} A_{\mathrm{zd}} \left[\frac{\partial B_{0,\mathrm{zd}}}{\partial \theta_{\mathrm{zd}}} \dot{\theta}_{\mathrm{zd}} \right]^{\mathrm{T}} \right) \dot{\theta} \leq 0, \end{aligned}$$

where we exploited Lemma 4, and the last step holds since it is a quadratic form of a positive defined matrix (hypothesis (ii)) is always positive. The thesis follows by applying the LaSalle Lemma [22].

□

B. Controlling with polynomial curvature: the PD-poly

Following the proof that controlling the constant approximation of κ produces a well defined problem, we propose here a controller to achieve this goal with zero error at steady state. We call it PD with polynomial terms compensation, or PD-poly.

Theorem 2. *All the equilibria of the closed loop of system* (34) *and the controller*

$$\begin{aligned} \tau = &\frac{[1, \ -B_{0,\mathrm{zd}} B_{\mathrm{zd},\mathrm{zd}}^{-1}] K \theta + [1, \ -B_{0,\mathrm{zd}} B_{\mathrm{zd},\mathrm{zd}}^{-1}] G(\theta)}{[1, \ -B_{0,\mathrm{zd}} B_{\mathrm{zd},\mathrm{zd}}^{-1}] A} \\ &- \gamma_{\mathrm{D}} \dot{\theta}_0 - \gamma_{\mathrm{P}} (\theta_0 - \bar{\theta}_0), \end{aligned} \tag{38}$$

are such that $\theta_0 = \bar{\theta}_0$*, as soon as* $B_{0,\mathrm{zd}} B_{\mathrm{zd},\mathrm{zd}}^{-1} A_{\mathrm{zd}} \neq 1$*, and* $\gamma_{\mathrm{P}} \neq 0$*.*

Proof. We start by expliciting the acceleration of the zero dynamics variables $\ddot{\theta}_{\mathrm{zd}}$ from the general form (36)

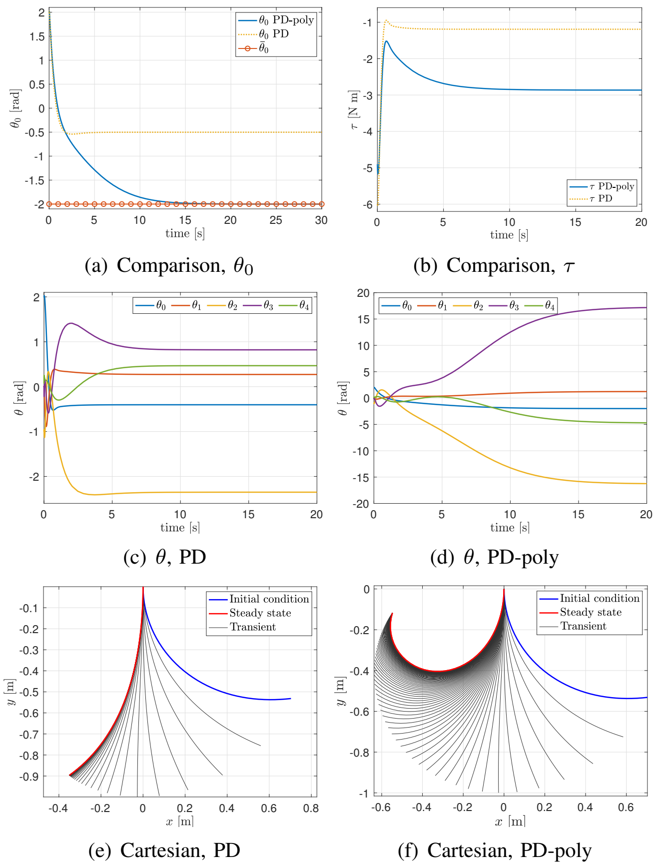
$$\begin{aligned} \ddot{\theta}_{\mathrm{zd}} = B_{\mathrm{zd},\mathrm{zd}}^{-1} \Bigg(&-B_{\mathrm{zd},0} \ddot{\theta}_0 - C_{\mathrm{zd}} - \begin{bmatrix} K_{\mathrm{zd},0} & K_{\mathrm{zd},\mathrm{zd}} \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_{\mathrm{zd}} \end{bmatrix} \\ &- \begin{bmatrix} D_{\mathrm{zd},0} & D_{\mathrm{zd},\mathrm{zd}} \end{bmatrix} \begin{bmatrix} \dot{\theta}_0 \\ \dot{\theta}_{\mathrm{zd}} \end{bmatrix} - G_{\mathrm{zd}} + A_{\mathrm{zd}} \tau \Bigg). \end{aligned}$$

This equation can now be plugged into the first line of (36), getting to

$$\begin{aligned} &(B_{0,0} - B_{0,\mathrm{zd}} B_{\mathrm{zd},\mathrm{zd}}^{-1} B_{\mathrm{zd},0}) \ddot{\theta}_0 + (C_0 - B_{0,\mathrm{zd}} B_{\mathrm{zd},\mathrm{zd}}^{-1} C_{\mathrm{zd}}) \\ &+ \begin{bmatrix} K_{0,0} - B_{0,\mathrm{zd}} B_{\mathrm{zd},\mathrm{zd}}^{-1} K_{\mathrm{zd},0} & K_{0,\mathrm{zd}} - B_{0,\mathrm{zd}} B_{\mathrm{zd},\mathrm{zd}}^{-1} K_{\mathrm{zd},\mathrm{zd}} \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_{\mathrm{zd}} \end{bmatrix} \\ &+ \begin{bmatrix} D_{0,0} - B_{0,\mathrm{zd}} B_{\mathrm{zd},\mathrm{zd}}^{-1} D_{\mathrm{zd},0} & D_{0,\mathrm{zd}} - B_{0,\mathrm{zd}} B_{\mathrm{zd},\mathrm{zd}}^{-1} D_{\mathrm{zd},\mathrm{zd}} \end{bmatrix} \begin{bmatrix} \dot{\theta}_0 \\ \dot{\theta}_{\mathrm{zd}} \end{bmatrix} \\ &+ (G_0 - B_{0,\mathrm{zd}} B_{\mathrm{zd},\mathrm{zd}}^{-1} G_{\mathrm{zd}}) = (1 - B_{0,\mathrm{zd}} B_{\mathrm{zd},\mathrm{zd}}^{-1} A_{\mathrm{zd}}) \tau. \end{aligned}$$

By getting rid of $\ddot{\theta}_{\mathrm{zd}}$ we obtained a scalar dynamics in the state variables, describing the evolution of the variable of interest θ_0. Substituting (36) into this dynamics, and imposing the equilibrium conditions $\ddot{\theta} = 0$, and $\dot{\theta} = 0$, yields

[2]Note that condition (iii) is not state dependent, and it can be shown holding always true. The remaining two conditions are LMIs, and can thus be very efficiently evaluated. Note also that the conditions are well posed since $B_{\mathrm{zd},\mathrm{zd}}, D_{\mathrm{zd},\mathrm{zd}}, K_{\mathrm{zd},\mathrm{zd}} \succ 0$, begin sub-matrices of positive defined matrices.



(a) Comparison, θ_0 (b) Comparison, τ

(c) θ, PD (d) θ, PD-poly

(e) Cartesian, PD (f) Cartesian, PD-poly

Fig. 3. Closed loop simulations, with $\kappa(0,s) \simeq 2\sin(s)$, $\dot{\kappa}(0,s) = 0$, $\rho = 1\text{Kg}$, $k = 1\text{Nm}$, $d = 1\text{Nms}$, $\phi = 0$, $\gamma_\text{P} = 1\text{Nm}$, $\gamma_\text{D} = 1\text{Nms}$, and $m = 4$. Panel (a) shows a comparison of constant curvature regulation when using a standard PD, and the here proposed PD-poly. Panel (b) shows the torques produced by the two controllers. Panel (c) shows the evolution of θ for the PD case, and panel (d) for the PD-poly case. Panel (e,f) show the resulting evolutions in Cartesian space. Note when using the simple PD the soft robot is not able to lift its own weight over a certain threshold.

$(1 - B_{0,\text{zd}}B_{\text{zd,zd}}^{-1}A_{\text{zd}})\gamma_\text{P}(\theta_0 - \bar{\theta}_0) = 0$. This implies $\theta_0 = \bar{\theta}_0$, since $1 - B_{0,\text{zd}}B_{\text{zd,zd}}^{-1}A_{\text{zd}} \neq 0$ and $\gamma_\text{P} \neq 0$ by hypothesis. □

Note that the theorem can be proven following the same steps even when the compensation is evaluated in feedforward.

Example 3. *We consider a soft robot as in example 2. This time two loops are closed on τ, with the aim of regulating θ_0; a standard PD, and the here proposed PD-poly. The gains are $\gamma_\text{P} = 1\text{Nm}$, $\gamma_\text{D} = 1\text{Nms}$. Fig. 3 shows the resulting evolutions for $m = 4$.*

VI. Conclusions and Future Work

This paper introduced a novel model, inspired by assumed mode technique, and specifically devised for control oriented applications. We developed the model, discussed its main characteristics, and used it to attack the control of the constant curvature approximation of the robot. Future work will focus on; i) high gain linear feedback controllers for soft robots, ii) designing new control goals using the theory of dummy outputs, iii) analyzing different series expansions to define the modal coordinates, iv) deriving controllers for the infinite dimensional case. The latter goal is eased by the fact that - under the hypothesis of constant curvature dominance - the zero dynamics is quasi linear [23].

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