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DEGENERATE PRINCIPAL SERIES FOR CLASSICAL AND ODD GSPIN GROUPS IN THE GENERAL CASE

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Abstract. Let G_n denote either the group $SO(2n+1,F)$, $Sp(2n,F)$, or $G\mathrm{Spin}(2n+1,F)$ over a non-archimedean local field of characteristic different from two. We determine all composition factors of degenerate principal series of G_n, using methods based on the Aubert involution and known results on irreducible subquotients of the generalized principal series of a particular type.

1. Introduction

Let F be a non-archimedean local field of characteristic different from two. Let G_n denote a symplectic, odd special orthogonal, or odd general spin group of split rank n defined over F, and $G_n = \mathrm{G}_n(F)$. The aim of this paper is to obtain a uniform description of reducibility and composition factors of degenerate principle series of G_n. This greatly generalizes and simplifies previous works of Jantzen [8], Kudla-Rallis [16], Gustafson [7], and others. We note that the degenerate principle series, besides being interesting by themselves, play an important role in the theory of automorphic forms, especially the extension of the Siegel-Weil formula, constructions of residual spectrum [12, 13], and in the local theta-correspondence.

Let σ denote an irreducible unitary cuspidal representation of some G_n. Also, let ρ_0 denote an irreducible unitary self-contragredient (resp., essentially self-contragredient, i.e., $\widetilde{\rho} \cong \rho \otimes \omega_\sigma$) cuspidal representation of $GL(n_{\rho_0}, F)$, and let ρ denote an irreducible unitary self-contragredient (resp., essentially self-contragredient) cuspidal representation of $GL(n_\rho, F)$ when G_n is a classical group (resp., $G_n = G\mathrm{Spin}(2n+1,F)$). Then there exist unique non-negative half-integers α, β such that $\nu^\alpha\rho \rtimes \sigma$, $\nu^\beta\rho_0 \rtimes \sigma$ are reducible (for more details regarding the notation we refer the reader to Section 2). For $x \geq \alpha > 0$ such that $x - \alpha \in \mathbb{Z}$, the induced representation $\nu^{-x}\rho \times \nu^{-x+1}\rho \times \cdots \times \nu^{-\alpha}\rho \rtimes \sigma$ contains a unique irreducible subrepresentation, which we denote by $\zeta(\rho, x; \sigma)$. A degenerate principal series is an induced representation of the form

$$\zeta([\nu^{-b}\rho_0, \nu^{-a}\rho_0]) \rtimes \zeta(\rho, x; \sigma), \tag{1}$$

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for a, b such that $b - a \in \mathbb{Z}$, where $\zeta([\nu^{-b}\rho_0, \nu^{-a}\rho_0])$ is a Zelevinsky segment representation, i.e., the unique irreducible subrepresentation of $\nu^{-b}\rho_0 \times \nu^{-b+1}\rho_0 \times \cdots \times \nu^{-a}\rho_0$. It has been explained in detail in [8, Section 2] that this definition generalizes the classical notion of the degenerate principal series, studied in [7] and [16]. We note that the composition series of the degenerate principal series (1) have been determined in [8] for $\alpha \in \{0, \frac{1}{2}, 1\}$, using Tadić's Jacquet modules method [27, 28], and here we treat the general case. Since the case $\alpha = 0$ is also handled in [5], and the results extend to the GSpin case in the same way, we consider the case $\alpha > 0$. Our results show that the degenerate principal series are multiplicity one representations of length up to four, and also provide a deeper insight into the structure of the irreducible subquotients.

Our approach to the determination of reducibility and composition factors of induced representations of the form (1) is completely different from one used in [8], and is based on the methods of the Aubert involution. The Aubert dual of the degenerate principal series is a special type of the generalized principal series, and the composition factors of such representations have been determined in [26] and [19, Proposition 3.2]. To determine the Aubert duals of composition factors in question, we use a further adjustment of the methods initiated in [20–22]. Eventually, it turns out that needed Aubert duals of tempered representations mostly follow directly from [20, 22]. On the other hand, to determine the Aubert duals of the involved non-tempered representations we use an inductive approach based on the detailed investigation of embeddings and Jacquet modules of such representations, using a case-by-case consideration. Let us also note that an algorithm for explicit determination of the Aubert duals for classical groups in the half-integral case has been recently provided in [11].

Let us now describe the contents of the paper in more detail. In the following section we present some preliminaries, while the first special case $\beta = 0$ is treated in the third section. The case $\beta > 0$ is studied in Sections 4 – 6, where in the fourth section we handle the case $a \geq 1$, in the fifth section the case $a \leq 0$, and in the sixth section we deal with the case $a = \frac{1}{2}$. To work effectively, from Lemma 2.5 to the end of Section 6, we mainly focus on the cases $G_n = Sp(2n, F)$ and $SO(2n + 1, F)$ (see Remark 2.4). In the final section we provide necessary adjustments in the odd GSpin case.

2. Preliminaries

Throughout the paper, F will denote a non-archimedean local field of characteristic different from two.

For a connected reductive p-adic group G defined over field F, let Σ denote the set of roots of G with respect to fixed minimal parabolic subgroup and let Δ stand for the corresponding subset of simple roots. For $\theta \subseteq \Delta$, we let P_θ denote the standard parabolic subgroup of G corresponding to θ and let M_θ denote a corresponding standard Levi subgroup. Let W denote the Weyl group of G.

For a parabolic subgroup P of G with the Levi subgroup M, and a representation σ of M, we denote by $i_M(\sigma)$ a normalized parabolically induced representation of G induced from σ. Also, let $r_M(\sigma)$ stand for the normalized Jacquet module of an admissible finite length representation σ of G, with respect to the standard parabolic subgroup having the Levi subgroup equal to M.

We take a moment to recall the definition of the Aubert involution and some of its basic properties [3, 4].

Theorem 2.1. *Define the operator on the Grothendieck group of admissible representations of finite length of G by*

$$D_G = \sum_{\theta \subseteq \Delta} (-1)^{|\theta|} i_{M_\theta} \circ r_{M_\theta}.$$

Operator D_G has the following properties:

(i) *D_G is an involution.*
(ii) *D_G takes irreducible representations to irreducible ones, up to $\pm$.*
(iii) *If σ is an irreducible cuspidal representation, then $D_G(\sigma) = (-1)^{|\Delta|}\sigma$.*
(iv) *For a standard Levi subgroup $M = M_\theta$, we have*

$$r_M \circ D_G = \mathrm{Ad}(w) \circ D_{w^{-1}(M)} \circ r_{w^{-1}(M)},$$

where w is the longest element of the set $\{w \in W : w^{-1}(\theta) > 0\}$.
(v) *For a standard Levi subgroup $M = M_\theta$, we have $D_G \circ i_M = i_M \circ D_M$.*

We look at the usual towers of symplectic or orthogonal groups $G_n = G(V_n)$, that are groups of isometries of F-spaces $(V_n, (\ ,\ )), n \geq 0$, where the form $(\ ,\ )$ is non-degenerate and it is skew-symmetric if the tower is symplectic and symmetric otherwise. In the final section, we also consider the odd general spin groups $G_n = G\mathrm{Spin}(2n+1, F)$ (see Section 7 for the definition). The set of standard parabolic subgroups of the group G_n will be fixed in the usual way.

Then the Levi subgroups of standard parabolic subgroups have the form $M \cong GL(n_1, F) \times \cdots \times GL(n_k, F) \times G_m$, where $GL(n_i, F)$ denotes a general linear group of rank n_i over F. For simplicity of exposition, if $\delta_i, i = 1, 2, \ldots, k$ denotes a representation of $GL(n_i, F)$, and if τ stands for a representation of G_m, we let $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$ stand for the induced representation $i_M(\delta_1 \otimes \delta_2 \otimes \cdots \otimes \delta_k \otimes \tau)$ of G_n, where M is the standard Levi subgroup isomorphic to $GL(n_1, F) \times \cdots \times GL(n_k, F) \times G_m$. Here $n = n_1 + n_2 + \cdots + n_k + m$.

Similarly, by $\delta_1 \times \delta_2 \times \cdots \times \delta_k$ we denote the induced representation $i_{M'}(\delta_1 \otimes \delta_2 \otimes \cdots \otimes \delta_k)$ of the group $GL(n', F)$, where the Levi subgroup M' equals $GL(n_1, F) \times GL(n_2, F) \times \cdots \times GL(n_k, F)$ and $n' = n_1 + n_2 + \cdots + n_k$.

Let $\mathrm{Irr}(GL(n, F))$ denote the set of all irreducible admissible representations of $GL(n, F)$, and let $\mathrm{Irr}(G_n)$ denote the set of all irreducible admissible representations of G_n. Let $R(GL(n, F))$ stand for the Grothendieck group of admissible representations of finite length of $GL(n, F)$ and define $R(GL) = \bigoplus_{n \geq 0} R(GL(n, F))$. Similarly, let $R(G_n)$ stand for the Grothendieck group of admissible representations of finite length of G_n and define $R(G) = \bigoplus_{n \geq 0} R(G_n)$.

If σ is an irreducible representation of G_n, we denote by $\hat{\sigma}$ the representation $\pm D_{G_n}(\sigma)$, taking the sign $+$ or $-$ such that $\hat{\sigma}$ is a positive element in $R(G_n)$. We call $\hat{\sigma}$ the Aubert dual of σ.

Using Jacquet modules for the maximal standard parabolic subgroups of $GL(n, F)$, one can define $m^*(\pi) = \sum_{k=0}^{n} (r_{(k)}(\pi)) \in R(GL) \otimes R(GL)$, for an irreducible representation π of $GL(n, F)$, and then extend m^* linearly to $R(GL)$. Here $r_{(k)}(\pi)$ denotes the normalized Jacquet module of π with respect to the standard parabolic subgroup having the Levi subgroup equal to $GL(k, F) \times GL(n-k, F)$, and we identify $r_{(k)}(\pi)$ with its semisimplification in $R(GL(k, F)) \otimes R(GL(n-k, F))$.

Let ν denote the composition of the determinant mapping with the normalized absolute value on F. Let $\rho \in \mathrm{Irr}(GL(k,F))$ denote a cuspidal representation. By a segment of cuspidal representations we mean a set of the form $\{\rho, \nu\rho, \dots, \nu^m\rho\}$, which we denote by $[\rho, \nu^m\rho]$.

By the results of [30], each irreducible essentially square-integrable representation $\delta \in \mathrm{Irr}(GL(n,F))$ is attached to a segment, and we set $\delta = \delta([\nu^a\rho, \nu^b\rho])$, which is the unique irreducible subrepresentation of $\nu^b\rho \times \nu^{b-1}\rho \times \cdots \times \nu^a\rho$, where $a, b \in \mathbb{R}$ are such that $b-a$ is a non-negative integer and ρ is an irreducible unitary cuspidal representation of some $GL(k,F)$. The induced representation $\nu^b\rho \times \nu^{b-1}\rho \times \cdots \times \nu^a\rho$ also contains a unique irreducible quotient, which we denote by $\zeta([\nu^a\rho, \nu^b\rho])$. Furthermore, $\zeta([\nu^a\rho, \nu^b\rho])$ is the unique irreducible subrepresentation of $\nu^a\rho \times \nu^{a+1}\rho \times \cdots \times \nu^b\rho$, and in $R(GL)$ we have

$$\nu^a\rho \times \nu^{a+1}\rho = \delta([\nu^a\rho, \nu^{a+1}\rho]) + \zeta([\nu^a\rho, \nu^{a+1}\rho])$$

and

$$\nu^a\rho \times \nu^{a+1}\rho \times \nu^{a+1}\rho = \delta([\nu^a\rho, \nu^{a+1}\rho]) \times \nu^{a+1}\rho + \zeta([\nu^a\rho, \nu^{a+1}\rho]) \times \nu^{a+1}\rho,$$

both representations $\delta([\nu^a\rho, \nu^{a+1}\rho]) \times \nu^{a+1}\rho$ and $\zeta([\nu^a\rho, \nu^{a+1}\rho]) \times \nu^{a+1}\rho$ being irreducible.

Let us briefly recall the Langlands classification for classical groups. We favor the subrepresentation version of this classification over the quotient one since it is more appropriate for our Jacquet module considerations.

For every irreducible essentially square-integrable representation $\delta \in R(GL)$, there is a unique $e(\delta) \in \mathbb{R}$ such that $\nu^{-e(\delta)}\delta$ is unitarizable. Note that $e(\delta([\nu^a\rho, \nu^b\rho])) = (a+b)/2$. Every non-tempered irreducible representation π of G_n can be written as the unique irreducible (Langlands) subrepresentation of an induced representation of the form $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$, where τ is a unitary tempered representation of some G_t, and $\delta_1, \delta_2, \dots, \delta_k \in R(GL)$ are irreducible essentially square-integrable representations such that $e(\delta_1) \le e(\delta_2) \le \cdots \le e(\delta_k) < 0$. In this case, we write $\pi = L(\delta_1, \delta_2, \dots, \delta_k; \tau)$. For a given π, the representations $\delta_1, \delta_2, \dots, \delta_k$ are unique up to a permutation among those δ_i having the same exponents.

Let $\tau \in R(G)$ denote an irreducible tempered representation. If $\delta_1, \delta_2, \dots, \delta_k \in R(GL)$ are irreducible essentially square-integrable representations such that $e(\delta_i) < 0$ for $i = 1, 2, \dots, k$, and $\delta_i \times \delta_j \cong \delta_j \times \delta_i$ for $i < j$ such that $e(\delta_i) > e(\delta_j)$, then the induced representation $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$ contains a unique irreducible subrepresentation, which will also be denoted by $L(\delta_1, \delta_2, \dots, \delta_k; \tau)$, for simplicity of the notation.

For a representation $\sigma \in R(G_n)$ and $1 \le k \le n$, we denote by $r_{(k)}(\sigma)$ the normalized Jacquet module of σ with respect to the parabolic subgroup $P_{(k)}$ having the Levi subgroup equal to $GL(k,F) \times G_{n-k}$. We identify $r_{(k)}(\sigma)$ with its semisimplification in $R(GL(k,F)) \otimes R(G_{n-k})$ and consider

$$\mu^*(\sigma) = 1 \otimes \sigma + \sum_{k=1}^{n} r_{(k)}(\sigma) \in R(GL) \otimes R(G).$$

We pause to state a result, derived in [27] ([14] for odd GSpin groups), which presents a crucial structural formula for our calculations of Jacquet modules of classical groups.

Lemma 2.2. *Let $\rho \in \mathrm{Irr}(GL(n,F))$ denote a cuspidal representation and let $k,l \in \mathbb{R}$ such that $k+l$ is a non-negative integer. Let $\sigma \in R(G)$ denote an admissible representation of finite length, and write $\mu^*(\sigma) = \sum_{\tau,\sigma'} \tau \otimes \sigma'$. If σ is a representation of the odd GSpin group, let ω_σ denote the central character of σ, otherwise let ω_σ be trivial. Then the following holds:*

$$\mu^*(\delta([\nu^{-k}\rho,\nu^l\rho]) \rtimes \sigma) = \sum_{i=-k-1}^{l} \sum_{j=i}^{l} \sum_{\tau,\sigma'} \delta([\nu^{-i}\widetilde{\rho} \otimes (\omega_\sigma \circ det), \nu^k \widetilde{\rho} \otimes (\omega_\sigma \circ det)]) \times \delta([\nu^{j+1}\rho,\nu^l\rho]) \times \tau \otimes \delta([\nu^{i+1}\rho,\nu^j\rho]) \rtimes \sigma'.$$

We omit $\delta([\nu^x\rho,\nu^y\rho])$ if $x > y$.

An irreducible representation $\sigma \in R(G)$ is called strongly positive if for every embedding

$$\sigma \hookrightarrow \nu^{s_1}\rho_1 \times \nu^{s_2}\rho_2 \times \cdots \times \nu^{s_k}\rho_k \rtimes \sigma_{cusp},$$

where $\rho_i \in R(GL(n_{\rho_i},F))$, $i = 1,2,\ldots,k$, are unitary cuspidal representations and $\sigma_{cusp} \in R(G)$ is an irreducible unitary cuspidal representation, we have $s_i > 0$ for each i.

Let us briefly recall an inductive description of non-cuspidal strongly positive discrete series, which has been obtained in [14, 17, 25].

Proposition 2.3. *Suppose that $\sigma_{sp} \in R(G)$ is an irreducible strongly positive representation and let $\rho \in R(GL)$ denote an irreducible unitary cuspidal representation such that some twist of ρ appears in the cuspidal support of σ_{sp}. We denote by σ_{cusp} the partial cuspidal support of σ_{sp}. Then there exist unique $a,b \in \mathbb{R}$ such that $a > 0, b > 0$, $b - a \in \mathbb{Z}_{\geq 0}$, and a unique irreducible strongly positive representation σ'_{sp} without $\nu^a\rho$ in the cuspidal support, with the property that σ_{sp} is the unique irreducible subrepresentation of $\delta([\nu^a\rho,\nu^b\rho]) \rtimes \sigma'_{sp}$. Furthermore, there is a non-negative integer l such that $a + l = s$ for $s > 0$ such that $\nu^s\rho \rtimes \sigma_{cusp}$ reduces. If $l = 0$, there are no twists of ρ appearing in the cuspidal support of σ'_{sp} and if $l > 0$ there exist unique $b' > b$ and a unique strongly positive discrete series σ''_{sp}, which contains neither $\nu^a\rho$ nor $\nu^{a+1}\rho$ in its cuspidal support, such that σ'_{sp} can be written as the unique irreducible subrepresentation of $\delta([\nu^{a+1}\rho,\nu^{b'}\rho]) \rtimes \sigma''_{sp}$.*

Throughout the paper, we fix an irreducible unitary cuspidal representation $\sigma \in R(G)$. Also, we fix an irreducible unitary cuspidal representation $\rho_0 \in R(GL)$ and an irreducible (essentially) self-contragredient unitary cuspidal representation $\rho \in R(GL)$, such that $\nu^\alpha\rho \rtimes \sigma$ reduces for some $\alpha > 0$. We note that $2\alpha \in \mathbb{Z}$, due to results of [1], [24, Théorème 3.1.1], and [6, Theorem 7.8], and that $\nu^s\rho \rtimes \sigma$ is irreducible for $s \notin \{\alpha, -\alpha\}$.

Let x stand for a half-integer such that $x \geq \alpha$ and $x - \alpha \in \mathbb{Z}$. Then the induced representation

$$\nu^{-x}\rho \times \nu^{-x+1}\rho \times \cdots \times \nu^{-\alpha}\rho \rtimes \sigma$$

has a unique irreducible subrepresentation, which we denote by $\zeta(\rho,x;\sigma)$. Using [20, Theorem 3.5], we deduce that the Aubert dual of $\zeta(\rho,x;\sigma)$ is the unique irreducible subrepresentation of $\nu^x\rho \times \nu^{x-1}\rho \times \cdots \times \nu^\alpha\rho \rtimes \sigma$. We note that this representation is strongly positive, and will be denoted by $\delta(\rho,x;\sigma)$.

Let a, b denote real numbers such that $b-a \in \mathbb{Z}$. We are interested in determining the composition factors of the degenerate principal series

$$\zeta([\nu^{-b}\rho_0, \nu^{-a}\rho_0]) \rtimes \zeta(\rho, x; \sigma).$$

Since in $R(G)$ we have

$$\begin{aligned}
&\zeta([\nu^{-b}\rho_0, \nu^{-a}\rho_0]) \rtimes \zeta(\rho, x; \sigma) = \zeta([\nu^{a}\widetilde{\rho_0}, \nu^{b}\widetilde{\rho_0}]) \rtimes \zeta(\rho, x; \sigma) \\
&\text{ if } G_n = Sp(2n, F), SO(2n+1, F), \\
&\zeta([\nu^{-b}\rho_0, \nu^{-a}\rho_0]) \rtimes \zeta(\rho, x; \sigma) = \zeta([\nu^{a}\widetilde{\rho_0} \otimes \omega_\sigma, \nu^{b}\widetilde{\rho_0} \otimes \omega_\sigma]) \rtimes \zeta(\rho, x; \sigma) \\
&\text{ if } G_n = G\mathrm{Spin}(2n+1, F),
\end{aligned}$$

we can assume that $-a \leq b$.

By properties of the Aubert involution, the Aubert dual of the degenerate principal series $\zeta([\nu^{-b}\rho_0, \nu^{-a}\rho_0]) \rtimes \zeta(\rho, x; \sigma)$ is the generalized principal series

$$\begin{aligned}
&\delta([\nu^{a}\widetilde{\rho_0}, \nu^{b}\widetilde{\rho_0}]) \rtimes \delta(\rho, x; \sigma) \text{ if } G_n = Sp(2n, F), SO(2n+1, F), \\
&\delta([\nu^{a}\widetilde{\rho_0} \otimes \omega_\sigma, \nu^{b}\widetilde{\rho_0} \otimes \omega_\sigma]) \rtimes \delta(\rho, x; \sigma) \text{ if } G_n = G\mathrm{Spin}(2n+1, F),
\end{aligned} \tag{2}$$

whose composition factors are completely described in [26] (this has been already noted in [9, Corollary 4.3]). We note that the results of [26] extend to the GSpin case by the last section of the paper. It follows from [26, Section 2] (or [14, Proposition 2.5] for GSpin groups) that the induced representation (2) is irreducible unless ρ_0 is (essentially) self-contragredient. Thus, in what follows we can assume that ρ_0 is (essentially) self-contragredient, and let us denote by β the unique non-negative real number such that $\nu^{\beta}\rho_0 \rtimes \sigma$ reduces. Again, it follows from [26, Section 2] that the induced representation (2) is irreducible if $a - \beta \notin \mathbb{Z}$ (the argument is similar for GSpin). So, we can also assume that $a - \beta \in \mathbb{Z}$.

Remark 2.4. (1) To work effectively, from now on until Section 6, G_n will only denote $Sp(2n, F)$ and $SO(2n+1, F)$. In Section 7, we will consider the case of $G_n = G\mathrm{Spin}(2n+1, F)$.

(2) All the lemmas and propositions in the rest of this section are also valid for the odd GSpin case (with the same statements, after replacing "self-contragredient" by "essentially self-contragredient", and adding the unitarity condition for the cuspidal representation σ); see Section 7 for more detailed comments.

We will use the following result [10, Lemma 5.5] several times.

Lemma 2.5. *Suppose that $\pi \in R(G_n)$ is an irreducible representation, λ an irreducible representation of the Levi subgroup M of G_n, and π is a subrepresentation of $\mathrm{Ind}_M^{G_n}(\lambda)$. If $L > M$, then there is an irreducible subquotient ρ of $\mathrm{Ind}_M^{L}(\lambda)$ such that π is a subrepresentation of $\mathrm{Ind}_L^{G_n}(\rho)$.*

The following result is a direct consequence of [20, Lemma 2.2].

Lemma 2.6. *Suppose that the Jacquet module of π with respect to the appropriate parabolic subgroup contains an irreducible cuspidal representation of the form $\nu^{a_1}\rho_1 \otimes \nu^{a_2}\rho_2 \otimes \cdots \otimes \nu^{a_k}\rho_k \otimes \sigma$, where $\rho_1, \ldots, \rho_k \in R(GL)$ are self-contragredient representations. Then $\widehat{\pi}$ is a subrepresentation of $\nu^{-a_1}\rho_1 \times \nu^{-a_2}\rho_2 \times \cdots \times \nu^{-a_k}\rho_k \rtimes \sigma$.*

We will now present a sequence of lemmas which enable us to use an inductive procedure when determining the Aubert duals.

For a non-negative integer m, real number t, and an irreducible cuspidal representation $\rho_1 \in R(GL)$, we denote by $(\nu^t\rho_1)^m$ the induced representation $\nu^t\rho_1 \times \cdots \times \nu^t\rho_1$, where $\nu^t\rho_1$ appears m times. Note that the induced representation $\zeta([\nu^c\rho_1, \nu^d\rho_1]) \times (\nu^t\rho_1)^m$ is irreducible for $t \in \{c, c+1, \dots, d\}$ [30].

Lemma 2.7. *Let c and d denote positive real numbers such that $d-c$ is a nonnegative integer. Let $\rho_1 \in R(GL)$ denote an irreducible cuspidal self-contragredient representation. Suppose that π is a subrepresentation of an induced representation of the form $\zeta([\nu^c\rho_1, \nu^d\rho_1]) \times (\nu^t\rho_1)^m \rtimes \pi_1$, where $t \in \{c, c+1, \dots, d\}$, π_1 is irreducible, and $\mu^*(\pi_1)$ does not contain an irreducible constituent of the form $\nu^i\rho_1 \otimes \pi_2$ for $i \in \{c, c+1, \dots, d\}$, with $\pi_2 \in R(G)$. Then $\widehat{\pi}$ is the unique irreducible subrepresentation of $\delta([\nu^{-d}\rho_1, \nu^{-c}\rho_1]) \times (\nu^{-t}\rho_1)^m \rtimes \widehat{\pi_1}$.*

Proof. We prove the lemma only in the case $m = 0$. The case $m > 0$ can be handled in the same way. From properties of the Aubert involution we conclude that $\widehat{\pi}$ is contained in $\delta([\nu^{-d}\rho_1, \nu^{-c}\rho_1]) \rtimes \widehat{\pi_1}$.

From embeddings

$$\pi \hookrightarrow \zeta([\nu^c\rho_1, \nu^d\rho_1]) \rtimes \pi_1 \hookrightarrow \nu^c\rho_1 \times \cdots \times \nu^d\rho_1 \rtimes \pi_1$$

and Frobenius reciprocity, it follows that the Jacquet module of π with respect to the appropriate parabolic subgroup contains $\nu^c\rho_1 \otimes \cdots \otimes \nu^d\rho_1 \otimes \pi_1$.

Using transitivity of Jacquet modules and Lemma 2.6, we obtain that the Jacquet module of $\widehat{\pi}$ with respect to the appropriate parabolic subgroup contains an irreducible constituent of the form $\nu^{-c}\rho_1 \otimes \cdots \otimes \nu^{-d}\rho_1 \otimes \pi'$.

Since $\mu^*(\pi_1)$ does not contain an irreducible constituent of the form $\nu^i\rho_1 \otimes \pi_2$ for $i \in \{c, c+1, \dots, d\}$, it follows from Lemma 2.6 that $\mu^*(\widehat{\pi_1})$ does not contain an irreducible constituent of the form $\nu^{-i}\rho_1 \otimes \pi_2$ for $i \in \{c, c+1, \dots, d\}$, with $\pi_2 \in R(G)$. Now it follows directly from the structural formula that $\nu^{-c}\rho_1 \otimes \cdots \otimes \nu^{-d}\rho_1 \otimes \widehat{\pi_1}$ is the unique irreducible constituent of the form $\nu^{-c}\rho_1 \otimes \cdots \otimes \nu^{-d}\rho_1 \otimes \pi'$ appearing in the Jacquet module of $\delta([\nu^{-d}\rho_1, \nu^{-c}\rho_1]) \rtimes \widehat{\pi_1}$ with respect to the appropriate parabolic subgroup, and it appears there with multiplicity one. It follows that $\delta([\nu^{-d}\rho_1, \nu^{-c}\rho_1]) \rtimes \widehat{\pi_1}$ contains a unique irreducible subrepresentation.

On the other hand, by Frobenius reciprocity every irreducible subrepresentation of $\delta([\nu^{-d}\rho_1, \nu^{-c}\rho_1]) \rtimes \widehat{\pi_1}$ contains $\nu^{-c}\rho_1 \otimes \cdots \otimes \nu^{-d}\rho_1 \otimes \widehat{\pi_1}$ in the Jacquet module with respect to the appropriate parabolic subgroup. Thus, $\widehat{\pi}$ has to be the unique irreducible subrepresentation of $\delta([\nu^{-d}\rho_1, \nu^{-c}\rho_1]) \rtimes \widehat{\pi_1}$. This ends the proof. □

Lemma 2.8. *Let c and d denote positive real numbers such that $d-c$ is a nonnegative integer. Let $\rho_1 \in R(GL)$ denote an irreducible cuspidal self-contragredient representation. Suppose that π is a subrepresentation of an induced representation of the form $\zeta([\nu^c\rho_1, \nu^d\rho_1]) \times (\nu^d\rho_1)^m \rtimes \pi_1$, where π_1 is an irreducible representation such that the Jacquet module of π_1 with respect to the appropriate parabolic subgroup does not contain an irreducible constituent of the form $\nu^{d-k}\rho_1 \otimes \cdots \otimes \nu^{d-1}\rho_1 \otimes \nu^d\rho_1 \otimes \pi'$ for a nonnegative integer $k < d$, with $\pi' \in R(G)$. Then $\widehat{\pi}$ is the unique irreducible subrepresentation of $\delta([\nu^{-d}\rho_1, \nu^{-c}\rho_1]) \times (\nu^{-d}\rho_1)^m \rtimes \widehat{\pi_1}$.*

Lemma 2.9. *Suppose that $\rho_0 \not\cong \rho$ and let π denote an irreducible subquotient of $\delta([\nu^a\rho_0, \nu^b\rho_0]) \rtimes \delta(\rho, x; \sigma)$. Then there is an irreducible representation $\pi_1 \in R(G)$ such that π is a subrepresentation of $\delta([\nu^\alpha\rho, \nu^x\rho]) \rtimes \pi_1$ and $\widehat{\pi}$ is the unique irreducible subrepresentation of $\nu^{-x}\rho \times \nu^{-x+1}\rho \times \cdots \times \nu^{-\alpha}\rho \rtimes \widehat{\pi_1}$. Furthermore, if*

$\widehat{\pi_1} \cong L(\delta_1, \delta_2, \dots, \delta_k; \tau_{temp})$, *where* $e(\delta_i) \leq e(\delta_j)$ *for* $i \leq j$, *then*

$$\widehat{\pi} \cong L(\nu^{-x}\rho, \nu^{-x+1}\rho, \dots, \nu^{-\alpha}\rho, \delta_1, \delta_2, \dots, \delta_k; \tau_{temp}).$$

Proof. By the results of [26], there is an irreducible tempered representation $\tau \in R(G)$ such that either $\pi \cong \tau$ or $\pi \cong L(\delta([\nu^c\rho_0, \nu^{-a}\rho_0]); \tau)$ for some $c \geq -b$ such that $c - a < 0$. Also, it is easy to see that there is an irreducible representation τ_1 such that τ is a subrepresentation of $\delta([\nu^\alpha\rho, \nu^x\rho]) \rtimes \tau_1$, and there are no twists of ρ appearing in the cuspidal support of τ_1. If $\pi \cong \tau$, we can take $\pi_1 \cong \tau_1$. Otherwise, since $\rho_0 \ncong \rho$ we have

$$\begin{aligned}\pi \hookrightarrow \delta([\nu^c\rho_0, \nu^{-a}\rho_0]) \rtimes \tau &\hookrightarrow \delta([\nu^c\rho_0, \nu^{-a}\rho_0]) \times \delta([\nu^\alpha\rho, \nu^x\rho]) \rtimes \tau_1 \\ &\cong \delta([\nu^\alpha\rho, \nu^x\rho]) \times \delta([\nu^c\rho_0, \nu^{-a}\rho_0]) \rtimes \tau_1,\end{aligned}$$

and by [25, Lemma 3.2] there is an irreducible representation π_1 such that π is a subrepresentation of $\delta([\nu^\alpha\rho, \nu^x\rho]) \rtimes \pi_1$. Since there are no twists of ρ appearing in the cuspidal support of π_1, it can be seen in the same way as in the proof of Lemma 2.7 that $\widehat{\pi}$ is the unique irreducible subrepresentation of $\nu^{-x}\rho \times \nu^{-x+1}\rho \times \cdots \times \nu^{-\alpha}\rho \rtimes \widehat{\pi_1}$.

If we write $\widehat{\pi_1} \cong L(\delta_1, \delta_2, \dots, \delta_k; \tau_{temp})$, then $\delta_i \cong \delta([\nu^{x_i}\rho_0, \nu^{y_i}\rho_0])$ for $i = 1, 2, \dots, k$, and we have $\nu^z\rho \times \delta_i \cong \delta_i \times \nu^z\rho$ for all $i = 1, 2, \dots, k$ and $z \in \mathbb{R}$. This ends the proof. □

The following result provides embeddings needed for an inductive determination of the Aubert duals.

Proposition 2.10. *Let* $\rho_1 \in R(GL)$ *denote an irreducible self-contragredient cuspidal representation, and let* $\sigma_{sp} \in R(G)$ *denote a strongly positive discrete series. Let* k, l *denote half-integers such that* $k - l$ *is a positive integer and* $k + l > 0$.

1. *If* $\nu^k\rho_1 \rtimes \sigma_{sp}$ *is irreducible and* $k \geq -l + 2$, *then* $L(\delta([\nu^{-k}\rho_1, \nu^{-l}\rho_1]); \sigma_{sp})$ *is a subrepresentation of* $\nu^k\rho_1 \rtimes L(\delta([\nu^{-k+1}\rho_1, \nu^{-l}\rho_1]); \sigma_{sp})$.
2. *If* $\mu^*(\sigma_{sp})$ *does not contain an irreducible constituent of the form* $\nu^{-l}\rho_1 \otimes \pi$, *with* $\pi \in R(G)$, *then* $L(\delta([\nu^{-k}\rho_1, \nu^{-l}\rho_1]); \sigma_{sp})$ *is a subrepresentation of* $\nu^{-l}\rho_1 \rtimes L(\delta([\nu^{-k}\rho_1, \nu^{-l-1}\rho_1]); \sigma_{sp})$.
3. *Suppose that* σ_{sp} *is a subrepresentation of* $\nu^t\rho_1 \rtimes \sigma'_{sp}$ *for some* $t \neq k$, $t \neq -l+1$ *and a strongly positive representation* σ'_{sp}. *Then* $L(\delta([\nu^{-k}\rho_1, \nu^{-l}\rho_1]); \sigma_{sp})$ *is a subrepresentation of* $\nu^t\rho_1 \rtimes L(\delta([\nu^{-k}\rho_1, \nu^{-l}\rho_1]); \sigma'_{sp})$.

Proof. We only prove the first part of the proposition, other parts can be proved in the same way but more easily. We have the following embeddings and isomorphisms:

$$\begin{aligned}L(\delta([\nu^{-k}\rho_1, \nu^{-l}\rho_1]); \sigma_{sp}) &\hookrightarrow \delta([\nu^{-k}\rho_1, \nu^{-l}\rho_1]) \rtimes \sigma_{sp} \\ &\hookrightarrow \delta([\nu^{-k+1}\rho_1, \nu^{-l}\rho_1]) \times \nu^{-k}\rho_1 \rtimes \sigma_{sp} \\ &\cong \delta([\nu^{-k+1}\rho_1, \nu^{-l}\rho_1]) \times \nu^k\rho_1 \rtimes \sigma_{sp} \\ &\cong \nu^k\rho_1 \times \delta([\nu^{-k+1}\rho_1, \nu^{-l}\rho_1]) \rtimes \sigma_{sp}.\end{aligned}$$

By Lemma 2.5, there is an irreducible subquotient π of $\delta([\nu^{-k+1}\rho_1, \nu^{-l}\rho_1]) \rtimes \sigma_{sp}$ such that $L(\delta([\nu^{-k}\rho_1, \nu^{-l}\rho_1]); \sigma_{sp})$ is a subrepresentation of $\nu^k\rho_1 \rtimes \pi$. Frobenius reciprocity implies that $\mu^*(\nu^k\rho_1 \rtimes \pi)$ contains $\delta([\nu^{-k}\rho_1, \nu^{-l}\rho_1]) \otimes \sigma_{sp}$.

Using the structural formula and a description of the Jacquet modules of strongly positive representations, provided in [18, Theorem 4.6] and [23, Section 7], we

deduce that $\mu^*(\delta([\nu^{-k+1}\rho_1,\nu^{-l}\rho_1])\rtimes\sigma_{sp})$ does not contain an irreducible constituent of the form $\delta([\nu^{-k}\rho_1,\nu^{-l}\rho_1])\otimes\pi_1$, with $\pi_1\in R(G)$. Thus, $\mu^*(\pi)$ contains $\delta([\nu^{-k+1}\rho_1,\nu^{-l}\rho_1])\otimes\sigma_{sp}$ and it is a direct consequence of the Langlands classification that $\pi\cong L(\delta([\nu^{-k+1}\rho_1,\nu^{-l}\rho_1]);\sigma_{sp})$. □

Note that both description of subquotients of $\delta([\nu^a\rho_0,\nu^b\rho_0])\rtimes\delta(\rho,x;\sigma)$ and their Aubert duals depend on the reduciblity point β of ρ_0 and σ [22,26]. The description of the Aubert duals happens to be slightly different in the case $\beta=0$. Accordingly we also consider two cases: Section 3 is the case $\beta=0$ (Section 5 of [22]) and Sections 4, 5, 6 is the case $\beta>0$ (Section 4 of [22]).

3. Case $\beta=0$

In this section we consider the $\beta=0$ case. Note that this implies $a\in\mathbb{Z}$.

The following irreducibility result is a direct consequence of [26, Proposition 3.1].

Proposition 3.1. *Degenerate principal series $\zeta([\nu^{-b}\rho_0,\nu^{-a}\rho_0])\rtimes\zeta(\rho,x;\sigma)$ is irreducible if and only if $a\geq 1$.*

We consider the remaining cases in the following proposition.

Proposition 3.2. *Suppose that $a\leq 0$, and write $\rho_0\rtimes\sigma=\tau_1+\tau_{-1}$, as a sum of mutually non-isomorphic irreducible tempered representations. If $-a<b$, then in $R(G)$ we have:*

$$\begin{aligned}&\zeta([\nu^{-b}\rho_0,\nu^{-a}\rho_0])\rtimes\zeta(\rho,x;\sigma)\\ &=L(\nu^{-x}\rho,\dots,\nu^{-\alpha}\rho,\nu^{-b}\rho_0,\dots,\nu^{a-1}\rho_0,\nu^a\rho_0,\nu^a\rho_0,\dots,\nu^{-1}\rho_0,\nu^{-1}\rho_0,\tau_1)\\ &+L(\nu^{-x}\rho,\dots,\nu^{-\alpha}\rho,\nu^{-b}\rho_0,\dots,\nu^{a-1}\rho_0,\nu^a\rho_0,\nu^a\rho_0,\dots,\nu^{-1}\rho_0,\nu^{-1}\rho_0,\tau_{-1})\\ &+L(\nu^{-x}\rho,\dots,\nu^{-\alpha}\rho,\nu^{-b}\rho_0,\dots,\nu^{a-2}\rho_0,\delta([\nu^{a-1}\rho_0,\nu^a\rho_0]),\dots,\delta([\nu^{-1}\rho_0,\rho_0];\sigma)).\end{aligned}$$

If $-a=b$, then in $R(G)$ we have:

$$\begin{aligned}&\zeta([\nu^{a}\rho_0,\nu^{-a}\rho_0])\rtimes\zeta(\rho,x;\sigma)\\ &=L(\nu^{-x}\rho,\dots,\nu^{-\alpha}\rho,\nu^{a}\rho_0,\nu^a\rho_0,\dots,\nu^{-1}\rho_0,\nu^{-1}\rho_0,\tau_1)\\ &+L(\nu^{-x}\rho,\dots,\nu^{-\alpha}\rho,\nu^{a}\rho_0,\nu^a\rho_0,\dots,\nu^{-1}\rho_0,\nu^{-1}\rho_0,\tau_{-1}).\end{aligned}$$

Proof. We will only comment on the case $-a<b$, since the case $-a=b$ can be handled in the same way as in the proof of [22, Theorem 5.1]. By [26, Theorem 2.1] and classification of discrete series [15,25], in $R(G)$ we have

$$\delta([\nu^a\rho_0,\nu^b\rho_0])\rtimes\delta(\rho,x;\sigma)=\sigma_1+\sigma_{-1}+L(\delta([\nu^{-b}\rho_0,\nu^{-a}\rho_0]);\delta(\rho,x;\sigma)),$$

where σ_i is a discrete series subrepresentation of $\delta([\nu^a\rho_0,\nu^b\rho_0])\rtimes\delta(\rho,x;\sigma)$ such that

$$\mu^*(\sigma_i)\geq\delta([\nu\rho_0,\nu^{-a}\rho_0])\times\delta([\nu\rho_0,\nu^b\rho_0])\times\delta([\nu^\alpha\rho,\nu^x\rho])\otimes\tau_i$$

and

$$\mu^*(\sigma_i)\ngeq\delta([\nu\rho_0,\nu^{-a}\rho_0])\times\delta([\nu\rho_0,\nu^b\rho_0])\times\delta([\nu^\alpha\rho,\nu^x\rho])\otimes\tau_{-i}$$

for $i\in\{1,-1\}$.

Since σ_i is a subrepresentation of $\delta([\nu^a\rho_0,\nu^b\rho_0])\rtimes\delta(\rho,x;\sigma)$, for $i\in\{1,-1\}$, we have

$$\begin{aligned}\sigma_i&\hookrightarrow\delta([\nu^a\rho_0,\nu^b\rho_0])\rtimes\delta(\rho,x;\sigma)\hookrightarrow\delta([\nu^a\rho_0,\nu^b\rho_0])\times\delta([\nu^\alpha\rho,\nu^x\rho])\rtimes\sigma\\ &\cong\delta([\nu^\alpha\rho,\nu^x\rho])\times\delta([\nu^a\rho_0,\nu^b\rho_0])\rtimes\sigma.\end{aligned}$$

By Lemma 2.5, there is an irreducible subquotient π_i of $\delta([\nu^a\rho_0,\nu^b\rho_0])\rtimes\sigma$ such that σ_i is a subrepresentation of $\delta([\nu^\alpha\rho,\nu^x\rho])\rtimes\pi_i$.

Using [26, Theorem 2.1] and classification of discrete series one more time, we obtain that in $R(G)$ we have

$$\delta([\nu^a\rho_0,\nu^b\rho_0])\rtimes\sigma=\sigma_1'+\sigma_{-1}'+L(\delta([\nu^{-b}\rho_0,\nu^{-a}\rho_0]);\sigma),$$

where σ_i' is a discrete series subrepresentation of $\delta([\nu^a\rho_0,\nu^b\rho_0])\rtimes\sigma$ such that

$$\mu^*(\sigma_i')\geq\delta([\nu\rho_0,\nu^{-a}\rho_0])\times\delta([\nu\rho_0,\nu^b\rho_0])\otimes\tau_i$$

and $\mu^*(\sigma_i')\not\geq\delta([\nu\rho_0,\nu^{-a}\rho_0])\times\delta([\nu\rho_0,\nu^b\rho_0])\otimes\tau_{-i}$, for $i\in\{1,-1\}$. Also, note that $\mu^*(L(\delta([\nu^{-b}\rho_0,\nu^{-a}\rho_0]);\sigma))$ does not contain $\delta([\nu^a\rho_0,\nu^b\rho_0])\otimes\sigma$, since both $\mu^*(\sigma_1')$ and $\mu^*(\sigma_{-1}')$ contain $\delta([\nu^a\rho_0,\nu^b\rho_0])\otimes\sigma$, and $\mu^*(\delta([\nu^a\rho_0,\nu^b\rho_0])\rtimes\sigma)$ contains $\delta([\nu^a\rho_0,\nu^b\rho_0])\otimes\sigma$ with multiplicity two.

Thus, $\pi_i\cong\sigma_i'$. Now Lemma 2.9 and [22, Theorem 5.1] imply that

$$\widehat{\sigma_i}\cong L(\nu^{-x}\rho,\dots,\nu^{-\alpha}\rho,\nu^{-b}\rho_0,\dots,\nu^{a-1}\rho_0,\nu^a\rho_0,\nu^a\rho_0,\dots,\nu^{-1}\rho_0,\nu^{-1}\rho_0,\tau_{-i}).$$

In the same way we obtain that $L(\delta([\nu^{-b}\rho_0,\nu^{-a}\rho_0]);\delta(\rho,x;\sigma))$ is a subrepresentation of $\delta([\nu^\alpha\rho,\nu^x\rho])\rtimes L(\delta([\nu^{-b}\rho_0,\nu^{-a}\rho_0]);\sigma)$. By Lemma 2.9, it remains to determine the Aubert dual of $L(\delta([\nu^{-b}\rho_0,\nu^{-a}\rho_0]);\sigma)$. Since $b>0$, if $b\geq -a+2$, then using the first part of Proposition 2.10 we get that $L(\delta([\nu^{-b}\rho_0,\nu^{-a}\rho_0]);\sigma)$ is a subrepresentation of $\nu^b\rho_0\rtimes L(\delta([\nu^{-b+1}\rho_0,\nu^{-a}\rho_0]);\sigma)$. Also, it follows from the structural formula that $\mu^*(L(\delta([\nu^{-b+1}\rho_0,\nu^{-a}\rho_0]);\sigma))$ does not contain an irreducible constituent of the form $\nu^b\rho_0\otimes\pi'$. Using Lemma 2.7 and repeating this procedure, we deduce that the Aubert dual of $L(\delta([\nu^{-b}\rho_0,\nu^{-a}\rho_0]);\sigma)$ is an irreducible subrepresentation of

$$\nu^{-b}\rho_0\times\cdots\times\nu^{a-2}\rho_0\rtimes L(\delta([\nu^{a-1}\widehat{\rho_0,\nu^{-a}}\rho_0]);\sigma).$$

The representation $L(\delta([\nu^{a-1}\rho_0,\nu^{-a}\rho_0]);\sigma)$ is the unique irreducible quotient of the induced representation $\delta([\nu^a\rho_0,\nu^{-a+1}\rho_0])\rtimes\sigma$. By [26, Theorem 2.1], we see that $\delta([\nu^a\rho_0,\nu^{-a+1}\rho_0])\rtimes\sigma$ contains two irreducible subrepresentations and Frobenius reciprocity implies that each of them contains an irreducible constituent of the form $\nu^{-a+1}\rho_0\otimes\pi$ in the Jacquet module with respect to the appropriate parabolic subgroup.

If $\nu^{-a+1}\rho_0\otimes\pi$ is an irreducible constituent of $\mu^*(\delta([\nu^a\rho_0,\nu^{-a+1}\rho_0])\rtimes\sigma)$, it follows from the structural formula that π is an irreducible subquotient of $\delta([\nu^a\rho_0,\nu^{-a}\rho_0])\rtimes\sigma$, which is a length two representation. Thus, there are only two irreducible constituents of the form $\nu^{-a+1}\rho_0\otimes\pi$ appearing in $\mu^*(\delta([\nu^a\rho_0,\nu^{-a+1}\rho_0])\rtimes\sigma)$, and $\mu^*(L(\delta([\nu^{a-1}\rho_0,\nu^{-a}\rho_0]);\sigma))$ does not contain any of them.

From the second part of Proposition 2.10 it follows that $L(\delta([\nu^{a-1}\rho_0,\nu^{-a}\rho_0]);\sigma)$ is a subrepresentation of $\nu^{-a}\rho_0\rtimes L(\delta([\nu^{a-1}\rho_0,\nu^{-a-1}\rho_0]);\sigma)$.

Since $a-1\leq -1$, using the first part of Proposition 2.10 we also obtain

$$L(\delta([\nu^{a-1}\rho_0,\nu^{-a-1}\rho_0]);\sigma)\hookrightarrow\nu^{-a+1}\rho_0\rtimes L(\delta([\nu^a\rho_0,\nu^{-a-1}\rho_0]);\sigma).$$

Consequently, $L(\delta([\nu^{a-1}\rho_0,\nu^{-a}\rho_0]);\sigma)$ is a subrepresentation of

$$\nu^{-a}\rho_0\times\nu^{-a+1}\rho_0\rtimes L(\delta([\nu^a\rho_0,\nu^{-a-1}\rho_0]);\sigma),$$

and there is an irreducible subquotient π_2 of

$$\nu^{-a}\rho_0\times\nu^{-a+1}\rho_0$$

such that $L(\delta([\nu^{a-1}\rho_0,\nu^{-a}\rho_0]);\sigma)$ is a subrepresentation of

$$\pi_2 \rtimes L(\delta([\nu^{a}\rho_0,\nu^{-a-1}\rho_0]);\sigma).$$

Since $\mu^*(L(\delta([\nu^{a-1}\rho_0,\nu^{-a}\rho_0]);\sigma))$ does not contain an irreducible constituent of the form $\nu^{-a-1}\rho_0\otimes\pi'$, it follows that $\pi_2\ncong\delta([\nu^{-a}\rho_0,\nu^{-a+1}\rho_0])$, so we have that $\pi_2\cong\zeta([\nu^{-a}\rho_0,\nu^{-a+1}\rho_0])$. It can also be seen, following the same arguments as for $L(\delta([\nu^{a-1}\rho_0,\nu^{-a}\rho_0]);\sigma)$, that $\mu^*(L(\delta([\nu^{a}\rho_0,\nu^{-a-1}\rho_0]);\sigma))$ does not contain an irreducible constituent of the form $\nu^{i}\rho_0\otimes\pi'$, for $i\in\{-a+1,-a\}$. Now Lemma 2.7 implies that $L(\delta([\nu^{a-1}\widehat{\rho_0,\nu^{-a}}\rho_0]);\sigma)$ is the unique irreducible subrepresentation of

$$\delta([\nu^{a-1}\rho_0,\nu^{a}\rho_0])\rtimes L(\delta([\nu^{a}\widehat{\rho_0,\nu^{-a}}\rho_0]);\sigma),$$

and a repeated application of this procedure ends the proof. □

4. Case $a\geq 1$

From now on, we assume that $\beta>0$. In this section we consider the case $a\geq 1$. Let us first consider the more complicated case $\rho_0\cong\rho$. Directly from [26, Proposition 3.1] we obtain the following reducibility criterion.

Proposition 4.1. *Degenerate principal series $\zeta([\nu^{-b}\rho,\nu^{-a}\rho])\rtimes\zeta(\rho,x;\sigma)$ reduces if and only if one of the following holds:*

- $a\leq\alpha-1\leq b<x$,
- $a\leq x+1$ *and* $x<b$.

Proposition 4.2. *If $a\leq\alpha-1\leq b<x$, then in $R(G)$ we have*

$$\begin{aligned}&\zeta([\nu^{-b}\rho,\nu^{-a}\rho])\rtimes\zeta(\rho,x;\sigma)\\ &=L(\nu^{-x}\rho,\dots,\nu^{-b-1}\rho,\nu^{-b}\rho,\nu^{-b}\rho,\dots,\nu^{-\alpha}\rho,\nu^{-\alpha}\rho,\nu^{-\alpha+1}\rho,\dots,\nu^{-a}\rho;\sigma)\\ &+L(\nu^{-x}\rho,\dots,\nu^{-b-2}\rho,\delta([\nu^{-b-1}\rho,\nu^{-b}\rho]),\dots,\delta([\nu^{-\alpha}\rho,\nu^{-\alpha+1}\rho]),\nu^{-\alpha+2}\rho,\dots,\nu^{-a}\rho;\sigma).\end{aligned}$$

Proof. In $R(G)$ we have

$$\begin{aligned}\delta([\nu^{a}\rho,\nu^{b}\rho])\rtimes\delta(\rho,x;\sigma)&=L(\delta([\nu^{-b}\rho,\nu^{-a}\rho]);\delta(\rho,x;\sigma))\\ &+L(\delta([\nu^{-\alpha+2}\rho,\nu^{-a}\rho]);\sigma_{sp}),\end{aligned}$$

where σ_{sp} is the unique irreducible subrepresentation of $\delta([\nu^{\alpha-1}\rho,\nu^{b}\rho])\rtimes\delta(\rho,x;\sigma)$. We note that σ_{sp} is a strongly positive discrete series.

Let us first determine the Aubert dual of $L(\delta([\nu^{-b}\rho,\nu^{-a}\rho]);\delta(\rho,x;\sigma))$. The third part of Proposition 2.10 implies that

$$L(\delta([\nu^{-b}\rho,\nu^{-a}\rho]);\delta(\rho,x;\sigma))\hookrightarrow\nu^{x}\rho\rtimes L(\delta([\nu^{-b}\rho,\nu^{-a}\rho]);\delta(\rho,x-1;\sigma)).$$

Using the structural formula and a description of the Jacquet modules of strongly positive representations, we deduce that $\mu^*(\delta([\nu^{-b}\rho,\nu^{-a}\rho])\rtimes\delta(\rho,x-1;\sigma))$ does not contain an irreducible constituent of the form $\nu^{x}\rho\otimes\pi_2$. Repeating this procedure and using Lemma 2.7, we obtain that the Aubert dual of $L(\delta([\nu^{-b}\rho,\nu^{-a}\rho]);\delta(\rho,x;\sigma))$ is an irreducible subrepresentation of

$$\nu^{-x}\rho\times\nu^{-x+1}\rho\times\cdots\times\nu^{-b-1}\rho\rtimes L(\delta([\nu^{-b}\rho,\widehat{\nu^{-a}\rho}]);\delta(\rho,b;\sigma)).$$

Since $\nu^b\rho \rtimes \delta(\rho, b; \sigma)$ is irreducible, by [26, Proposition 3.1], we have

$$\begin{aligned} L(\delta([\nu^{-b}\rho, \nu^{-a}\rho]); \delta(\rho, b; \sigma)) &\hookrightarrow \delta([\nu^{-b+1}\rho, \nu^{-a}\rho]) \times \nu^{-b}\rho \rtimes \delta(\rho, b; \sigma) \\ &\cong \delta([\nu^{-b+1}\rho, \nu^{-a}\rho]) \times \nu^{b}\rho \rtimes \delta(\rho, b; \sigma) \\ &\hookrightarrow \delta([\nu^{-b+1}\rho, \nu^{-a}\rho]) \times \nu^{b}\rho \times \nu^{b}\rho \rtimes \delta(\rho, b-1; \sigma) \\ &\cong \nu^{b}\rho \times \nu^{b}\rho \times \delta([\nu^{-b+1}\rho, \nu^{-a}\rho]) \rtimes \delta(\rho, b-1; \sigma). \end{aligned}$$

Note that $\delta([\nu^{-b+1}\rho, \nu^{-a}\rho]) \rtimes \delta(\rho, b-1; \sigma)$ is irreducible, thus isomorphic to $L(\delta([\nu^{-b+1}\rho, \nu^{-a}\rho]); \delta(\rho, b-1; \sigma))$ and that $\mu^*(\delta([\nu^{-b+1}\rho, \nu^{-a}\rho]) \rtimes \delta(\rho, b-1; \sigma))$ does not contain an irreducible constituent of the form $\nu^b\rho \otimes \pi$. A repeated application of Lemma 2.7 and the previous procedure implies that the Aubert dual of $L(\delta([\nu^{-b}\rho, \nu^{-a}\rho]); \delta(\rho, b; \sigma))$ is an irreducible subrepresentation of

$$\nu^{-b}\rho \times \nu^{-b}\rho \times \cdots \times \nu^{-\alpha}\rho \times \nu^{-\alpha}\rho \rtimes L(\delta(\widehat{[\nu^{-\alpha+1}\rho, \nu^{-a}\rho]}); \sigma).$$

Since the induced representation $\delta([\nu^{-\alpha+1}\rho, \nu^{-a}\rho]) \rtimes \sigma$ is also irreducible, its Jacquet module with respect to the appropriate parabolic subgroup contains $\nu^{\alpha-1}\rho \otimes \cdots \otimes \nu^a\rho \otimes \sigma$. Now Lemma 2.6 implies that the Aubert dual of $L(\delta([\nu^{-\alpha+1}\rho, \nu^{-a}\rho]); \sigma)$ is the unique irreducible subrepresentation of $\nu^{-\alpha+1}\rho \times \cdots \times \nu^{-a}\rho \rtimes \sigma$. Altogether, the Aubert dual of $L(\delta([\nu^{-b}\rho, \nu^{-a}\rho]); \delta(\rho, x; \sigma))$ is isomorphic to

$$L(\nu^{-x}\rho, \dots, \nu^{-b-1}\rho, \nu^{-b}\rho, \nu^{-b}\rho, \dots, \nu^{-\alpha}\rho, \nu^{-\alpha}\rho, \nu^{-\alpha+1}\rho, \dots, \nu^{-a}\rho; \sigma).$$

It remains to determine the Aubert dual of $L(\delta([\nu^{-\alpha+2}\rho, \nu^{-a}\rho]); \sigma_{sp})$.

If $x > b+1$, it follows from [18, Section 3] that σ_{sp} is a subrepresentation of $\nu^x\rho \rtimes \sigma'_{sp}$, where σ'_{sp} is the unique irreducible subrepresentation of $\delta([\nu^{\alpha-1}\rho, \nu^b\rho]) \rtimes \delta(\rho, x-1; \sigma)$. The third part of Proposition 2.10 implies $L(\delta([\nu^{-\alpha+2}\rho, \nu^{-a}\rho]); \sigma_{sp})$ is a subrepresentation of $\nu^x\rho \rtimes L(\delta([\nu^{-\alpha+2}\rho, \nu^{-a}\rho]); \sigma'_{sp})$. Using Lemma 2.7 and continuing in the same way, we deduce the Aubert dual of $L(\delta([\nu^{-\alpha+2}\rho, \nu^{-a}\rho]); \sigma_{sp})$ is a subrepresentation of

$$\nu^{-x}\rho \times \cdots \times \nu^{-b-2}\rho \rtimes L(\delta(\widehat{[\nu^{-\alpha+2}\rho, \nu^{-a}\rho]}); \sigma^{(1)}_{sp}),$$

where $\sigma^{(1)}_{sp}$ is the unique irreducible subrepresentation of $\delta([\nu^{\alpha-1}\rho, \nu^b\rho]) \rtimes \delta(\rho, b+1; \sigma)$. From embeddings of strongly positive representations ([18, Section 3]), using Proposition 2.10(3) twice, we get

$$L(\delta([\nu^{-\alpha+2}\rho, \nu^{-a}\rho]); \sigma^{(1)}_{sp}) \hookrightarrow \nu^b\rho \times \nu^{b+1}\rho \rtimes L(\delta([\nu^{-\alpha+2}\rho, \nu^{-a}\rho]); \sigma^{(2)}_{sp}),$$

where $\sigma^{(2)}_{sp}$ is the unique irreducible subrepresentation of $\delta([\nu^{\alpha-1}\rho, \nu^{b-1}\rho]) \rtimes \delta(\rho, b; \sigma)$. Now [18, Theorem 3.4] implies

$$L(\delta([\nu^{-\alpha+2}\rho, \nu^{-a}\rho]); \sigma^{(1)}_{sp}) \hookrightarrow \zeta([\nu^b\rho, \nu^{b+1}\rho]) \rtimes L(\delta([\nu^{-\alpha+2}\rho, \nu^{-a}\rho]); \sigma^{(2)}_{sp}).$$

Using a repeated application of Lemma 2.7 and continuing in the same way, we obtain that the Aubert dual of $L(\delta([\nu^{-\alpha+2}\rho, \nu^{-a}\rho]); \sigma^{(1)}_{sp})$ is a subrepresentation of

$$\delta([\nu^{-b-1}\rho, \nu^{-b}\rho]) \times \cdots \times \delta([\nu^{-\alpha}\rho, \nu^{-\alpha+1}\rho]) \rtimes L(\delta(\widehat{[\nu^{-\alpha+2}\rho, \nu^{-a}\rho]}); \sigma),$$

and it can be seen in the same way as in the case of $L(\delta([\nu^{-\alpha+1}\rho, \nu^{-a}\rho]); \sigma)$ that the Aubert dual of $L(\delta([\nu^{-\alpha+2}\rho, \nu^{-a}\rho]); \sigma)$ is the unique irreducible subrepresentation of $\nu^{-\alpha+2}\rho \times \cdots \times \nu^{-a}\rho \rtimes \sigma$. This ends the proof. □

Proposition 4.3. *Suppose that $a \leq x+1$ and $x < b$. If $a > \alpha$, then in $R(G)$ we have*

$$\begin{aligned}&\zeta([\nu^{-b}\rho, \nu^{-a}\rho]) \rtimes \zeta(\rho, x; \sigma)\\ &= L(\nu^{-b}\rho, \dots, \nu^{-x-1}\rho, \nu^{-x}\rho, \nu^{-x}\rho, \dots, \nu^{-a}\rho, \nu^{-a}\rho, \nu^{-a+1}\rho, \dots, \nu^{-\alpha}\rho; \sigma)\\ &+L(\nu^{-b}\rho, \dots, \nu^{-x-2}\rho, \delta([\nu^{-x-1}\rho, \nu^{-x}\rho]), \dots, \delta([\nu^{-a}\rho, \nu^{-a+1}\rho]), \nu^{-a+2}\rho, \dots, \nu^{-\alpha}\rho; \sigma).\end{aligned}$$

If $a \leq \alpha$, then in $R(G)$ we have

$$\begin{aligned}&\zeta([\nu^{-b}\rho, \nu^{-a}\rho]) \rtimes \zeta(\rho, x; \sigma)\\ &= L(\nu^{-b}\rho, \dots, \nu^{-x-1}\rho, \nu^{-x}\rho, \nu^{-x}\rho, \dots, \nu^{-\alpha}\rho, \nu^{-\alpha}\rho, \nu^{-\alpha+1}\rho, \dots, \nu^{-a}\rho; \sigma)\\ &+L(\nu^{-b}\rho, \dots, \nu^{-x-2}\rho, \delta([\nu^{-x-1}\rho, \nu^{-x}\rho]), \dots, \delta([\nu^{-\alpha-1}\rho, \nu^{-\alpha}\rho]); \sigma_{sp}),\end{aligned}$$

where σ_{sp} is the unique irreducible subrepresentation of $\nu^a\rho \times \cdots \times \nu^\alpha\rho \rtimes \sigma$.

Proof. Under the assumptions of the proposition, in $R(G)$ we have

$$\begin{aligned}\delta([\nu^a\rho, \nu^b\rho]) \rtimes \delta(\rho, x; \sigma) &= L(\delta([\nu^{-b}\rho, \nu^{-a}\rho]); \delta(\rho, x; \sigma))\\ &+L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]); \delta(\rho, b; \sigma)).\end{aligned}$$

Let us first determine the Aubert dual of $L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]); \delta(\rho, b; \sigma))$. Using the third part of Proposition 2.10 and Lemma 2.7, we obtain that it is an irreducible subrepresentation of

$$\nu^{-b}\rho \times \cdots \times \nu^{-x-1}\rho \rtimes L(\delta([\nu^{-x}\rho, \widehat{\nu^{-a}\rho}]); \delta(\rho, x; \sigma)).$$

Note that the induced representation $\nu^x\rho \rtimes \delta(\rho, x; \sigma)$ is irreducible. Using the second part of Proposition 2.10 we deduce that $L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]); \delta(\rho, x; \sigma))$ is a subrepresentation of $\nu^x\rho \rtimes L(\delta([\nu^{-x+1}\rho, \nu^{-a}\rho]); \delta(\rho, x; \sigma))$, and then the third part of the same proposition gives an embedding

$$L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]); \delta(\rho, x; \sigma)) \hookrightarrow \nu^x\rho \times \nu^x\rho \rtimes L(\delta([\nu^{-x+1}\rho, \nu^{-a}\rho]); \delta(\rho, x-1; \sigma)).$$

We can continue in the same way to obtain the Aubert dual of $L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]); \delta(\rho, x; \sigma))$ using Lemma 2.7.

If $a = \alpha$, it follows that the Aubert dual of $L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]); \delta(\rho, x; \sigma))$ is an irreducible subrepresentation of

$$\nu^{-x}\rho \times \nu^{-x}\rho \times \cdots \times \nu^{-a}\rho \times \nu^{-a}\rho \rtimes \sigma.$$

If $a > \alpha$, it follows that the Aubert dual of $L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]); \delta(\rho, x; \sigma))$ is an irreducible subrepresentation of

$$\nu^{-x}\rho \times \nu^{-x}\rho \times \cdots \times \nu^{-a}\rho \times \nu^{-a}\rho \rtimes \delta(\rho, \widehat{a-1}; \sigma),$$

and it follows from [20, Theorem 3.5] that $\delta(\rho, \widehat{a-1}; \sigma) \cong L(\nu^{-a+1}\rho, \dots, \nu^{-\alpha}\rho; \sigma)$. Finally, if $a < \alpha$, it follows that the Aubert dual of $L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]); \delta(\rho, x; \sigma))$ is an irreducible subrepresentation of

$$\nu^{-x}\rho \times \nu^{-x}\rho \times \cdots \times \nu^{-\alpha}\rho \times \nu^{-\alpha}\rho \rtimes L(\delta([\widehat{\nu^{-\alpha+1}\rho, \nu^{-a}\rho}]); \sigma),$$

and the Aubert dual of $L(\delta([\nu^{-\alpha+1}\rho, \nu^{-a}\rho]); \sigma)$ is the unique irreducible subrepresentation of $\nu^{-\alpha+1}\rho \times \cdots \times \nu^{-a}\rho \rtimes \sigma$, as before.

Let us now determine the Aubert dual of $L(\delta([\nu^{-b}\rho,\nu^{-a}\rho]);\delta(\rho,x;\sigma))$. First, using Lemma 2.7, together with the first part of Proposition 2.10, we obtain that it is an irreducible subrepresentation of

$$\nu^{-b}\rho\times\cdots\times\nu^{-x-2}\rho\rtimes L(\delta([\nu^{-x-1}\rho,\widehat{\nu^{-a}\rho}]);\delta(\rho,x;\sigma)).$$

Note that, by [26, Proposition 3.1], in $R(G)$ we have

$$\begin{aligned}\delta([\nu^{a}\rho,\nu^{x+1}\rho])\rtimes\delta(\rho,x;\sigma)) &= L(\delta([\nu^{-x-1}\rho,\nu^{-a}\rho]);\delta(\rho,x;\sigma))\\ &+L(\delta([\nu^{-x}\rho,\nu^{-a}\rho]);\delta(\rho,x+1;\sigma)).\end{aligned}$$

Since $\delta([\nu^{a}\rho,\nu^{x}\rho])\rtimes\delta(\rho,x;\sigma)$ is irreducible, the structural formula directly implies that $\nu^{x+1}\rho\otimes\delta([\nu^{a}\rho,\nu^{x}\rho])\rtimes\delta(\rho,x;\sigma)$ is the unique irreducible constituent of the form $\nu^{x+1}\rho\otimes\pi$ appearing in $\mu^*(\delta([\nu^{a}\rho,\nu^{x+1}\rho])\rtimes\delta(\rho,x;\sigma))$, which appears there with multiplicity one, and it obviously appears in $\mu^*(L(\delta([\nu^{-x}\rho,\nu^{-a}\rho]);\delta(\rho,x+1;\sigma)))$. Thus, $\mu^*(L(\delta([\nu^{-x-1}\rho,\nu^{-a}\rho]);\delta(\rho,x;\sigma)))$ does not contain an irreducible constituent of the form $\nu^{x+1}\rho\otimes\pi$.

Now, using the third part of Proposition 2.10, and then the first part of the same proposition, we obtain an embedding

$$\begin{aligned}&L(\delta([\nu^{-x-1}\rho,\nu^{-a}\rho]);\delta(\rho,x;\sigma))\\ \hookrightarrow\ &\zeta([\nu^{x}\rho,\nu^{x+1}\rho])\rtimes L(\delta([\nu^{-x}\rho,\nu^{-a}\rho]);\delta(\rho,x-1;\sigma)).\end{aligned}$$

Also, in the same way as before we conclude that $\mu^*(L(\delta([\nu^{-x}\rho,\nu^{-a}\rho]);\delta(\rho,x-1;\sigma)))$ does not contain an irreducible constituent of the form $\nu^{i}\rho\otimes\pi$ for $i\in\{x,x+1\}$. Using Lemma 2.7 and repeating this procedure, we obtain an embedding of the Aubert dual of $L(\delta([\nu^{-x-1}\rho,\nu^{-a}\rho]);\delta(\rho,x;\sigma))$.

If $a=\alpha$, it follows that the Aubert dual of $L(\delta([\nu^{-x-1}\rho,\nu^{-a}\rho]);\delta(\rho,x;\sigma))$ is an irreducible subrepresentation of

$$\delta([\nu^{-x-1}\rho,\nu^{-x}\rho])\times\cdots\times\delta([\nu^{-a-1}\rho,\nu^{-a}\rho])\rtimes L(\widehat{\nu^{-\alpha}\rho};\sigma),$$

and it follows from [20, Theorem 3.5] that the Aubert dual of $L(\nu^{-\alpha}\rho;\sigma)$ is isomorphic to $\delta(\rho,\alpha;\sigma)$. Note that for $a=\alpha$ we have $\sigma_{sp}\cong\delta(\rho,\alpha;\sigma)$.

If $a>\alpha$, it follows that the Aubert dual of $L(\delta([\nu^{-x-1}\rho,\nu^{-a}\rho]);\delta(\rho,x;\sigma))$ is an irreducible subrepresentation of

$$\delta([\nu^{-x-1}\rho,\nu^{-x}\rho])\times\cdots\times\delta([\nu^{-a-1}\rho,\nu^{-a}\rho])\rtimes\delta(\rho,\widehat{a-2};\sigma),$$

and it follows from [20, Theorem 3.5] that the Aubert dual of $\delta(\rho,a-2;\sigma)$ is the unique irreducible subrepresentation of $\nu^{-a+2}\rho\times\cdots\times\nu^{-\alpha}\rho\rtimes\sigma$.

If $a<\alpha$, it follows that the Aubert dual of $L(\delta([\nu^{-x-1}\rho,\nu^{-a}\rho]);\delta(\rho,x;\sigma))$ is an irreducible subrepresentation of

$$\delta([\nu^{-x-1}\rho,\nu^{-x}\rho])\times\cdots\times\delta([\nu^{-\alpha-1}\rho,\nu^{-\alpha}\rho])\rtimes L(\delta([\widehat{\nu^{-\alpha}\rho,\nu^{-a}}\rho]);\sigma),$$

and it follows from [20, Theorem 3.5] that the Aubert dual of $L(\delta([\nu^{-\alpha}\rho,\nu^{-a}\rho]);\sigma)$ is the unique irreducible subrepresentation of $\nu^{a}\rho\times\cdots\times\nu^{\alpha}\rho\rtimes\sigma$, which is strongly positive. This proves the proposition. □

Let us now consider the case $\rho_0\ncong\rho$. The following proposition can be proved in the same way as Proposition 4.3, using Lemma 2.9, details being left to the reader.

Proposition 4.4. *Degenerate principal series $\zeta([\nu^{-b}\rho_0,\nu^{-a}\rho_0])\rtimes\zeta(\rho,x;\sigma)$ is irreducible if and only if either $a>\beta$ or $b<\beta$. If $\zeta([\nu^{-b}\rho_0,\nu^{-a}\rho_0])\rtimes\zeta(\rho,x;\sigma)$ reduces, in $R(G)$ we have*

$$\begin{aligned}&\zeta([\nu^{-b}\rho_0,\nu^{-a}\rho_0])\rtimes\zeta(\rho,x;\sigma)\\ &=L(\nu^{-b}\rho_0,\dots,\nu^{-a}\rho_0,\nu^{-x}\rho,\dots,\nu^{-\alpha}\rho;\sigma)\\ &+L(\nu^{-b}\rho_0,\dots,\nu^{-\beta-1}\rho_0,\nu^{-x}\rho,\dots,\nu^{-\alpha}\rho;\sigma_{sp}),\end{aligned}$$

where σ_{sp} is the unique irreducible subrepresentation of $\nu^a\rho_0\times\cdots\times\nu^\beta\rho_0\rtimes\sigma$.

5. Case $a\le 0$

In this section we analyze the case when $a\le 0$. To make the notation uniform, we let $\tau^{(1)}=\rho_0\rtimes\sigma$ if $a\in\mathbb{Z}$ and $\tau^{(1)}=\sigma$ if $a\notin\mathbb{Z}$. Also, if $a\notin\mathbb{Z}$, let $\tau^{(2)}$ denote the unique irreducible (strongly positive) subrepresentation of $\nu^{\frac12}\rho_0\times\nu^{\frac32}\rho_0\times\cdots\times\nu^\beta\rho_0\rtimes\sigma$. If $a\in\mathbb{Z}$, let τ' denote the unique irreducible (strongly positive) subrepresentation of $\nu\rho_0\times\cdots\times\nu^\beta\rho_0\rtimes\sigma$ and let $\tau^{(2)}$ denote an irreducible (tempered) subrepresentation of $\rho_0\rtimes\tau'$ which does not contain an irreducible representation of the form $\nu\rho_0\otimes\pi$ in the Jacquet module with respect to the appropriate parabolic subgroup. We note that such a subrepresentation of $\rho_0\rtimes\tau'$ is unique by [29, Section 4].

For an irreducible self-contragredient cuspidal representation $\rho_1\in R(GL)$ and an irreducible cuspidal representation $\sigma_1\in R(G)$ such that $\nu^{\frac12}\rho_1\rtimes\sigma_1$ reduces, we denote by $\tau(\rho_1,\sigma_1)$ the unique irreducible tempered subrepresentation of $\delta([\nu^{-\frac12}\rho_1,\nu^{\frac12}\rho_1])\rtimes\sigma_1$ which is not a subrepresentation of $\nu^{\frac12}\rho_1\times\nu^{\frac12}\rho_1\rtimes\sigma_1$, Also, for a real number y let $\lceil y\rceil$ stand for the smallest integer which is not smaller than y.

We will again first consider the more complicated case $\rho_0\cong\rho$. Let us first assume that $-a=b$.

Proposition 5.1. *Degenerate principal series $\zeta([\nu^{-a}\rho,\nu^a\rho])\rtimes\zeta(\rho,x;\sigma)$ is irreducible if and only if either $-a\le\alpha-2$ or $-a=x$. If $\alpha-2<-a<x$, in $R(G)$ we have*

$$\begin{aligned}&\zeta([\nu^{-a}\rho,\nu^a\rho])\rtimes\zeta(\rho,x;\sigma)\\ &=L(\nu^{-x}\rho,\dots,\nu^{a-1}\rho,\nu^a\rho,\nu^a\rho,\nu^a\rho,\dots,\nu^{-\alpha}\rho,\nu^{-\alpha}\rho,\nu^{-\alpha}\rho,\nu^{-\alpha+1}\rho,\nu^{-\alpha+1}\rho,\dots,\\ &\nu^{\lceil\alpha\rceil-\alpha-1}\rho,\nu^{\lceil\alpha\rceil-\alpha-1}\rho;\tau^{(1)})+\pi,\end{aligned}$$

where

$$\begin{aligned}\pi\cong L(&\nu^{-x}\rho,\dots,\nu^{a-2}\rho,\delta([\nu^{a-1}\rho,\nu^a\rho]),\nu^a\rho,\dots,\delta([\nu^{-\alpha}\rho,\nu^{-\alpha+1}\rho]),\nu^{-\alpha+1}\rho,\\ &\nu^{-\alpha+2}\rho,\nu^{-\alpha+2}\rho,\dots,\nu^{\lceil\alpha\rceil-\alpha-1}\rho,\nu^{\lceil\alpha\rceil-\alpha-1}\rho;\tau^{(1)})\end{aligned}$$

if $\alpha\ge\frac32$,

$$\pi\cong L(\nu^{-x}\rho,\dots,\nu^{a-2}\rho,\delta([\nu^{a-1}\rho,\nu^a\rho]),\nu^a\rho,\dots,\delta([\nu^{-2}\rho,\nu^{-1}\rho]),\nu^{-1}\rho,\delta([\nu^{-1}\rho,\rho]);\sigma)$$

if $\alpha=1$,

$$\begin{aligned}\pi\cong L(&\nu^{-x}\rho,\dots,\nu^{a-2}\rho,\delta([\nu^{a-1}\rho,\nu^a\rho]),\nu^a\rho,\dots,\delta([\nu^{-\frac52}\rho,\nu^{-\frac32}\rho]),\nu^{-\frac32}\rho,\\ &\delta([\nu^{-\frac32}\rho,\nu^{-\frac12}\rho]);\tau(\rho,\sigma))\end{aligned}$$

if $\alpha=\frac12$.

If $-a > x$, in $R(G)$ we have

$$\begin{aligned}&\zeta([\nu^{-a}\rho,\nu^{a}\rho])\rtimes\zeta(\rho,x;\sigma)\\ =\ &L(\nu^{a}\rho,\nu^{a}\rho,\dots,\nu^{-x-1}\rho,\nu^{-x-1}\rho,\nu^{-x}\rho,\nu^{-x}\rho,\nu^{-x}\rho,\dots,\nu^{-\alpha}\rho,\nu^{-\alpha}\rho,\nu^{-\alpha}\rho,\\ &\nu^{-\alpha+1}\rho,\nu^{-\alpha+1}\rho,\dots,\nu^{\lceil\alpha\rceil-\alpha-1}\rho,\nu^{\lceil\alpha\rceil-\alpha-1}\rho;\tau^{(1)})\\ +\ &L(\nu^{a}\rho,\nu^{a}\rho,\dots,\nu^{-x-2}\rho,\nu^{-x-2}\rho,\nu^{-x-1}\rho,\delta([\nu^{-x-1}\rho,\nu^{-x}\rho]),\dots,\\ &\nu^{-\alpha-1}\rho,\delta([\nu^{-\alpha-1}\rho,\nu^{-\alpha}\rho]),\nu^{-\alpha}\rho,\dots,\nu^{\lceil\alpha\rceil-\alpha-1}\rho;\tau^{(2)}).\end{aligned}$$

Proof. Reducibility of $\delta([\nu^{-a}\rho,\nu^{a}\rho])\rtimes\delta(\rho,x;\sigma)$ is an integral part of the classification of discrete series. If such an induced representation reduces, it is a direct sum of two mutually non-isomorphic irreducible tempered representations, whose Aubert duals can be easily obtained from [22, Theorem 4.11, Theorem 4.16, Theorem 4.21]. □

Now we deal with the case $-a < b$. The reducibility criterion follows from [26, Theorem 4.1(i)].

Proposition 5.2. *Degenerate principal series $\zeta([\nu^{-b}\rho,\nu^{-a}\rho])\rtimes\zeta(\rho,x;\sigma)$ is irreducible if and only if one of the following holds:*

- $b < \alpha - 1$,
- $-a < \alpha - 1$ *and* $b = x$.

Other possibilities will be studied using a case-by-case consideration.

Proposition 5.3. *Suppose that $\alpha - 1 \le -a < b < x$. Let*

$$\begin{aligned}\pi_1 \cong\ &L(\nu^{-x}\rho,\dots,\nu^{-b-2}\rho,\delta([\nu^{-b-1}\rho,\nu^{-b}\rho]),\dots,\delta([\nu^{a-2}\rho,\nu^{a-1}\rho]),\\ &\delta([\nu^{a-1}\rho,\nu^{a}\rho]),\nu^{a}\rho,\dots,\delta([\nu^{-\alpha}\rho,\nu^{-\alpha+1}\rho]),\nu^{-\alpha+1}\rho,\\ &\nu^{-\alpha+2}\rho,\nu^{-\alpha+2}\rho,\dots,\nu^{\lceil\alpha\rceil-\alpha-1}\rho,\nu^{\lceil\alpha\rceil-\alpha-1}\rho;\tau^{(1)})\end{aligned}$$

if $\alpha \ge \frac{3}{2}$,

$$\begin{aligned}\pi_1 \cong\ &L(\nu^{-x}\rho,\dots,\nu^{-b-2}\rho,\delta([\nu^{-b-1}\rho,\nu^{-b}\rho]),\dots,\delta([\nu^{a-2}\rho,\nu^{a-1}\rho]),\\ &\delta([\nu^{a-1}\rho,\nu^{a}\rho]),\nu^{a}\rho,\dots,\delta([\nu^{-2}\rho,\nu^{-1}\rho]),\nu^{-1}\rho,\delta([\nu^{-1}\rho,\rho]);\sigma)\end{aligned}$$

if $\alpha = 1$, and

$$\begin{aligned}&\pi_1 \cong L(\nu^{-x}\rho,\dots,\nu^{-b-2}\rho,\delta([\nu^{-b-1}\rho,\nu^{-b}\rho]),\dots,\delta([\nu^{a-2}\rho,\nu^{a-1}\rho]),\\ &\delta([\nu^{a-1}\rho,\nu^{a}\rho]),\nu^{a}\rho,\dots,\delta([\nu^{-\frac{5}{2}}\rho,\nu^{-\frac{3}{2}}\rho]),\nu^{-\frac{3}{2}}\rho,\delta([\nu^{-\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho]);\tau(\rho,\sigma)),\end{aligned}$$

if $\alpha = \frac{1}{2}$.

Also, let

$$\begin{aligned}&\pi_2 \cong L(\nu^{-x}\rho,\dots,\nu^{-b-1}\rho,\nu^{-b}\rho,\nu^{-b}\rho,\dots,\nu^{a-2}\rho,\nu^{a-2}\rho,\nu^{a-1}\rho,\delta([\nu^{a-1}\rho,\nu^{a}\rho]),\dots,\\ &\nu^{-\alpha}\rho,\delta([\nu^{-\alpha}\rho,\nu^{-\alpha+1}\rho]),\nu^{-\alpha+1}\rho,\nu^{-\alpha+2}\rho,\nu^{-\alpha+2}\rho,\dots,\nu^{\lceil\alpha\rceil-\alpha-1}\rho,\nu^{\lceil\alpha\rceil-\alpha-1}\rho;\tau^{(1)})\end{aligned}$$

if $\alpha \ge \frac{3}{2}$,

$$\begin{aligned}&\pi_2 \cong L(\nu^{-x}\rho,\dots,\nu^{-b-1}\rho,\nu^{-b}\rho,\nu^{-b}\rho,\dots,\nu^{a-2}\rho,\nu^{a-2}\rho,\nu^{a-1}\rho,\delta([\nu^{a-1}\rho,\nu^{a}\rho]),\dots,\\ &\nu^{-1}\rho,\delta([\nu^{-1}\rho,\rho]),\sigma),\end{aligned}$$

if $\alpha = 1$, *and*

$$\pi_2 \cong L(\nu^{-x}\rho, \dots, \nu^{-b-1}\rho, \nu^{-b}\rho, \nu^{-b}\rho, \dots, \nu^{a-2}\rho, \nu^{a-2}\rho, \nu^{a-1}\rho, \delta([\nu^{a-1}\rho, \nu^{a}\rho]), \dots, \nu^{-\frac{3}{2}}\rho, \delta([\nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]); \tau(\rho, \sigma))$$

if $\alpha = \frac{1}{2}$.

Then in $R(G)$ *we have*

$$\zeta([\nu^{-b}\rho, \nu^{-a}\rho]) \rtimes \zeta(\rho, x; \sigma) = L(\nu^{-x}\rho, \dots, \nu^{-b-1}\rho, \nu^{-b}\rho, \nu^{-b}\rho, \dots, \nu^{a-1}\rho, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \nu^{a}\rho, \dots, \nu^{-\alpha}\rho, \nu^{-\alpha}\rho, \nu^{-\alpha}\rho, \nu^{-\alpha+1}\rho, \nu^{-\alpha+1}\rho, \dots, \nu^{\lceil\alpha\rceil-\alpha-1}\rho, \nu^{\lceil\alpha\rceil-\alpha-1}\rho; \tau^{(1)}) + \pi_1 + \pi_2.$$

Proof. By [26, Theorem 2.1], in $R(G)$ we have

$$\delta([\nu^{a}\rho, \nu^{b}\rho]) \rtimes \delta(\rho, x; \sigma) = L(\delta([\nu^{-b}\rho, \nu^{-a}\rho]); \delta(\rho, x; \sigma)) + \sigma_1 + \sigma_2,$$

where σ_1, σ_2 are mutually non-isomorphic discrete series representations. Aubert duals of σ_1 and σ_2 have been obtained in [22, Theorems 4.11, 4.16]. It remains to determine the Aubert dual of $L(\delta([\nu^{-b}\rho, \nu^{-a}\rho]); \delta(\rho, x; \sigma))$. Using Proposition 2.10(3) and Lemma 2.7, we deduce that the Aubert dual of $L(\delta([\nu^{-b}\rho, \nu^{-a}\rho]); \delta(\rho, x; \sigma))$ is an irreducible subrepresentation of

$$\nu^{-x}\rho \times \cdots \times \nu^{-b-1}\rho \rtimes L(\delta([\nu^{-b}\rho, \widehat{\nu^{-a}\rho}]); \delta(\rho, b; \sigma)).$$

If $b > -a + 1$, we have the following embeddings and isomorphisms:

$$\begin{aligned} L(\delta([\nu^{-b}\rho, \nu^{-a}\rho]); \delta(\rho, b; \sigma)) &\hookrightarrow \delta([\nu^{-b+1}\rho, \nu^{-a}\rho]) \times \nu^{-b}\rho \rtimes \delta(\rho, b; \sigma) \\ &\cong \delta([\nu^{-b+1}\rho, \nu^{-a}\rho]) \times \nu^{b}\rho \rtimes \delta(\rho, b; \sigma) \\ &\cong \nu^{b}\rho \times \delta([\nu^{-b+1}\rho, \nu^{-a}\rho]) \rtimes \delta(\rho, b; \sigma) \\ &\hookrightarrow \nu^{b}\rho \times \delta([\nu^{-b+1}\rho, \nu^{-a}\rho]) \times \nu^{b}\rho \rtimes \delta(\rho, b-1; \sigma) \\ &\cong \nu^{b}\rho \times \nu^{b}\rho \times \delta([\nu^{-b+1}\rho, \nu^{-a}\rho]) \rtimes \delta(\rho, b-1; \sigma). \end{aligned}$$

Thus, there is an irreducible subquotient π of $\delta([\nu^{-b+1}\rho, \nu^{-a}\rho]) \rtimes \delta(\rho, b-1; \sigma)$ such that $L(\delta([\nu^{-b}\rho, \nu^{-a}\rho]); \delta(\rho, b; \sigma))$ is a subrepresentation of $\nu^{b}\rho \times \nu^{b}\rho \rtimes \pi$. Since $\mu^*(L(\delta([\nu^{-b}\rho, \nu^{-a}\rho]); \delta(\rho, b; \sigma))) \geq \delta([\nu^{-b}\rho, \nu^{-a}\rho]) \otimes \delta(\rho, b; \sigma)$, it follows that $\pi \cong L(\delta([\nu^{-b+1}\rho, \nu^{-a}\rho]); \delta(\rho, b-1; \sigma))$. Obviously, $\mu^*(L(\delta([\nu^{-b+1}\rho, \nu^{-a}\rho]); \delta(\rho, b-1; \sigma)))$ does not contain an irreducible constituent of the form $\nu^{b}\rho \otimes \pi_1$. Repeated application of this procedure and Lemma 2.7 lead us to an embedding

$$L(\delta([\nu^{-b}\rho, \widehat{\nu^{-a}\rho}]); \delta(\rho, b; \sigma)) \hookrightarrow \nu^{-b}\rho \times \nu^{-b}\rho \times \cdots \times \nu^{a-2}\rho \times \nu^{a-2}\rho \rtimes L(\delta([\nu^{a-1}\rho, \widehat{\nu^{-a}\rho}]); \delta(\rho, -a+1; \sigma)).$$

Thus, it remains to determine $L(\delta([\nu^{a-1}\rho, \widehat{\nu^{-a}\rho}]); \delta(\rho, -a+1; \sigma))$. Proposition 2.10(2) implies that $L(\delta([\nu^{a-1}\rho, \nu^{-a}\rho]); \delta(\rho, -a+1; \sigma))$ is a subrepresentation of $\nu^{-a}\rho \rtimes L(\delta([\nu^{a-1}\rho, \nu^{-a-1}\rho]); \delta(\rho, -a+1; \sigma))$, and in the same way as before we get

$$L(\delta([\nu^{a-1}\rho, \nu^{-a}\rho]); \delta(\rho, -a+1; \sigma)) \hookrightarrow \nu^{-a}\rho \times \nu^{-a+1}\rho \times \nu^{-a+1}\rho \rtimes L(\delta([\nu^{a}\rho, \nu^{-a-1}\rho]); \delta(\rho, -a; \sigma)).$$

By [26, Theorem 4.1], in $R(G)$ we have

$$\begin{aligned}&\delta([\nu^a\rho,\nu^{-a+1}\rho])\rtimes\delta(\rho,-a+1;\sigma)\\&=L(\delta([\nu^{a-1}\rho,\nu^{-a}\rho]);\delta(\rho,-a+1;\sigma))+\tau_{temp},\end{aligned}$$

where τ_{temp} is the unique common irreducible subrepresentation of

$$\delta([\nu^{a-1}\rho,\nu^{-a+1}\rho])\rtimes\delta(\rho,-a;\sigma)$$

and

$$\delta([\nu^{a}\rho,\nu^{-a+1}\rho])\rtimes\delta(\rho,-a+1;\sigma).$$

From the structural formula we obtain that

$$\nu^{-a+1}\rho\times\nu^{-a+1}\rho\otimes\delta([\nu^a\rho,\nu^{-a}\rho])\rtimes\delta(\rho,-a;\sigma)$$

is the unique irreducible constituent of $\mu^*(\delta([\nu^a\rho,\nu^{-a+1}\rho])\rtimes\delta(\rho,-a+1;\sigma))$ of the form $\nu^{-a+1}\rho\times\nu^{-a+1}\rho\otimes\pi'$, which appears there with multiplicity one, and by Frobenius reciprocity it is contained in $\mu^*(\tau_{temp})$. Thus,

$$\mu^*(L(\delta([\nu^{a-1}\rho,\nu^{-a}\rho]);\delta(\rho,-a+1;\sigma)))$$

does not contain an irreducible constituent of the form $\nu^{-a+1}\rho\times\nu^{-a+1}\rho\otimes\pi'$, which yields

$$\begin{aligned}&L(\delta([\nu^{a-1}\rho,\nu^{-a}\rho]);\delta(\rho,-a+1;\sigma))\\&\hookrightarrow\zeta([\nu^{-a}\rho,\nu^{-a+1}\rho])\times\nu^{-a+1}\rho\rtimes L(\delta([\nu^a\rho,\nu^{-a-1}\rho]);\delta(\rho,-a;\sigma)).\end{aligned}$$

Also, $\mu^*(L(\delta([\nu^a\rho,\nu^{-a-1}\rho]);\delta(\rho,-a;\sigma)))$ does not contain an irreducible constituent of the form $\nu^{-a+1}\rho\otimes\pi_1'$, so using Lemma 2.8 and a repeated application of this procedure, we get that the Aubert dual of $L(\delta([\nu^{a-1}\rho,\nu^{-a}\rho]);\delta(\rho,-a+1;\sigma))$ is an irreducible subrepresentation of

$$\begin{aligned}&\nu^{a-1}\rho\times\delta([\nu^{a-1}\rho,\nu^a\rho])\times\cdots\times\nu^{-\alpha-1}\rho\times\delta([\nu^{-\alpha-1}\rho,\nu^{-\alpha}\rho])\\&\rtimes L(\delta([\nu^{-\alpha}\rho,\widehat{\nu^{\alpha-1}\rho}]);\delta(\rho,\alpha;\sigma)).\end{aligned}$$

If $\alpha=\frac{1}{2}$, by [22, Lemma 4.10] we have $L(\delta([\nu^{-\alpha}\rho,\widehat{\nu^{\alpha-1}\rho}]);\delta(\rho,\alpha;\sigma))\cong\tau(\rho,\sigma)$. If $\alpha>\frac{1}{2}$, in the same way as before we get

$$\begin{aligned}&L(\delta([\nu^{-\alpha}\rho,\widehat{\nu^{\alpha-1}\rho}]);\delta(\rho,\alpha;\sigma))\\&\hookrightarrow\nu^{-\alpha}\rho\times\delta([\nu^{-\alpha}\rho,\nu^{-\alpha+1}\rho])\rtimes L(\delta([\nu^{-\alpha+1}\rho,\widehat{\nu^{\alpha-2}\rho}]);\sigma).\end{aligned}$$

For $\alpha=1$, we have $L(\delta([\nu^{-\alpha+1}\rho,\nu^{\alpha-2}\rho]);\sigma)\cong\sigma$, and for $\alpha\geq\frac{3}{2}$ we have

$$\begin{aligned}&L(\delta([\nu^{-\alpha+1}\rho,\widehat{\nu^{\alpha-2}\rho}]);\sigma)\\&\hookrightarrow\nu^{-\alpha+1}\rho\times\nu^{-\alpha+2}\rho\times\nu^{-\alpha+2}\rho\times\cdots\times\nu^{\lceil\alpha\rceil-\alpha-1}\rho\times\nu^{\lceil\alpha\rceil-\alpha-1}\rho\rtimes\tau^{(1)}.\end{aligned}$$

This ends the proof. □

Proposition 5.4. *If $-a > x$, in $R(G)$ we have*

$$\begin{aligned}
&\zeta([\nu^{-b}\rho,\nu^{-a}\rho])\rtimes\zeta(\rho,x;\sigma)\\
&=L(\nu^{-b}\rho,\dots,\nu^{a-1}\rho,\nu^{a}\rho,\nu^{a}\rho,\dots,\nu^{-x-1}\rho,\nu^{-x-1}\rho,\nu^{-x}\rho,\nu^{-x}\rho,\nu^{-x}\rho,\dots,\\
&\nu^{-\alpha}\rho,\nu^{-\alpha}\rho,\nu^{-\alpha}\rho,\nu^{-\alpha+1}\rho,\nu^{-\alpha+1}\rho,\dots,\nu^{\lceil\alpha\rceil-\alpha-1}\rho,\nu^{\lceil\alpha\rceil-\alpha-1}\rho;\tau^{(1)})\\
&+L(\nu^{-b}\rho,\dots,\nu^{a-1}\rho,\nu^{a}\rho,\nu^{a}\rho,\dots,\nu^{-x-2}\rho,\nu^{-x-2}\rho,\nu^{-x-1}\rho,\delta([\nu^{-x-1}\rho,\nu^{-x}\rho]),\dots,\\
&\nu^{-\alpha-1}\rho,\delta([\nu^{-\alpha-1}\rho,\nu^{-\alpha}\rho]),\nu^{-\alpha}\rho,\dots,\nu^{\lceil\alpha\rceil-\alpha-1}\rho;\tau^{(2)})\\
&+L(\nu^{-b}\rho,\dots,\nu^{a-2}\rho,\delta([\nu^{a-1}\rho,\nu^{a}\rho]),\dots,\delta([\nu^{-x-2}\rho,\nu^{-x-1}\rho]),\\
&\delta([\nu^{-x-1}\rho,\nu^{-x}\rho]),\nu^{-x}\rho,\dots,\delta([\nu^{-\alpha-1}\rho,\nu^{-\alpha}\rho]),\nu^{-\alpha}\rho,\nu^{-\alpha+1}\rho,\dots,\nu^{\lceil\alpha\rceil-\alpha-1}\rho;\tau^{(2)}).
\end{aligned}$$

Proof. Again, by [26, Theorem 2.1], in $R(G)$ we have

$$\delta([\nu^{a}\rho,\nu^{b}\rho])\rtimes\delta(\rho,x;\sigma)=L(\delta([\nu^{-b}\rho,\nu^{-a}\rho]);\delta(\rho,x;\sigma))+\sigma_1+\sigma_2,$$

where σ_1,σ_2 are mutually non-isomorphic discrete series representations. It is enough to determine the Aubert dual of $L(\delta([\nu^{-b}\rho,\nu^{-a}\rho]);\delta(\rho,x;\sigma))$, which can be determined in a similar way as in the proof of the previous proposition, details being left to the reader. □

Proposition 5.5. *Suppose that $\alpha-1\le -a<x\le b$. Let*

$$\begin{aligned}
\pi_1\cong L(&\nu^{-b}\rho,\dots,\nu^{-x-1}\rho,\nu^{-x}\rho,\nu^{-x}\rho,\dots,\nu^{a-2}\rho,\nu^{a-2}\rho,\nu^{a-1}\rho,\delta([\nu^{a-1}\rho,\nu^{a}\rho]),\\
&\dots,\nu^{-\frac{3}{2}}\rho,\delta([\nu^{-\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho]);\tau(\rho,\sigma))
\end{aligned}$$

if $\alpha=\frac{1}{2}$,

$$\begin{aligned}
\pi_1\cong L(&\nu^{-b}\rho,\dots,\nu^{-x-1}\rho,\nu^{-x}\rho,\nu^{-x}\rho,\dots,\nu^{a-2}\rho,\nu^{a-2}\rho,\nu^{a-1}\rho,\delta([\nu^{a-1}\rho,\nu^{a}\rho]),\\
&\dots,\nu^{-1}\rho,\delta([\nu^{-1}\rho,\rho]);\sigma)
\end{aligned}$$

if $\alpha=1$, and

$$\begin{aligned}
\pi_1\cong L(&\nu^{-b}\rho,\dots,\nu^{-x-1}\rho,\nu^{-x}\rho,\nu^{-x}\rho,\dots,\nu^{a-2}\rho,\nu^{a-2}\rho,\nu^{a-1}\rho,\delta([\nu^{a-1}\rho,\nu^{a}\rho]),\dots,\\
&\nu^{-\alpha}\rho,\delta([\nu^{-\alpha}\rho,\nu^{-\alpha+1}\rho]),\nu^{-\alpha+1}\rho,\nu^{-\alpha+2}\rho,\nu^{-\alpha+2}\rho,\dots,\nu^{\lceil\alpha\rceil-\alpha-1}\rho,\nu^{\lceil\alpha\rceil-\alpha-1}\rho;\tau^{(1)})
\end{aligned}$$

if $\alpha\ge\frac{3}{2}$.
Let

$$\begin{aligned}
\pi_2\cong L(&\nu^{-b}\rho,\dots,\nu^{-x-2}\rho,\delta([\nu^{-x-1}\rho,\nu^{-x}\rho]),\dots,\delta([\nu^{a-3}\rho,\nu^{a-2}\rho]),\\
&\delta([\nu^{a-2}\rho,\nu^{a}\rho]),\dots,\delta([\nu^{-\frac{3}{2}}\rho,\nu^{\frac{1}{2}}\rho]);\sigma)
\end{aligned}$$

if $\alpha=\frac{1}{2}$,

$$\begin{aligned}
\pi_2\cong L(&\nu^{-b}\rho,\dots,\nu^{-x-2}\rho,\delta([\nu^{-x-1}\rho,\nu^{-x}\rho]),\dots,\delta([\nu^{a-3}\rho,\nu^{a-2}\rho]),\\
&\delta([\nu^{a-2}\rho,\nu^{a}\rho]),\dots,\delta([\nu^{-2}\rho,\rho]);\delta(\rho,1;\sigma))
\end{aligned}$$

if $\alpha=1$, and

$$\begin{aligned}
\pi_2\cong L(&\nu^{-b}\rho,\dots,\nu^{-x-2}\rho,\delta([\nu^{-x-1}\rho,\nu^{-x}\rho]),\dots,\delta([\nu^{a-3}\rho,\nu^{a-2}\rho]),\\
&\delta([\nu^{a-2}\rho,\nu^{a}\rho]),\dots,\delta([\nu^{-\alpha-1}\rho,\nu^{-\alpha+1}\rho]),\nu^{-\alpha+2}\rho,\dots,\nu^{\lceil\alpha\rceil-\alpha-1}\rho;\tau^{(2)})
\end{aligned}$$

if $\alpha\ge\frac{3}{2}$.

If $x < b$, in $R(G)$ we have

$$\begin{aligned}&\zeta([\nu^{-b}\rho,\nu^{-a}\rho])\rtimes\zeta(\rho,x;\sigma)\\ &=L(\nu^{-b}\rho,\dots,\nu^{-x-1}\rho,\nu^{-x}\rho,\nu^{-x}\rho,\dots,\nu^{a-1}\rho,\nu^{a-1}\rho,\nu^{a}\rho,\nu^{a}\rho,\nu^{a}\rho,\dots,\\ &\nu^{-\alpha}\rho,\nu^{-\alpha}\rho,\nu^{-\alpha}\rho,\nu^{-\alpha+1}\rho,\nu^{-\alpha+1}\rho,\dots,\nu^{\lceil\alpha\rceil-\alpha-1}\rho,\nu^{\lceil\alpha\rceil-\alpha-1}\rho;\tau^{(1)})\\ +L(\nu^{-b}\rho,\dots,\nu^{-x-2}\rho,&\delta([\nu^{-x-1}\rho,\nu^{-x}\rho]),\dots,\delta([\nu^{a-2}\rho,\nu^{a-1}\rho]),\delta([\nu^{a-1}\rho,\nu^{a}\rho]),\nu^{a}\rho,\\ &\dots,\delta([\nu^{-\alpha-1}\rho,\nu^{-\alpha}\rho]),\nu^{-\alpha}\rho,\nu^{-\alpha+1}\rho,\dots,\nu^{\lceil\alpha\rceil-\alpha-1}\rho;\tau^{(2)})\\ &+\pi_1+\pi_2.\end{aligned}$$

If $x = b$, in $R(G)$ we have

$$\begin{aligned}&\zeta([\nu^{-b}\rho,\nu^{-a}\rho])\rtimes\zeta(\rho,x;\sigma)\\ &=L(\nu^{-b}\rho,\nu^{-b}\rho,\dots,\nu^{a-1}\rho,\nu^{a-1}\rho,\nu^{a}\rho,\nu^{a}\rho,\nu^{a}\rho,\dots,\nu^{-\alpha}\rho,\nu^{-\alpha}\rho,\nu^{-\alpha}\rho,\\ &\nu^{-\alpha+1}\rho,\nu^{-\alpha+1}\rho,\dots,\nu^{\lceil\alpha\rceil-\alpha-1}\rho,\nu^{\lceil\alpha\rceil-\alpha-1}\rho;\tau^{(1)})+\pi_1.\end{aligned}$$

Proof. Let us first consider the case $x < b$ By [19, Proposition 3.2], in $R(G)$ we have

$$\begin{aligned}\delta([\nu^{a}\rho,\nu^{b}\rho])\rtimes\delta(\rho,x;\sigma)&=L(\delta([\nu^{-b}\rho,\nu^{-a}\rho]);\delta(\rho,x;\sigma))+\sigma_1\\ +L(\delta([\nu^{-b}\rho,\nu^{x}\rho]);\delta(\rho,-a;\sigma))&+L(\delta([\nu^{-x}\rho,\nu^{-a}\rho]);\delta(\rho,b;\sigma)),\end{aligned}$$

where σ_1 is the unique common discrete series subrepresentation of both $\delta([\nu^{x}\rho,\nu^{b}\rho])\rtimes\delta(\rho,a;\sigma)$ and $\delta([\nu^{a}\rho,\nu^{x}\rho])\rtimes\delta(\rho,b;\sigma)$.

The Aubert duals of σ_1 and of $L(\delta([\nu^{-b}\rho,\nu^{x}\rho]);\delta(\rho,-a;\sigma))$ can be obtained from Proposition 5.4, interchanging the roles of a and x. Also, the Aubert dual of $L(\delta([\nu^{-x}\rho,\nu^{-a}\rho]);\delta(\rho,b;\sigma))$ can be obtained from Proposition 5.3, interchanging the roles of b and x.

It remains to determine the Aubert dual of $L(\delta([\nu^{-b}\rho,\nu^{-a}\rho]);\delta(\rho,x;\sigma))$. First, in the same way as in the previously considered cases we obtain that $L(\delta([\nu^{-b}\rho,\widehat{\nu^{-a}\rho}]);\delta(\rho,x;\sigma))$ is a subrepresentation of

$$\nu^{-b}\rho\times\cdots\times\nu^{-x-2}\rho\rtimes L(\delta([\nu^{-x-1}\rho,\widehat{\nu^{-a}\rho}]);\delta(\rho,x;\sigma)).$$

Also, if $x > -a+1$, we have

$$L(\delta([\nu^{-x-1}\rho,\nu^{-a}\rho]);\delta(\rho,x;\sigma))\hookrightarrow\nu^{x}\rho\times\nu^{x+1}\rho\rtimes L(\delta([\nu^{-x}\rho,\nu^{-a}\rho]);\delta(\rho,x-1;\sigma)),$$

and there is an irreducible subquotient π_1 of $\nu^{x}\rho\times\nu^{x+1}\rho$ such that $L(\delta([\nu^{-x-1}\rho,\nu^{-a}\rho]);\delta(\rho,x;\sigma))$ is a subrepresentation of $\pi_1\rtimes L(\delta([\nu^{-x}\rho,\nu^{-a}\rho]);\delta(\rho,x-1;\sigma))$.

The induced representation $\delta([\nu^{a}\rho,\nu^{x+1}\rho])\rtimes\delta(\rho,x;\sigma)$ is a length four representation, again by [19, Proposition 3.2]. If $\nu^{x+1}\rho\otimes\pi$ is an irreducible constituent of $\mu^*(\delta([\nu^{a}\rho,\nu^{x+1}\rho])\rtimes\delta(\rho,x;\sigma))$, using the structural formula we easily obtain that π is an irreducible subquotient of $\delta([\nu^{a}\rho,\nu^{x}\rho])\rtimes\delta(\rho,x;\sigma)$. From [26, Theorem 4.1] we conclude that $\mu^*(\delta([\nu^{a}\rho,\nu^{x+1}\rho])\rtimes\delta(\rho,x;\sigma))$ contains two irreducible constituents of the form $\nu^{x+1}\rho\otimes\pi$, which have to be contained in $\mu^*(L(\delta([\nu^{-x}\rho,\nu^{-a}\rho]);\delta(\rho,x+1;\sigma)))$ and in $\mu^*(\sigma_2)$, where σ_2 is a discrete series subrepresentation of $\delta([\nu^{a}\rho,\nu^{x+1}\rho])\rtimes\delta(\rho,x;\sigma)$. Thus, $\mu^*(L(\delta([\nu^{-x-1}\rho,\nu^{-a}\rho]);\delta(\rho,x;\sigma)))$ does not contain irreducible constituents of the form $\nu^{x+1}\rho\otimes\pi$, so $\pi_1\cong\zeta([\nu^{x}\rho,\nu^{x+1}\rho])$.

This can be used to conclude that the Aubert dual of $L(\delta([\nu^{-x-1}\rho,\nu^{-a}\rho]);\delta(\rho,x;\sigma))$ is a subrepresentation of

$$\delta([\nu^{-x-1}\rho,\nu^{-x}\rho])\times\cdots\times\delta([\nu^{a-3}\rho,\nu^{a-2}\rho])\rtimes L(\delta([\nu^{a-2}\rho,\widehat{\nu^{-a}\rho}]);\delta(\rho,-a+1;\sigma)).$$

Using Proposition 2.10(2), (3), and (1), respectively, we get

$$L(\delta([\nu^{a-2}\rho,\nu^{-a}\rho]);\delta(\rho,-a+1;\sigma))$$
$$\hookrightarrow\nu^{-a}\rho\times\nu^{-a+1}\rho\times\nu^{-a+2}\rho\rtimes L(\delta([\nu^{a-1}\rho,\nu^{-a-1}\rho]);\delta(\rho,-a;\sigma)).$$

We have already seen that $\mu^*(L(\delta([\nu^{a-2}\rho,\nu^{-a}\rho]);\delta(\rho,-a+1;\sigma)))$ does not contain an irreducible constituent of the form $\nu^{-a+2}\rho\otimes\pi$. If $\nu^{-a+1}\rho\otimes\pi$ is an irreducible constituent of $\mu^*(\delta([\nu^a\rho,\nu^{-a+2}\rho])\rtimes\delta(\rho,-a+1;\sigma))$, then π is an irreducible subquotient of $\delta([\nu^a\rho,\nu^{-a+2}\rho])\rtimes\delta(\rho,-a;\sigma)$, which is a length two representation. Thus, the Frobenius reciprocity can be used to deduce that $\mu^*(L(\delta([\nu^{a-2}\rho,\nu^{-a+1}\rho]);\delta(\rho,-a;\sigma)))$ and $\mu^*(\sigma_3)$, where σ_3 is a discrete series subrepresentation of $\delta([\nu^{a-2}\rho,\nu^{-a}\rho])\rtimes\delta(\rho,-a+1;\sigma)$, contain all irreducible constituents of the form $\nu^{-a+1}\rho\otimes\pi$ appearing in $\mu^*(\delta([\nu^a\rho,\nu^{-a+2}\rho])\rtimes\delta(\rho,-a+1;\sigma))$. So, $L(\delta([\nu^{a-2}\rho,\nu^{-a}\rho]);\delta(\rho,-a+1;\sigma))$ is a subrepresentation of

$$\zeta([\nu^{-a}\rho,\nu^{-a+2}\rho])\rtimes L(\delta([\nu^{a-1}\rho,\nu^{-a-1}\rho]);\delta(\rho,-a;\sigma)).$$

In the same way it can be seen that $\mu^*(L(\delta([\nu^{a-1}\rho,\nu^{-a-1}\rho]);\delta(\rho,-a;\sigma)))$ does not contain irreducible constituents of the form $\nu^y\rho\otimes\pi$ for $\pi\in\{-a,-a+1\}$. Using Lemma 2.7 and continuing in the same way, we get that the Aubert dual of $L(\delta([\nu^{a-1}\rho,\nu^{-a-1}\rho]);\delta(\rho,-a;\sigma))$ is a subrepresentation of

$$\delta([\nu^{a-2}\rho,\nu^{a}\rho])\times\cdots\times\delta([\nu^{-\alpha-2}\rho,\nu^{-\alpha}\rho])\rtimes L(\delta([\nu^{-\alpha-1}\rho,\widehat{\nu^{\alpha-1}\rho}]);\delta(\rho,\alpha;\sigma)).$$

Let us first consider the case $\alpha=\frac{1}{2}$. Then it can be seen, using the intertwining operators method, that $L(\delta([\nu^{-\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho]);\delta(\rho,\frac{1}{2};\sigma))$ is a subrepresentation of $\nu^{-\frac{1}{2}}\rho\times\nu^{\frac{1}{2}}\rho\times\nu^{\frac{3}{2}}\rho\rtimes\sigma$. Thus, there is an irreducible subquotient π_1 of $\nu^{-\frac{1}{2}}\rho\times\nu^{\frac{1}{2}}\rho\times\nu^{\frac{3}{2}}\rho$ such that $L(\delta([\nu^{-\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho]);\delta(\rho,\frac{1}{2};\sigma))$ is a subrepresentation of $\pi_1\rtimes\sigma$.

By [26, Theorem 5.1(ii)], in $R(G)$ we have

$$\delta([\nu^{\frac{1}{2}}\rho,\nu^{\frac{3}{2}}\rho])\rtimes\delta(\rho,\frac{1}{2};\sigma)=L(\delta([\nu^{-\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho]);\delta(\rho,\frac{1}{2};\sigma))+\sigma_4$$
$$+L(\delta([\nu^{-\frac{3}{2}}\rho,\nu^{\frac{1}{2}}\rho]);\sigma)+L(\nu^{-\frac{1}{2}}\rho;\delta(\rho,\frac{3}{2};\sigma)),$$

where σ_4 is the unique discrete series subrepresentation of $\delta([\nu^{\frac{1}{2}}\rho,\nu^{\frac{3}{2}}\rho])\rtimes\delta(\rho,\frac{1}{2};\sigma)$.

Since both induced representations $\delta([\nu^{\frac{1}{2}}\rho,\nu^{\frac{3}{2}}\rho])\rtimes\sigma$ and $\nu^{\frac{1}{2}}\rho\rtimes\delta(\rho,\frac{1}{2};\sigma)$ are of length two (by [26, Theorem 5.1]), it follows from the structural formula that $\mu^*(\delta([\nu^{\frac{1}{2}}\rho,\nu^{\frac{3}{2}}\rho])\rtimes\delta(\rho,\frac{1}{2};\sigma))$ contains exactly two irreducible constituents of the form $\nu^{\frac{3}{2}}\rho\otimes\pi$ and exactly two irreducible constituents of the form $\nu^{\frac{1}{2}}\rho\otimes\pi$. Now Frobenius reciprocity and transitivity of the Jacquet modules imply that all irreducible constituents of the form $\nu^{\frac{3}{2}}\rho\otimes\pi$ are contained in $\mu^*(\sigma_4)$ and in $\mu^*(L(\nu^{-\frac{1}{2}}\rho;\delta(\rho,\frac{3}{2};\sigma)))$, while all irreducible constituents of the form $\nu^{\frac{1}{2}}\rho\otimes\pi$ are contained in $\mu^*(\sigma_4)$ and in $\mu^*((L(\delta([\nu^{-\frac{3}{2}}\rho,\nu^{\frac{1}{2}}\rho]);\sigma))$.

Consequently, $\mu^*(L(\delta([\nu^{-\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho]);\delta(\rho,\frac{1}{2};\sigma)))$ does not contain irreducible constituents of the form $\nu^y\rho\otimes\pi$ for $y\in\{\frac{1}{2},\frac{3}{2}\}$.

Thus, it follows that $\pi_1 \cong \zeta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho])$, so $L(\delta([\nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]); \delta(\rho, \frac{1}{2}; \sigma))$ is a subrepresentation of $\zeta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma$. Now Lemma 2.7 can be used to obtain the Aubert dual of $L(\delta([\nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]); \delta(\rho, \frac{1}{2}; \sigma))$ is isomorphic to $L(\delta([\nu^{-\frac{3}{2}}\rho, \nu^{\frac{1}{2}}\rho]); \sigma)$.

If $\alpha > \frac{1}{2}$, in the same way as before we deduce that the Aubert dual of $L(\delta([\nu^{-\alpha-1}\rho, \nu^{\alpha-1}\rho]); \delta(\rho, \alpha; \sigma))$ is a subrepresentation of

$$\delta([\nu^{-\alpha-1}\rho, \nu^{-\alpha+1}\rho]) \rtimes L(\delta(\widehat{[\nu^{-\alpha}\rho, \nu^{\alpha-2}\rho]}); \sigma).$$

If $\alpha = 1$, from [20, Theorem 3.5] we deduce that $L(\delta(\widehat{[\nu^{-\alpha}\rho, \nu^{\alpha-2}\rho]}); \sigma) \cong \delta(\rho, 1; \sigma)$. If $\alpha \geq \frac{3}{2}$, from [22, Lemma 4.10] we get that $L(\delta(\widehat{[\nu^{-\alpha}\rho, \nu^{\alpha-2}\rho]}); \sigma)$ is the unique irreducible subrepresentation of $\nu^{-\alpha+2}\rho \times \cdots \times \nu^{\lceil\alpha\rceil-\alpha-1}\rho \rtimes \tau^{(2)}$.

If $x = b$, in $R(G)$ we have

$$\delta([\nu^a\rho, \nu^b\rho]) \rtimes \delta(\rho, b; \sigma) = L(\delta([\nu^{-b}\rho, \nu^{-a}\rho]); \delta(\rho, b; \sigma)) + \tau,$$

where τ is the unique common irreducible tempered subrepresentation of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \delta(\rho, b; \sigma)$ and $\delta([\nu^{-b}\rho, \nu^b\rho]) \rtimes \delta(\rho, a; \sigma)$. The Aubert dual of the representation $L(\delta([\nu^{-b}\rho, \nu^{-a}\rho]); \delta(\rho, b; \sigma))$ has been determined in the proof of Proposition 5.3, while the Aubert dual of τ can be obtained from [22, Theorem 4.16]. □

Proposition 5.6. *If $-a < \alpha - 2$ and $\alpha - 1 \leq b < x$, in $R(G)$ we have*

$$\begin{aligned}
&\zeta([\nu^{-b}\rho, \nu^{-a}\rho]) \rtimes \zeta(\rho, x; \sigma) \\
&= L(\nu^{-x}\rho, \dots, \nu^{-b-1}\rho, \nu^{-b}\rho, \nu^{-b}\rho, \dots, \nu^{-\alpha}\rho, \nu^{-\alpha}\rho, \nu^{-\alpha+1}\rho, \dots, \nu^{a-1}\rho, \\
&\qquad \nu^a\rho, \nu^a\rho, \dots, \nu^{\lceil\alpha\rceil-\alpha-1}\rho, \nu^{\lceil\alpha\rceil-\alpha-1}\rho; \tau^{(1)}) \\
&+ L(\nu^{-x}\rho, \dots, \nu^{-b-2}\rho, \delta([\nu^{-b-1}\rho, \nu^{-b}\rho]), \dots, \delta([\nu^{-\alpha}\rho, \nu^{-\alpha+1}\rho]), \nu^{-\alpha+2}\rho, \dots \\
&\qquad \nu^{a-1}\rho, \nu^a\rho, \nu^a\rho, \dots, \nu^{\lceil\alpha\rceil-\alpha-1}\rho, \nu^{\lceil\alpha\rceil-\alpha-1}\rho; \tau^{(1)}).
\end{aligned}$$

If $-a = \alpha - 2$ and $\alpha - 1 \leq b < x$, in $R(G)$ we have

$$\begin{aligned}
&\zeta([\nu^{-b}\rho, \nu^{-a}\rho]) \rtimes \zeta(\rho, x; \sigma) \\
&= L(\nu^{-x}\rho, \dots, \nu^{-b-1}\rho, \nu^{-b}\rho, \nu^{-b}\rho, \dots, \nu^{-\alpha}\rho, \nu^{-\alpha}\rho, \nu^{-\alpha+1}\rho, \nu^{-\alpha+2}\rho, \nu^{-\alpha+2}\rho, \\
&\qquad \dots, \nu^{\lceil\alpha\rceil-\alpha-1}\rho, \nu^{\lceil\alpha\rceil-\alpha-1}\rho; \tau^{(1)}) \\
&+ L(\nu^{-x}\rho, \dots, \nu^{-b-2}\rho, \delta([\nu^{-b-1}\rho, \nu^{-b}\rho]), \dots, \delta([\nu^{-\alpha}\rho, \nu^{-\alpha+1}\rho]), \\
&\qquad \nu^{-\alpha+2}\rho, \nu^{-\alpha+2}\rho, \dots, \nu^{\lceil\alpha\rceil-\alpha-1}\rho, \nu^{\lceil\alpha\rceil-\alpha-1}\rho; \tau^{(1)}).
\end{aligned}$$

Proof. We discuss only the case $-a = \alpha - 2$, since the case $-a < \alpha - 2$ can be handled in the same way, but more easily. Let us denote by σ_{sp} a strongly positive discrete series subrepresentation of $\delta([\nu^{\alpha-1}\rho, \nu^b\rho]) \rtimes \delta(\rho, x; \sigma)$ ([17, Section 4] or Proposition 2.3). Note that we have $\alpha \geq \frac{5}{2}$.

By [26, Theorem 4.1], in $R(G)$ we have

$$\delta([\nu^{-\alpha+2}\rho, \nu^b\rho]) \rtimes \delta(\rho, x; \sigma) = L(\delta([\nu^{-b}\rho, \nu^{\alpha-2}\rho]); \delta(\rho, x; \sigma)) + \tau,$$

where τ is the unique common irreducible (tempered) subrepresentation of induced representations $\delta([\nu^{-\alpha+2}\rho, \nu^b\rho]) \rtimes \delta(\rho, x; \sigma)$ and $\delta([\nu^{-\alpha+2}\rho, \nu^{\alpha-2}\rho]) \rtimes \sigma_{sp}$.

Using the same reasoning as in the previously considered cases, we deduce that the Aubert dual of $L(\delta([\nu^{-b}\rho, \nu^{\alpha-2}\rho]); \delta(\rho, x; \sigma))$ is isomorphic to

$$L(\nu^{-x}\rho, \dots, \nu^{-b-1}\rho, \nu^{-b}\rho, \nu^{-b}\rho, \dots, \nu^{-\alpha}\rho, \nu^{-\alpha}\rho, \nu^{-\alpha+1}\rho, \nu^{-\alpha+2}\rho, \nu^{-\alpha+2}\rho, \dots, \nu^{\lceil\alpha\rceil-\alpha-1}\rho, \nu^{\lceil\alpha\rceil-\alpha-1}\rho; \tau^{(1)}).$$

Let us determine the Aubert dual of τ. If $x > b+1$, it follows from the classification provided in [17, Section 4] that σ_{sp} is a subrepresentation $\nu^x\rho \rtimes \sigma_{sp}^{(1)}$, where $\sigma_{sp}^{(1)}$ is the unique irreducible subrepresentation of $\delta([\nu^{\alpha-1}\rho, \nu^b\rho]) \rtimes \delta(\rho, x-1; \sigma)$. Then τ is a subrepresentation of $\nu^x\rho \rtimes \tau_1$, where τ_1 is a common irreducible subrepresentation of both $\delta([\nu^{-\alpha+2}\rho, \nu^b\rho]) \rtimes \delta(\rho, x-1; \sigma)$ and $\delta([\nu^{-\alpha+2}\rho, \nu^{\alpha-2}\rho]) \rtimes \sigma_{sp}^{(1)}$. Continuing in this way we obtain that the Aubert dual of τ is a subrepresentation of

$$\nu^{-x}\rho \times \cdots \times \nu^{-b-2}\rho \rtimes \widehat{\tau_2},$$

where τ_2 is the unique common irreducible subrepresentation of $\delta([\nu^{-\alpha+2}\rho, \nu^b\rho]) \rtimes \delta(\rho, b+1; \sigma)$ and $\delta([\nu^{-\alpha+2}\rho, \nu^{\alpha-2}\rho]) \rtimes \sigma_{sp}^{(2)}$, where $\sigma_{sp}^{(2)}$ is the unique irreducible subrepresentation of $\delta([\nu^{\alpha-1}\rho, \nu^b\rho]) \rtimes \delta(\rho, b+1; \sigma)$. Since $\sigma_{sp}^{(2)}$ is a subrepresentation of $\zeta([\nu^{b-1}\rho, \nu^b\rho]) \rtimes \sigma_{sp}^{(3)}$, where $\sigma_{sp}^{(3)}$ is the unique irreducible subrepresentation of $\delta([\nu^{\alpha-1}\rho, \nu^{b-1}\rho]) \rtimes \delta(\rho, b; \sigma)$, and $\mu^*(\sigma_{sp}^{(3)})$ does not contain an irreducible constituent of the form $\nu^b\rho \otimes \pi$ by [18, Theorem 4.6], we can continue in the same way to obtain that $\widehat{\tau_2}$ is an irreducible subrepresentation of

$$\delta([\nu^{-b-1}\rho, \nu^{-b}\rho]) \times \cdots \times \delta([\nu^{-\alpha-1}\rho, \nu^{-\alpha}\rho]) \rtimes \widehat{\tau_3},$$

where τ_3 is the unique common irreducible subrepresentation of $\delta([\nu^{-\alpha+2}\rho, \nu^{\alpha-1}\rho]) \rtimes \delta(\rho, \alpha; \sigma)$ and $\delta([\nu^{-\alpha+2}\rho, \nu^{\alpha-2}\rho]) \rtimes \sigma_{sp}^{(4)}$, where $\sigma_{sp}^{(4)}$ is the unique irreducible subrepresentation of $\nu^{\alpha-1}\rho \rtimes \delta(\rho, \alpha; \sigma)$.

It follows at once that τ_3 is a subrepresentation of the induced representation $\nu^{\alpha-1}\rho \times \nu^\alpha\rho \rtimes \delta([\nu^{-\alpha+2}\rho, \nu^{\alpha-2}\rho]) \rtimes \sigma$. Since $\delta([\nu^{-\alpha+2}\rho, \nu^{\alpha-2}\rho]) \rtimes \sigma$ is irreducible and $\mu^*(\sigma_{sp}^{(4)})$ does not contain an irreducible constituent of the form $\nu^\alpha\rho \otimes \pi$, it follows that τ_3 is a subrepresentation of $\zeta([\nu^{\alpha-1}\rho, \nu^\alpha\rho]) \times \delta([\nu^{-\alpha+2}\rho, \nu^{\alpha-2}\rho]) \rtimes \sigma$. Now the rest of the proof follows in the same way as in the previously considered cases. We note that the Aubert dual of τ_3 can also be obtained using [22, Lemma 4.13, Lemma 4.15]. □

Proposition 5.7. *If $-a < \alpha - 1$ and $x < b$, in $R(G)$ we have*

$$\begin{aligned}
&\zeta([\nu^{-b}\rho, \nu^{-a}\rho]) \rtimes \zeta(\rho, x; \sigma) \\
&= L(\nu^{-b}\rho, \dots, \nu^{-x-2}\rho, \delta([\nu^{-x-1}\rho, \nu^{-x}\rho]), \dots, \delta([\nu^{-\alpha-1}\rho, \nu^{-\alpha}\rho]), \\
&\qquad \nu^a\rho, \dots, \nu^{\lceil\alpha\rceil-\alpha-1}\rho; \tau^{(2)}) \\
&+ L(\nu^{-b}\rho, \dots, \nu^{-x-1}\rho, \nu^{-x}\rho, \nu^{-x}\rho, \dots, \nu^{-\alpha}\rho, \nu^{-\alpha}\rho, \nu^{-\alpha+1}\rho, \dots, \nu^{a-1}\rho, \\
&\qquad \nu^a\rho, \nu^a\rho, \dots, \nu^{\lceil\alpha\rceil-\alpha-1}\rho, \nu^{\lceil\alpha\rceil-\alpha-1}\rho; \tau^{(1)}).
\end{aligned}$$

If $-a = x$, in $R(G)$ we have

$$\begin{aligned}&\zeta([\nu^{-b}\rho, \nu^{-a}\rho]) \rtimes \zeta(\rho, x; \sigma)\\ = L(&\nu^{-b}\rho, \dots, \nu^{a-2}\rho, \delta([\nu^{a-1}\rho, \nu^{a}\rho]), \nu^{a}\rho, \dots, \delta([\nu^{-\alpha-1}\rho, \nu^{-\alpha}\rho]), \nu^{-\alpha}\rho\\ &\nu^{-\alpha+1}\rho, \dots, \nu^{\lceil\alpha\rceil-\alpha-1}\rho; \tau^{(2)})\\ +L(&\nu^{-b}\rho, \dots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \nu^{a}\rho, \dots, \nu^{-\alpha}\rho, \nu^{-\alpha}\rho, \nu^{-\alpha}\rho,\\ &\nu^{-\alpha+1}\rho, \nu^{-\alpha+1}\rho, \dots, \nu^{\lceil\alpha\rceil-\alpha-1}\rho, \nu^{\lceil\alpha\rceil-\alpha-1}\rho; \tau^{(1)}).\end{aligned}$$

Proof. If $-a < \alpha - 1$ and $x < b$, in $R(G)$ we have

$$\begin{aligned}&\delta([\nu^{a}\rho, \nu^{b}\rho]) \rtimes \delta(\rho, x; \sigma)\\ &= L(\delta([\nu^{-b}\rho, \nu^{-a}\rho]); \delta(\rho, x; \sigma)) + L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]); \delta(\rho, b; \sigma)).\end{aligned}$$

In the same way as in the previously considered cases, we deduce that the Aubert dual of $L(\delta([\nu^{-b}\rho, \nu^{-a}\rho]); \delta(\rho, x; \sigma))$ is a subrepresentation of

$$\begin{aligned}&\nu^{-b}\rho \times \cdots \times \nu^{-x-2}\rho \times \delta([\nu^{-x-1}\rho, \nu^{-x}\rho]) \times \cdots \times \delta([\nu^{-\alpha-1}\rho, \nu^{-\alpha}\rho])\\ &\rtimes L(\delta([\widehat{\nu^{-\alpha}\rho, \nu^{-a}}\rho]); \sigma),\end{aligned}$$

and it has been already proved that the Aubert dual of $L(\delta([\nu^{-\alpha}\rho, \nu^{-a}\rho]); \sigma)$ is isomorphic to $L(\nu^{a}\rho, \dots, \nu^{\lceil\alpha\rceil-\alpha-1}\rho; \tau^{(2)})$.

Next, the Aubert dual of $L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]); \delta(\rho, b; \sigma))$ is an irreducible subrepresentation of

$$\nu^{-b}\rho \times \cdots \times \nu^{-x-1}\rho \rtimes L(\delta([\nu^{-x}\rho, \widehat{\nu^{-a}\rho}]); \delta(\rho, x; \sigma)).$$

Since the induced representation $\delta([\nu^{a}\rho, \nu^{x}\rho]) \rtimes \delta(\rho, x; \sigma)$ is irreducible, the Jacquet module of $L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]); \delta(\rho, x; \sigma))$ with respect to the appropriate parabolic subgroup contains

$$\begin{aligned}&\nu^{x}\rho \otimes \nu^{x}\rho \otimes \cdots \otimes \nu^{\alpha}\rho \otimes \nu^{\alpha}\rho \otimes \nu^{\alpha-1}\rho \otimes \cdots \otimes \nu^{-a+1}\rho\\ &\otimes \nu^{-a}\rho \otimes \nu^{-a}\rho \otimes \cdots \otimes \nu^{\alpha-\lceil\alpha\rceil+1}\rho \otimes \nu^{\alpha-\lceil\alpha\rceil+1}\rho \otimes \tau',\end{aligned}$$

where $\tau' \cong \sigma$ if $a \notin \mathbb{Z}$ and $\tau' \cong \rho \otimes \sigma$ otherwise. Now, using Lemma 2.6 we obtain the Aubert dual of $L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]); \delta(\rho, x; \sigma))$.

If $-a = x$, in $R(G)$ we have

$$\delta([\nu^{a}\rho, \nu^{b}\rho]) \rtimes \delta(\rho, x; \sigma) = L(\delta([\nu^{-b}\rho, \nu^{-a}\rho]); \delta(\rho, -a; \sigma)) + \tau,$$

where τ is the unique irreducible (tempered) common subrepresentation of

$$\delta([\nu^{a}\rho, \nu^{b}\rho]) \rtimes \delta(\rho, -a; \sigma)$$

and $\delta([\nu^{a}\rho, \nu^{-a}\rho]) \rtimes \delta(\rho, b; \sigma)$. The Aubert dual of $L(\delta([\nu^{-b}\rho, \nu^{-a}\rho]); \delta(\rho, -a; \sigma))$ can be obtained in the same way as before.

In a standard way we obtain that the Aubert dual of τ is a subrepresentation of

$$\nu^{-b}\rho \times \cdots \times \nu^{a-1}\rho \rtimes \widehat{\tau'},$$

where $\tau' \cong \delta([\nu^{a}\rho, \nu^{-a}\rho]) \rtimes \delta(\rho, -a; \sigma)$, and now $\widehat{\tau'}$ can be directly obtained using Lemma 2.6. This ends the proof. □

Now we turn our attention to the case $\rho_0 \ncong \rho$. We assume that $\beta \neq 0$, since the case $\beta = 0$ has been treated in the third section. We omit the proofs, since all the results can be obtained in the same way as in the $\rho_0 \cong \rho$ case, enhanced by Lemma 2.9.

Proposition 5.8. *Suppose that $\rho_0 \ncong \rho$. Then $\zeta([\nu^{-b}\rho_0, \nu^{-a}\rho_0]) \rtimes \zeta(\rho, x; \sigma)$ is irreducible if and only if $b < \beta$. If $b \geq \beta$ and $-a = b$, in $R(G)$ we have*

$$\begin{aligned}&\zeta([\nu^{-b}\rho_0, \nu^{b}\rho_0]) \rtimes \zeta(\rho, x; \sigma)\\ &= L(\nu^{-x}\rho, \dots, \nu^{-\alpha}\rho, \nu^{-b}\rho_0, \nu^{-b}\rho_0, \dots, \nu^{\lceil\beta\rceil-\beta-1}\rho_0, \nu^{\lceil\beta\rceil-\beta-1}\rho_0; \tau^{(1)})\\ &+L(\nu^{-x}\rho, \dots, \nu^{-\alpha}\rho, \nu^{-b}\rho_0, \nu^{-b}\rho_0, \dots, \nu^{-\beta}\rho_0, \nu^{-\beta}\rho_0, \nu^{-\beta+1}\rho_0, \dots, \nu^{\lceil\beta\rceil-\beta-1}\rho_0; \tau^{(2)}).\end{aligned}$$

If $\beta \leq -a < b$, in $R(G)$ we have

$$\begin{aligned}&\zeta([\nu^{-b}\rho_0, \nu^{-a}\rho_0]) \rtimes \zeta(\rho, x; \sigma)\\ &= L(\nu^{-x}\rho, \dots, \nu^{-\alpha}\rho, \nu^{-b}\rho_0, \dots, \nu^{a-1}\rho_0, \nu^{a}\rho_0, \nu^{a}\rho_0, \dots, \nu^{\lceil\beta\rceil-\beta-1}\rho_0, \nu^{\lceil\beta\rceil-\beta-1}\rho_0; \tau^{(1)})\\ &+L(\nu^{-x}\rho, \dots, \nu^{-\alpha}\rho, \nu^{-b}\rho_0, \dots, \nu^{a-1}\rho_0, \nu^{a}\rho_0, \nu^{a}\rho_0, \dots, \nu^{-\beta}\rho_0, \nu^{-\beta}\rho_0,\\ &\nu^{-\beta+1}\rho_0, \dots, \nu^{\lceil\beta\rceil-\beta-1}\rho_0; \tau^{(2)})\\ &+L(\nu^{-x}\rho, \dots, \nu^{-\alpha}\rho, \nu^{-b}\rho_0, \dots, \nu^{a-2}\rho_0, \delta([\nu^{a-1}\rho_0, \nu^{a}\rho_0]), \dots, \delta([\nu^{-\beta-1}\rho_0, \nu^{-\beta}\rho_0]),\\ &\nu^{-\beta+1}\rho_0, \dots, \nu^{\lceil\beta\rceil-\beta-1}\rho_0; \tau^{(2)}).\end{aligned}$$

If $-a < \beta = b$, in $R(G)$ we have

$$\begin{aligned}&\zeta([\nu^{-b}\rho_0, \nu^{-a}\rho_0]) \rtimes \zeta(\rho, x; \sigma)\\ &= L(\nu^{-x}\rho, \dots, \nu^{-\alpha}\rho, \nu^{-b}\rho_0, \dots, \nu^{a-1}\rho_0, \nu^{a}\rho_0, \nu^{a}\rho_0, \dots, \nu^{\lceil\beta\rceil-\beta-1}\rho_0, \nu^{\lceil\beta\rceil-\beta-1}\rho_0; \tau^{(1)})\\ &+L(\nu^{-x}\rho, \dots, \nu^{-\alpha}\rho, \nu^{a}\rho_0, \dots, \nu^{\lceil\beta\rceil-\beta-1}\rho_0; \tau^{(2)}).\end{aligned}$$

If $-a < \beta < b$, in $R(G)$ we have

$$\begin{aligned}&\zeta([\nu^{-b}\rho_0, \nu^{-a}\rho_0]) \rtimes \zeta(\rho, x; \sigma)\\ &= L(\nu^{-x}\rho, \dots, \nu^{-\alpha}\rho, \nu^{-b}\rho_0, \dots, \nu^{a-1}\rho_0, \nu^{a}\rho_0, \nu^{a}\rho_0, \dots, \nu^{\lceil\beta\rceil-\beta-1}\rho_0, \nu^{\lceil\beta\rceil-\beta-1}\rho_0; \tau^{(1)})\\ &+L(\nu^{-x}\rho, \dots, \nu^{-\alpha}\rho, \nu^{-b}\rho_0, \dots, \nu^{-\beta-1}\rho_0, \nu^{a}\rho_0, \dots, \nu^{\lceil\beta\rceil-\beta-1}\rho_0; \tau^{(2)}).\end{aligned}$$

6. Case $a = \frac{1}{2}$

This section is devoted to the case $a = \frac{1}{2}$. Again, we first consider the more complicated case $\rho_0 \cong \rho$, and let $\tau(\rho_1, \sigma_1)$ be as in the previous section.

Irreducibility criterion is a direct consequence of [26, Theorem 5.1].

Proposition 6.1. *Degenerate principal series $\zeta([\nu^{-b}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \zeta(\rho, x; \sigma)$ is irreducible if and only if one of the following holds:*

- *$\alpha > \frac{1}{2}$ and $b = x$,*
- *$b < \alpha - 1$.*

The composition factors in other cases are given in the following sequence of propositions.

Proposition 6.2. *If $\alpha > \frac{1}{2}$ and $x < b$, in $R(G)$ we have*

$$\begin{aligned}&\zeta([\nu^{-b}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \zeta(\rho, x; \sigma)\\ &= L(\nu^{-b}\rho, \dots, \nu^{-x-2}\rho, \delta([\nu^{-x-1}\rho, \nu^{-x}\rho]), \dots, \delta([\nu^{-\alpha-1}\rho, \nu^{-\alpha}\rho]); \tau^{(2)})\\ &+L(\nu^{-b}\rho, \dots, \nu^{-x-1}\rho, \nu^{-x}\rho, \nu^{-x}\rho, \dots, \nu^{-\alpha}\rho, \nu^{-\alpha}\rho, \nu^{-\alpha+1}\rho, \dots, \nu^{-\frac{1}{2}}\rho; \sigma).\end{aligned}$$

Proof. By [26, Theorem 5.1], in $R(G)$ we have:

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \delta(\rho, x; \sigma)$$
$$= L(\delta([\nu^{-b}\rho, \nu^{-\frac{1}{2}}\rho]); \delta(\rho, x; \sigma)) + L(\delta([\nu^{-x}\rho, \nu^{-\frac{1}{2}}\rho]); \delta(\rho, b; \sigma)).$$

First, in a standard way, using the intertwining operators methods, Proposition 2.10(1), and Lemma 2.7, we get that the Aubert dual of $L(\delta([\nu^{-b}\rho, \nu^{-\frac{1}{2}}\rho]); \delta(\rho, x; \sigma))$ is a subrepresentation of

$$\nu^{-b}\rho \times \cdots \times \nu^{-x-2}\rho \times \delta([\nu^{-x-1}\rho, \nu^{-x}\rho]) \times \cdots \times \delta([\nu^{-\alpha-1}\rho, \nu^{-\alpha}\rho])$$
$$\rtimes L(\delta(\widehat{[\nu^{-\alpha}\rho, \nu^{-\frac{1}{2}}\rho]}); \sigma),$$

and by [20, Theorem 3.5] the Aubert dual of $L(\delta([\nu^{-\alpha}\rho, \nu^{-\frac{1}{2}}\rho]); \sigma)$ is isomorphic to $\tau^{(2)}$.

Using Proposition 2.10(3) and Lemma 2.7, we deduce that the Aubert dual of $L(\delta([\nu^{-x}\rho, \nu^{-\frac{1}{2}}\rho]); \delta(\rho, b; \sigma))$ is a subrepresentation of

$$\nu^{-b}\rho \times \cdots \times \nu^{-x-1}\rho \rtimes L(\delta(\widehat{[\nu^{-x}\rho, \nu^{-\frac{1}{2}}\rho]}); \delta(\rho, x; \sigma)).$$

Now by irreducibility of $\delta([\nu^{\frac{1}{2}}\rho, \nu^x\rho]) \rtimes \delta(\rho, x; \sigma)$, the rest of the proof follows in the same way as in the proof of Proposition 5.7. □

The following result can be obtained following the same lines as in the proofs of Propositions 6.2 and 4.2.

Proposition 6.3. *If $\alpha > \frac{1}{2}$ and $\alpha - 1 \leq b < x$, in $R(G)$ we have*

$$\zeta([\nu^{-b}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \zeta(\rho, x; \sigma)$$
$$= L(\nu^{-x}\rho, \dots, \nu^{-b-1}\rho, \nu^{-b}\rho, \nu^{-b}\rho, \dots, \nu^{-\alpha}\rho, \nu^{-\alpha}\rho, \nu^{-\alpha+1}\rho, \dots, \nu^{-\frac{1}{2}}\rho; \sigma)$$
$$+L(\nu^{-x}\rho, \dots, \nu^{-b-2}\rho, \delta([\nu^{-b-1}\rho, \nu^{-b}\rho]), \dots, \delta([\nu^{-\alpha}\rho, \nu^{-\alpha+1}\rho]), \nu^{-\alpha+2}\rho, \dots, \nu^{-\frac{1}{2}}\rho; \sigma).$$

Proposition 6.4. *If $\alpha = \frac{1}{2}$ and $x < b$, in $R(G)$ we have*

$$\zeta([\nu^{-b}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \zeta(\rho, x; \sigma)$$
$$= L(\nu^{-b}\rho, \dots, \nu^{-x-1}\rho, \nu^{-x}\rho, \nu^{-x}\rho, \dots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho; \sigma)$$
$$+L(\nu^{-b}\rho, \dots, \nu^{-x-2}\rho, \delta([\nu^{-x-1}\rho, \nu^{-x}\rho]), \dots, \delta([\nu^{-\frac{5}{2}}\rho, \nu^{-\frac{3}{2}}\rho]), \delta([\nu^{-\frac{3}{2}}\rho, \nu^{\frac{1}{2}}\rho]); \sigma)$$
$$+L(\nu^{-b}\rho, \dots, \nu^{-x-2}\rho, \delta([\nu^{-x-1}\rho, \nu^{-x}\rho]), \dots, \delta([\nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]); \delta(\rho, \frac{1}{2}; \sigma))$$
$$+L(\nu^{-b}\rho, \dots, \nu^{-x-1}\rho, \nu^{-x}\rho, \nu^{-x}\rho, \dots, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho; \tau(\rho; \sigma)).$$

Proof. By [26, Theorem 5.1], in $R(G)$ we have:

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \delta(\rho, x; \sigma) = L(\delta([\nu^{-b}\rho, \nu^{-\frac{1}{2}}\rho]); \delta(\rho, x; \sigma)) + \sigma_{ds}$$
$$+L(\delta([\nu^{-x}\rho, \nu^{-\frac{1}{2}}\rho]); \delta(\rho, b; \sigma)) + L(\delta([\nu^{-b}\rho, \nu^x\rho]); \sigma),$$

where σ_{ds} is the unique common irreducible (discrete series) subrepresentation of both $\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \delta(\rho, x; \sigma)$ and $\delta([\nu^{-x}\rho, \nu^b\rho]) \rtimes \sigma$. Note that $\widehat{\sigma_{ds}}$ has been determined in [22, Theorem 5.2.(i)].

Let us determine the Aubert duals of representations $L(\delta([\nu^{-b}\rho, \nu^x\rho]); \sigma)$ and $L(\delta([\nu^{-b}\rho, \nu^{-\frac{1}{2}}\rho]); \delta(\rho, x; \sigma))$. The Aubert dual of $L(\delta([\nu^{-x}\rho, \nu^{-\frac{1}{2}}\rho]); \delta(\rho, b; \sigma))$ can

be obtained in a similar way, but more easily. Using the same arguments as before, we obtain the following embeddings:

$$L(\delta([\widehat{\nu^{-b}\rho,\nu^{x}\rho}]);\sigma)\hookrightarrow \nu^{-b}\rho\times\cdots\times\nu^{-x-2}\rho\rtimes L(\delta([\widehat{\nu^{-x-1}\rho,}\nu^{x}\rho]);\sigma),$$

$$L(\delta([\nu^{-b}\rho,\widehat{\nu^{-\frac{1}{2}}\rho}]);\delta(\rho,x;\sigma))$$

$$\hookrightarrow \nu^{-b}\rho\times\cdots\times\nu^{-x-2}\rho\rtimes L(\delta([\nu^{-x-1}\rho,\widehat{\nu^{-\frac{1}{2}}\rho}]);\delta(\rho,x;\sigma)).$$

Since $\delta([\nu^{\frac{1}{2}}\rho,\nu^{x}\rho])\rtimes\delta(\rho,x;\sigma)$ is a length two representation by [26, Theorem 5.1], it follows at once from the structural formula that $\mu^*(\delta([\nu^{\frac{1}{2}}\rho,\nu^{x+1}\rho])\rtimes\delta(\rho,x;\sigma))$ contains exactly two irreducible constituents of the form $\nu^{x+1}\rho\otimes\pi$, which have to be contained in $\mu^*(L(\delta([\nu^{-x}\rho,\nu^{-\frac{1}{2}}\rho]);\delta(\rho,x+1;\sigma)))$ and in $\mu^*(\sigma'_{ds})$, where σ'_{ds} is the unique discrete series subquotient of $\delta([\nu^{\frac{1}{2}}\rho,\nu^{x+1}\rho])\rtimes\delta(\rho,x;\sigma)$. Thus, neither $\mu^*(L(\delta([\nu^{-x-1}\rho,\nu^{x}\rho]);\sigma))$, nor $\mu^*(L(\delta([\nu^{-x-1}\rho,\nu^{-\frac{1}{2}}\rho]);\ \delta(\rho,x;\sigma)))$ contains an irreducible constituent of the form $\nu^{x+1}\rho\otimes\pi$. This leads to an embedding

$$L(\delta([\nu^{-x-1}\rho,\nu^{x}\rho]);\sigma)\hookrightarrow\zeta([\nu^{x}\rho,\nu^{x+1}\rho])\rtimes L(\delta([\nu^{-x}\rho,\nu^{x-1}\rho]);\sigma)$$

and, if $x\geq\frac{3}{2}$, to an embedding

$$L(\delta([\nu^{-x-1}\rho,\nu^{-\frac{1}{2}}\rho]);\delta(\rho,x;\sigma))$$

$$\hookrightarrow\zeta([\nu^{x}\rho,\nu^{x+1}\rho])\rtimes L(\delta([\nu^{-x}\rho,\nu^{-\frac{1}{2}}\rho]);\delta(\rho,x-1;\sigma)).$$

Using Lemma 2.8 and repeating the same arguments, we obtain

$$L(\delta([\widehat{\nu^{-x-1}\rho,}\nu^{x}\rho]);\sigma)$$

$$\hookrightarrow\delta([\nu^{-x-1}\rho,\nu^{-x}\rho])\times\cdots\times\delta([\nu^{-\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho])\rtimes L(\widehat{\nu^{-\frac{1}{2}}\rho};\sigma),$$

and

$$L(\delta([\nu^{-x-1}\rho,\widehat{\nu^{-\frac{1}{2}}\rho}]);\delta(\rho,x;\sigma))$$

$$\hookrightarrow\delta([\nu^{-x-1}\rho,\nu^{-x}\rho])\times\cdots\times\delta([\nu^{-\frac{5}{2}}\rho,\nu^{-\frac{3}{2}}\rho])\rtimes L(\delta([\nu^{-\frac{3}{2}}\rho,\widehat{\nu^{-\frac{1}{2}}\rho}]);\delta(\rho,\frac{1}{2};\sigma)).$$

We have already seen that $L(\widehat{\nu^{-\frac{1}{2}}\rho};\sigma)\cong\delta(\rho,\frac{1}{2};\sigma)$ and that the Aubert dual of $L(\delta([\nu^{-\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho]);\delta(\rho,\frac{1}{2};\sigma))$ is isomorphic to $L(\delta([\nu^{-\frac{3}{2}}\rho,\nu^{\frac{1}{2}}\rho]);\sigma)$. This ends the proof. □

The remaining cases are covered in the following propositions, a detailed verification being left to the reader.

Proposition 6.5. *If $\alpha=\frac{1}{2}$ and $b\leq x$, in $R(G)$ we have*

$$\zeta([\nu^{-b}\rho,\nu^{-\frac{1}{2}}\rho])\rtimes\zeta(\rho,x;\sigma)$$

$$=L(\nu^{-x}\rho,\dots,\nu^{-b-1}\rho,\nu^{-b}\rho,\nu^{-b}\rho,\dots,\nu^{-\frac{3}{2}}\rho,\nu^{-\frac{3}{2}}\rho;\tau(\rho;\sigma))$$

$$+L(\nu^{-x}\rho,\dots,\nu^{-b-1}\rho,\nu^{-b}\rho,\nu^{-b}\rho,\dots,\nu^{-\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho;\sigma).$$

Proposition 6.6. *Suppose that $\rho_0\not\cong\rho$. Then the degenerate principal series $\zeta([\nu^{-b}\rho_0,\nu^{-\frac{1}{2}}\rho_0])\rtimes\zeta(\rho,x;\sigma)$ is irreducible if and only if $b<\beta$. If $b\geq\beta$, in*

$R(G)$ we have

$$\zeta([\nu^{-b}\rho_0, \nu^{-\frac{1}{2}}\rho_0]) \rtimes \zeta(\rho, x; \sigma) = L(\nu^{-x}\rho, \dots, \nu^{-\alpha}\rho, \nu^{-b}\rho, \dots, \nu^{-\frac{1}{2}}\rho; \sigma) + L(\nu^{-x}\rho, \dots, \nu^{-\alpha}\rho, \nu^{-b}\rho, \dots, \nu^{-\beta-1}\rho; \tau^{(2)}).$$

7. The odd GSpin case

In this section we consider the odd GSpin case.

Remark 7.1. All the propositions in Sections 3 – 6 are valid for the odd GSpin case with exactly the same statements. More precisely, all the arguments used in [20, 22, 26] (except [26, Theorem 2.1]), as well as those used in the previous sections, can be directly carried out to the odd GSpin case, since they completely rely on properties of the Aubert involution which hold for general reductive groups, the structural formula and classifications of discrete series provided for the odd GSpin groups in [14, 15] (see also Lemma 2.2 for the structure formula for odd GSpin groups). In the following, we will comment on the generalizations of the results in [26] to odd GSpin groups and give the proof for the odd GSpin case of [26, Theorem 2.1].

Let us first recall the definition of odd GSpin groups. Let ν_m be the $m \times m$ matrix with ones on the second diagonal and zeros elsewhere. Let

$$J_{2m} = \begin{pmatrix} 0 & \nu_m \\ -\nu_m & 0 \end{pmatrix}.$$

Then the similitude symplectic groups are defined as follows:

$$GSp(2n, F) = \{g \in GL(2n, F) : {}^t g J_{2n} g = \lambda(g) J_{2n} \text{ for some } \lambda(g) \in F^*\}.$$

Let $T = \{t = \mathrm{diag}(t_1, \dots, t_n, at_n^{-1}, \dots, at_1^{-1}) : t_i, a \in F^*\}$; then T is a maximal torus for $GSp(2n, F)$. For $t = \mathrm{diag}(t_1, \dots, t_n, at_n^{-1}, \dots, at_1^{-1}) \in T$, let $e_0(t) = a$, and let $e_i(t) = t_i$ for $i = 1, \dots, n$. Let $X = \mathrm{Hom}(T, F^*)$ be the character lattice of T. Then $X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$. Let $X^\vee = \mathrm{Hom}(F^*, T)$ be the cocharacter lattice of X, and let $\{e_0^*, e_1^*, \dots, e_n^*\}$ be the basis of $X^\vee$ dual to the basis $\{e_0, e_1, \dots, e_n\}$ of X. Then $X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \cdots \oplus \mathbb{Z}e_n^*$. Let $\Delta = \{e_i - e_{i+1}, i = 1, \dots, n-1, 2e_n - e_0\}$, $\Delta^\vee = \{e_i^* - e_{i+1}^*, i = 1, \dots, n-1, e_n^*\}$. Then the root datum of $GSp(2n)$ is $(X, \Delta, X^\vee, \Delta^\vee)$.

Definition 7.2. $G\mathrm{Spin}(2n+1, F)$ is F-points of the unique split F-group having root datum $(X^\vee, \Delta^\vee, X, \Delta)$ which is dual to that of $GSp(2n, F)$.

Remark 7.3. Let $\mathrm{Spin}(2n+1, F)$ be the double covering of special orthogonal group $SO(2n+1, F)$. Then by [2, Proposition 2.2], the derived group of the split $G\mathrm{Spin}(2n+1, F)$ is $\mathrm{Spin}(2n+1, F)$ and $G\mathrm{Spin}(2n+1, F)$ is isomorphic to

$$(GL(1, F) \times \mathrm{Spin}(2n+1, F))/\{(1, 1), (-1, c)\},$$

where $c = (2e_n - e_0)(-1)$.

We now briefly summarize the main results in [26]. Let H_n be either a symplectic group or a special odd orthogonal group defined over a non-archimedean local field F of characteristic different from two, having split rank n. In [26], Muić studies the reducibility of $\delta \rtimes \sigma$, where σ is a strongly positive representation in $H_n(F)$ and $\delta := \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho])$ is an irreducible essentially square integrable representation of $GL_m(F)$ (Here, ρ is an irreducible unitary cuspidal representation of $GL(F)$ and $l_1, l_2 \in \mathbb{R}$ is such that $l_1 + l_2 \in \mathbb{Z}_{\geq 0}$.) Muić, in [26], further describes the

composition series of $\delta \rtimes \sigma$ if it is reducible. Chapters 3, 4, and 5 in [26] describe the cases $l_1 \leq -1, l_1 \geq 0$, and $l_1 = -1/2$ (Proposition 3.1, Theorem 4.1, and Theorem 5.1), respectively. The main ingredients for the proofs of those propositions and theorems are Tadić's structure formula for H_n [27] (he mainly uses the information from the GL cuspidal part in the Jacquet modules of the representations) and the classification of discrete series of H_n [25]. All those ingredients are now available for odd GSpin groups (Lemma 2.2 and [15]). However, we note that the proof of [26, Theorem 2.1] cannot be applied to the GSpin groups. We will reprove this theorem below (Theorem 7.5), in the case which we use when determining the composition factors of the degenerate principal series. Then, for odd GSpin groups, all the results in Chapters 3, 4, and 5 in [26], together with the correction of [26, Theorem 4.1.(iv), Lemma 4.9] obtained in [19, Proposition 3.2], follow in the same way as in those two papers. Therefore, our results on the composition factors of the degenerate principal series also hold in the odd GSpin case.

Remark 7.4. To prove [26, Theorem 2.1], two lemmas ([26, Lemma 2.1, 2.2]: description of non-tempered subquotients and tempered but non-square integrable subquotients of generalized principal series) are needed. The main ingredients in the proofs of those lemmas are again Tadić's structure formula (especially the information about GL cuspidal support), Casselman's square-integrability criterion, and classification of discrete series representations, which all can be applied directly to $G\mathrm{Spin}(2n+1, F)$, so we skip the proofs of those lemmas for $G\mathrm{Spin}(2n+1, F)$.

Recall that α (resp., β) is the reducibility point of ρ (resp., ρ_0) and σ, i.e., $\nu^s\rho \rtimes \sigma$ (resp., $\nu^s\rho_0 \rtimes \sigma$) is irreducible if and only if $s \notin \{\alpha, -\alpha\}$ (resp., $s \notin \{\beta, -\beta\}$).

Theorem 7.5. *Suppose that σ is an irreducible unitary cuspidal representation of $G\mathrm{Spin}(2n+1, F)$, and that one of the following holds:*

(1) $\rho_0 \ncong \rho$, $\beta \leq -a < b$, *and* $b - \beta \in \mathbb{Z}$,
(2) $\rho_0 \cong \rho$, $b > -a > x$, *and* $b - \alpha \in \mathbb{Z}$,
(3) $\rho_0 \cong \rho$, $\alpha - 1 \leq -a < b < x$, $-a \geq 0$, *and* $b - \alpha \in \mathbb{Z}$.

Then in an appropriate Grothendieck group we have

$$\delta([\nu^a\rho_0, \nu^b\rho_0]) \rtimes \delta(\rho, x; \sigma) = L(\delta([\nu^{-b}\rho_0, \nu^{-a}\rho_0]); \delta(\rho, x; \sigma)) + \sigma_{ds}^{(1)} + \sigma_{ds}^{(2)},$$

where $\sigma_{ds}^{(1)}$ and $\sigma_{ds}^{(2)}$ are mutually non-isomorphic discrete series subrepresentations of $\delta([\nu^a\rho_0, \nu^b\rho_0]) \rtimes \delta(\rho, x; \sigma)$.

Proof. We prove only the part (3), other parts can be proved in the same way, but more easily. It can be seen in the same way as in the proof of [26, Theorem 2.1] that $L(\delta([\nu^{-b}\rho_0, \nu^{-a}\rho_0]); \delta(\rho, x; \sigma))$ is the unique non-tempered irreducible subquotient of $\delta([\nu^a\rho_0, \nu^b\rho_0]) \rtimes \delta(\rho, x; \sigma)$. Also, representations $\sigma_{ds}^{(1)}$ and $\sigma_{ds}^{(2)}$ have been constructed in [15, Theorem 3.14]. Let us prove that there are no other irreducible tempered subquotients of $\delta([\nu^a\rho_0, \nu^b\rho_0]) \rtimes \delta(\rho, x; \sigma)$.

Let π denote an irreducible tempered subquotient of $\delta([\nu^a\rho_0, \nu^b\rho_0]) \rtimes \delta(\rho, x; \sigma)$. From the cuspidal support considerations one can conclude that π has to be square-integrable and non-strongly positive. Thus, by the classification given in [15], if $\alpha \geq 2$, π can be written as a subrepresentation of one of the following induced representations:

$$\delta([\nu^a\rho, \nu^b\rho]) \rtimes \delta(\rho, x; \sigma), \delta([\nu^{-b}\rho, \nu^x\rho]) \rtimes \delta(\rho, -a; \sigma), \delta([\nu^{-\alpha+2}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp},$$

where σ_{sp} stands for the unique irreducible subrepresentation of $\delta([\nu^{\alpha-1}\rho, \nu^b\rho]) \rtimes \delta(\rho, x; \sigma)$. Thus, $\mu^*(\pi)$ contains one of the following irreducible constituents:

$$\delta([\nu^a\rho, \nu^b\rho]) \otimes \delta(\rho, x; \sigma), \delta([\nu^{-b}\rho, \nu^x\rho]) \otimes \delta(\rho, -a; \sigma), \delta([\nu^{-\alpha+2}\rho, \nu^{-a}\rho]) \otimes \sigma_{sp}.$$

If $\alpha < 2$, π can be written as a subrepresentation of one of the following induced representations:

$$\delta([\nu^a\rho, \nu^b\rho]) \rtimes \delta(\rho, x; \sigma), \delta([\nu^{-b}\rho, \nu^x\rho]) \rtimes \delta(\rho, -a; \sigma),$$

and $\mu^*(\pi)$ contains one of the following irreducible constituents:

$$\delta([\nu^a\rho, \nu^b\rho]) \otimes \delta(\rho, x; \sigma), \delta([\nu^{-b}\rho, \nu^x\rho]) \otimes \delta(\rho, -a; \sigma).$$

By [15, Theorem 3.14], only irreducible subrepresentations of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \delta(\rho, x; \sigma)$ are $\sigma_{ds}^{(1)}$ and $\sigma_{ds}^{(2)}$. Also, it is easy to see, using the odd GSpin version of the structural formula given in [14], together with the classification of strongly positive discrete series, that $\delta([\nu^{-b}\rho, \nu^x\rho]) \otimes \delta(\rho, -a; \sigma)$ appears with multiplicity one in $\mu^*(\delta([\nu^a\rho, \nu^b\rho]) \rtimes \delta(\rho, x; \sigma))$, and that $\delta([\nu^{-\alpha+2}\rho, \nu^{-a}\rho]) \otimes \sigma_{sp}$ also appears with multiplicity one in $\mu^*(\delta([\nu^a\rho, \nu^b\rho]) \rtimes \delta(\rho, x; \sigma))$ if $\alpha \geq 2$.

Let τ_i, for $i \in \{1, 2\}$, denote an irreducible tempered subrepresentation of $\delta([\nu^a\rho, \nu^{-a}\rho]) \rtimes \delta(\rho, x; \sigma)$ such that $\sigma_{ds}^{(i)}$ is the unique irreducible subrepresentation of $\delta([\nu^{-a+1}\rho, \nu^b\rho]) \rtimes \tau_i$. By [29, Section 4], there is a unique $j \in \{1, 2\}$ such that τ_j is a subrepresentation of $\delta([\nu^{-a+1}\rho, \nu^x\rho]) \times \delta([\nu^a\rho, \nu^{-a}\rho]) \rtimes \delta(\rho, -a; \sigma)$. It follows from the proof of [15, Theorem 3.15] that $\sigma_{ds}^{(j)}$ is a subrepresentation of $\delta([\nu^{-b}\rho, \nu^x\rho]) \rtimes \delta(\rho, -a; \sigma)$, so $\mu^*(\sigma_{ds}^{(j)})$ contains $\delta([\nu^{-b}\rho, \nu^x\rho]) \otimes \delta(\rho, -a; \sigma)$.

Similarly, if $\alpha \geq 2$, then there is a unique $k \in \{1, 2\}$ such that τ_k is a subrepresentation of $\delta([\nu^{\alpha-1}\rho, \nu^{-a}\rho]) \times \delta([\nu^{\alpha-1}\rho, \nu^{-a}\rho]) \times \delta([\nu^{-\alpha+2}\rho, \nu^{\alpha-2}\rho]) \rtimes \delta(\rho, x; \sigma)$. It follows from the proof of [15, Theorem 3.15] that $\sigma_{ds}^{(k)}$ is a subrepresentation of $\delta([\nu^{-\alpha+2}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}$. Frobenius reciprocity implies that $\mu^*(\sigma_{ds}^{(k)})$ contains $\delta([\nu^{-\alpha+2}\rho, \nu^{-a}\rho]) \otimes \sigma_{sp}$.

From the multiplicities of $\delta([\nu^a\rho, \nu^b\rho]) \otimes \delta(\rho, x; \sigma)$, $\delta([\nu^{-b}\rho, \nu^x\rho]) \otimes \delta(\rho, -a; \sigma)$, and $\delta([\nu^{-\alpha+2}\rho, \nu^{-a}\rho]) \otimes \sigma_{sp}$ in $\mu^*(\delta([\nu^a\rho, \nu^b\rho]) \rtimes \delta(\rho, x; \sigma))$, we conclude that π is isomorphic either to $\sigma_{ds}^{(1)}$ or to $\sigma_{ds}^{(2)}$, and the theorem is proved. □

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