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Methods

Submodularity in Conic Quadratic Mixed 0–1 Optimization

Alper Atamtürk,[a] Andrés Gómez[b]

[a] Department of Industrial Engineering and Operations Research, University of California, Berkeley, Berkeley, California 94720;
[b] Department of Industrial and Systems Engineering, University of Southern California, Los Angeles, California, 90089

Contact: atamturk@berkeley.edu, http://orcid.org/0000-0003-1220-808X (AA); gomezand@usc.edu,
http://orcid.org/0000-0003-3668-0653 (AG)

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Abstract. We describe strong convex valid inequalities for conic quadratic mixed 0–1 optimization. These inequalities can be utilized for solving numerous practical nonlinear discrete optimization problems from value-at-risk minimization to queueing system design, from robust interdiction to assortment optimization through appropriate conic quadratic mixed 0–1 relaxations. The inequalities exploit the submodularity of the binary restrictions and are based on the polymatroid inequalities over binaries for the diagonal case. We prove that the convex inequalities completely describe the convex hull of a single conic quadratic constraint as well as the rotated cone constraint over binary variables and unbounded continuous variables. We then generalize and strengthen the inequalities by incorporating additional constraints of the optimization problem. Computational experiments on mean-risk optimization with correlations, assortment optimization, and robust conic quadratic optimization indicate that the new inequalities strengthen the convex relaxations substantially and lead to significant performance improvements.

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1. Introduction

Submodular set functions play an important role in many fields and have received substantial interest in the literature as they can be minimized in polynomial time (Grötschel et al. 1981, Schrijver 2000, Orlin 2009). Combinatorial optimization problems, such as the min-cut problem, entropy minimization, matroids, binary and quadratic function minimization with a nonpositive matrix, are special cases of submodular minimization (Fujishige 2005). The utilization of submodularity, however, has been mainly restricted to 0–1 optimization problems although many practical problems involve continuous variables as well.

The goal in this paper is to exploit submodularity to derive valid inequalities for mixed 0–1 minimization problems with a conic quadratic objective:

$$\min a'x + \Omega\sqrt{x'Qx} : x \in X \subseteq \{0,1\}^n \times \mathbb{R}^m_+, \tag{1}$$

or a conic quadratic constraint:

$$a'x + \Omega\sqrt{x'Qx} \le r,\ x \in X \subseteq \{0,1\}^n \times \mathbb{R}^m_+, \tag{2}$$

where $\Omega \in \mathbb{R}_+$, $r \in \mathbb{R}$, and Q is a symmetric positive semidefinite matrix. Formulations (1) and (2) are frequently used to model mean-risk problems. In particular, (1) is value-at-risk minimization, and (2) is a probabilistic constraint for a random variable $\tilde{p}'x$ with $\tilde{p} \sim N(a, Q)$. They are also used to model conservative robust formulations with an appropriate value of Ω if $\tilde{p}$ is not normally distributed (Ben-Tal et al. 2009). See Atamtürk and Gómez (2019b) and Buchheim and De Santis (2019) for specialized algorithms for solving problem (1).

Introducing an auxiliary variable z to represent the square root term $\sqrt{x'Qx}$ in (1) and (2), we write

$$f(x) = \sqrt{x'Qx} \le z,\ x \in X \subseteq \{0,1\}^n \times \mathbb{R}^m_+.$$

The motivation for this study stems from the fact that f is submodular for the simplest nontrivial nonconvex case: when Q is diagonal and $m = 0$ (Shen et al. 2003). Therefore, one may expect submodularity to play a significant role in analyzing and solving optimization problems with a general conic quadratic objective or constraint as submodularity is contained in a basic form.

Toward this goal, we consider the conic quadratic mixed-binary set

$$H_X = \left\{(x,y) \in X, z \in \mathbb{R}_+ : \sigma + \sum_{i=1}^n c_i x_i + \sum_{i=1}^m d_i y_i^2 \le z^2\right\},$$

where $X \subseteq \mathbb{D} = \{0,1\}^n \times \mathbb{R}_+^m$, $c \in \mathbb{R}_+^n$, $d \in \mathbb{R}_+^m$, and $\sigma \geq 0$, and derive strong inequalities for it. Note that $H_{\mathbb{D}}$ is the mixed-integer epigraph of the function

$$f(x,y) = \sqrt{\sigma + \sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{m} d_i y_i^2}.$$

The set H_X arises frequently in mixed-integer optimization models, well beyond the natural extension to mixed 0–1 mean-risk minimization or chance-constrained optimization with uncorrelated random variables. In particular, in Section 2, we describe applications on optimization with *correlated* random variables, inventory and scheduling problems, assortment optimization, fractional linear binary optimization, Sharpe ratio maximization, facility location problems, and conic quadratic interdiction problems.

Let $H_{\mathbb{B}}$ denote the pure binary case of $H_{\mathbb{D}}$ with $m = 0$ for which f is submodular. Although the convex hull of $H_{\mathbb{B}}$, conv$(H_{\mathbb{B}})$, is a polyhedral set and well understood, that is not the case for the mixed-integer set $H_{\mathbb{D}}$. Note, however, that, for a fixed y, f is submodular in x. By exploiting this partial submodularity for the mixed-integer case, in this paper, we give a complete nonlinear inequality description of conv$(H_{\mathbb{D}})$. We review the polymatroid inequalities for the pure binary case in Section 3.

Moreover, we show that the resulting nonlinear inequalities are also strong for the rotated conic quadratic mixed 0–1 set

$$R_X = \left\{(x,y) \in X, (w,z) \in \mathbb{R}_+^2 : \sigma + \sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{m} d_i y_i^2 \leq 4wz\right\}.$$

Observe that, even for the binary case ($m = 0$), the definition of R_X has the product of two continuous variables w, z on the right-hand side. Therefore, the existing polymatroid inequalities from the binary case cannot be directly applied to R_X. Several of the applications in Section 2 are modeled using the rotated cone set R_X.

1.1. Literature Review

A major difficulty in developing strong formulations for mixed-integer nonlinear sets such as H_X is that the corresponding convex hulls are not polyhedral even though most of the theory and methodology developed for mixed-integer optimization focuses on the polyhedral case. Recently, there has been an increasing effort to generalize methods from the linear case to the nonlinear case, including Gomory cuts (Çezik and Iyengar 2005), MIR cuts (Atamtürk and Narayanan 2007), cut-generating functions (Santana and Dey 2017), minimal valid inequalities (Kılınç-Karzan 2015), conic lifting (Atamtürk and Narayanan 2011), intersection cuts, disjunctive cuts, and lift-and-project cuts (Ceria and Soares 1999, Stubbs and Mehrotra 1999). Kılınç et al. (2010) and Bonami (2011) discuss the separation of split cuts using outer approximations and nonlinear programming. Additionally, some classes of nonlinear sets have been studied in detail: Belotti et al. (2015) study the intersection of a convex set and a linear disjunction, Modaresi and Vielma (2014) study intersections of quadratic and conic quadratic inequalities, Kılınç-Karzan and Yıldız (2015) study disjunctions on the second-order cone, Burer and Kılınç-Karzan (2017) study the intersection of a nonconvex quadratic and a conic quadratic inequality, Dadush et al. (2011a, 2014) investigate the the Chvátal–Gomory closure of convex sets, and Dadush et al. (2011b) investigate the split closure of a convex set. These inequalities are general and do not exploit any special structure.

Another stream of research for mixed-integer nonlinear optimization involves generating strong cuts by exploiting structured sets as they are common for the linear integer case. Although the applicability of such cuts is restricted to certain classes of problems, they tend to be far more effective than the general cuts that ignore any problem structure. Aktürk et al. (2009, 2010) give second-order representable perspective cuts for a nonlinear scheduling problem with variable upper bounds, which are generalized by Günlük and Linderoth (2010) and Dong et al. (2015) to problems with separable nonlinear functions and indicator variables; although describing convex hulls of mixed-integer sets with *nonseparable nonlinear* functions, such as $f(x,y)$, is significantly more difficult, there has been recent progress in the context of quadratic optimization (e.g., Jeon et al. 2017; Atamtürk and Gómez 2018, 2019a; Atamtürk et al. 2018; Frangioni et al. 2020). Ahmed and Atamtürk (2011) give strong lifted inequalities for maximizing a submodular concave utility function. Atamtürk and Narayanan (2009) and Atamtürk and Bhardwaj (2015) study binary knapsack sets defined by a single second-order conic constraint. Modaresi et al. (2016) derive closed-form intersection cuts for a number of structured sets. Atamtürk and Jeon (2017) and Gómez (2018) give strong valid inequalities for mean-risk minimization with variable upper bounds.

Closely related to this paper, Atamtürk and Narayanan (2008) study $H_{\mathbb{B}}$ in the context of mean-risk minimization. Yu and Ahmed (2017) study the generalization with a cardinality constraint, that is, H_Y, where $Y = \left\{x \in \{0,1\}^n : \sum_{i=1}^{n} x_i \leq k\right\}$. However, more general sets have not been considered in the literature. More importantly perhaps, the valid inequalities derived for the pure binary case have limited use for mixed-integer problems or even for pure binary problems with correlated random variables (nondiagonal matrix Q).

1.2. Notation

Let x denote an n-dimensional vector of binary variables, y denote an m-dimensional vector of continuous variables, and c and d be *nonnegative* vectors of dimension n and m, respectively. Define $N = \{1, \dots, n\}$ and $M = \{1, \dots, m\}$. Let $\text{conv}(X)$ denote the convex hull of X. Given a vector $a \in \mathbb{R}^n$ and $S \subseteq \{1, \dots, n\}$, let $\text{diag}(a)$ denote the $n \times n$ diagonal matrix A with $A_{ii} = a_i$, and let $a(S) = \sum_{i \in S} a_i$. Let $\mathbb{B} = \{0,1\}^n$ and $\mathbb{G} = \{0,1\}^n \times [0,1]^m$.

1.3. Outline

The rest of the paper is organized as follows. In Section 2, we discuss applications in which sets H_X and R_X arise naturally. In Section 3, we review the existing results for $H_{\mathbb{B}}$ and $H_{\mathbb{G}}$. In Section 4, we show that a nonlinear generalization of the polymatroid inequalities is sufficient to describe the convex hull of $H_{\mathbb{D}}$. In Section 5, we study the bounded set $H_{\mathbb{G}}$, give an explicit convex hull description for the case $n = m = 1$, and propose strong valid inequalities for the general case. In Section 6, we describe a strengthening procedure for the nonlinear polymatroid inequalities for any mixed-integer set X; the approach generalizes the lifting method of Yu and Ahmed (2017) for the pure binary cardinality-constrained case. In Section 7, we discuss the implementation of the proposed inequalities using off-the-shelf conic quadratic solvers. In Section 8, we test the effectiveness of the proposed inequalities for a variety of problems discussed in Section 2. Section 9 concludes the paper.

2. Applications

In this section, we present seven mixed 0–1 optimization problems in which sets H_X and R_X arise naturally.

2.1. Mean-Risk Minimization and Chance Constraints with Uncorrelated Random Variables

Conic quadratic constraints are frequently used to model probabilistic optimization with Gaussian distributions (e.g., Birge and Louveaux 2011). In particular, if a_i, c_i denote the mean and variance of random variables $\tilde{p}_i$, $i \in N$, and b_i, d_i the mean and variance of random variables $\tilde{q}_i$, $i \in M$, and all variables are independent, then

$$\min_{(x,y,z) \in H_X} a'x + b'y + \Phi^{-1}(\alpha) z$$

corresponds to the value-at-risk minimization problem over X, where Φ is the cumulative distribution function of the standard normal distribution and $0.5 < \alpha < 1$. Alternatively, the chance constraint $\mathbf{Pr}(\tilde{p}'x + \tilde{q}'y \le r) \ge \alpha$ is equivalent to $a'x + b'y + \Phi^{-1}(\alpha) z \le r$, $(x, y, z) \in H_X$. Models with H_X also arise in robust and distributionally robust optimization problems with ellipsoidal uncertainty sets (Ben-Tal and Nemirovski 1998, 1999; El Ghaoui et al. 2003; Ben-Tal et al. 2009; Zhang et al. 2018).

2.2. Mean-Risk Minimization and Chance Constraints with Correlated Random Variables

If $\tilde{p} \sim \mathcal{N}(a, Q)$, where a is the mean vector and $Q \succeq 0$ is the covariance matrix, then the value-at-risk minimization or chance-constrained optimization with 0–1 variables involve constraints of the form $\sqrt{x'Qx} \le z$.

A standard technique in quadratic optimization consists in utilizing the diagonal entries of matrices to construct strong convex relaxations (e.g., Poljak and Wolkowicz 1995, Anstreicher 2012). In particular, for $x \in \{0,1\}^n$, we have

$$x'Qx \le z \Longleftrightarrow x'(Q - \text{diag}(c))x + c'x \le z$$

with $c \in \mathbb{R}^n_+$ such that $Q - \text{diag}(c) \succeq 0$. This transformation is based on the ideal (convex hull) representation of the separable quadratic term $x' \text{diag}(c) x$ as $c'x$ for $x \in \{0,1\}^n$. Using a similar idea and introducing a continuous variable $y \in \mathbb{R}_+$, we get

$$\sqrt{x'Qx} \le z \Leftrightarrow (x, y, z) \in H_X \text{ and } \sqrt{x'(Q - \text{diag}(c))x} \le y. \tag{3}$$

The approach presented here can also be used for mixed binary sets X.

2.3. Robust Conic Quadratic Interdiction

Given a set of potential adverse event (e.g., natural disasters, disruptions, enemy attacks) scenarios C, consider the problem of minimizing the worst-case cost when only a subset of the events can occur simultaneously. If the nominal problem—when no adverse event occurs—is a mixed-integer linear optimization problem, then the worst-case minimization problem can be formulated as

$$\min_{x \in X} \max_{u \in \mathcal{U}} a_0'x + \sum_{j \in C} \left(a_j'x\right) u_j, \tag{LI}$$

where $\mathcal{U} = \left\{u \in \{0,1\}^C : \sum_{j \in E} u_j \le \Gamma\right\}$ is the uncertainty set, $\Gamma \in \mathbb{Z}_+$ is the maximum number of events that may occur simultaneously, a_0 is the nominal cost vector and $a_j \in \mathbb{R}^n_+$ is the additional cost vector if event j occurs. Problem (LI) arises naturally in robust optimization (Bertsimas and Sim 2003, 2004), and it has received a vast amount of attention in the context of interdiction (e.g., Wood 1993, Cormican et al. 1998, Israeli and Wood 2002, Lim and Smith 2007).

We now consider the generalization, where the nominal problem is a mixed-integer conic quadratic optimization problem, for example, with a value-at-risk minimization objective, considered in Atamtürk et al. (2019). In this case, the worst-case minimization problem is

$$\omega^* = \min_{x\in X} \max_{u\in\mathcal{U}} a_0'x + \sum_{j\in C}\left(a_j'x\right)u_j + \sqrt{x'Q_0x + \sum_{j\in C}(x'Q_jx)u_j}, \tag{CQI}$$

where $Q_0 \succeq 0$ is the nominal covariance matrix and $Q_j \succeq 0$ is the matrix of increased covariances if event j happens.

Problem (CQI) was studied by Atamtürk and Gómez (2017) for a convex feasible set X. They show that solving the inner maximization problem is $\mathcal{NP}$-hard for a fixed value of x and that feasible solutions with objective values within 25% of the optimal can be obtained by solving the optimization problem

$$\begin{aligned} \omega_a = \min\ & \frac{1}{4}w + a_0'x + z_0 + \Gamma\gamma \\ \text{s.t. } & \gamma \ge a_j'x + z_j && \forall j\in C \\ & x'Q_jx \le z_jw && \forall j\in\{0\}\cup C \\ & x\in X,\ z\in\mathbb{R}_+^{|C|+1},\ w\in\mathbb{R}_+,\ \gamma\in\mathbb{R}_+. \end{aligned} \tag{IA}$$

Formulations for the generalization in which $\mathcal{U}$ is set of extreme points of an integral polytope are also proposed but are omitted here for brevity.

If the set X is conic quadratic representable, then (IA) can be tackled with off-the-shelf mixed-integer conic quadratic solvers. Moreover, if all x variables are continuous, then (IA) is a convex optimization problem, thus polynomial time solvable. In contrast, if some variables are discrete, then (IA) is much more challenging, especially because of the rotated cone constraints $x'Q_jx \le z_jw$. Observe that, in this case, we can introduce an additional variable $y\in\mathbb{R}_+$ and then utilize the decomposition

$$\begin{aligned} & x'Q_jx \le z_jw \Leftrightarrow \left(x, y, w, z_j\right) \in R_X \text{ and} \\ & x'\left(Q - \operatorname{diag}(c)\right)x \le y \end{aligned}$$

to derive stronger formulations.

2.4. Lot-Sizing and Scheduling Problems

Inventory problems with economic order quantity involve expressions of the form $k\frac{p}{q}$, where $p\in\mathbb{R}_+$ is the demand, $q\in\mathbb{R}_+$ is the lot size, and $k\in\mathbb{R}_+$ is a fixed cost for ordering inventory. In simple settings, the optimal lot size q^* can be expressed explicitly (Nahmias and Cheng 2005), but in more complex settings in which the demand is a linear function of discrete variables, for example, in joint location inventory problems (Özsen et al. 2008, Atamtürk et al. 2012), this is not possible. In such cases, the order costs involve expressions of the form

$$\frac{c'x}{q} \le z \Leftrightarrow (x, q, z) \in R_{\mathbb{B}}. \tag{4}$$

The ratio (4) also arises in scheduling, specifically in the *economic lot scheduling problem* (Sahinidis and Grossmann 1991, Bollapragada and Rao 1999, Pesenti and Ukovich 2003, Bulut and Tasgetiren 2014). In this context, c is the vector to setup costs/times and q denotes a production cycle length; thus, z in (4) corresponds to setup costs/times per unit time. Expression (4) also arises in the plant design and scheduling problems to model the profitability or productivity of the plant (Castro et al. 2005, 2009).

2.5. Queueing System Design

The service system design problem, also referred to as the facility location problem with stochastic demand and congestion (Amiri 1997; Berman and Krass 2001; Elhedhli 2005, 2006), aims to locate a set of service facilities while balancing operational costs and service quality. If a facility services too many customers, it may become overly congested, resulting in long waiting times for the customers and poor service quality overall. Specifically, congestion is often modeled using queueing theory. Given an M/M/1 queue with mean demand λ and mean service rate $\mu > \lambda$, the average time in the system is $\frac{1}{\mu-\lambda}$. Additionally, in the service system design problem, the demand at location i is of the form $\lambda_i = c_i'x$, where x are binary decision variables modeling the assignments of customers to facilities; moreover, the service rates are of the form $\mu_i = d_i'y$, where y are variables representing the servers installed at location j. Thus, the service system design problem is of the form

$$\min_{(x,y,t)\in X} a'x + b'y + \Omega\sum_i \frac{c_i'x}{a_i'y - c_i'x}, \tag{SSDP}$$

where $\Omega > 0$ is the weight given to the service quality, and each term $\frac{c_i'x}{a_i'y-c_i'x}$ is the total time of servicing the customers at location j. Observe that

$$\frac{c_i'x}{a_i'y - c_i'x} \le z \Leftrightarrow (x, \mu-\lambda, z) \in R_{\mathbb{B}},$$

thus, strong formulations for $R_{\mathbb{B}}$ can be directly used in the context of (SSDP).

2.6. Binary Linear Fractional Problems

Generalizing the models in Sections 2.4 and 2.5, binary linear fractional problems are optimization problems with constraints of the form

$$\frac{c_0 + \sum_{i=1}^n c_ix_i}{a_0 + \sum_{i=1}^n a_ix_i} \le z \Leftrightarrow c_0 + \sum_{i=1}^n c_ix_i^2 \le zw,\ w = a_0 + \sum_{i=1}^n a_ix_i$$

$$\Leftrightarrow (x, w, z) \in R_B, \text{ with } w = a_0 + \sum_{i=1}^n a_ix_i,$$

where $a_i, c_i \geq 0$ for $i = 0, \dots, n$. Note that a lower bound on the ratio can also be expressed similarly by complementing variables. Binary fractional optimization arises in numerous applications, including assortment optimization with mixtures of multinomial logits (Désir et al. 2014, Méndez-Díaz et al. 2014, Şen et al. 2018), WLAN design (Amaldi et al. 2011), facility location problems with market share considerations (Tawarmalani et al. 2002), and cutting stock problems (Gilmore and Gomory 1963), among others; see also the survey Borrero et al. (2016a) and the references therein.

Applications of binary linear fractional optimization are abundant in network problems. For example, given a graph $G = (V, E)$, problems of the form

$$\min\left\{\frac{\sum_{(i,j)\in E} c_{ij}x_{ij}}{\sum_{i\in V} a_i x_i} : x_{ij} \geq |x_i - x_j|, (i,j)\in E, x \in X \subseteq \{0,1\}^{V+E}\right\} \tag{5}$$

arise in the study of expander graphs (Davidoff et al. 2003); in particular, the optimal value of (5) with $c = 1$, $a = 1$, and $X = \left\{x \in \{0,1\}^{V+E} : 1 \leq \sum_{i\in V} x_i \leq 0.5|V|\right\}$ corresponds to the Cheeger constant of the graph. See Hochbaum (2010) and Hochbaum et al. (2013) for other fractional cut problems arising in image segmentation, and see Prokopyev et al. (2009) for a discussion of other ratio problems in graphs arising in facility location.

2.7. Sharpe Ratio Maximization

Let a_i, c_i be the mean and variance of normally distributed independent random variables $\tilde{p}_i$, $i \in N$ as in Section 2.1. A natural alternative to mean-risk minimization for a risk-adverse decision maker is, given a budget r, to maximize the probability of meeting the budget; that is,

$$\max_{x\in X} \mathbf{Pr}\left(\tilde{p}'x \leq r\right). \tag{6}$$

Problems of the form (6) are considered in Nikolova et al. (2006) in the context of the stochastic shortest path problem.

Assuming there is a solution $x \in X$ satisfying $a'x \leq r$, note that

$$\mathbf{Pr}\left(\tilde{p}'x \leq r\right) = \mathbf{Pr}\left(\frac{\tilde{p}'x - a'x}{\sqrt{c'x}} \leq \frac{r - a'x}{\sqrt{c'x}}\right) = \Phi\left(\frac{r - a'x}{\sqrt{c'x}}\right).$$

Because Φ is monotone nondecreasing and $r - a'x \geq 0$ for any optimal solution, we see that (6) is equivalent to maximizing $\frac{r-a'x}{\sqrt{c'x}}$. Observe that the resulting objective corresponds to maximizing the reward-to-volatility or Sharpe ratio (Sharpe 1994), a commonly used risk-adjusted performance measure in finance.

Maximizing the Sharpe ratio is equivalent to minimizing $\frac{\sqrt{c'x}}{r-a'x}$. Therefore, we can restate (6) as

$$\begin{aligned}
&\min z\\
&\text{s.t. } w = r - a'x\\
&\qquad \sqrt{c'x} \leq wz \qquad (7)\\
&\qquad x \in X,\ w, z \geq 0. \qquad (8)
\end{aligned}$$

Constraint (7) is not conic quadratic. Note, however, for $w, z \geq 0$, we have

$$\sqrt{c'x} \leq wz \Leftrightarrow \sqrt{4\left(\sqrt[4]{c'x}\right)^2 + (w-z)^2} \leq w + z.$$

Then one gets a convex relaxation by replacing the nonconvex term $\sqrt[4]{c'x}$ by its convex lower bound $\sqrt[4]{\sum_{i\in N} c_i x_i^4}$. The resulting conic quadratic representable inequality can be written as

$$\sqrt{\sum_{i\in N} c_i x_i^4} \leq wz.$$

As we show in Section 4.2, a nonlinear version of the extended polymatroid inequalities corresponding to the submodular function $\bar{h}(x) = 2\sqrt[4]{c'x}$ is sufficient to describe the convex hull of the set given by (7) and (8) for $X = \mathbb{B}$, see Section 4.2.

3. Preliminaries

In this section, we review earlier results for the binary and mixed 0–1 cases. Given $\sigma \geq 0$ and $c_i > 0$, $i \in N$, consider the set

$$H_{\mathbb{B}} = \left\{(x,z) \in B \times \mathbb{R}_+ : \sqrt{\sigma + \sum_{i\in N} c_i x_i} \leq z\right\}. \tag{9}$$

Observe that $H_{\mathbb{B}}$ is the binary restriction of $H_{\mathbb{D}}$ obtained by setting $y = 0$, and it is the union of a finite number of line segments; therefore, its convex hull is polyhedral. For a given permutation $((1), (2), \dots, (n))$ of N, let

$$\begin{aligned}
&\sigma_{(k)} = c_{(k)} + \sigma_{(k-1)}, \text{ and } \sigma_{(0)} = \sigma,\\
&\pi_{(k)} = \sqrt{\sigma_{(k)}} - \sqrt{\sigma_{(k-1)}},
\end{aligned} \tag{10}$$

and define the polymatroid inequality as

$$\sum_{i=1}^{n} \pi_{(i)} x_{(i)} \leq z - \sqrt{\sigma}. \tag{11}$$

Let Π_σ be the set of such coefficient vectors π for *all* permutations of N. The set function defining H_B is nondecreasing submodular; therefore, Π_σ is the set of extreme points of the extended polymatroid (Edmonds 1970) associated with the submodular function $f(x) = \sqrt{\sigma + \sum_{i\in N} c_i x_i}$. Then it follows from Lovász (1983) that the convex hull of $H_{\mathbb{B}}$ is given by the set of all polymatroid inequalities and the bounds of the variables.

Proposition 1 (Convex Hull of H_B).

$$\operatorname{conv}(H_B) = \Big\{(x,z) \in [0,1]^N \times \mathbb{R}_+ : \pi' x \le z - \sqrt{\sigma},\ \forall \pi \in \Pi_\sigma\Big\}.$$

Proposition 2 is a direct consequence of a result by Edmonds (1970), showing the maximization of a linear function over a polymatroid can be solved by the greedy algorithm. Therefore, a point $(\bar{x},\bar{z})\in[0,1]^N\times\mathbb{R}_+$ can be separated from $\operatorname{conv}(H_{\mathbb{B}})$ via the greedy algorithm by sorting $\bar{x}_i, i\in N$ in nonincreasing order in $O(n\log n)$ time.

Proposition 2 (Separation). *A point $(\bar{x},\bar{z}) \notin \operatorname{conv}(H_B)$ such that $\bar{x}_{(1)} \ge \bar{x}_{(2)} \ge \cdots \ge \bar{x}_{(n)}$ is separated from $\operatorname{conv}(H_{\mathbb{B}})$ by inequality (11).*

Atamtürk and Narayanan (2008) consider the mixed-integer version of $H_{\mathbb{B}}$:

$$H_{\mathbb{G}} = \left\{(x,y,z) \in C \times \mathbb{R}_+ : \sqrt{\sigma + \sum_{i\in N} c_i x_i + \sum_{i\in M} d_i y_i^2} \le z\right\},$$

where $d_i > 0,\ i \in M$ and give valid inequalities for $H_{\mathbb{G}}$ based on the polymatroid inequalities. Without loss of generality, the upper bounds of the continuous variables in $H_{\mathbb{G}}$ are set to one by scaling.

Proposition 3 (Valid Inequalities for $H_{\mathbb{G}}$). *For $T \subseteq M$, inequalities*

$$\pi' x + \sqrt{\sigma + \sum_{i\in T} d_i y_i^2} \le z, \quad \pi \in \Pi_{\sigma + d(T)} \tag{12}$$

are valid for $H_{\mathbb{G}}$.

Inequalities (12) are obtained by setting the subset T of the continuous variables to their upper bounds and relaxing the rest, and they dominate any inequality of the form

$$\xi' x + \sqrt{\sigma + \sum_{i\in T} d_i y_i^2} \le z$$

with $\xi \in \mathbb{R}^n$. Although inequalities (12) are the strongest possible among inequalities that are linear in x and conic quadratic in y, they may be weak or dominated by other classes of nonlinear inequalities. In this paper, we introduce stronger and more general inequalities than (12) for $H_{\mathbb{G}}$.

4. The Case of Unbounded Continuous Variables

In this section, we focus on the case with unbounded continuous variables, that is, on $H_{\mathbb{D}}$, where $\mathbb{D} = \{0,1\}^n \times \mathbb{R}^n_+$. In this case, because the continuous variables have no upper bound, the only class of valid inequalities of type (12) are the polymatroid inequalities

$$\sqrt{\sigma} + \pi' x \le z, \quad \forall \pi \in \Pi_\sigma \tag{13}$$

themselves from the "binary-only" relaxation by letting $T = \emptyset$. Inequalities (13) ignore the continuous variables and are, consequently, weak for $H_{\mathbb{D}}$. Here, we define a new class of *nonlinear* valid inequalities and prove that they are sufficient to define the convex hull of $H_{\mathbb{D}}$.

Consider the inequalities

$$\left(\sqrt{\sigma} + \pi' x\right)^2 + \sum_{i\in M} d_i y_i^2 \le z^2, \quad \pi \in \Pi_\sigma. \tag{14}$$

Proposition 4. *Inequalities (14) are valid for $H_{\mathbb{D}}$.*

Proof. Consider the extended formulation of $H_{\mathbb{D}}$ given by

$$\widehat{H}_{\mathbb{D}} = \left\{(x,y) \in \mathbb{D}, (z,s) \in \mathbb{R}^2_+ : s^2 + \sum_{i\in M} d_i y_i^2 \le z^2,\ \sigma + \sum_{i\in N} c_i x_i \le s^2\right\}.$$

The validity of inequalities (14) for $H_{\mathbb{D}}$ follows directly from the validity of the polymatroid inequality $\sqrt{\sigma} + \pi' x \le s$, $\pi \in \Pi_\sigma$ (Proposition 1) for $\widehat{H}_{\mathbb{D}}$. □

Remark 1. For $M = \emptyset$, inequalities (14) reduce to the polymatroid inequalities (11).

Remark 2. Because inequalities (14) correspond to polymatroid inequalities in an extended formulation, the separation for them is the same as in the binary case and can be done by sorting in $O(n \log n)$ (Proposition 2).

Inequalities (14) are obtained simply by extracting a submodular component from function f. The approach can be generalized to sets of the form

$$U = \left\{x \in X, (y,z) \in \mathbb{R}^{m+1}_+ : h(x) + \sum_{i\in M} d_i y_i^2 \le z^2\right\},$$

and $h : \{0,1\}^n \to \mathbb{R}_+$ is an arbitrary nonnegative function. Define

$$U_0 = \left\{x \in X, s \ge 0 : \sqrt{h(x)} \le s\right\}$$

and observe that, because U_0 is a finite union of line segments, $\operatorname{conv}(U_0)$ is a polyhedron. Moreover, valid inequalities for $\operatorname{conv}(U_0)$ of the form $\xi' x \le s$, $\xi \in \Xi$, can be lifted into valid nonlinear inequalities for U of the form

$$(\xi' x)^2 + \sum_{i\in M} d_i y_i^2 \le z^2. \tag{15}$$

Proposition 5 implies inequalities of the form (15) are sufficient to describe $\text{conv}(U)$ if $\xi' x \le s$, $\xi \in \Xi$ are sufficient to describe $\text{conv}(U_0)$.

Proposition 5. *The convex hull of U is described as*

$$\text{conv}(U) = \left\{(x,y,z) \in \mathbb{R}_+^{n+m+1} : \exists s \text{ s.t. } (x,s) \in \text{conv}(U_0) \right.$$
$$\left. \textit{and } s^2 + \sum_{i \in M} d_i y_i^2 \le z^2 \right\}.$$

Proof. Consider the optimization of an arbitrary linear function over the extended formulation of U obtained by adding a variable $s \ge 0$ and the constraint $\sqrt{h(x)} \le s$,

$$\begin{aligned} \min\ & -a'x - b'y + rz \\ \text{s.t. } & s^2 + \sum_{i \in M} d_i y_i^2 \le z^2,\ (x,s) \in U_0,\ y \in \mathbb{R}_+^m, z \ge 0 \end{aligned} \tag{BP}$$

and over its convex relaxation,

$$\begin{aligned} \min\ & -a'x - b'y + rz \\ \text{s.t. } & s^2 + \sum_{i \in M} d_i y_i^2 \le z^2, (x,s) \in \text{conv}(U_0), y \in \mathbb{R}_+^m, z \ge 0. \end{aligned} \tag{P1}$$

We prove that, for any linear objective both (BP) and (P1) are unbounded or (P1) has an optimal solution that is integer in x. Without loss of generality, we can assume that $r > 0$ (if $r < 0$, then both problems are unbounded, and if $r = 0$, then (P1) reduces to a linear program over an integral polyhedron by setting z sufficiently large and is equivalent to (BP)), $r = 1$ (by scaling), $b_i > 0$ (otherwise $y_i = 0$ in any optimal solution), and $d_i = 1$ for all $i \in M$ (by scaling y_i).

Eliminating the variable z from (P1) we restate the problem as

$$\min\left\{-a'x - b'y + \sqrt{s^2 + \sum_{i \in M} y_i^2} : (x,s) \in \text{conv}(U_0),\ y \in \mathbb{R}_+^m \right\}. \tag{P2}$$

Note that, if $y = 0$ in an optimal solution of (P2), then (P2) reduces to a linear optimization over $\text{conv}(U_0)$, which has an optimal integer solution. Thus, we assume that $\sqrt{s^2 + \sum_{i \in M} y_i^2} > 0$, and in that case, the objective function is differentiable, and by convexity of (P2), optimal solutions correspond to Karush–Kuhn–Tucker (KKT) points. Let $\mu \in \mathbb{R}_+^m$ be the dual variables for constraints $y \ge 0$. From the KKT conditions of (P2) with respect to y, we see that

$$-\mu_k = b_k - \frac{y_k}{\sqrt{s^2 + \sum_{i \in M} y_i^2}}, \ \forall k \in M.$$

However, the complementary slackness conditions $y_k \mu_k = 0$ imply that $\mu_k = 0$ for all k as, otherwise, $-\mu_k = b_k$ contradicts the assumption that $b_k > 0$. Therefore, it holds that

$$y_k = b_k \sqrt{s^2 + \sum_{i \in M} y_i^2}, \quad \forall k \in M.$$

Defining $\beta = \sum_{i=1}^m b_i^2$, we have

$$\sum_{i \in M} b_i y_i = \beta \sqrt{s^2 + \sum_{i \in M} y_i^2}$$

and

$$\sum_{i \in M} y_i^2 = \beta \left(s^2 + \sum_{i \in M} y_i^2\right). \tag{16}$$

Observe that, if $\beta \ge 1$, equality (16) cannot be satisfied (unless $\beta = 1$ and $s = 0$), and the feasible (P2) is dual infeasible. Indeed, let $\lambda > 0$ and $\bar{y}_i = \lambda b_i$ for all $i \in M$ and observe that, for any value of s,

$$\lim_{\lambda \to \infty} -b'\bar{y} + \sqrt{s^2 + \sum_{i \in M} \bar{y}_i^2} = \begin{cases} -\infty & \text{if } \beta > 1 \\ 0 & \text{if } \beta = 1. \end{cases}$$

Thus, if $\beta > 1$, then both problems (BP) and (P2) are unbounded. Moreover, if $\beta = 1$, let

$$(x^*, s^*) \in \underset{(x,s) \in \text{conv}(U_0)}{\arg\min} \ -a'x$$

with a minimal value of s^*; if $s^* = 0$, then $(x^*, \bar{y}, s^*)$ is an optimal solution of both (BP) and (P2) for any $\lambda > 0$, and if $s^* > 0$, then there does not exist an optimal solution for problems (BP) and (P2), but infima of the objective functions are attained at x^*, s^*, and $y = \bar{y}$ as $\lambda \to \infty$.

If $\beta < 1$, then we deduce from (16) that

$$\sum_{i \in M} y_i^2 = \frac{\beta}{1 - \beta} s^2.$$

Replacing the summands in the objective, we rewrite (P2) as

$$\begin{aligned} \min\ & -a'x + s\sqrt{1 - \beta} \\ \text{s.t. } & (x,s) \in \text{conv}(U_0). \end{aligned} \tag{P3}$$

As $\beta < 1$, (P3) has an optimal solution, and it is integral in x. By projecting out the additional variable s, we obtain the desired result. □

Remark 3. From Proposition 5, we see that, with no constraints on the continuous variables, describing the mixed-integer set $\text{conv}(H_X)$ reduces to describing a polyhedral set. Moreover, strong inequalities from pure binary sets (e.g., Yu and Ahmed 2017) can be naturally lifted into strong inequalities for H_X.

Corollary 1. *Inequalities* (14) *and bound constraints completely describe* $\text{conv}(H_{\mathbb{D}})$.

Proof. Follows from Proposition 5 with $U_0 = H_{\mathbb{B}}$, where the convex hull of $H_{\mathbb{B}}$ is given in Proposition 1 and substituting out the auxiliary variable s. □

4.1. Comparison with Inequalities in the Literature

As seen in this section, inequalities (14) give the convex hull of $H_{\mathbb{D}}$. Therefore, they are the strongest possible inequalities for $H_{\mathbb{D}}$. It is of interest to study the relationships to inequalities previously given in the literature. It turns out that, for the case of a single binary variable, they can be obtained as either split cuts or conic MIR inequalities based on a single disjunction. The equivalence does not hold in higher dimensions as, in such cases, $H_{\mathbb{D}}$ is a disjunction of 2^n sets, and neither split cuts nor conic MIR inequalities based on single disjunctions are sufficient to describe $\operatorname{conv}(H_{\mathbb{D}})$.

To see the equivalence, we now consider the special case of conic quadratic constraint with a single binary variable x:

$$H^1 = \left\{(x, y, z) \in \{0,1\} \times \mathbb{R}_+^{m+1} : \sqrt{\sigma + cx + \sum_{i\in M} d_i y_i^2} \le z\right\}.$$

4.1.1. Comparison with Split Cuts.

We first compare inequalities (14) with the split cuts given in Modaresi et al. (2016). Following the notation used by the authors, let

$$B = \left\{(y, z) \in \mathbb{R}_+^{m+2} : \sqrt{\sigma + y_0^2 + \sum_{i\in M} d_i y_i^2} \le z\right\}$$

be the base set, let $F = \{y \in \mathbb{R}_+^{m+1} : 0 \le y_0 \le c\}$ be the forbidden set, and define $K = B \setminus \operatorname{int}(F)$, where $\operatorname{int}(F)$ denotes the interior of F. Letting $y_0 := \sqrt{c}x$, we see that H^1 and K are equivalent.

First, consider the case $\sigma = 0$. From Corollary 1, we see that that

$$\operatorname{conv}\left(H^1\right) = \left\{(x, y, z) \in [0,1] \times \mathbb{R}_+^{m+1} : \sqrt{cx^2 + \sum_{i\in M} d_i y_i^2} \le z\right\}.$$

Moreover, from corollary 5 of Modaresi et al. (2016), because $0 \notin (0, c)$, we find that $\operatorname{conv}(K) = B$. Thus, the results coincide in that the convex hulls of H^1 and K are the natural convex relaxations of the sets.

Now consider the case $\sigma > 0$. From Corollary 1, we see that

$$\operatorname{conv}\left(H^1\right) = \left\{(x, y, z) \in [0,1] \times \mathbb{R}_+^{m+1} : \left(\sqrt{\sigma} + \left(\sqrt{c+\sigma} - \sqrt{\sigma}\right)x\right)^2 + \sum_{i\in M} d_i y_i^2 \le z^2\right\}. \tag{17}$$

Moreover, from proposition 8 of Modaresi et al. (2016), we find that

$$\operatorname{conv}(K) = \left\{(y, z) \in \mathbb{R}_+^{m+2} : \left(\sqrt{\sigma} + \frac{\sqrt{\sigma + c} - \sqrt{\sigma}}{\sqrt{c}} y_0\right)^2 + \sum_{i\in M} d_i y_i^2 \le z^2\right\}.$$

Thus, the results coincide again.

4.1.2. Comparison with Conic MIR Inequalities.

We now compare inequalities (14) with the *simple nonlinear conic mixed-integer rounding inequality* given in Atamtürk and Narayanan (2010). Letting $a = \sqrt{\sigma} + \sqrt{\sigma + c}$ and $b = \frac{\sqrt{\sigma}}{a}$, we can write

$$H^1 = \left\{(x, y, z) \in \{0,1\} \times \mathbb{R}_+^{m+1} : (x - b)^2 + \sum_{i\in M} d_i \frac{y_i^2}{a^2} \le \frac{z^2}{a^2}\right\}.$$

Note that, if $\sigma = 0$, then $b = 0$, and the MIR inequalities reduce to the original inequality, which defines the convex hull of H^1. If $\sigma > 0$, then $\lfloor b \rfloor = 0$, and the simple mixed integer rounding inequality is

$$\begin{aligned}
&\left((1 - 2b)x + b\right)^2 + \sum_{i\in M} d_i \frac{y_i^2}{a^2} \le \frac{z^2}{a^2} \\
\Leftrightarrow &\left(\left(1 - 2\frac{\sqrt{\sigma}}{\sqrt{\sigma} + \sqrt{\sigma + c}}\right)x + \frac{\sqrt{\sigma}}{a}\right)^2 + \sum_{i\in M} d_i \frac{y_i^2}{a^2} \le \frac{z^2}{a^2} \\
\Leftrightarrow &\left(\left(\frac{\sqrt{\sigma + c} - \sqrt{\sigma}}{a}\right)x + \frac{\sqrt{\sigma}}{a}\right)^2 + \sum_{i\in M} d_i \frac{y_i^2}{a^2} \le \frac{z^2}{a^2},
\end{aligned}$$

and multiplying both sides by a^2 we get (17).

4.2. Set R_X with Rotated Cone

Here, we consider the set R_X and, more generally, sets of the form written in conic quadratic form

$$U_R = \left\{x \in X, (y, w, z) \in \mathbb{R}_+^{m+2} : h(x) + \sum_{i\in M} d_i y_i^2 + (w - z)^2 \le (w + z)^2\right\},$$

where $h : X \to \mathbb{R}_+$.

Observe that the approach discussed in Section 4 can be used for R_X and U_R. For example, using inequalities (14) for R_X results in the valid inequalities

$$\left(\sqrt{\sigma} + \pi' x\right)^2 + \sum_{i\in M} d_i y_i^2 + (w - z)^2 \le (w + z)^2, \quad \pi \in \Pi_\sigma. \tag{18}$$

We can also write inequalities (18) in rotated cone form:

$$\left(\sqrt{\sigma} + \pi' x\right)^2 + \sum_{i\in M} d_i y_i^2 \le 4wz, \quad \pi \in \Pi_\sigma.$$

Note, however, that the second-order cone constraint defining R_X and U_R has additional structure, namely the continuous nonnegative variables w and z in both sides of the inequality. Nevertheless, as Proposition 6 states, inequalities (18) are sufficient to characterize $\text{conv}(R_X)$. The proof of Proposition 6 is provided in Appendix A.

Proposition 6. *The convex hull of U_R is described as*

$$\text{conv}(U_R) = \left\{(x,y,w,z) \in \mathbb{R}_+^{n+m+2} : \exists s \text{ s.t. } (x,s) \in \text{conv}(U_0) \text{ and } s^2 + \sum_{i\in M} d_i {y_i}^2 \le 4wz\right\}.$$

Remark 4. Consider again the set given by (7) and (8) in Section 2.7 and observe that it corresponds to U_R with $m=0$ and $U_0 = \{x \in X,\ s \in \mathbb{R}_+ : 2\sqrt[4]{c'x} \le s\}$. Thus, if $X = \{0,1\}^n$, then $\text{conv}(U_R)$ is given by bound constraints and inequalities

$$(\xi' x)^2 \le wz,\ \xi \in \Pi(\bar{h}),$$

where $\Pi(\bar{h})$ is the set of extreme points of the extended polymatroid associated with the submodular function $\bar{h}(s) = 2\sqrt[4]{c'x}$.

5. The Case of Bounded Continuous Variables

In this section, we study $H_{\mathbb{G}}$ with bounded continuous variables, that is, by scaling $\mathbb{G} = \{0,1\}^n \times [0,1]^m$. We first give a description of $\text{conv}(H_{\mathbb{G}})$ for the case $n=m=1$ and discuss the difficulties in obtaining the convex hull description for the general case (Section 5.1). Then we describe valid conic quadratic inequalities that can be used with off-the-shelf solvers (Section 5.2).

5.1. Two-Variable Case with a Bounded Continuous Variable

In this section, we study the three-dimensional set

$$L = \left\{(x,y,z) \in \{0,1\} \times [0,1] \times \mathbb{R}_+ : \sqrt{\sigma + cx + dy^2} \le z\right\},$$

where $\sigma \ge 0$ is a constant. First, we give its convex hull description.

Proposition 7. *The convex hull of L is described as*

$$\text{conv}(L) = \{(x,y,z) \in [0,1] \times [0,1] \times \mathbb{R}_+ : g(x,y) \le z\},$$

where

$$g(x,y) = \begin{cases} g_1(x,y) = \sqrt{\left(\sqrt{\sigma} + x\left(\sqrt{c+\sigma} - \sqrt{\sigma}\right)\right)^2 + dy^2} & \text{if } y \le x + (1-x)\sqrt{\frac{\sigma}{\sigma+c}} \\ g_2(x,y) = \sqrt{\sigma(1-x)^2 + d(y-x)^2} + x\sqrt{\sigma+c+d} & \text{otherwise.} \end{cases}$$

Proof. A point (x,y,z) belongs to $\text{conv}(L)$ if and only if there exist $x_1, x_2, y_1, y_2, z_1, z_2$, and $0 \le \lambda \le 1$ such that the system

$$x = (1-\lambda)x_1 + \lambda x_2 \tag{19}$$
$$y = (1-\lambda)y_1 + \lambda y_2 \tag{20}$$
$$z = (1-\lambda)z_1 + \lambda z_2 \tag{21}$$
$$z_1 \ge \sqrt{\sigma + dy_1^2} \tag{22}$$
$$z_2 \ge \sqrt{\sigma + c + dy_2^2} \tag{23}$$
$$0 \le y_1, y_2 \le 1,\ x_1 = 0,\ x_2 = 1 \tag{24}$$

is feasible. Observe that, from (19) and (24), we can conclude that $\lambda = x$. Also observe that, from (19), (22), and (23), we have that

$$z = (1-x)z_1 + xz_2$$
$$\Leftrightarrow z \ge (1-x)\sqrt{\sigma + dy_1^2} + x\sqrt{\sigma + c + dy_2^2}.$$

Therefore, the system is feasible if and only if

$$\text{(CH)} \quad \begin{aligned} z \ge \min_{y_1,y_2}\ & (1-x)\sqrt{\sigma + dy_1^2} + x\sqrt{\sigma + c + dy_2^2} && (25) \\ \text{s.t. } & y = (1-x)y_1 + xy_2 && (\gamma) \\ & y_1 \le 1 && (\alpha_1) \\ & y_2 \le 1 && (\alpha_2) \\ & y_1 \ge 0 && (\beta_1) \\ & y_2 \ge 0, && (\beta_2) \end{aligned}$$

and let γ, $\alpha = (\alpha_1, \alpha_2)$, and $\beta = (\beta_1, \beta_2)$ be the dual variables of the optimization problem. Note that the objective function is differentiable even if $\sigma = 0$ because, in that case, the function $\sqrt{\sigma + dy_1^2}$ reduces to the linear function $\sqrt{d}y_1$. Moreover, the optimization problem is convex, and from KKT conditions for variables y_1 and y_2, we find that

$$-(1-x)\frac{dy_1}{\sqrt{\sigma + dy_1^2}} = \gamma(1-x) + \alpha_1 - \beta_1$$
$$-x\frac{dy_2}{\sqrt{\sigma + c + dy_2^2}} = \gamma x + \alpha_2 - \beta_2$$
$$\Longrightarrow \frac{y_1}{\sqrt{\sigma + dy_1^2}} + \bar{\alpha}_1 - \bar{\beta}_1 = \frac{y_2}{\sqrt{\sigma + c + dy_2^2}} + \bar{\alpha}_2 - \bar{\beta}_2, \tag{26}$$

where $\bar{\alpha}, \bar{\beta}$ correspond to α and β after scaling. We deduce from (26) and complementary slackness that $y_1, y_2 > 0$ (unless $y = 0$) and that $y_1 \le y_2$; if $y_1 = 0$ and $y_2 > 0$, then $\bar{\alpha}_1 = \bar{\beta}_2 = 0$, and (26) reduces to $-\bar{\beta}_1 = y_2/\sqrt{\sigma + c + dy_2^2} + \bar{\alpha}_2$, which has no solution because the right-hand side is positive. Letting $y_2 = 0$ and $y_1 > 0$ results in a similar contradiction, and if $0 < y_2 < y_1$, then $\bar{\beta}_1 = \bar{\alpha}_2 = \bar{\beta}_2 = 0$ and (26) reduces to $y_1/\sqrt{\sigma + dy_1^2} + \bar{\alpha}_1 = y_2/\sqrt{\sigma + c + dy_2^2}$, which has no solution because $y_1 > y_2$ implies that $y_1/\sqrt{\sigma + dy_1^2} > y_2/\sqrt{\sigma + dy_2^2} > y_2/\sqrt{\sigma + c + dy_2^2}$.

Therefore, for an optimal solution, either $0 < y_1, y_2 < 1$ (and $\bar{\alpha} = \bar{\beta} = 0$) or $y_2 = 1$ (and $\bar{\alpha}_2 \geq 0$). If $\bar{\alpha} = \bar{\beta} = 0$, then

$$y_1^* = y\frac{\sqrt{\sigma}}{x\sqrt{c+\sigma} + (1-x)\sqrt{\sigma}} \quad \text{and}$$
$$y_2^* = y\frac{\sqrt{c+\sigma}}{x\sqrt{c+\sigma} + (1-x)\sqrt{\sigma}}$$

satisfy conditions (20) and (26). Thus, if $y_2^* \leq 1$, then y_1^*, y_2^* also satisfy the bound constraints and correspond to an optimal solution to problem (CH). Replacing (y_1, y_2) by their optimal values (y_1^*, y_2^*) in (25), we find that

$$z \geq \sqrt{\left(\sqrt{\sigma} + x\left(\sqrt{c+\sigma} - \sqrt{\sigma}\right)\right)^2 + dy^2}.$$

The condition $y_2^* \leq 1$ is equivalent to

$$y \leq \frac{x\sqrt{c+\sigma} + (1-x)\sqrt{\sigma}}{\sqrt{c+\sigma}} = x + (1-x)\sqrt{\frac{\sigma}{c+\sigma}}.$$

On the other hand, if $y_2^* > 1$, an optimal solution to the optimization problem (CH) is given by $\bar{y}_2 = 1$ and $\bar{y}_1 = \frac{y-x}{1-x}$. Substituting (y_1, y_2) by their optimal values in (25),

$$z \geq \sqrt{\sigma(1-x)^2 + d(y-x)^2} + x\sqrt{\sigma + c + d}$$

when $y \geq x + (1-x)\sqrt{\frac{\sigma}{\sigma+c}}$. □

Note that inequality $g_1(x,y) \leq z$ is a special case of inequalities (14). If $\sigma = 0$, then we find that $g_2(x,y) \leq z$ reduces to $\sqrt{d}y + x(\sqrt{c+d} - \sqrt{d}) \leq z$, which is a special case of inequalities (12). However, inequality $g_2(x,y) \leq z$ is not valid if $\sigma > 0$. In particular, it cuts off the feasible point $(x,y,z) = (1, 0, \sqrt{\sigma + c})$. Moreover, it can be shown that the inequality $g_2(x,y) \leq z$ cuts off portions of conv(L) whenever $y \leq x + (1-x)\frac{\sqrt{\sigma}}{\sqrt{\sigma+c}}$.

Example 1. Consider the set L with $\sigma = d = 1$ and $c = 2$. Figure 1 shows functions g_1 and g_2 when $x = 0.5$ is fixed and illustrates the comments. We see that the function g_2 is always "above" the function g_1 and cuts the convex hull of L (the shaded region) whenever $y \leq x + (1-x)\frac{\sqrt{\sigma}}{\sqrt{\sigma+c}}$.

Unfortunately, Proposition 7 does not help to describe the convex hull of $H_{\mathbb{G}}$ with more than one bounded variable. Additionally, piecewise valid functions such as $g(x,y)$ in Proposition 7 cannot be directly used with standard algorithms for convex mixed-integer optimization. Thus, we now turn our attention to deriving inequalities that are valid and can be implemented as conic quadratic cuts if not sufficient to describe conv$(H_{\mathbb{G}})$ in general.

5.2. The General (Multivariable) Case

To obtain valid inequalities for $H_{\mathbb{G}}$, we write the conic quadratic constraint in extended form for a subset $T \subseteq M$ of the continuous variables:

$$\begin{aligned} & s^2 + \sum_{i \in M \setminus T} d_i y_i^2 \leq z^2, \\ & \sigma + \sum_{i \in N} c_i x_i + \sum_{i \in T} d_i y_i^2 \leq s^2, \\ & x \in \{0,1\}^n, y \in [0,1]^M, s \geq 0. \end{aligned} \tag{27}$$

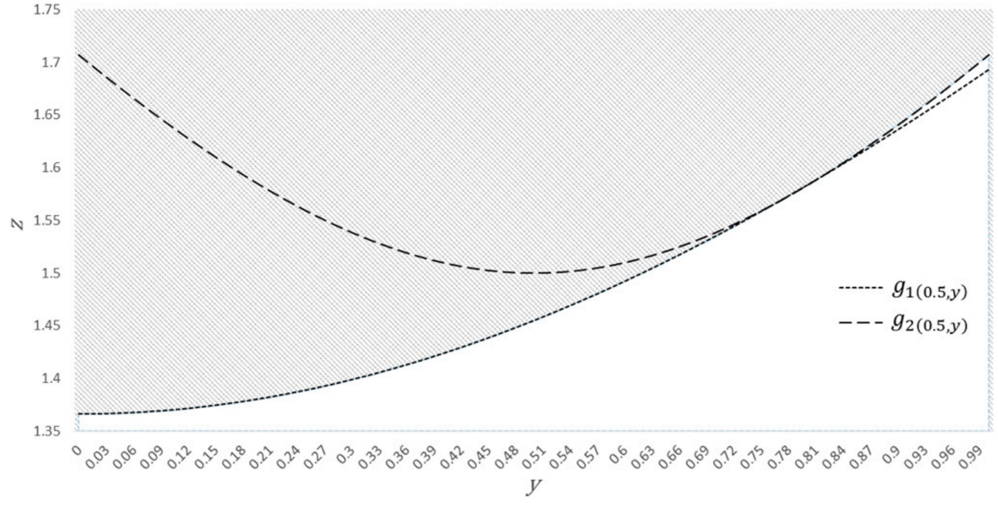
Applying inequality (12) to (27) and eliminating the auxiliary variable s, we obtain the inequalities

$$\left(\sqrt{\sigma + \sum_{i \in T} d_i y_i^2} + \pi' x\right)^2 + \sum_{i \in M \setminus T} d_i y_i^2 \leq z^2, \quad \pi \in \Pi_{\sigma + d(T)}. \tag{28}$$

Proposition 8. *For $T \subseteq M$, inequalities (28) are valid for $H_{\mathbb{G}}$.*

Note that inequalities (28) generalize or strengthen the previous valid inequalities proposed in this paper and other inequalities in the literature.

Figure 1. Functions g_1, g_2 with $\sigma = d = 1$, $c = 2$ $(x = 0.5)$



Remark 5. For $T = \emptyset$, inequalities (28) coincide with inequalities (14). For $T = M$, inequalities (28) coincide with inequalities (12). If $T \subset M$, then inequalities (28) dominate inequalities (12).

Example 1 (Continued). We obtain from (28) the valid inequality

$$g_3(x, y) = \sqrt{\sigma + dy^2} + x\left(\sqrt{\sigma + c + d} - \sqrt{\sigma + d}\right) \le z$$

for L. As Figure 2 shows, the inequality provides a good approximation of L for the example considered.

6. Valid Inequalities for General H_X

In this section, we derive inequalities that exploit the structure for an arbitrary set $X \subseteq \mathbb{D}$. We first describe a lifting procedure for obtaining valid inequalities for *any* mixed binary set X, where computing each coefficient requires solving an integer optimization problem (Section 6.1). Then, we propose an approach based on linear programming to efficiently compute weaker valid inequalities (Section 6.2).

6.1. General Mixed Binary Set X

We now consider valid inequalities for H_X, where $X \subseteq \mathbb{D}$. The inequalities described here have a structure similar to the nonlinear extended polymatroid inequalities (14) and (28). For a given a permutation $((1), (2), \ldots, (n))$ of N and $T \subseteq M$, let

$$h_k(x, y) = \sigma + \sum_{i=1}^{k-1} c_{(i)} x_{(i)} + \sum_{i \in T} d_i y_i^2,$$

$$\bar{\sigma}_{(k)} = \max\{h_k(x, y) : (x, y) \in X, x_k = 1\}, \text{ and} \quad (29)$$

$$\rho_{(k)} = \begin{cases} \sqrt{c_{(k)} + \bar{\sigma}_{(k)}} - \sqrt{\bar{\sigma}_{(k)}} & \text{if } \bar{\sigma}_{(k)} < \infty \\ 0 & \text{otherwise.} \end{cases} \quad (30)$$

Consider the inequality

$$\left(\sqrt{\sigma + \sum_{i \in T} d_i y_i^2} + \sum_{i=1}^{n} \rho_{(i)} x_{(i)}\right)^2 + \sum_{i \in M \setminus T} d_i y_i^2 \le z^2. \quad (31)$$

Proposition 9. *For $T \subseteq M$, inequalities (31) are valid for H_X.*

Proof. Let

$$H_X(T) = \left\{(x, y) \in X, s \ge 0 : \sqrt{\sigma + \sum_{i \in N} c_i x_i + \sum_{i \in T} d_i y_i^2} \le s\right\},$$

and consider the extended formulation of H_X given by

$$\hat{H}_X = \left\{(x, y, s) \in H_X(T), z \ge 0 : \sqrt{s^2 + \sum_{i \in M \setminus T} d_i y_i^2} \le z\right\}.$$

To prove the validity of (31) for H_X, it is sufficient to show that

$$\sqrt{\sigma + \sum_{i \in T} d_i y_i^2} + \sum_{i=1}^{n} \rho_{(i)} x_{(i)} \le s \quad (32)$$

is valid for $H_X(T)$. In particular, we prove by induction that

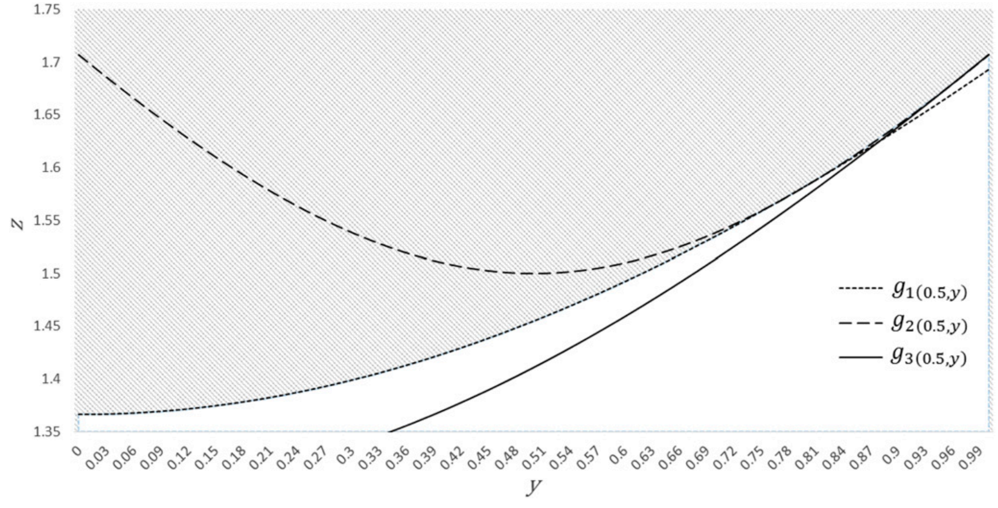
$$\sqrt{\sigma + \sum_{i \in T} d_i y_i^2} + \sum_{i=1}^{k} \rho_{(i)} x_{(i)} \le \sqrt{\sigma + \sum_{i=1}^{k} c_{(i)} x_{(i)} + \sum_{i \in T} d_i y_i^2} \quad (33)$$

for all $(x, y) \in X$ and $k = 0, \ldots, n$.

Base case: $k = 0$. Inequality (33) holds trivially.

Inductive step: Let $(\bar{x}, \bar{y}) \in X$, and suppose inequality (33) holds for $k - 1$. Observe that, if $\bar{x}_{(k)} = 0$ or

Figure 2. Functions g_1, g_2, and g_3 with $\sigma = d = 1$, $c = 2$ $(x = 0.5)$



$\rho_{(k)} = 0$, then inequality (33) clearly holds for k. Therefore, assume that $\bar{x}_{(k)} = 1$ and $\bar{\sigma}_{(k)} < \infty$. We have

$$\begin{aligned}
&\sqrt{\sigma + \sum_{i=1}^{k} c_{(i)}\bar{x}_{(i)} + \sum_{i\in T} d_i \bar{y}_i^2} \\
&= \sqrt{h_k(\bar{x},\bar{y}) + c_{(k)}} \\
&= \sqrt{h_k(\bar{x},\bar{y})} + \left(\sqrt{h_k(\bar{x},\bar{y}) + c_{(k)}} - \sqrt{h_k(\bar{x},\bar{y})}\right) \\
&\geq \sqrt{h_k(\bar{x},\bar{y})} + \left(\sqrt{\bar{\sigma}_{(k)} + c_{(k)}} - \sqrt{\bar{\sigma}_{(k)}}\right) \qquad (34) \\
&\geq \sqrt{\sigma + \sum_{i\in T} d_i \bar{y}_i^2} + \sum_{i=1}^{k} \rho_{(i)}\bar{x}_{(i)}, \qquad (35)
\end{aligned}$$

where (34) follows from $\bar{\sigma}_{(k)} \geq h_k(\bar{x},\bar{y})$ (by definition of $\bar{\sigma}_{(k)}$) and from the concavity of the square root function, and (35) follows from $\sqrt{h_k(\bar{x},\bar{y})} \geq \sqrt{\sigma + \sum_{i\in T} d_i \bar{y}_i^2} + \sum_{i=1}^{k-1} \rho_{(i)}\bar{x}_{(i)}$ (induction hypothesis) and from the definition of $\rho_{(k)}$. □

Remark 6. Note that $\sigma_{(k-1)} = \sigma + \sum_{i=1}^{k-1} c_{(i)}$, and if $T = \emptyset$, then

$$\bar{\sigma}_{(k)} = \max_{\substack{x\in X \\ x_{(k)}=1}} \sigma + \sum_{i=1}^{k-1} c_{(i)} x_i.$$

In particular, if $T = \emptyset$, then $\bar{\sigma}_{(k)} \leq \sigma_{(k-1)}$ and $\rho_{(k)} \geq \pi_{(k)}$. Thus, for $T = \emptyset$ and $X = \mathbb{D}$, inequalities (31) reduce to inequalities (14); for $T = \emptyset$ and $X \subset \mathbb{D}$, inequalities (31) dominate inequalities (14).

Remark 7. For $X = \mathbb{G}$, inequalities (31) reduce to inequalities (28). For $X \subset \mathbb{G}$, inequalities (31) dominate inequalities (28).

Remark 8. For the case of the pure binary set defined by a cardinality constraint, that is, $Y = \{x \in \{0,1\}^n : \sum_{i=1}^{n} x_i \leq k\}$ and $\sigma = 0$, Yu and Ahmed (2017) give facets for $\text{conv}(H_Y)$. However, noting the computation burden of constructing them, they propose approximate lifted inequalities of the form $\sum_{i\leq k} \pi_{(i)} x_{(i)} + \sum_{i>k} \rho_{(i)} x_{(i)} \leq z$, where π are computed according to (10), and

$$\rho_{(i)} = \sqrt{c\big(T_{(i)}\big) + c_{(i)}} - \sqrt{c\big(T_{(i)}\big)}$$

with $T_{(i)} = \arg\max\{c(T) : T \subseteq \{(1),\ldots,(i-1)\},\ |T| = k-1\}$. Thus, their approximate lifted inequalities coincide with inequalities (31) and can be computed in $O(n \log n)$. If the set X has additional constraints, then inequalities (31) are stronger than the approximate lifted inequalities of Yu and Ahmed (2017).

Remark 9. The strengthened extended polymatroid inequalities described in this section can be used with rotated cone constraints as well. In particular, for the set

$$R_X = \left\{(x,y) \in X, w \geq 0, z \geq 0 : \sigma + \sum_{i\in N} c_i x_i + \sum_{i\in M} d_i y_i^2 \leq 4wz\right\},$$

we find that inequalities

$$\left(\sqrt{\sigma + \sum_{i\in T} d_i y_i^2} + \sum_{i=1}^{n} \rho_{(i)} x_{(i)}\right)^2 + \sum_{i\in M\setminus T} d_i y_i^2 \leq 4wz \qquad (36)$$

are valid for R_X.

6.2. Relaxed Inequalities

Note that computing each coefficient of inequality (31) requires solving a nonconvex mixed 0–1 optimization problem (29), which may not be practical in most cases. However, observe from Remarks 6 and 7 that solving the optimization problem over *any* relaxation of X that includes the bound constraints results in valid inequalities at least as strong as the ones resulting from using only the bound constraints.

In particular, assume in problem (29) that, for $i \in T$, y_i has a finite upper bound (otherwise the problem is unbounded and $\rho_i = 0$) and $u_i = 1$ (by scaling). Moreover let X_P be a polyhedron such that $X \subseteq X_P$. Convex constraints can also be included in X_P by using a suitable linear outer approximation (Ben-Tal and Nemirovski 2001, Tawarmalani and Sahinidis 2005, Hijazi et al. 2013, Lubin et al. 2018).

Given X_P, the approximate coefficients

$$\begin{aligned}
&\hat{\rho}_{(k)} = \sqrt{c_{(k)} + \hat{\sigma}_{(k)}} - \sqrt{\hat{\sigma}_{(k)}}, \text{ with} \\
&\hat{\sigma}_{(k)} = \sigma + \max\left\{\sum_{i=1}^{k-1} c_{(i)} x_{(i)} + \sum_{i\in T} d_i y_i : (x,y) \in X_P, x_k = 1\right\}
\end{aligned} \qquad (37)$$

can be computed efficiently by solving n linear programs. Moreover, the linear program required to compute $\hat{\sigma}_{(k)}$ differs from the one required for $\hat{\sigma}_{(k-1)}$ in two bound constraints, corresponding to $x_{(k-1)}$ and $x_{(k)}$, and one objective coefficient, corresponding to $x_{(k-1)}$. Therefore, using the simplex method with warm starts, each $\hat{\sigma}_{(k)}$ can be computed efficiently, using only a small number of simplex pivots.

7. Computational Considerations

Table 1 presents a classification of the proposed inequalities, depending on whether the continuous variables are bounded or not, on whether the inequalities are for the set with the conic quadratic cone H_X or the rotated cone R_X and on whether additional constraints are used to strengthen the inequalities (strengthened) or not (polymatroid). Note that there is a direct correspondence between the inequalities for conic quadratic cones and for rotated cones, and although not explicitly shown in the paper, it is easy to construct the rotated cone version of inequality (28).

We now consider the implementation of the proposed inequalities in branch-and-cut algorithms. First, in Section 7.1, we discuss the difficulties in using the inequalities for the (more general) bounded case; then, in Section 7.2, we show how to efficiently use the cuts for the unbounded case.

7.1. Bounded Case

For brevity, we only discuss inequalities (28) of the form $\varphi(x, y) \le z$, where

$$\varphi(x,y) = \sqrt{\left(\sqrt{\sigma + \sum_{i\in T} d_i y_i^2} + \pi' x\right)^2 + \sum_{i\in M\setminus T} d_i y_i^2}.$$

All other inequalities for the bounded case have a similar structure, so the discussion extends directly to those cases as well. Inequalities (28) are nonlinear, and can be added to the formulation as nonlinear inequalities or can be implemented via linear cutting planes using outer approximations. Unfortunately, both approaches have drawbacks that may limit the effectiveness of the inequalities in practice when used with current off-the-shelf solvers.

7.1.1. Implementation as Nonlinear Cuts.

The function φ is conic quadratic representable; in particular, the inequality $\varphi(x, y) \le z$ is equivalent to the system

$$s_1^2 \ge \sigma + \sum_{i\in T} d_i y_i^2 \tag{38}$$

$$s_2 = s_1 + \pi' x$$

$$z \ge s_2^2 + \sum_{i\in M\setminus T} d_i y_i^2 \tag{39}$$

$$0 \le s_1, s_2,$$

where (38) and (39) are conic quadratic inequalities accepted by most solvers.

Observe that adding each inequality (28) requires two additional variables and conic constraints; thus, adding even a modest number of inequalities may substantially increase the difficulty of solving the convex relaxations at each node of the branch-and-bound tree. Additionally, solvers rely on the dual simplex method to solve the subproblems arising in branch and bound (by constructing a linear approximation of nonpolyhedral sets) because of its warm start capabilities; adding nonlinear cuts such as (38) and (39) may render the existing simplex tableau ineffective and require solving the subproblems from scratch. Finally, commercial solvers, currently, do not allow adding nonlinear cuts during branching, and inequalities (28) need to be added by the user at the root node explicitly, giving up the benefits of built-in cut-management strategies.

Table 1. Classification of the Proposed Inequalities

Continuous variables	Polymatroid: Conic quad	Polymatroid: Rotated	Strengthened: Conic quad	Strengthened: Rotated
Unbounded	(14)	(16)	(29), $T=\emptyset$	(34), $T=\emptyset$
Bounded	(26)		(29)	(34)

7.1.2. Implementation as Linear Outer Approximations.

Cutting planes based on a linear outer approximation of the convex function φ can be added using gradients. Given a fractional solution $(\bar{x}, \bar{y})$, the linear underestimator $\bar{\varphi}(x, y) \le z$, where

$$\bar{\varphi}(x,y) = \varphi(\bar{x},\bar{y}) + \nabla_x\varphi(\bar{x})'(x-\bar{x}) + \nabla_y\varphi(\bar{y})'(y-\bar{y})$$

is valid. In particular, we find

$$\bar{\varphi}(x,y) = \psi + \frac{1}{\psi}\Bigg(\eta\pi'(x-\bar{x}) + \zeta\sum_{i\in T} d_i\bar{y}_i(y_i-\bar{y}_i) + \sum_{i\in M\setminus T} d_i\bar{y}_i(y_i-\bar{y}_i)\Bigg),$$

where

$$\eta = \sqrt{\sigma + \sum_{i\in T} d_i\bar{y}_i^2} + \pi'\bar{x}; \quad \zeta = \frac{\eta}{\sqrt{\sigma + \sum_{i\in T} d_i\bar{y}_i^2}};$$

$$\psi = \sqrt{\eta^2 + \sum_{i\in M\setminus T} d_i\bar{y}_i^2}.$$

An implementation based on the linear cuts $\bar{\varphi}(x, y) \le z$ leverages the existing capabilities of current commercial solvers, including warm starts and cut-management strategies. Nevertheless, each linear inequality $\bar{\varphi}(x, y) \le z$ is often weak, and constructing a suitable approximation of the original nonlinear inequality $\varphi(x, y) \le z$ may require a prohibitive number of cuts.

In Appendix B we provide a comparison of both approaches for a simple mean-risk minimization problem with bounded continuous variables and no correlations. Adding the nonlinear inequalities directly, as discussed in Section 7.1.1, results in significantly better performance, both in terms of the relaxation quality and the solution times. These results are consistent with the recent experience by the authors using other classes of nonlinear inequalities; see Atamtürk and Gómez (2018) and Gómez (2018).

7.2. Unbounded Case

In most of the applications discussed in Section 2, the continuous variables are used to model covariance terms, rotated cone constraints, or denominators in fractional optimization. In such cases, the continuous variables are unbounded, and the proposed inequalities can be implemented efficiently in such settings. Observe that the conic quadratic inequality arising in set H_X can be written in an extended formulation as

$$\begin{aligned} s^2 &\geq \sum_{i\in N} c_i x_i^2 \\ z^2 &\geq s^2 + \sum_{i\in M} d_i y_i^2 \\ 0 &\leq s. \end{aligned}$$

Similarly, the rotated cone inequality arising in set R_X can be written as

$$\begin{aligned} s^2 &\geq \sum_{i\in N} c_i x_i^2 \\ t^2 &\geq s^2 + \sum_{i\in M} d_i y_i^2 \\ t^2 &\leq wz \\ 0 &\leq s, t. \end{aligned}$$

In both cases, the polymatroid and strengthened inequalities can be added as linear cuts, $\pi' x \leq s$ and $\rho' x \leq s$, respectively. Thus, when adding the nonlinear inequalities as linear cuts in an extended formulation, optimization algorithms benefit from the warm starts and cut-management strategies without sacrificing the strength of the inequalities. Such a formulation cannot be used effectively for the bounded case because an additional variable would be needed for each subset T of M.

8. Experiments

In this section we report computational experiments performed to test the effectiveness of the polymatroid inequalities in solving second-order cone optimization with a branch-and-cut algorithm. In Section 8.1, we solve instances with general covariance matrices (see application in Section 2.2); in Section 8.2, we solve conic quadratic interdiction problems (see application in Section 2.3); and in Section 8.3, we solve binary linear fractional problems (see applications in Section 2.6).

All experiments are done using CPLEX 12.6.2 solver on a workstation with a 2.93 GHz Intel®Core™ i7 CPU and 8 GB main memory and with a single thread. We compare using default CPLEX without adding any cuts (cpx), using the inequalities in Section 4 (polymatroid) and using the strengthened inequalities in Section 6 (strengthened). Because, in all cases, the continuous variables are unbounded, we implement the inequalities as discussed in Section 7.2. The time limit is set to two hours, and CPLEX's default settings are used. The inequalities are added only at the root node using callback functions, and all times reported include the time required to add cuts.

8.1. Mean-Risk Minimization with Correlated Random Variables

In this section, we test the effectiveness of the polymatroid inequalities in instances with correlated random variables. In particular, we solve mean-risk minimization problems

$$\min_{x\in\{0,1\}^n} \left\{ -a' x + \Omega\sqrt{x' Q x} : \sum_{i=1}^n x_i \leq k \right\}, \tag{40}$$

where the matrix Q is generated according to a factor model, that is, $Q = ZFZ' + D$, where $F \in \mathbb{R}^{r\times r}$ is the factor covariance matrix, $Z \in \mathbb{R}^{n\times r}$ is the exposure matrix, and $D \in \mathbb{R}^{n\times n}$ is the diagonal matrix with the specific covariances. Observe that, in such instances, we can set $\text{diag}(c) = D$ in Equation (3).

In our experiments $F = GG'$ with $G \in \mathbb{R}^{r\times r}$ and $G_{ij} \sim U[-1, 1]$; $Z_{ij} \sim U[0, 1]$ with probability 0.2 and $Z_{ij} = 0$ otherwise; $D_{ii} \sim U[0, \delta\bar{q}]$, where $\delta \geq 0$ is a diagonal dominance parameter and $\bar{q} = \frac{1}{N}\sum_{i\in N} Q_{0ii}$; and $a_i \sim U[0.85\sqrt{Q_{ii}}, 1.15\sqrt{Q_{ii}}]$. We set the parameter $\Omega = \Phi^{-1}(\alpha)$, where Φ is the cumulative distribution function of the normal distribution and $\alpha \in \{0.95, 0.975, 0.99\}$. We let $n = 200$, $r = 40$, and k equal 10%, 15%, and 20% of the number of the variables.

Tables 2 and 3 present the results for different values of the diagonal dominance parameter δ. Each row represents the average over five instances generated with the same parameters and shows the initial gap (igap), the root gap improvement (rimp), the number of nodes explored (nodes), the time elapsed in seconds (time), and the end gap (egap) [in brackets, the number of instances solved to optimality (#)]. The initial gap is computed as $\texttt{igap} = \frac{t_{\text{opt}} - t_{\text{relax}}}{|t_{\text{opt}}|} \times 100$, where t_{opt} is the objective value of the best feasible solution at termination and t_{relax} is the objective value of the continuous relaxation. The end gap is computed as $\texttt{egap} = \frac{t_{\text{opt}} - t_{\text{bb}}}{|t_{\text{opt}}|} \times 100$, where t_{bb} is the objective value of the best lower bound at termination. The root improvement is computed as $\texttt{rimp} = \frac{t_{\text{root}} - t_{\text{relax}}}{t_{\text{opt}} - t_{\text{relax}}} \times 100$, where t_{root} is the value of the continuous relaxation after adding the valid inequalities to the formulation. Figure 3 shows the corresponding performance profiles.

Observe that adding inequalities polymatroid or strengthened closes the initial integrality gaps by

Table 2. Experiments with General Covariance Matrices ($\delta = 0.5$)

			cpx				Polymatroid				Strengthened			
k	α	Igap	Rimp	Nodes	Time	Egap[#]	Rimp	Nodes	Time	Egap[#]	Rimp	Nodes	Time	Egap[#]
20	0.95	1.7	22.6	9,557	74	0.0[5]	53.3	3,957	23	0.0[5]	55.6	2,367	17	0.0[5]
	0.975	3.0	21.3	33,468	242	0.0[5]	53.5	13,316	86	0.0[5]	55.9	5,839	40	0.0[5]
	0.99	5.2	15.2	164,568	1,845	0.0[5]	52.8	80,735	730	0.0[5]	55.3	23,577	269	0.0[5]
	Average		**19.7**	**69,198**	**720**	**0.0[15]**	**53.2**	**32,669**	**280**	**0.0[15]**	**55.6**	**10,594**	**109**	**0.0[15]**
30	0.95	0.8	15.5	7,115	57	0.0[5]	53.3	1,656	11	0.0[5]	52.4	1,159	9	0.0[5]
	0.975	1.3	14.9	18,901	135	0.0[5]	53.1	2,800	20	0.0[5]	54.0	2,095	15	0.0[5]
	0.99	2.3	5.7	76,675	1,005	0.0[5]	61.1	8,265	48	0.0[5]	62.1	5,131	30	0.0[5]
	Average		**12.0**	**34,230**	**399**	**0.0[15]**	**55.8**	**4,240**	**26**	**0.0[15]**	**56.2**	**2,795**	**18**	**0.0[15]**
40	0.95	0.4	23.3	2,910	18	0.0[5]	48.5	611	6	0.0[5]	50.5	577	6	0.0[5]
	0.975	0.7	20.0	4,216	30	0.0[5]	54.3	884	7	0.0[5]	55.5	839	7	0.0[5]
	0.99	1.1	13.5	46,030	514	0.0[5]	55.9	2,493	18	0.0[5]	56.7	2,144	14	0.0[5]
	Average		**18.9**	**17,719**	**187**	**0.0[15]**	**52.9**	**1,329**	**10**	**0.0[15]**	**54.2**	**1,187**	**9**	**0.0[15]**

Notes. Bold indicates the average of the rows for each k. Igap, initial gap; cpx, default CPLEX without adding any cuts; Rimp, root gap improvement; Egap, end gap.

45%–75%, resulting in significant performance improvement over default CPLEX. In particular, using inequalities strengthened for instances with $k = 20$ leads to a seven time speed-up with $\delta = 0.5$ and two time speed-up with $\delta = 1$) and lower end gaps. Moreover, for instances with $k \geq 30$ using inequalities strengthened results in at least an order of magnitude speed-up over default CPLEX. The impact of both inequalities increases with higher diagonal dominance as expected. In Figure 3, we see that, for $\delta = 1.0$, cpx requires close to 3,000 seconds to solve 70% of the instances, polymatroid requires 110 seconds, and strengthened requires 50 seconds to solve a similar number of instances; that is, strengthened is 50 times faster than cpx. In fact, strengthened solves in 60 seconds 73% of the instances, the same quantity that cpx solves in two hours. Finally, we see that the strengthened inequalities result in consistently better performance than the simpler polymatroid inequalities.

8.2. Conic Quadratic Interdiction Instances

In this section, we test the effectiveness of the proposed inequalities for the interdiction problem (CQI) discussed in Section 2.3. In our computations, we model a decision maker that seeks a path with minimal value at risk. After the decision maker decides on a path, an adversary may attack a limited number of arcs on the path, increasing the expectation and/or covariance of travel times/costs.

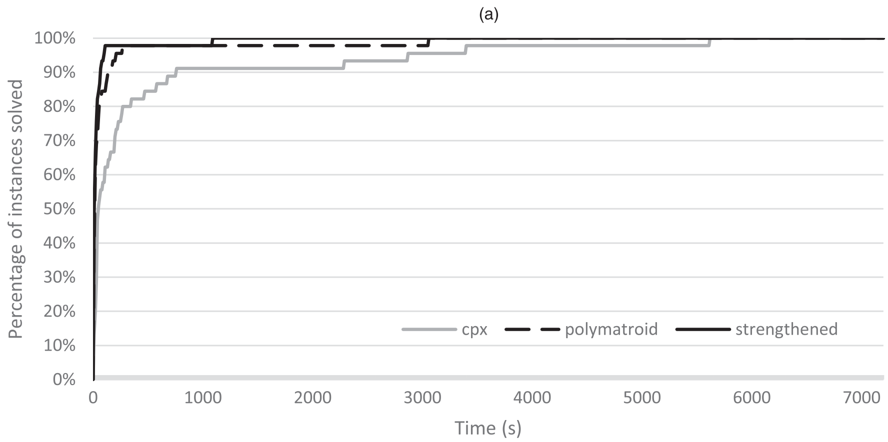
The feasible region X is given by path constraints on a 40×40 grid network. There is a potential adverse event corresponding to each arc, and each event results in an increase in the nominal duration/cost and variance of that arc: in particular, for $i = 1, \ldots, n$, $a_i \sim U[0,2]e^i$, where e^i is the vector that has value one

Table 3. Experiments with General Covariance Matrices ($\delta = 1.0$)

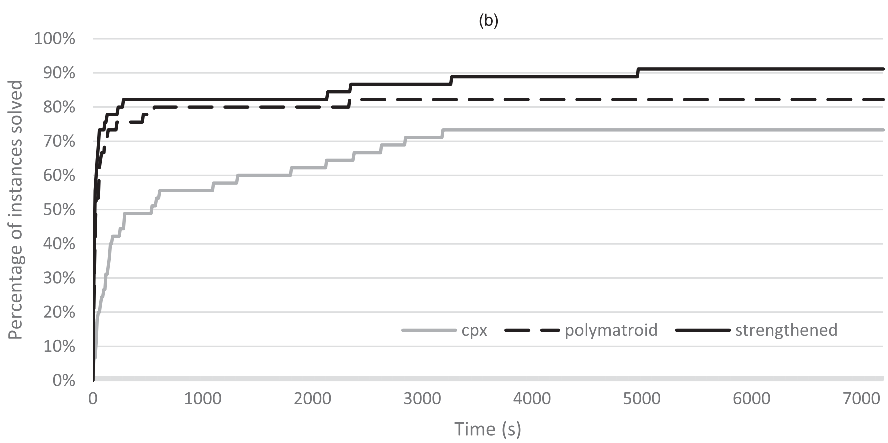
			cpx				Polymatroid				Strengthened			
k	α	Igap	Rimp	Nodes	Time	Egap[#]	Rimp	Nodes	Time	Egap[#]	Rimp	Nodes	Time	Egap[#]
20	0.95	2.9	21.6	64,283	927	0.0[5]	55.1	14,984	165	0.0[5]	59.1	6,233	68	0.0[5]
	0.975	5.0	15.5	240,224	3,975	0.4[3]	44.4	189,826	3,390	0.4[3]	50.9	102,053	1,915	0.1[4]
	0.99	9.0	6.4	378,116	7,200	2.2[0]	35.7	477,553	7,200	1.9[0]	43.1	430,707	5,966	0.6[2]
	Average		**14.5**	**227,541**	**4,034**	**0.9[8]**	**45.1**	**227,454**	**3,585**	**0.8[8]**	**51.0**	**179,664**	**2,650**	**0.2[11]**
30	0.95	1.1	17.1	32,629	316	0.0[5]	77.2	1,082	12	0.0[5]	78.2	682	10	0.0[5]
	0.975	2.0	12.5	150,756	2,046	0.1[4]	72.9	12,202	107	0.0[5]	75.5	4,896	39	0.0[5]
	0.99	3.5	10.5	258,866	3,679	0.5[3]	67.8	115,507	1,510	0.1[4]	70.6	59,106	511	0.0[5]
	Average		**13.4**	**147,417**	**2,014**	**0.2[12]**	**72.6**	**42,930**	**543**	**0.0[14]**	**74.8**	**21,561**	**187**	**0.0[15]**
40	0.95	0.6	23.9	6,522	64	0.0[5]	72.3	270	9	0.0[5]	74.8	192	8	0.0[5]
	0.975	1.0	24.0	31,022	414	0.0[5]	71.0	823	12	0.0[5]	72.1	695	11	0.0[5]
	0.99	1.6	17.6	122,568	2,907	0.2[3]	73.9	4,416	37	0.0[5]	75.1	2,543	26	0.0[5]
	Average		**21.8**	**53,371**	**1,128**	**0.1[13]**	**72.4**	**1,836**	**19**	**0.0[15]**	**74.0**	**1,143**	**15**	**0.0[15]**

Notes. Bold indicates the average of the rows for each k. Igap, initial gap; cpx, default CPLEX without adding any cuts; Rimp, root gap improvement; Egap, end gap.

Figure 3. Percentage of Instances Solved Within a Given Time Limit for Mean-Risk Minimization with Correlated Random Variables



$\delta = 0.5.$



$\delta = 1.0.$

in the ith position and zero elsewhere, and the ith row and column of Q_i is drawn from $U[0, 2]$ and Q_i has zero entries elsewhere. Each element of the nominal cost vector a_0 is drawn from $U[0, 1]$, and the squared roots of every diagonal element of Q_0 are also generated from $U[0, 1]$. The parameter Ω is set as in Section 8.1.

Table 4 shows the results for different values of α and the parameter controlling the number of attacks Γ, and Figure 4 shows the corresponding performance profile. Observe that the strengthened cuts result in a better root improvement of 55%—compared with 30%–37% achieved by default CPLEX. Moreover, when using the strengthened inequalities, 37 instances are solved to optimality, and default CPLEX is able to solve only 22 instances. We also see that, in these path instances, the polymatroid inequalities result in longer solution times than cpx (despite better root improvements). On the other hand, the strengthened inequalities are effective in reducing both the integrality gaps and solution times.

8.3. Binary Fractional Optimization Instances

We now test the inequalities in a binary fractional problem arising in assortment optimization with cardinality constraint:

$$\text{(FP)} \quad \max\left\{\sum_{j=1}^{m} \frac{\sum_{i=1}^{n} c_{ij}x_i}{a_{0j} + \sum_{i=1}^{n} a_{ij}x_i} : \sum_{i=1}^{n} x_i \le k,\ x \in \{0,1\}^n\right\}.$$

The data are generated as in the assortment optimization problems considered in Şen et al. (2018): $a_{ij} \sim U[0, 1]$ for all i, j, $c_{ij} = a_{ij}r_{ij}$ with $r_{ij} \sim U[1, 3]$, $n = 200$, $m = 20$ and $a_{0j} = a_0$ for all $j = 1, \ldots, m$ with $a_0 \in \{5, 10\}$, and $k \in \{10, 20, 50\}$.

Binary fractional problems (FP) are usually solved by linearizing the fractional terms (see Tawarmalani et al. 2002, Prokopyev et al. 2005, Bront et al. 2009, Méndez-Díaz et al. 2014, Borrero et al. 2016b, Şen et al. 2018), which requires the addition of $O(nm)$ additional variables and big-M constraints. On the other hand, the rotated cone reformulation outlined in

Table 4. Experiments with Robust Conic Instances

			cpx				Polymatroid				Strengthened			
Γ	α	Igap	Rimp	Nodes	Time	Egap[#]	Rimp	Nodes	Time	Egap[#]	Rimp	Nodes	Time	Egap[#]
4	0.95	22.6	35.1	65,533	3,124	0.6[4]	44.3	72,322	5,220	1.2[3]	56.8	17,057	917	0.0[5]
	0.975	24.1	30.2	95,337	4,239	0.8[4]	41.1	87,697	7,200	3.5[0]	55.2	53,022	2,648	0.0[5]
	0.99	25.7	26.6	153,481	7,200	2.2[0]	37.9	80,160	7,200	7.6[0]	53.5	102,578	4,452	0.0[5]
	Average		**30.6**	**104,117**	**4,854**	**1.2[8]**	**41.1**	**80,060**	**6,540**	**4.1[3]**	**55.2**	**57,552**	**2,672**	**0.0[15]**
6	0.95	26.9	38.8	73,898	3,422	0.4[4]	45.0	89,319	5,771	2.0[3]	56.2	33,364	1,644	0.0[5]
	0.975	28.1	34.6	138,231	5,676	1.8[2]	41.3	96,917	7,200	5.5[0]	54.1	113,745	4,895	0.0[5]
	0.99	29.7	32.0	160,074	6,823	4.2[1]	38.9	94,762	7,200	7.2[0]	52.2	113,954	6,091	2.0[1]
	Average		**35.1**	**124,068**	**5,307**	**2.1[7]**	**41.7**	**93,666**	**6,704**	**4.9[2]**	**54.2**	**87,021**	**4,210**	**0.7[11]**
8	0.95	30.2	40.9	143,946	5,474	0.8[4]	46.4	112,279	6,822	1.6[1]	55.2	53,942	2,234	0.0[5]
	0.975	31.3	36.3	145,582	5,967	1.9[2]	42.7	107,432	7,200	4.8[0]	53.4	99,904	4,679	0.4[4]
	0.99	32.7	34.2	123,325	6,512	3.5[1]	39.5	94,691	7,200	8.2[0]	51.1	136,632	6,162	2.4[2]
	Average		**37.1**	**137,618**	**5,984**	**2.1[7]**	**42.8**	**104,801**	**7,055**	**4.9[1]**	**53.2**	**96,826**	**4,358**	**0.9[11]**

Notes. Bold indicates the average of the rows for each Γ. Igap, initial gap; cpx, default CPLEX without adding any cuts; Rimp, root gap improvement; Egap, end gap.

Figure 4. Percentage of Instances Solved Within a Given Time Limit for Interdiction Problems

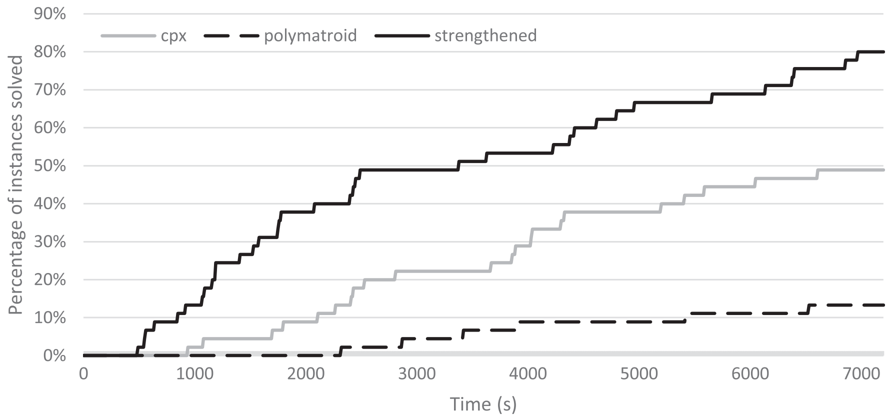


Table 5. Experiments with Binary Fractional Optimization

		cpx-milo				cpx-conic				Polymatroid			
a_0	k	Rgap	Nodes	Time	Egap[#]	Rgap	Nodes	Time	Egap[#]	Rgap	Nodes	Time	Egap[#]
5	10	50.9	20,737	7,200	43.6[0]	3.1	24,073	572	0.0[5]	0.1	46	19	0.0[5]
	20	18.0	51,180	7,200	17.0[0]	2.7	123,655	7,200	1.9[0]	0.0	118	54	0.0[5]
	50	0.9	621,742	6,010	0.5[1]	4.9	55,155	7,200	4.5[0]	0.1	15,465	263	0.0[5]
Average		**23.3**	**231,220**	**6,803**	**20.4[1]**	**3.2**	**67,628**	**4,991**	**2.1[5]**	**0.1**	**5,210**	**112**	**0.0[15]**
10	10	46.8	380,700	7,200	15.9[0]	2.2	48,541	972	0.0[5]	0.0	6	14	0.0[5]
	20	39.8	23,770	7,200	37.4[0]	3.7	206,603	7,200	1.4[0]	0.0	61	37	0.0[5]
	50	5.6	136,382	7,200	5.2[0]	5.1	52,700	7,200	4.6[0]	0.1	36,959	396	0.0[5]
Average		**30.7**	**180,284**	**7,200**	**19.5[0]**	**4.3**	**102,615**	**5,124**	**2.0[5]**	**0.0**	**12,342**	**149**	**0.0[15]**

Notes. Bold indicates the average of the rows for each $a0$. cpx-milo, classical big M linear formulation used in Bront et al. (2009) and Méndez-Díaz et al. (2014); cpx-conic, conic formulation without adding inequalities; Rgap, root gap; Egap, end gap.

Section 2.6 requires adding only m additional variables and avoids big M constraints altogether.

We test the classical big M linear formulation used in Bront et al. (2009) and Méndez-Díaz et al. (2014) (cpx-milo), the conic formulation without adding inequalities (cpx-conic), and the conic formulation strengthened with polymatroid inequalities.[1] Table 5 shows the results. Each row represents the average over five instances generated with the same parameters and for each combination of the parameters a_0 and k and for each formulation, the root gap (rgap), the number of nodes explored (nodes), the time elapsed in seconds (time), and the end gap (egap)[in brackets, the number of instances solved to optimality (#)]. The root gap is computed as $\texttt{rgap} = \frac{t_{\text{opt}}-t_{\text{root}}}{|t_{\text{opt}}|} \times 100$, where t_{opt} is the objective value of the best feasible solution at termination, and t_{root} is the objective value of the relaxation obtained after processing the root node (i.e., after user cuts and cuts added by CPLEX).

We see that the conic formulation with polymatroid inequalities results in substantially faster solution times than the other formulations. In particular, CPLEX with the classical big M linear optimization formulation cpx-milo can only solve 1/30 instances after two hours of branch and bound, and the average end gaps are 20%; the conic formulation with extended polymatroid cuts is able to solve all instances to optimality in less than three minutes (on average). We see that root gaps for polymatroid are very small in all instances (less than 0.1%), and optimality can be proven in instances with small cardinality parameter k after few branch-and-bound nodes (e.g., in instances with $k = 10$ and $a_0 = 5$ optimality is proven after 46 nodes, and cpx-conic requires 24,000 nodes to prove optimality).

9. Conclusions

We propose new convex valid inequalities that exploit submodularity for conic quadratic mixed 0–1 sets. The studied sets arise in a variety of risk-adverse decision-making problems (e.g., chance-constrained optimization with correlated variables, robust optimization with ellipsoidal or discrete uncertainty sets) as well as in models of other problems commonly arising in operations research (e.g., lot sizing, scheduling, assortment, fractional linear optimization). The unbounded version of the convex inequalities, which arise naturally in most applications, can be efficiently implemented as linear cuts in an extended space, which make them particularly effective. Moreover, the inequalities can be strengthened to take advantage of other constraints in a problem through approximate lifting without affecting this convenient property. Computational experiments performed on correlated mean-risk minimization, robust interdiction, and assortment optimization problems indicate that the proposed inequalities improve the performance of branch-and-bound solvers substantially; in some cases, problems for which no efficient algorithms were known are now solved in seconds.

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Appendix A

Proof of Proposition 6. Consider the optimization of an arbitrary linear function over the convex relaxation of the extended formulation of U_R given by

$$(\mathrm{P_R}) \quad \begin{aligned} \min\ & a'x + b'y + pw + qz \\ \text{s.t. } & s^2 + \sum_{i\in M} d_i y_i^2 + (w-z)^2 \le (w+z)^2 \\ & (x,s) \in \operatorname{conv}(U_0) \\ & y \in \mathbb{R}_+^m, w \ge 0, z \ge 0. \end{aligned} \tag{A.1}$$

Without loss of generality, we can assume that $p > 0$ and $q > 0$ (if $p < 0$ or $q < 0$, then the problem is unbounded, and if $p = 0$ or $q = 0$, then (P_R) reduces to a linear program over an integral polyhedron). Moreover, observe that, if $w = z$ in an optimal solution, then the problem reduces to a linear optimization over conv(U), which has an optimal integral solution (Proposition 5). Thus, we can assume that $w \neq z$, in which case, the left-hand side of (A.1) is differentiable, and we infer from KKT conditions with respect to w and z that

$$-p = -\lambda + \lambda \frac{w-z}{\sqrt{s^2 + \sum_{i\in M} d_i y_i^2 + (w-z)^2}}, \tag{A.2}$$

$$-q = -\lambda - \lambda \frac{w-z}{\sqrt{s^2 + \sum_{i\in M} d_i y_i^2 + (w-z)^2}}, \tag{A.3}$$

where λ is the dual variable associated with constraint (A.1). We deduce from (A.2) that

$$w - z = \frac{\lambda - p}{\lambda}\sqrt{s^2 + \sum_{i\in M} d_i y_i^2 + (w-z)^2},$$

and from (A.3) that

$$w - z = \frac{q - \lambda}{\lambda}\sqrt{s^2 + \sum_{i\in M} d_i y_i^2 + (w-z)^2}. \tag{A.4}$$

In particular, we find that $\lambda = \frac{p+q}{2}$. Moreover, we obtain from (A.4) that

$$\begin{aligned}(w-z)^2 &= \left(\frac{q-\lambda}{\lambda}\right)^2 \left(s^2 + \sum_{i\in M} d_i y_i^2 + (w-z)^2\right) \\ &= \left(\frac{q-p}{q+p}\right)^2 \left(s^2 + \sum_{i\in M} d_i y_i^2 + (w-z)^2\right).\end{aligned}$$

Letting

$$\beta = \frac{\left(\frac{q-p}{q+p}\right)^2}{1 - \left(\frac{q-p}{q+p}\right)^2},$$

we deduce that

$$(w-z)^2 = \beta\left(s^2 + \sum_{i\in M} d_i y_i^2\right).$$

Therefore, we have that

$$\sqrt{s^2+\sum_{i\in M} d_i y_i^2 + (w-z)^2} = \sqrt{1+\beta}\sqrt{s^2+\sum_{i\in M} d_i y_i^2}.$$

Moreover, because, in any optimal solution of (P_R), constraint (A.1) is binding, we have

$$w+z = \sqrt{1+\beta}\sqrt{s^2+\sum_{i\in M} d_i y_i^2}.$$

Multiplying equality (A.2) by w in both sides, and multiplying equality (A.3) by z in both sides, we find that

$$\begin{aligned} pw+qz &= \lambda(w+z) - \lambda\frac{(w-z)^2}{\sqrt{s^2+\sum_{i\in M} d_i y_i^2+(w-z)^2}}\\ &= \lambda\sqrt{1+\beta}\sqrt{s^2+\sum_{i\in M} d_i y_i^2} - \lambda\frac{\beta\left(s^2+\sum_{i\in M} d_i y_i^2\right)}{\sqrt{1+\beta}\sqrt{s^2+\sum_{i\in M} d_i y_i^2}}\\ &= \lambda\frac{s^2+\sum_{i\in M} d_i y_i^2}{\sqrt{1+\beta}\sqrt{s^2+\sum_{i\in M} d_i y_i^2}}\\ &= \frac{\lambda}{\sqrt{1+\beta}}\sqrt{s^2+\sum_{i\in M} d_i y_i^2}. \end{aligned} \tag{A.5}$$

Therefore, substituting $pw+qz$ in the objective function of (P_R) by (A.5) and using that $\lambda=\frac{p+q}{2}$, we see that problem (P_R) reduces to

$$(P'_R) \qquad \min\ a'x + b'y + \frac{p+q}{2\sqrt{1+\beta}}\sqrt{s^2+\sum_{i\in M} d_i y_i^2}$$
$$\text{s.t. } (x,s)\in \operatorname{conv}(U_0),\ y\in\mathbb{R}^m_+.$$

Moreover, (P'_R) is of the form of (P_2) in Proposition 5 (after scaling) and, thus, has an optimal integer solution. Therefore, after projecting out the additional variable s, we find the desired result. □

Appendix B

Here, we test the effectiveness of the unbounded polymatroid inequalities (14) and bounded inequalities (28) in solving optimization problems with bounded continuous variables of the form

$$\min\{-a'x - b'y + \Omega z : (x,y,z)\in H_{\mathbb{G}}\}. \tag{B.1}$$

For two numbers $\ell < u$, let $U[\ell,u]$ denote the continuous uniform distribution between ℓ and u. The data for the model is generated as follows: $a_i \sim U[0,1]$, $\sqrt{c_i}\sim U[0.85a_i, 1.15a_i]$ for $i\in N$, $b_j\sim U[0,1]$, $\sqrt{d_j}\sim U[0.85b_j, 1.15b_j]$ for $j\in M$, and Ω is the solution[2] of

$$-a(N) - b(M) + \Omega\sqrt{c(N)+d(M)} = 0.$$

These instances have large integrality gaps with a single conic quadratic constraint.

The unbounded inequalities are added as linear cuts in an extended formulation as described in Section 7.2. The bounded inequalities are either added directly as nonlinear inequalities as described in Section 7.1.1 (bounded-nonlinear) or using outer approximations as described in Section 7.1.2 (bounded gradient). A greedy heuristic is used to choose $T\subseteq M$ for inequalities (28): given a fractional point $(\bar{x},\bar{y},\bar{z})$ with $\bar{y}_{(1)}\ge\bar{y}_{(2)}\ge\ldots\ge\bar{y}_{(m)}$, we check for violation

Table B.1. Experiments with Bounded Continuous Variables

			cpx				Unbounded				Bounded gradient				Bounded nonlinear			
n	m	Igap	Rimp	Nodes	Time	Egap[#]	Rimp	Nodes	Time	Egap[#]	Rimp	Nodes	Time	Egap[#]	Rimp	Nodes	Time	Egap[#]
100	20	1,554.7	0.0	441,520	162	0.0[5]	90.0	9,617	112	0.0[5]	99.7	25	74	0.0[5]	100.0	1	45	0.0[5]
	50	724.6	0.0	2,126,713	1,644	0.0[5]	76.0	853,671	7,200	72.5[0]	99.2	1,985	4,375	0.0[5]	99.9	30	219	0.0[5]
	100	267.8	0.0	8,922,545	6,850	16.6[1]	62.1	726,361	7,200	83.3[0]	81.0	—	7,200	53.8[0]	99.9	55	84	0.0[5]
Average			**0.0**	**3,830,259**	**2,885**	**5.6[11]**	**76.0**	**529,883**	**4,804**	**51.9[5]**	**93.3**	**670**	**3,874**	**17.9[10]**	**100.0**	**29**	**116**	**0.0[15]**
200	40	987.1	0.0	15,133,028	7,200	352.7[0]	89.3	127,408	7,200	72.9[0]	99.5	85	6,253	3.5[3]	100.0	52	475	0.0[5]
	100	396.6	0.0	11,650,607	7,200	397.3[0]	73.9	57,742	7,200	100.7[0]	79.7	—	7,200	133.2[0]	99.9	140	395	0.0[5]
	200	217.6	0.0	4,970,327	7,200	114.4[0]	18.3	1,647,845	7,200	690.5[0]	2.2	2,034,862	7,200	181.6[0]	99.8	183	710	0.0[5]
Average			**0.0**	**10,584,654**	**7,200**	**205.2[0]**	**60.5**	**610,998**	**7,200**	**213.1[0]**	**64.6**	**581,419**	**6,845**	**64.6[3]**	**99.9**	**125**	**527**	**0.0[15]**

Notes. Bold indicates the average of the rows for each n. Igap, initial gap; cpx, default CPLEX without adding any cuts; Rimp, root gap improvement; Egap, end gap.

inequalities for each T_i of the form $T_i = \{(1), (2), \ldots, (i)\}$ for $i = 0, \ldots, m$. When using the nonlinear inequalities bounded nonlinear, we iteratively solve the continuous relaxations and explicitly add the most violated inequality (28) found, and the process is repeated until the relative violation of the inequality found is less than 10^{-3}, that is,

$$\frac{\sqrt{c\left(\sqrt{\sigma + \sum_{i\in T} d_i \bar{y}_i^2} + \pi'\bar{x}\right)^2 + \sum_{i\in M\setminus T} d_i \bar{y}_i^2}}{\bar{z}} - 1 \leq 10^{-3}.$$

This process requires solving many continuous relaxations of (B.1) using the barrier algorithm (which is the default algorithm for convex conic quadratic optimization). For bounded gradient, the inequalities are added at the root node of the branch-and-bound tree using CPLEX callbacks.

Table B.1 presents the results. Each row represents the average over five instances generated with the same parameters and shows the number of discrete (n) and continuous (m) variables, the initial gap (igap), the root gap improvement (rimp), the number of nodes explored (nodes), the time elapsed (including the time used adding the inequalities) in seconds (time), and the end gap (egap)[in brackets, the number of instances solved to optimality (#)]. The initial gap is computed as $\texttt{igap} = \frac{t_{\text{opt}} - t_{\texttt{relax}}}{|t_{\text{opt}}|} \times 100$, where t_{opt} is the objective value of the best feasible solution at termination and t_{relax} is the objective value of the continuous relaxation. The end gap is computed as $\texttt{egap} = \frac{t_{\text{opt}} - t_{\text{bb}}}{|t_{\text{opt}}|} \times 100$, where t_{bb} is the objective value of the best lower bound at termination. The root improvement is computed as $\texttt{rimp} = \frac{t_{\texttt{root}} - t_{\texttt{relax}}}{t_{\text{opt}} - t_{\texttt{relax}}} \times 100$, where t_{root} is the value of the continuous relaxation after adding the valid inequalities to the formulation.

Observe in Table B.1 that the use of the unbounded inequalities, which do not exploit the upper bounds of the continuous variables, closes 68.2% of the initial gap on average, but the gap improvement does not necessarily translate to better solution times or end gaps. The performance of the bounded inequalities, when added as gradients, is adequate when m is small, achieving close to 100% root gap improvement. However, the performance degrades substantially as m increases; in particular, for $m = 100$, the full two hours are spent at the root node adding cuts, and the root improvement of close to 80% is still far from 99.9%, achieved by bounded nonlinear. Moreover, for $n = 200$ and $m = 200$, both unbounded and bounded-gradient inequalities are ineffective at closing the root gap, with root improvements of 18.3% and 2.2%, respectively. In contrast, adding the bounded inequalities as nonlinear inequalities results in all cases in the best performance with root improvements close to 100%, significantly fewer branch-and-bound nodes explored, and better solution times than the other formulations.

Endnotes

[1] For these instances, the strengthened inequalities perform very similarly to polymatroid because the simpler inequalities already achieve close to 100% root gap improvements. Therefore, we only present the results with inequalities polymatroid.

[2] This choice of Ω ensures that the linear and nonlinear components are well balanced, resulting in challenging instances with large integrality gap.

References

Ahmed S, Atamtürk A (2011) Maximizing a class of submodular utility functions. *Math. Programming* 128(1–2):149–169.

Aktürk MS, Atamtürk A, Gürel S (2009) A strong conic quadratic reformulation for machine-job assignment with controllable processing times. *Oper. Res. Lett.* 37(3):187–191.

Aktürk MS, Atamtürk A, Gürel S (2010) Parallel machine match-up scheduling with manufacturing cost considerations. *J. Scheduling* 13(1):95–110.

Amaldi E, Bosio S, Malucelli F, Yuan D (2011) Solving nonlinear covering problems arising in wlan design. *Oper. Res.* 59(1): 173–187.

Amiri A (1997) Solution procedures for the service system design problem. *Comput. Oper. Res.* 24(1):49–60.

Anstreicher KM (2012) On convex relaxations for quadratically constrained quadratic programming. *Math. Programming* 136(2): 233–251.

Atamtürk A, Bhardwaj A (2015) Supermodular covering knapsack polytope. *Discrete Optim.* 18:74–86.

Atamtürk A, Gómez A (2017) Maximizing a class of utility functions over the vertices of a polytope. *Oper. Res.* 65(2):433–445.

Atamtürk A, Gómez A (2018) Strong formulations for quadratic optimization with M-matrices and indicator variables. *Math. Programming* 170(1):141–176.

Atamtürk A, Gómez A (2019a) Rank-one convexification for sparse regression. Preprint, submitted January 29, https://arxiv.org/abs/1901.10334.

Atamtürk A, Gómez A (2019b) Simplex QP-based methods for minimizing a conic quadratic objective over polyhedra. *Math. Programming Comput.* 11(2):311–340.

Atamtürk A, Jeon H (2017) Lifted polymatroid inequalities for mean-risk optimization with indicator variables. *J. Global Optim.* 73(4): 677–699.

Atamtürk A, Narayanan V (2007) Cuts for conic mixed-integer programming. Fischetti M, Williamson DP, eds. *Integer Programming and Combinatorial Optimization* (Springer, Berlin, Heidelberg), 16–29.

Atamtürk A, Narayanan V (2008) Polymatroids and mean-risk minimization in discrete optimization. *Oper. Res. Lett.* 36(5):618–622.

Atamtürk A, Narayanan V (2009) The submodular knapsack polytope. *Discrete Optim.* 6(4):333–344.

Atamtürk A, Narayanan V (2010) Conic mixed-integer rounding cuts. *Math. Programming* 122(1):1–20.

Atamtürk A, Narayanan V (2011) Lifting for conic mixed-integer programming. *Math. Programming* 126(2):351–363.

Atamtürk A, Berenguer G, Shen Z-J (2012) A conic integer programming approach to stochastic joint location-inventory problems. *Oper. Res.* 60(2):366–381.

Atamtürk A, Deck C, Jeon H (2019) Successive quadratic upper-bounding for discrete mean-risk minimization and network interdiction. *INFORMS J. Comput.*, ePub ahead of print, October 19, https://doi.org/10.1287/ijoc.2018.0870.

Atamtürk A, Gómez A, Han S (2018) Sparse and smooth signal estimation: Convexification of l0 formulations. BCOL Research Report 18.05, IEOR, University of California, Berkeley, Berkeley.

Belotti P, Góez JC, Pólik I, Ralphs TK, Terlaky T (2015) A conic representation of the convex hull of disjunctive sets and conic cuts for integer second order cone optimization. Al-Baali M, Grandinetti L, Purnama A, eds. *Numerical Analysis and Optimization*, Springer Proceedings in Mathematics and Statistics, vol. 134 (Springer, Cham, Switzerland), 1–35.

Ben-Tal A, Nemirovski A (1998) Robust convex optimization. *Math. Oper. Res.* 23(4):769–805.

Ben-Tal A, Nemirovski A (1999) Robust solutions of uncertain linear programs. *Oper. Res. Lett.* 25(1):1–13.

Ben-Tal A, Nemirovski A (2001) On polyhedral approximations of the second-order cone. *Math. Oper. Res.* 26(2):193–205.

Ben-Tal A, El Ghaoui L, Nemirovski A (2009) *Robust Optimization*, Princeton Series in Applied Mathematics. (Princeton University Press, Princeton, NJ).

Berman O, Krass D (2001) Facility location problems with stochastic demands and congestion. Drezner Z, Hamacher HW, eds. *Facility Location Applications and Theory* (Springer, Berlin), 329–372.

Bertsimas D, Sim M (2003) Robust discrete optimization and network flows. *Math. Programming* 98(1–3):49–71.

Bertsimas D, Sim M (2004) The price of robustness. *Oper. Res.* 52(1): 35–53.

Birge JR, Louveaux F (2011) *Introduction to Stochastic Programming* (Springer Science & Business Media, New York).

Bollapragada R, Rao U (1999) Single-stage resource allocation and economic lot scheduling on multiple, nonidentical production lines. *Management Sci.* 45(6):889–904.

Bonami P (2011) Lift-and-project cuts for mixed integer convex programs. Günlük O, Woeginger GJ, eds. *Integer Programming and Combinatoral Optimization—IPCO 2011*, Lecture Notes in Computer Science, vol. 6655 (Springer, Berlin, Heidelberg), 52–64.

Borrero JS, Gillen C, Prokopyev OA (2016a) Fractional 0–1 programming: Applications and algorithms. *J. Global Optim.* 69(1):1–28.

Borrero JS, Gillen C, Prokopyev OA (2016b) A simple technique to improve linearized reformulations of fractional (hyperbolic) 0–1 programming problems. *Oper. Res. Lett.* 44(4):479–486.

Bront JJM, Méndez-Díaz I, Vulcano G (2009) A column generation algorithm for choice-based network revenue management. *Oper. Res.* 57(3):769–784.

Buchheim C, De Santis M (2019) An active set algorithm for robust combinatorial optimization based on separation oracles. *Math. Programming Comput.* Forthcoming.

Bulut O, Tasgetiren MF (2014) An artificial bee colony algorithm for the economic lot scheduling problem. *Internat. J. Production Res.* 52(4):1150–1170.

Burer S, Kılınç-Karzan F (2017) How to convexify the intersection of a second order cone and a nonconvex quadratic. *Math. Programming* 162(1–2):393–429.

Castro PM, Barbosa-Póvoa AP, Novais AQ (2005) Simultaneous design and scheduling of multipurpose plants using resource task network based continuous-time formulations. *Indust. Engrg. Chemical Res.* 44(2):343–357.

Castro PM, Westerlund J, Forssell S (2009) Scheduling of a continuous plant with recycling of byproducts: A case study from a tissue paper mill. *Comput. Chemical Engrg.* 33(1):347–358.

Ceria S, Soares J (1999) Convex programming for disjunctive convex optimization. *Math. Programming* 86(3):595–614.

Çezik MT, Iyengar G (2005) Cuts for mixed 0-1 conic programming. *Math. Programming* 104(1):179–202.

Cormican KJ, Morton DP, Wood RK (1998) Stochastic network interdiction. *Oper. Res.* 46(2):184–197.

Dadush D, Dey SS, Vielma JP (2011a) The Chvátal-Gomory closure of a strictly convex body. *Math. Oper. Res.* 36(2):227–239.

Dadush D, Dey SS, Vielma JP (2011b) The split closure of a strictly convex body. *Oper. Res. Lett.* 39(2):121–126.

Dadush D, Dey S, Vielma JP (2014) On the Chvátal-Gomory closure of a compact convex set. *Math. Programming* 145(1–2):327–348.

Davidoff G, Sarnak P, Valette A (2003) *Elementary Number Theory, Group Theory and Ramanujan Graphs*, London Mathematical Society Student Texts, vol. 55 (Cambridge University Press, Cambridge, UK).

Désir A, Goyal V, Zhang J (2014) Near-optimal algorithms for capacity constrained assortment optimization. Working paper, Columbia University, New York.

Dong H, Chen K, Linderoth J (2015) Regularization vs. relaxation: A conic optimization perspective of statistical variable selection. Preprint, submitted October 20, https://arxiv.org/abs/1510.06083.

Edmonds J (1970) Submodular functions, matroids, and certain polyhedra. Guy R, Hanani H, Sauer N, Schönenheim J, eds. *Combinatorial Structures and Their Applications* (Gordon and Breach, New York), 69–87.

El Ghaoui L, Oks M, Oustry F (2003) Worst-case value-at-risk and robust portfolio optimization: A conic programming approach. *Oper. Res.* 51(4):543–556.

Elhedhli S (2005) Exact solution of a class of nonlinear knapsack problems. *Oper. Res. Lett.* 33(6):615–624.

Elhedhli S (2006) Service system design with immobile servers, stochastic demand, and congestion. *Manufacturing Service Oper. Management* 8(1):92–97.

Frangioni A, Claudio G, James H (2020) Decompositions of semidefinite matrices and the perspective reformulation of nonseparable quadratic programs. *Math. Oper. Res.* 45(1):15–33.

Fujishige S (2005) *Submodular Functions and Optimization*, vol. 58 (Elsevier, Amsterdam).

Gilmore PC, Gomory RE (1963) A linear programming approach to the cutting stock problem—part II. *Oper. Res.* 11(6):863–888.

Gómez A (2018) Strong formulations for conic quadratic optimization with indicator variables. *Optim. Online* (May 9), http://www.optimization-online.org/DB_HTML/2018/05/6616.html.

Grötschel M, Lovász L, Schrijver A (1981) The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica* 1(2):169–197.

Günlük O, Linderoth J (2010) Perspective reformulations of mixed integer nonlinear programs with indicator variables. *Math. Programming* 124(1–2):183–205.

Hijazi H, Bonami P, Ouorou A (2013) An outer-inner approximation for separable mixed-integer nonlinear programs. *INFORMS J. Comput.* 26(1):31–44.

Hochbaum DS (2010) Polynomial time algorithms for ratio regions and a variant of normalized cut. *IEEE Trans. Pattern Anal. Machine Intelligence* 32(5):889–898.

Hochbaum DS, Lyu C, Bertelli E (2013) Evaluating performance of image segmentation criteria and techniques. *EURO J. Comput. Optim.* 1(1–2):155–180.

Israeli E, Wood RK (2002) Shortest-path network interdiction. *Networks* 40(2):97–111.

Jeon H, Linderoth J, Miller A (2017) Quadratic cone cutting surfaces for quadratic programs with on–off constraints. *Discrete Optim.* 24:32–50.

Kılınç M, Linderoth J, Luedtke J (2010) Effective separation of disjunctive cuts for convex mixed integer nonlinear programs. *Optim. Online* (August 21), http://www.optimization-online.org/DB_FILE/2010/11/2808.pdf.

Kılınç-Karzan F (2015) On minimal valid inequalities for mixed integer conic programs. *Math. Oper. Res.* 41(2):477–510.

Kılınç-Karzan F, Yıldız S (2015) Two-term disjunctions on the second-order cone. *Math. Programming* 154(1–2):463–491.

Lim C, Smith JC (2007) Algorithms for discrete and continuous multicommodity flow network interdiction problems. *IIE Trans.* 39(1):15–26.

Lovász L (1983) Submodular functions and convexity. Bachem A, Korte B, Grötschel M, eds. *Mathematical Programming The State of the Art: Bonn 1982* (Springer, Berlin, Heidelberg), 235–257.

Lubin M, Yamangil E, Bent R, Vielma JP (2018) Polyhedral approximation in mixed-integer convex optimization. *Math. Programming* 172(1–2):139–168.

Méndez-Díaz I, Miranda-Bront JJ, Vulcano G, Zabala P (2014) A branch-and-cut algorithm for the latent-class logit assortment problem. *Discrete Appl. Math.* 164(1):246–263.

Modaresi S, Vielma JP (2014) Convex hull of two quadratic or a conic quadratic and a quadratic inequality. *Math. Programming* 164(1–2): 383–409.

Modaresi S, Kılınç MR, Vielma JP (2016) Intersection cuts for nonlinear integer programming: Convexification techniques for structured sets. *Math. Programming* 155(1–2):575–611.

Nahmias S, Cheng Y (2005) *Production and Operations Analysis*, vol. 6 (McGraw-Hill, New York).

Nikolova E, Kelner J, Brand M, Mitzenmacher M (2006) Stochastic shortest paths via quasi-convex maximization. Azar Y, Erlebach T, eds. *Algorithms—ESA 2006*, Lecture Notes in Computer Science, vol. 4168 (Springer, Berlin, Heidelberg), 552–563.

Orlin JB (2009) A faster strongly polynomial time algorithm for submodular function minimization. *Math. Programming* 118(2):237–251.

Özsen L, Coullard CR, Daskin MS (2008) Capacitated warehouse location model with risk pooling. *Naval Res. Logist.* 55(4):295–312.

Pesenti R, Ukovich W (2003) Economic lot scheduling on multiple production lines with resource constraints. *Internat. J. Production Econom.* 81:469–481.

Poljak S, Wolkowicz H (1995) Convex relaxations of (0, 1)-quadratic programming. *Math. Oper. Res.* 20(3):550–561.

Prokopyev OA, Kong N, Martinez-Torres DL (2009) The equitable dispersion problem. *Eur. J. Oper. Res.* 197(1):59–67.

Prokopyev OA, Meneses C, Oliveira CA, Pardalos PM (2005) On multiple-ratio hyperbolic 0–1 programming problems. *Pacific J. Optim.* 1(2):327–345.

Sahinidis N, Grossmann IE (1991) MINLP model for cyclic multiproduct scheduling on continuous parallel lines. *Comput. Chemical Engrg.* 15(2):85–103.

Santana A, Dey SS (2017) Some cut-generating functions for second-order conic sets. *Discrete Optim.* 24:51–65.

Schrijver A (2000) A combinatorial algorithm minimizing submodular functions in strongly polynomial time. *J. Combin. Theory Ser. B* 80(2):346–355.

Şen A, Atamtürk A, Kaminsky P (2018) A conic integer programming approach to constrained assortment optimization under the mixed multinomial logit model. *Oper. Res.* 66(4):994–1003.

Sharpe WF (1994) The Sharpe ratio. *J. Portfolio Management* 21: 49–58.

Shen Z-JM, Coullard C, Daskin MS (2003) A joint location-inventory model. *Transportation Sci.* 37(1):40–55.

Stubbs AR, Mehrotra S (1999) A branch-and-cut method for 0-1 mixed convex programming. *Math. Programming* 86(3):515–532.

Tawarmalani M, Sahinidis NV (2005) A polyhedral branch-and-cut approach to global optimization. *Math. Programming* 103(2): 225–249.

Tawarmalani M, Ahmed S, Sahinidis NV (2002) Global optimization of 0-1 hyperbolic programs. *J. Global Optim.* 24(4):385–416.

Wood RK (1993) Deterministic network interdiction. *Math. Comput. Model.* 17(2):1–18.

Yu J, Ahmed S (2017) Polyhedral results for a class of cardinality constrained submodular minimization problems. *Discrete Optim.* 24:87–102.

Zhang Y, Jiang R, Shen S (2018) Ambiguous chance-constrained binary programs under mean-covariance information. *SIAM J. Optim.* 28(4):2922–2944.

Alper Atamtürk is a professor of industrial engineering and operations research at the University of California, Berkeley. His research interests are in optimization, integer programming, optimization under uncertainty with applications to energy, machine learning, portfolio and network design. He is a National Security Science and Engineering Faculty Fellow (NSSEFF) of the U.S. Department of Defense.

Andrés Gómez is an assistant professor in the Department of Industrial and Systems Engineering at the University of Southern California. His research interests are in nonlinear mixed-integer optimization and its applications, with special focus in applications in machine learning, finance, and revenue management.